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Stable Components of Wild Tilted Algebras

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DEDICATED TO MY TEACHER A. BERGMANN ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

If A is a finite dimensional, connected, wild, hereditary algebra and T is a tilting module in $A-mod$ with corresponding tilted algebra $B=\text{End}_A(T)$, then T defines a torsion theory $(\mathscr{F}(T), \mathscr{G}(T))$ in $A-mod$ and a splitting torsion theory $(\mathscr{Y}(T), \mathscr{X}(T))$ in $B-mod$. The torsion-free class $\mathscr{F}(T)$ ($\mathscr{Y}(T)$, respectively) is defined by

$$\mathscr{F}(T)=\{X \mid \text{Hom}_A(T,X)=0\} \qquad (\mathscr{Y}(T)=\{Y \mid \text{Tor}_1^B(T,X)=0\}).$$

The torsion classes are defined by

$$\mathscr{G}(T)=\{X \mid \text{Ext}_A^1(T,X)=0\} \qquad \text{and} \qquad \mathscr{X}(T)=\{Y \mid T\otimes_B Y=0\}.$$

By the theorem of Brenner–Butler, the functor $F=\text{Hom}_A(T,-)$ defines an equivalence between $\mathscr{G}(T)$ and $\mathscr{Y}(T)$ whereas the functor $F'=\text{Ext}_A^1(T,-)$ defines an equivalence from $\mathscr{F}(T)$ to $\mathscr{X}(T)$. Normally we assume T to be square-free. Our basic references on tilting theory are [1, 5, 13] and we will use the standard results without comments.

Ringel proved in [15] that all regular components in the Auslander–Reiten quiver $\Gamma(B)$ of B, except the connecting component which also may be regular (if T is a regular tilting module), are quasiserial and tubes or of type $\mathbb{Z}A_\infty$. In [17] it was proved that B has preprojective and preinjective components. Since the torsion theory $(\mathscr{Y}(T), \mathscr{X}(T))$ splits, we may restrict ourselves to $\mathscr{Y}(T)$ (or $\mathscr{X}(T)$). By [8] it is enough for the study of $\mathscr{Y}(T)$ to consider the case that T has no preinjective direct summands. All stable components (in $\mathscr{Y}$) then are of type $\mathbb{Z}A_\infty$. More precise: If $\mathscr{D}$ is a stable component in $\mathscr{Y}(T)$, then there exists an indecomposable regular A-module X such that $(\to X)$ is mapped by F to $(\to F(X))$, with $F(X)\in\mathscr{D}$. The symbol

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$(\to Y)$ $((Y\to)$, respectively) denotes the full subquiver of predecessors (successors, respectively) of Y in the component containing $[Y]$.

The tilting module T has a decomposition $T = T_p \otimes T_1$ such that $F(T_p)$ is preprojective in $B-mod$ and $F(T_1)$ has no preprojective direct summand. By [16, 17], $\mathrm{End}(T_p) = C$ is a concealed wild connected algebra and the preprojective component of $\Gamma(C)$ is the preprojective component of $\Gamma(B)$. The main result of the paper is the following

THEOREM 1. *If A is a wild hereditary connected algebra, if $T = T_p \oplus T_1$ is a tilting module without preinjective direct summands, we set $B = \mathrm{End}_A(T)$ and $C = \mathrm{End}_A(T_p)$. Then we have that if $\mathscr{C}$ is a component in $\mathscr{Y}(T) \subset B-mod$ different from the connecting component, then there exists an indecomposable module $Z \in \mathscr{C}$ such that $(Z\to)$ is in $C-mod$.*

As a consequence of Theorem 1 we will see, that with the help of tilting modules we can construct bijections between the regular components of any two wild hereditary connected algebras (Theorem 3 and Corollary 4.1). C. M. Ringel's pertinent comments and suggestions, based on my talks, given at Bielefeld and on preliminary versions of this paper have influenced this work essentially.

Notations. The word algebra always denotes a finite dimensional, unitary (basic, connected) algebra over some commutative field k. If A is an algebra, $A-mod$ denotes the category of finitely generated left A-modules. The standard duality $\mathrm{Hom}_k(-, k)$ is denoted by D and morphisms are written on the opposite side of the scalars.

If $\mathscr{X}$ is a class of modules, we denote by add $\mathscr{X}$ the full subcategory of $A-mod$ consisting of direct sums of summands of elements of $\mathscr{X}$.

We call a module X a *brick* if $\mathrm{End}(X)$ is a division-ring. Bricks without self-extensions and projective dimension at most 1 are called *stones*. By $\Gamma(A)$ we denote the Auslander–Reiten quiver of A. The vertices of $\Gamma(A)$ are isomorphism classes of indecomposable A-modules; if the context is clear, we will not distinguish between an indecomposable module X and its class $[X]$. The Auslander–Reiten translations will be denoted by τ and τ^-; to emphasise the algebra we sometimes add a subscript, for example τ_A, τ_C^-. In general we follow the notations used in [13].

1. MORPHISMS IN REGULAR COMPONENTS

LEMMA 1.1. *Let A be a finite dimensional, connected wild hereditary algebra and X a regular indecomposable A-module.*

(a) *For positive r there exist neither monomorphisms nor epimorphisms in* $\mathrm{Hom}(X, \tau^{-r}X)$.

(b) *If* $\operatorname{Hom}(X, \tau^{-r}X)$ *is nonzero, then* $\operatorname{Hom}(X, r^{-s}X) \neq 0$ *for all* $0 \leqslant s \leqslant r$.

Proof. (a) Since A is hereditary, τ is a left exact functor. If $f: X \to \tau^{-r}X$ is injective, also $\tau^{lr}f: \tau^{lr}X \to \tau^{(l-1)r}X$ is injective for all $l \in \mathbb{N}$, which gives a strictly decreasing chain

$$\underline{\dim}\, \tau^{-r}X > \underline{\dim}\, X > \underline{\dim}\, \tau^{r}X > \cdots > \underline{\dim}\, \tau^{lr}X > \cdots$$

a contradiction. For $f: X \to \tau^{-r}X$ surjective we dually consider the right exact functors τ^{-lr}.

(b) We prove that $\operatorname{Hom}(X, \tau^{-r}X) \neq 0$ implies $\operatorname{Hom}(X, \tau^{-r+1}X) \neq 0$: If $\operatorname{Hom}(X, \tau^{-r+1}X) = 0$, we get from [5, (4.1)] that any nonzero map $f: X \to \tau^{-r}X$ is an epimorphism or a monomorphism, a contradiction to part (a).

LEMMA 1.2. *Let X be a quasi-simple regular brick. Then* $\operatorname{Hom}(X, \tau^{-l}X) = 0$ *for all* $l \geqslant 1$.

Proof. The proof, which is due to C. M. Ringel, uses an idea, given in [5, (4.1)]. By (1.1) it is enough show that $\operatorname{Hom}(X, \tau^{-}X) = 0$. If there is an $0 \neq f: X \to \tau^{-}X$ then again by (1.1), f is neither a monomorphism nor an epimorphism. Let $Z = (X)f$ and $u: X \to Z$ the induced map, $U = \tau^{-}X/Z$ and consider the short exact sequence

$$0 \longrightarrow Z \xrightarrow{\ e\ } \tau^{-}X \xrightarrow{\ \pi\ } U \longrightarrow 0.$$

Since A is hereditary $\operatorname{Ext}^1(U, u)$ is epic, so we get the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\ \alpha\ } & L & \longrightarrow & U & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \| & & \\
0 & \longrightarrow & Z & \xrightarrow{\ e\ } & \tau^{-}X & \longrightarrow & U & \longrightarrow & 0
\end{array}$$

The short exact sequence

$$0 \longrightarrow X \xrightarrow{(\alpha, u)} L \oplus Z \xrightarrow{\binom{-v}{e}} \tau^{-}X \longrightarrow 0$$

is non-split with decomposable middle term. Since the Auslander–Reiten sequence $0 \to X \to E \to \tau^{-}X \to 0$ is in the socle of $\operatorname{Ext}(\tau^{-}X, X)$, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & L \oplus Z & \longrightarrow & \tau^{-}X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & \tau^{-}X & \longrightarrow & 0
\end{array}$$

f is nonzero, so f is an automorphism as X is a brick. Thus we have $L \oplus Z \cong E$, but E is indecomposable, a contradiction.

If X is a quasi-simple regular module, we denote by $X(m)$ the indecomposable regular module with quasi-length m and regular socle X, whereas $[m]X$ denotes the regular module with regular top X and quasi-length m. If $[m]X$ is contained in a regular component C of the Auslander–Reiten quiver $\Gamma(A)$ of A, we denote, following [13, (3.3)] by $\mathscr{W}([m]X)$ the wing of $[m]X$ (of length m), that is, the mesh-complete full subquiver of $\mathscr{C}$, defined by the vertices $\tau^l([s]X)$ with $1 \leqslant s \leqslant m$ and $0 \leqslant l \leqslant m-s$. By add $\mathscr{W}([m]X)$ we denote the full additive subcategory of $A-mod$ defined by the indecomposable modules corresponding to the vertices of the wing $\mathscr{W}([m]X)$.

A basic tool for the sequel will be Ringel's results on extensions of homomorphisms between regular modules, see [12, Sect. 4]. Quite frequently we will use his result (4.1) or its dual version. For the convenience of the reader we formulate this result:

LEMMA 1.3. *Let $X(s)$ be an indecomposable regular module and Y indecomposable with $Y \not\cong X(i)/X(j)$ for $0 \leqslant j < i$ and $i \geqslant i_0$. Then any homomorphism $f\colon X(i_0) \to Y$ has an extension to $g\colon X(s) \to Y$.*

Immediately from this result it follows:

LEMMA 1.4. *Let $U = [r]X$ be a indecomposable regular module of quasi-length r and regular top X and let be $\mathscr{W}(U)$ the wing of U. Then we have*

(a) *For an indecomposable module Y not in* add $\mathscr{W}(U)$ *the following conditions are equivalent*:

(1) $\operatorname{Hom}(Y, U) = 0$,

(2) $\operatorname{Hom}(Y, \tau^i X) = 0$ *for* $i = 0, ..., r-1$

(3) $\operatorname{Hom}(Y, W) = 0$ *for all* $W \in$ add $\mathscr{W}(U)$.

(b) *For an indecomposable module Z not in* add $\mathscr{W}(U)$ *there are equivalent*:

(1) $\operatorname{Hom}(U, Z) = 0$

(2) $\operatorname{Hom}(\tau^i X, Z) = 0$ *for* $i = 0, ..., r-1$

(3) $\operatorname{Hom}(W, Z) = 0$ *for all* $W \in$ add $\mathscr{W}(U)$.

Let us now consider wings with special properties

LEMMA 1.5. *Let X be a quasi-simple regular module and $m \in \mathbb{N}$. For the wing $\mathscr{W}([m]X)$ there are equivalent*:

(a) $X, \tau X, ..., \tau^{m-1}X$ *are pairwise orthogonal.*

(b) *If* $Z, Y \in \operatorname{add} \mathscr{W}([m]X)$, *then* $rad^{\infty}(Z, Y)=0$.

Proof. (b) $\Rightarrow$ (a) is trivial.

(a) $\Rightarrow$ (b). Let be $f: Z \to Y$. We may assume that Y and Z are indecomposable, say $Z=U(r)=[r]U'$ and $Y=[s]V=V'(s)$ with U, U', V, V' quasi-simple. We prove the assertion by induction on r and s.

$r=1$. The case $s=1$ follows from (a). For $s>1$ consider the canonical projection $\pi: Y \to V$ with $\pi \in rad^{s}(Y, V)$. If $f\pi: U \to V$ is nonzero, it is an isomorphism and therefore π splits, a contradiction to $s>1$. If $f\pi=0$, then f has a factorisation $f=\bar{f}\varepsilon$ with $\bar{f}: Z \to V'(s-1)$ and $\varepsilon: V'(s-1) \to Y$ irreducible. By induction we have $rad^{\infty}(Z, V'(s-1))=0$, which gives the assertion.

Notice that for $r=1$ this proof also says that $V'=U$ and that $\operatorname{Hom}(Z, V)$ consists of the $\operatorname{End}(Z)$-scalar multiples of the inclusion $Z=V' \to V'(s)$.

$r>1$. For $\varepsilon: U \to U(r)=Z$, $\varepsilon \in rad^{r}(U, Z)$ the canonical injection, we consider the composition $\varepsilon f: U \to Y=V'(s)$. If $\varepsilon f \neq 0$ then by the first part we necessarily have $V'=U$ and $\varepsilon f \in rad^{s}(U, Y)$ and thus $f \in rad^{s-r}(Z, Y)$. For $\varepsilon f=0$ the map f has the factorisation $f=\pi\bar{f}$ with irreducible map $\pi: Z \to [r-1]U'$. By induction we know that $rad^{\infty}([r-1]U', Y)=0$, and thus the proof is finished.

We call a wing $\mathscr{W}([m]X)$ a *standard wing* if it satisfies the equivalent conditions of (1.5). Let us emphasise that all indecomposables in a standard wing $\mathscr{W}([m]X)$ are bricks and moreover, except $[m]X$ possibly, they are yet stones—by the Auslander–Reiten formula $\operatorname{Ext}(Y, Y) \cong D\operatorname{Hom}(Y, \tau Y)$.

If X is an indecomposable regular stone, it was proved in [7, (2.6)] that the quasi-length $l(X)$ of X is at most $n-2$, where n is the number of simple A-modules and it was remarked there, that there exist algebras with stones of quasi-length $n-2$. An example is given in [17]; Strauss also mentioned that for the wild hereditary *m-subspace* algebra all regular stones are quasi-simple. Let us now consider an indecomposable regular module $Y=[r]X$ with X quasi-simple. We get

PROPOSITION 1.6. *Let* $Y=[r]X$ *be a regular module. Then* $\mathscr{W}([r+1]X)$ *is a standard wing, if and only if* Y *is a stone.*

Proof. If $\operatorname{Ext}(Y, Y) \neq 0$ then we get

$$0 \neq \operatorname{Hom}(Y, \tau Y)=rad^{\infty}(Y, \tau Y).$$

Y and τY are both contained in the wing $\mathscr{W}([r+1]X)$, which therefore is not standard.

Assume now $\operatorname{Ext}(Y, Y)=0$. It follows for example from [7] that also the modules $[s]Y$ with $1 \leqslant s \leqslant r$ are stones. So we can use induction on r.

If $r=1$, that is, $Y=X$, we get $\operatorname{Hom}(X, \tau X)=0$ by the Auslander–Reiten formula and $\operatorname{Hom}(\tau X, X)=0$ by Lemma 1.2. Therefore by (1.5), $\mathscr{W}([2]X)$ is a standard wing.

If $r>1$ then $[r-1]X$ and $\tau([r-1]X)$ are stones and thus by induction $\mathscr{W}([r]X)$ and $\mathscr{W}(\tau([r]X))$ are standard wings. Therefore by Lemma 1.5 we know that $\operatorname{Hom}(\tau^i X, \tau^j X)=0$ for $|j-i|<r$.

By (1.2) we have $\operatorname{Hom}(\tau^j X, \tau^i X)=0$ for all $j>i$. Therefore it remains to show that $\operatorname{Hom}(X, \tau^r X)=0$:

LEMMA 1.7. *Let X be a regular quasi-simple module such that $[r]X=Y$ is a stone. Then we have*

$$\operatorname{Hom}(X, \tau^{r+1}X) \cong \operatorname{Hom}([r+1]X, \tau([r+1]X)) \cong D\operatorname{Ext}([r+1]X, [r+1]X).$$

Proof. Lemma 1.7 is only a slight variation of [7, (2.7)]. We fix the following notations: For $Z=[r+1]X$, $p: Z \to X$ denotes the surjective irreducible path and $e: \tau^{r+1}X \to \tau Z$ the injective irreducible path. Let

$$\alpha: \operatorname{Hom}(X, \tau^{r+1}X) \to \operatorname{Hom}(Z, \tau Z)$$

be defined by $\alpha(f)=pfe$; clearly α is injective. The surjectivity of α is proved in [7, (2.7)].

COROLLARY 1.8. *Let $X=[r]Y$ be a regular brick with quasi-top Y. Then $rad^{\infty}(X, \tau^{-l}X)=0$ for all $l \geqslant 0$. Especially we get* $\operatorname{Hom}(X, \tau^{-l}X)=0$ *for all $l \geqslant r$; for $0 \leqslant s < r$ the* $\operatorname{End}(X)$*-vectorspace* $\operatorname{Hom}(X, \tau^{-s}X)$ *is one-dimensional and generated by an irreducible path from X to $\tau^{-s}X$.*

Proof. Consider the standard wings $\mathscr{W}_1=\mathscr{W}(X)$ and $\mathscr{W}_2=\mathscr{W}(\tau^{-l}X)$. By (1.4), $\operatorname{Hom}(X, \tau^{-l}X) \neq 0$ if and only if there exist quasi-simple modules $X_i \in \mathscr{W}_i$ such that $\operatorname{Hom}(X_1, X_2) \neq 0$. By Lemma 1.2 and Proposition 1.6 this exactly occurs if $X_1=X_2$, that is, if and only if $\mathscr{W}=\mathscr{W}_1 \cap \mathscr{W}_2$ is non-empty. This especially implies that $\operatorname{Hom}(X, \tau^{-l}X)=0$ for all $l \geqslant r$. For $s<r$ one easily checks that each $f \in \operatorname{Hom}(X, \tau^{-s}X)$ is an $\operatorname{End}(X)$-multiple of the irreducible path from X to $\tau^{-s}X$ passing the top of the wing $\mathscr{W}$.

2. NON-REGULAR COMPONENTS

If A is hereditary, T a tilting module, and $B=\operatorname{End}_A(T)$ a tilted algebra (of type A), then the torsion-theory $(\mathscr{Y}(T), \mathscr{X}(T))$ in $B-mod$ splits. Therefore the study of $B-mod$ can be reduced to the study of $\mathscr{Y}(T)$ ($\mathscr{X}(T)$, respectively) and the study of morphisms from $\mathscr{Y}$ to $\mathscr{X}$.

Let us consider the first step: By [8] for $\mathscr{Y}(T)$ it is enough, to consider a tilting module T (more precise, a family of tilting modules) without non-zero preinjective direct summands. Thus let A be connected, wild, and hereditary with $n=n(A)$ simple modules, T a tilting module without pre-injective direct summands and $B=\mathrm{End}_A(T)$. Additionally we assume that T has regular direct summands, otherwise B is a concealed algebra.

In $\mathscr{Y}$ we have the following Auslander–Reiten components: First, the preinjective component of $\Gamma(A)$ is contained in $\mathscr{Y}$—it is part of the connecting component in $\Gamma(B)$.

By [16, 17], $\Gamma(B)$ has exactly one preprojective component $\mathscr{P}(B)$ with $r<n$ projective vertices. $\mathscr{P}(B)$ is the preprojective component of a wild concealed algebra. It can be expressed in the following way: T has a decomposition $T=T_p\oplus T_1$, such that $T_p\in T_1^{\perp}$ is a preprojective $T_1^{\perp}$-tilting module, and a projective B-module $F(X)$ is preprojective if and only if X is in add T_p, see [17, Theorem 9.5]. It should be mentioned that T_1 is a regular A-module. $T_1^{\perp}$ hereby denotes the *right perpendicular category* of T_1, that is, $T_1^{\perp}$ is the full subcategory of $A-mod$ defined by the modules X with $\mathrm{Hom}_A(T_1, X)=0$ and $\mathrm{Ext}_A^1(T_1, X)=0$, see [4]. Since T_1 is regular, $T_1^{\perp}\cong C'-mod$, with C' connected wild hereditary, see [17, (9.5)].

Further, there exist regular components in $\mathscr{Y}$. If $\mathscr{C}$ is a regular component in $\mathscr{Y}$, then $\mathscr{C}$ is a quasi-serial component of type $\mathbb{Z}A_{\infty}$, see [8, 15].

Finally we have additional components besides the preprojective component containing projective vertices, since T is not preprojective. By [8] the stable part of such a component is either empty or a disjoint union of components of type $\mathbb{Z}A_{\infty}$. If $\mathscr{D}$ is a regular component in $\Gamma(A)$, then there exists a quasi-simple module $S\in\mathscr{D}$ such that $(\to S)\subset\mathscr{G}$ and $F(\to S)=(\to F(S))$ is a full part of a stable component of $\mathscr{Y}(T)$. Moreover each stable component of $\mathscr{Y}(T)$ is of this type; that is, we have a bijection between the regular components in $A-mod$ and the stable components in $\mathscr{Y}(T)$, see [8].

If W is an indecomposable regular direct summand of a tilting module T with quasi-length s, then by [13, (4.4)] in the wing $\mathscr{W}(W)$ of W there are always s different indecomposable direct summands of T and they form a branch of length s. Thus we have a decomposition $T=T(W)\oplus T_2$, where $T(W)$ is a tilting set of the wing $\mathscr{W}(W)$. Moreover we have:

LEMMA 2.1. *Let W be an indecomposable regular stone of quasi-length s and $T(W)$ a tilting set in the wing $\mathscr{W}$ of W. Then the category $T(W)^{\perp}$ depends only on W: $T(W)^{\perp}$ is generated by the indecomposable modules $X\in W^{\perp}$ which are not in* add $\mathscr{W}(W)$.

Proof. If X is indecomposable in $T(W)^{\perp}$, then by [13, (4.4)], X clearly is neither in add $\mathcal{W}$ nor in add $\mathcal{W}(\tau W)$. Since W is a direct summand of $T(W)$, by (1.4) we see that $\operatorname{Hom}(T(W), X)=0$ if and only if $\operatorname{Hom}(W, X)=0$ and $\operatorname{Ext}(T(W), X) \cong D\operatorname{Hom}(X, \tau T(W))=0$ if and only if $\operatorname{Ext}(W, X)=0$.

In the sequel we consider a square-free tilting module T without preinjective direct summand. By [17] we have the decomposition $T=T_p \oplus T_1$, with $T_p \in T_1^{\perp}$. By Lemma 2.1 we see that for an indecomposable direct summand W of T_1 no summand of T_p is contained in the wing $\mathcal{W}(W)$. If W_1 and W_2 are different summands of T_1 such that $\mathcal{W}(W_i) \not\subset \mathcal{W}(W_j)$ for $i \neq j$, then $\mathcal{W}(W_1) \cap \mathcal{W}(W_2)=\varnothing$, since all epimorphic images of W_1 and W_2 are contained in $\mathcal{G}(T)$ and all submodules of τW_i are torsion-free. Therefore for T we have the decomposition

$$T=T_p \oplus \bigoplus_{i=1}^{l} T(M_i)$$

such that $T(M_i)$ is a tilting set in the wing $\mathcal{W}(M_i)$ with pairwise different wings $\mathcal{W}(M_i)$.

Defining $T'(M_j)=T_p \oplus \bigoplus_{i \neq j} T(M_i)$, we see from Lemma 2.1 that there exist some M_j such that $T'(M_j) \in T(M_j)^{\perp}$, since the ordinary quiver $\mathcal{Q}(B)$ of B has no oriented cycles. Let $\{W_i \mid 1 \leqslant i \leqslant r\}$ be the set of these M_j's and let $\{V_1 \cdots V_s\}$ be the others. Then we have

$$T=T_p \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j)$$

with $T'(W_j) \in T(W_j)^{\perp}$, $T'(V_i) \notin T(V_i)^{\perp}$.

For each V_i then there exists a sequence of nonzero maps $(f_l)_{1 \leqslant l \leqslant t}$ with

$$V_i=V_{i_1} \xrightarrow{f_1} V_{i_2} \longrightarrow \cdots \longrightarrow V_{i_t} \xrightarrow{f_t} W_j$$

for some W_j.

Let us call the above decomposition of T the *wing decomposition* of T. Since the indecomposable direct summands of $T(V_i)$ ($T(W_j)$, respectively) form a branch, we see that the component of $\Gamma(B)$ containing $F(V_i)$ ($F(W_j)$, respectively) contains the whole branch $F(T(V_i))$ ($F(T(W_j))$, respectively).

EXAMPLE. Let A be wild connected hereditary with three simple modules and T a tilting module without preinjective direct summand, not preprojective. Then the wing decomposition of T is $T=W \oplus T_p$ with W quasi-simple, see [10, 18].

The main result of this section will be a more detailed description of some of the components of $\Gamma(B)$ containing a branch $T(W)$. This was done for wild hereditary algebras with three simple modules in [10] and we will use similar arguments here.

If X is an indecomposable module in $\mathscr{G}(T)$, not Ext-projective in $\mathscr{G}(T)$, and $0 \to \tau X \to E \to X \to 0$ is the Auslander–Reiten sequence in $A-mod$ ending in X, then it is well known that $0 \to t(\tau X) \to t(E) \to X \to 0$, where t denotes the torsion-radical, is the relative Auslander–Reiten sequence in $\mathscr{G}(T)$ with end X, see, for example, [6]. Moreover we proved in [10]:

PROPOSITION 2.2. *If X is an indecomposable module in $\mathscr{G}(T)$, not* Ext-*projective and Q is the cokernel of the inclusion-map $t(\tau X) \to \tau X$, then Q is in* $\operatorname{add}(\tau T)$.

For $M \in \{V_i, W_j \mid 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant r\}$ consider the wing $\mathscr{W}(M)$. We define $\bar{M}$ to be the direct sum of those indecomposable direct summands of $T(M)$ not contained in $\mathscr{W}(\tau M)$, that is, $\bar{M}$ is the direct sum of those indecomposable summands X of $T(M)$ with $\operatorname{Hom}(M, X) \neq 0$. Then we get:

LEMMA 2.3. *Let T be a tilting module with wing decomposition*

$$T = T_p \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j).$$

If X is indecomposable in $\mathscr{G}(T)$, not Ext-*projective, such that $F(X)$ is not preprojective in $B-mod$ and X is not in $\mathscr{W}(V_i)$ and $\mathscr{W}(W_j)$ for $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant r$ then the cokernel Q of the inclusion map $t(\tau X) \to \tau X$ is in* $\operatorname{add} \tau(\bigoplus \bar{V}_i \oplus \bigoplus \bar{W}_j)$.

Proof. Since τ^- is a right-exact functor the epimorphism $\tau X \to Q$ induces an epimorphism $X \to \tau^- Q$ with $\tau^- Q \in \operatorname{add} T$. If $\tau^- Q$ has a direct summand in add T_p, $F(X)$ maps to a preprojective B-module and is thus preprojective too.

If $\tau^- Q$ has a direct summand Z in add $T(M)$, which is contained in the wing $\mathscr{W}(\tau M)$, then consider the quasi-composition series $Y_1 \subset Y_2 \subset \cdots \subset Y_r = \tau M$ of τM. By Lemma 1.3, the epimorphism $X \to Z$ can be lifted to $X \to Y_i$ for some i. But Y_i is torsion-free and X is a torsion module, so $\operatorname{Hom}(X, Y_i) = 0$.

THEOREM 2. *Let A be a finite-dimensional connected wild hereditary algebra and T a tilting module without preinjective direct summand. Let*

$$T = T_p \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j)$$

be its wing decomposition, let X_j be the quasi-socle and Z_j the quasi-top of W_j and let be $R_j \to W_j$ be the irreducible epimorphisms. Then we have

(a) *$R_i \in \mathcal{G}(T)$ for all $1 \leqslant i \leqslant r$.*

(b) *There exist some j such that*

(i) *$\tau^2 X_j \in \mathcal{G}(T)$.*

(ii) *$(\to \tau^2 X_j) \subset \mathcal{G}(T)$ and $F(\to \tau^2 X_j) = (\to F(\tau^2 X_j))$ is a full subquiver of the non-regular component $\mathcal{D}_j$ of $\Gamma(B)$, with $B = \mathrm{End}(T)$, containing the projective module $F(W_j)$.*

(iii) *$\mathrm{Hom}(\tau^m X_j, T) = 0$ for all $m \geqslant 2$.*

(iv) *All indecomposable modules $[m]\, Z_j$ ($m \in \mathbb{N}$) are in $\mathcal{G}(T)$.*

Proof. Let q_i be the quasi-length of W_i, that is, $W_i = [q_i]\, Z_i = X_i(q_i)$. First we show that $R_i \in \mathcal{G}(T)$, that is, $\mathrm{Hom}(R_i, \tau T) = 0$. Since $\mathcal{W}(R_i)$ is a standard wing by (1.6), we have $\mathrm{Hom}(R_i, \tau T(W_i)) = 0$. Let again $T'(W_i)$ be the complement of $T(W_i)$ in T and consider the short exact sequence

$$0 \to \tau W_i \to R_i \to Z_i \to 0.$$

Since Z_i is a torsion module, we have $\mathrm{Hom}(Z_i, \tau T'(W_i)) = 0$. As $T'(W_i) \in T(W_i)^{\perp}$, we have $\mathrm{Hom}(\tau W_i, \tau T'(W_i)) = 0$ and therefore also $\mathrm{Hom}(R_i, \tau T'(W_i)) = 0$.

For the proof of part (b) we first consider the following special case. For all wings $\mathcal{W}(V_i)$ and $\mathcal{W}(W_j)$ we assume that the indecomposable direct summands of $T(V_i)$ ($T(W_j)$, respectively) are situated at the projective vertices of the wings, that is, $T(W_j) = \bigoplus_{m=1}^{q_j} X_j(m)$ and $T(V_i) = \bigoplus_{m=1}^{p_i} Y_i(m)$, where Y_i is the quasi-socle and p_i the quasi-length of V_i.

By Lemma 2.3 we then see that for an indecomposable module $X \in \mathcal{G}(T)$ satisfying the conditions of (2.3) we have a short exact sequence

$$0 \to t(\tau X) \to \tau X \to Q \to 0 \qquad \text{with} \quad Q \in \mathrm{add}(\bigoplus \tau V_i \oplus \bigoplus \tau W_i).$$

For the proof we introduce the following auxiliary quiver $\Xi(n_1, ..., n_r)$: The vertices of $\Xi(n_1, ..., n_r)$ are the modules $\tau^{n_i} X_i$, with $n_i \geqslant 1$. We draw an arrow $\tau^{n_i} X_i \to \tau^{n_j} X_j$ if and only if there exists an epimorphism $f_{n_i}\colon \tau^{n_i} X_i \to \tau^{-l} X_j$ for some $l \geqslant 0$. By (1.1), $\Xi(n_1, ..., n_r)$ has no loops; since the functor τ^- is right-exact, it also has no oriented cycles. Assume

$$\tau^{n_1} X_1 \to \cdots \to \tau^{n_s} X_s \to \tau^{n_1} X_1$$

is an oriented cycle with corresponding epimorphisms $f_i\colon \tau^{n_i} X_i \to \tau^{-l_i} X_{i+1}$ (modulo r). Then $f_1 \circ (\tau^{-(n_1 + l_1)} f_2) \circ \cdots (\tau^{-(\sum n_i + \sum l_i)} f_s)$ is an epimorphism from $\tau^{n_1} X_1$ to $\tau^{-m} X_1$ with $m = \sum (n_i + l_i)$, again contradicting (1.1). But this immediately implies that for each $(n_1, ..., n_r)$ there exists a non-empty set of vertices of $\Xi(n_1, ..., n_r)$ which are not starting points of arrows.

LEMMA 2.4. *Suppose $\tau^{n_1}X_1, ..., \tau^{n_l}X_l$ are not starting points of the quiver $\Xi(n_1, ..., n_r)$ and for $1 \leqslant t \leqslant l$ either $\tau^{n_t}X_t$ is contained in $\mathscr{G}(T)$ or $n_t = 1$ holds. Then there exists no epimorphism from $\tau^{n_t}X_t$ to Z with $Z \in \operatorname{add}(\bigoplus V_i \oplus \bigoplus W_j)$.*

Proof. We may assume Z is indecomposable.

If $n_t = 1$, then $\operatorname{Hom}(\tau X_t, \tau V_i) = 0 = \operatorname{Hom}(\tau X_t, \tau W_j)$ for all $i = 1 \cdots s$ and for all j with $j \neq t$, since otherwise we would have nonzero maps $\tau W_t \to \tau V_i$ or $\tau W_t \to \tau W_j$ by (1.3) and (1.4) which contradicts our assumption on T. For $n_t > 1$ the module $\tau^{n_t}X_t$ is torsion and thus there are no maps to the torsion-free modules τV_i and τW_j. By [5, (4.1)] therefore each nonzero map $\tau^{n_t}X_t \to V_i$ and $\tau^{n_t}X_t \to W_j$ is either injective or surjective.

Assume there exists a surjective map $f: \tau^{n_t}X_t \to Z$. Clearly Z is not in $\operatorname{add}(\bigoplus W_j)$, since $Z_j = \tau^{-q_j+1}X_j$ is the quasi-top of W_j and there are no arrows starting at the vertices $\tau^{n_t}X_t$ for $1 \leqslant t \leqslant l$. If $Z = V_i$, we consider the sequence

$$V_i = V_{i_1} \xrightarrow{f_1} V_{i_2} \longrightarrow \cdots \longrightarrow V_{i_l} \xrightarrow{f_l} W_j.$$

Since $f: \tau^{n_t}X_t \to V_{i_1}$ is surjective and not an isomorphism by definition the map $f \circ f_1 : \tau^{n_t}X_t \to V_{i_2}$ cannot be injective, thus it is surjective. Repeating this procedure, we finally get a surjection $f \circ f_1 \cdots \circ f_l: \tau^{n_t}X_t \to W_j$, a contradiction.

Let us finish now the proof of the special case of the theorem. By part (a) all the modules R_l are in $\mathscr{G}(T)$. As the irreducible maps $R_l \to W_l$ are also relatively irreducible in $\mathscr{G}(T)$ none of the modules $F(R_l)$ is preprojective and we can apply (2.3).

From the Auslander–Reiten sequence

$$0 \to \tau R_l \to \tau W_l \oplus [q_l + 1]\, Z_l \to R_l \to 0$$

we get the relative Auslander–Reiten sequence

$$0 \to t(\tau R_l) \to t([q_l + 1]\, Z_l) \to R_l \to 0$$

and the universal sequence

$$0 \to t(\tau R_l) \to \tau R_l \to Q_l \to 0$$

with $0 \neq Q_l \in \operatorname{add}(\bigoplus \tau V_i \oplus \bigoplus \tau W_j)$.

If $Q_l = (\tau W_l)^m$, then we deduce from Proposition 1.6 that $m = 1$ holds and thus we get $t(\tau R_l) = \tau^2 X_l$ and $[q_l + 1]\, Z_l \in \mathscr{G}(T)$. So assume, the module $\tau^- Q_l$ has an indecomposable direct summand $Z \in \operatorname{add}(\bigoplus V_i \oplus \bigoplus_{j \neq l} W_j)$. Consider the exact sequence

$$0 \to \tau X_l \to R_l \to W_l \to 0.$$

Since $\mathrm{Hom}(W_l, Z)=0$ by definition of W_l and $\mathrm{Ext}(W_l, Z)=0$ as T is a tilting module, the epimorphism $R_l \to Z$ is induced by a surjection $\tau X_l \to Z$.

If the map $\tau X_l \to Z$ would be injective, $\tau^2 X_l$ would be torsionfree and thus R_l would be Ext-projective in $\mathscr{G}(T)$, a contradiction. If $\tau X_1, ..., \tau X_{l_1}$ are exactly these vertices of the quiver $\Xi(1, ..., 1)$ which are not starting points of arrows, we know by Lemma 2.4 there exist no such surjections, that is, $t(\tau R_l)=\tau^2 X_l$ for $1 \leqslant l \leqslant l_1$. Since

$$0 \to \tau^2 X_l \to [t_l+1]\, Z_l \to R_l \to 0$$

are relative Auslander–Reiten sequences for $1 \leqslant l \leqslant l_1$ the modules $F(\tau^2 X_l)$ are not preprojective in $B-mod$.

For the proof of (ii) we first consider the quiver $\Xi(\underbrace{2 \cdots 2}_{l_1}, 1 \cdots 1)$. Say $\tau^2 X_1, ..., \tau^2 X_{l_2}$ with $1 \leqslant l_2 \leqslant l_1$ are not starting points of arrows. Considering for $1 \leqslant l \leqslant l_2$ the universal sequences

$$0 \to t(\tau^3 X_l) \to \tau^3 X_l \to Q_l \to 0$$

with $Q_l \in \mathrm{add}(\bigoplus \tau V_i \oplus \bigoplus \tau W_j)$ we can apply the right-exact functor τ^- and get, as the modules $\tau^2 X_l$ are not sources of any arrow in $\Xi(2 \cdots 2, 1 \cdots 1)$, from Lemma 2.4 that $Q_l=0$, that is, $\tau^3 X_l \in \mathscr{G}(T)$.

For $1 \leqslant l \leqslant l_2$ the Auslander–Reiten sequences in $A-mod$

$$0 \to \tau^3 X_l \to E \to \tau^2 X_l \to 0$$

are relative Auslander–Reiten sequences in $\mathscr{G}(T)$ at the same time and thus the B-modules $F(\tau^3 X_l)$ are not preprojective.

Repeating this procedure, after finitely many steps this process becomes stationary; that is, we finally get a number l_∞ with $1 \leqslant l_\infty \leqslant \cdots \leqslant l_2 \leqslant l_1$ such that $\tau^m X_l \in \mathscr{G}(T)$ for all $m \geqslant 2$ and all $1 \leqslant l \leqslant l_\infty$. Then of course also the cones $(\to \tau^2 X_l)$ are in $\mathscr{G}(T)$. Since

$$0 \to \tau^2 X_l \to [q_l+1]\, Z_l \to R_l \to 0$$

is the relative Auslander–Reiten sequence starting in $\tau^2 X_l$ and $R_l \to W_l$ is irreducible in $\mathscr{G}(T)$, the cone $(\to F(\tau^2 X_l))$ is in the same component $\mathscr{D}$ as $F(W_l)$.

Part (iii) is clear, since $\mathrm{Hom}(\tau^m X_l, T)$ is isomorphic to $\mathrm{Hom}(\tau^{m+1} X_l, \tau T)=0$. Trivially $[m]\, Z_l$ is in $\mathscr{G}(T)$ for $1 \leqslant m \leqslant q_l$ and it was proved above for $m=q_l+1$. Using Proposition 2.2, it can be shown by induction that the Auslander–Reiten sequence in $A-mod$

$$0 \to \tau[m+j]\, Z_l \to [m+j+1]\, Z_l \oplus \tau[m+j-1]\, Z_l \to [m+j]\, Z_l \to 0$$

induces the relative Auslander–Reiten sequence

$$0 \to [j](\tau^2 X_l) \to [j-1](\tau^2 X_l) \oplus [m+j+1]\, Z_l \to [m+j]\, Z_l \to 0$$

which proves (iv).

Now we will consider the general case: If T is a tilting module with wing decomposition

$$T = T_p \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j)$$

then we consider the changed tilting module

$$\bar{T} = T_p \oplus \bigoplus_{i=1}^{s} \overline{T(V_i)} \oplus \bigoplus_{j=1}^{r} \overline{T(W_j)}$$

with $\overline{T(W_j)} = \bigoplus_{m=1}^{q_j} X_j(m)$ and $\overline{T(V_i)} = \bigoplus_{m=1}^{p_i} Y_i(m)$, where Y_i is the quasi-socle and p_i the quasi-length of V_i.

Let us call $\bar{T}$ the *normalised form* or the *normalisation* of T. By [13, (4.4)], $\bar{T}$ is a tilting module and $\bar{T} = T_p \oplus \bigoplus_{i=1}^{s} \overline{T(V_i)} \oplus \bigoplus_{j=1}^{r} \overline{T(W_j)}$ is its wing decomposition.

The general case of the proof follows immediately from

LEMMA 2.5. *Let T be a tilting module with wing decomposition*

$$T = T_p \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j)$$

and let $\bar{T}$ be the normalisation of T. For an indecomposable module X we have:

(a) *If X is not contained in the wings $\mathcal{W}(\tau V_i)$ and $\mathcal{W}(\tau W_j)$ for all i, j then X is in $\mathcal{G}(T)$ if and only if $X \in \mathcal{G}(\bar{T})$.*

(b) *If X is not contained in the wings $\mathcal{W}(V_i)$ and $\mathcal{W}(W_j)$ for all i, j then X is in $\mathcal{F}(T)$ if and only if $X \in \mathcal{F}(\bar{T})$.*

Proof. By Lemma (1.4) we have for example $\operatorname{Hom}(X, \tau V_i)) = 0$ if and only if $\operatorname{Hom}(X, \tau T(V_i)) = 0$ or $\operatorname{Hom}(X, \tau\overline{T(V_i)}) = 0$. Thus we get $\operatorname{Ext}(T, X) = 0$ if and only if $\operatorname{Ext}(\bar{T}, X) = 0$, which proves (a). The proof of (b) is similar.

For inductive procedures it sometimes is useful, to consider the special case $r + s = 1$, that is, $T = T_p + T(W)$. Let $R \to W$ be the irreducible surjection. Then we have:

COROLLARY 2.6. *R is quasi-simple in $C-$mod.*

Proof. Suppose R is not quasi-simple. Let

$$0 \longrightarrow \tau R \xrightarrow{\ f\ } U_1 \oplus U_2 \xrightarrow{\ g\ } R \longrightarrow 0$$

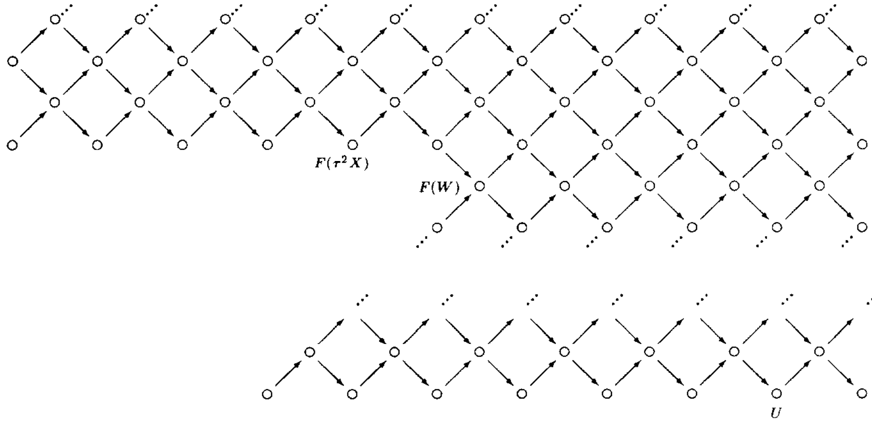

FIGURE 1

be the Auslander–Reiten sequence in $C-mod$ ending in R, $C=\mathrm{End}(T_p)$ with $U_1 \to R$ injective.

Suppose for simplicity that $T(W)=X(1)\oplus \cdots \oplus X(q-1)\oplus X(q)$, where X is the quasi-socle of W. Then B is a one-point extension of the non-connected algebra $C\oplus \mathrm{End}(\bigoplus_{1\leqslant i<q} X(i))$ by the module $R\oplus I$, where I is the injective-projective module of the second algebra (of type A_{q-1}). Thus by [13, (2.5.6)] the Auslander–Reiten sequence with end R in $B-mod$ is

$$0 \longrightarrow \overline{\tau R} \xrightarrow{(f,1)} (U_1\oplus U_2, |\tau R|, |f|) \xrightarrow{(g,0)} R \longrightarrow 0.$$

Since R is a brick, by Proposition 1.6 we get $\mathrm{Hom}(R, U_1)=0$, that is,

$$(U_1\oplus U_2, |\tau R|, |f|)=(U_1, 0, 0)\oplus (U_2, |\tau R|, |f|)$$

but we have seen that the relative Auslander–Reiten sequence

$$0\to \tau^2X\to [t+2]\,Y\to R\to 0$$

has indecomposable middle term.

Let us finally visualise the component $\mathscr{D}$ containing the module $F(W)$. Again we assume for simplicity that $T(W)=\bigoplus X(i)$ (see Fig. 1).

One easily checks that the module U is $\tau_C^- R$ and by (1.2) in the same way as in [10] we see that $(U\!\to)$ is in $C-mod$, since $\mathrm{Hom}(R, \tau_C^- R)=0$.

3. Proof of Theorem 1

Before going into the proof, let us formulate an easy consequence of [8, (2.3)]:

LEMMA 3.1. *Let A be a connected, wild hereditary algebra, T a tilting module without preinjective direct summands and $B = \operatorname{End}(T)$. Let X and Y be indecomposable modules in $\mathscr{Y}(T)$, not preprojective in $B-mod$ and not in the connecting component. Suppose further that $\tau_B^r X$ is defined for all $r \in \mathbb{N}$ and Y is the image of a regular A-module under the functor F. Then we have:*

(i) *There exists an integer N_1 such that* $\operatorname{Hom}(Y, \tau^r X) \neq 0$ *for all* $r \geqslant N_1$.

(ii) *There exists an integer N_2 such that* $\operatorname{Hom}(\tau^r X, Y) = 0$ *for all* $r \geqslant N_2$.

Proof. Let be $\mathscr{C}$ the component of $\Gamma(B)$ containing X. By [8, (2.3)] there exists $U = F(U')$ in $\mathscr{C}$ such that $(\rightarrow U) = F(\rightarrow U')$. Take a $t \in \mathbb{N}$ such that $Z = \tau_B^t X \in (\rightarrow U)$. If $Z = F(Z')$ with $Z' \in \mathscr{G}$, then we have $\tau_B^r Z = F(\tau_A^r Z')$. Since $\operatorname{Hom}_B(Y, \tau_B^r Z) = \operatorname{Hom}_A(Y', \tau_A^r Z')$ and $\operatorname{Hom}_B(\tau_B^r Z, Y) = \operatorname{Hom}_A(\tau_A^r Z', Y')$ the assertion (i) follows immediately from [2, (3.1)] and (ii) is a consequence of [8, (1.1)].

Let us now start the proof of Theorem 1. If

$$T = T_p \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j)$$

is the wing decomposition of T we will use induction on the number $r+s$. If $r+s=0$, then T is preprojective, B is concealed, and the assertion is obvious. So assume $r+s>0$ and thus also $r>0$. Since all preprojective B-modules are in $C-mod$ we can suppose that $\mathscr{C}$ is either a regular component in $\mathscr{Y}(T)$ or one of those components containing (at least one) projective module from a branch.

Let W be one of the modules W_j satisfying the conditions (b) of Theorem 2, that is, for the quasi-socle X of W we have:

1. $\tau^l X \in \mathscr{G}(T)$ for all $l \geqslant 2$,
2. $\operatorname{Hom}(\tau^l X, T) = 0$ for all $l \geqslant 2$.

We will first prove that for any indecomposable module $Z \in \mathscr{C}$, there exists a number N such that

$$\operatorname{Hom}_B(F(T(W)), \tau_B^{-m} Z) = 0 \qquad \text{for all} \quad m \geqslant N,$$

that is, $(\tau^{-N} Z\rightarrow)$ is in $\operatorname{End}_A(T'(W)) - mod$, where $T'(W)$ is defined as before by $T(W) \oplus T'(W) = T$.

Again, as in the proof of Theorem 2 it is enough to treat the special case that T is normalised.

By (1.4) it is enough to prove the following: Let Z be any indecomposable module in $\mathscr{C}$; then there exists $N \in \mathbb{N}$ such that $\operatorname{Hom}(F(W), \tau_B^{-t} Z) = 0$ for all $t \geqslant N$.

Consider the regular component $\mathscr{D}$ in $\Gamma(A)$ containing W. To simplify notations, we write $(i, j) \in \mathbb{N} \times \mathbb{N}$ for the indecomposable module $\tau^i(X(j))$, where X is the quasi-socle of W. Let q be the quasi-length of W, that is, $W = (0, q)$.

Let further $Z_t = \tau_B^{-t} Z$ for $t \geqslant 0$ and $Y_t \in \mathscr{G}(T)$ indecomposable with $F(Y_t) = Z_t$. Since there exist only finitely many indecomposable preprojective A-modules in $\mathscr{G}(T)$ (as T has regular direct summands) and their images under the functor F remain preprojective and additionally the preinjective component of $\Gamma(A)$ is mapped to the connecting component under F, we may assume that all Y_t are regular.

By (3.1) then there exists N such that

$$\operatorname{Hom}_B(F(i, 1), Z) = 0 \qquad \text{for all} \quad i \geqslant N \geqslant 2. \tag{1}$$

Notice that $F(i, 1) = \tau_B^{i-2} F(\tau_A^2 X)$.

As $(i, 1)$ and Y_t are regular, we have $\operatorname{Hom}_A((i, 1), Y_t) = \overline{\operatorname{Hom}}((i, 1), Y_t)$ where $\overline{\operatorname{Hom}}$ denotes the injectively stable morphisms. Since the injective B-modules are in $\mathscr{X}(T)$, we also have

$$\overline{\operatorname{Hom}}_B(F(i, 1), Z_t) = \operatorname{Hom}_B(F(i, 1), Z_t).$$

We know already that $\operatorname{Hom}((i, 1), T) = 0$ for all $i \geqslant 2$, which implies that

$$\underline{\operatorname{Hom}}_B(F(i, 1), Z_t) = \operatorname{Hom}_B(F(i, 0), Z_t),$$

where $\underline{\operatorname{Hom}}$ denotes the projectively stable maps. Thus the formula

$$\overline{\operatorname{Hom}}_B(F(i, 1), Z_t) \cong \underline{\operatorname{Hom}}_B(\tau_B^- F(i, 1), Z_{t+1})$$

implies $\operatorname{Hom}(F(i, 1), Z_t) \cong \operatorname{Hom}(F(i-1, 1), Z_{t+1})$. Iterating the application of τ_B^- we get from formula (1),

$$\operatorname{Hom}(F(i, 1), Z_t) = 0 \qquad \text{for all} \quad i \geqslant 2 \text{ and all } t \geqslant N. \tag{2}$$

Let us now prove that

$$\operatorname{Hom}_A((i, 1), Y_t) = 0 \qquad \text{for all} \quad i \geqslant 0 \text{ and all } t \geqslant N+2. \tag{3}$$

By [6] we have $Y_t \cong t(\tau Y_{t+1})$, so we get the short exact sequence

$$0 \longrightarrow Y_t \xrightarrow{\ e\ } \tau Y_{t+1} \xrightarrow{\ p\ } Q_t \longrightarrow 0 \tag{4}$$

with $Q_t \in \operatorname{add}(\tau T)$, see Proposition 2.2.

For $t \geqslant N$ we get $\mathrm{Hom}((2,1), Y_t) = 0$ by formula (2) and $\mathrm{Hom}((2,1), Q_t) = 0$ since $(2,1)$ is torsion and Q_t is torsion-free. Thus also $\mathrm{Hom}((2,1), \tau Y_{t+1}) = 0$ holds true. Since τ_A is functorial, we get $\mathrm{Hom}((1,1), Y_{t+1}) = 0 \, \forall t \geqslant N$. Notice that $(1,1) = \tau X$.

Suppose for some $t \geqslant N+1$ there is a nonzero map $f \in \mathrm{Hom}_A((1,1), \tau Y_{t+1})$. From $\mathrm{Hom}((1,1), Y_t) = 0$ we see that the composition $fp\colon (1,1) \to Q_t$ is nonzero; since it factorises over τY_{t+1}, fp is in $rad^{\infty}((1,1), Q_t)$. Let be $Q_t = E_1 \oplus E_2$ with $E_1 \in \mathrm{add}(\tau W)$, E_2 without summand isomorphic to τW and $p = (p_1, p_2)$. Then we have $fp_2 = 0$, by definition of $T(W)$. So we have $0 \neq fp_1 \in rad^{\infty}((1,1), E_1)$. But the wing $\mathcal{W}(\tau W)$ is a standard wing containing the vertex $(1,1)$ by (1.6) and thus we have $rad^{\infty}((1,1), E_2) = 0$. Therefore we have $\mathrm{Hom}((1,1), \tau_A Y_{t+1}) = 0$ and application of τ^{-} proves formula (3).

Suppose now by induction that for $2 \leqslant u \leqslant q$ there exists $N' \in \mathbb{N}$ such that $\mathrm{Hom}_A((i,j), Y_t) = 0$ for all (i,j) with $i \geqslant 0$ and $j < u$ and for all $t \geqslant N'$. Considering the Auslander–Reiten sequences

$$0 \to (i, u-1) \to (i, u) \oplus (i-1, u-1) \to (i-1, u-1) \to 0$$

we then get $\mathrm{Hom}((i,u), Y_t) = 0$ for all $i \geqslant 1$ and all $t \geqslant N'$.

Again we use formula (4). If $\mathrm{Hom}((1,u), Y_t) = 0$ and $f \neq 0$ is an element of $\mathrm{Hom}((1,u), \tau Y_{t+1})$, then as above fp is nonzero; using once more the decomposition $Q_t = E_1 \oplus E_2$, $p = (p_1, p_2)$, we have $0 \neq fp_1 \in rad^{\infty}((1,u), E_1)$, which contradicts the property $rad^{\infty}((1,u), E_1) = 0$ of standard wings. So we have $\mathrm{Hom}((1,u), \tau Y_{t+1}) = 0$ which gives $\mathrm{Hom}((0,u), Y_{t+1}) = 0$.

Especially for $u = q$ we get $\mathrm{Hom}(W, Y_{t+1}) = 0$ for all $t \geqslant N'$, which proves the first assertion: For $\bar{Z} = Z_{N'+1}$ the cone $(\bar{Z}\to)$ is contained in $\mathrm{End}_A(T'(W)) - mod$. It additionally implies that $(\bar{Z}\to)$ is in $F(T(W)^{\perp})$.

If $r + s = 1$, that is, $T'(W) = T_p$, we are finished. Notice that $\tau^{-l}\bar{Z}$ is defined for all natural l, that is, $\bar{Z}$ is regular in $C - mod$ (we have $\mathrm{End}(T'(W)) = C$ for $r + s = 1$). Thus $(\bar{Z}\to)$ is a successor-closed cone of a regular component in $\Gamma(C)$ and especially the stable component $\mathscr{C}_{st}$ of the component $\mathscr{C}$ containing Z is of type $\mathbb{Z}A_{\infty}$.

For $r + s > 1$ by definition of $T(W)$, the partial tilting module $T'(W)$ is a tilting module in $T(W)^{\perp}$ and $T(W)^{\perp}$ is equivalent to $A' - mod$, where A' is a connected wild hereditary algebra.

The above proof says that $(\bar{Z}\to)$ is a full successor-closed part of a component $\tilde{\mathscr{C}}$ in $\mathrm{End}_{T(W)^{\perp}}(T'(W)) = B'$ contained in the torsion-free part $\mathscr{Y}_{T(W)^{\perp}}(T'(W))$. Since $\tau_{B'}^{-l}\bar{Z} = \tau_{B}^{-l}\bar{Z}$ is in $\mathscr{Y}_{T(W)^{\perp}}(T'(W))$ for all l, the component $\tilde{\mathscr{C}}$ is not the connecting component in $\Gamma(B')$. But

$$T_p \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{W_j \neq W} T(W_j)$$

is the wing decomposition of $T'(W)$ in $T(W)^{\perp}$. This can be checked easily, as T is normalised: Irreducible maps (in $A-mod$) between summands of $T'(W)$ remain irreducible in $T(W)^{\perp}$. Since the whole wings $\mathscr{W}(V_i)$ and $\mathscr{W}(W_j)$ are in $T(W)^{\perp}$, the quasi-socles of V_i and W_j are quasi-simple in $T(W)^{\perp}$, too. Since the number of wings in this decomposition is $s+r-1$, we know by induction that there exists a indecomposable module $\tilde{Z}\in\tilde{\mathscr{C}}$ such that $(\tilde{Z}\rightarrow)$ is in $C-mod$ and Theorem 1 is proved.

COROLLARY 3.2. *Let X and Y be indecomposable modules in $\mathscr{Y}(T)$, not preprojective and not contained in the connecting component. Then we have*:

(a) *There exists N_1 such that* $\mathrm{Hom}(\tau^{-r}Y, X)\neq 0\ \forall r\geqslant N_1$.

(b) *There exists N_2 such that* $\mathrm{Hom}(X, \tau^{-r}Y)=0\ \forall r\geqslant N_2$.

Proof. (a) Let be $C=\mathrm{End}_A(T_p)$ and $\mathrm{res}(X)$ the restriction of X to C. One easily checks that $\mathrm{res}(X)$ has no indecomposable direct summand which is preprojective in $C-mod$. By Theorem 2 there exists an integer N such that $\tau_B^{-r}Y$ is a C-module for all $r\geqslant N$ with $\tau_C^{-s}(\tau_B^{-N}Y)=\tau_B^{-(N+s)}Y$. Now we have $\mathrm{Hom}_B(\tau_B^{-(N+s)}Y, X)=\mathrm{Hom}_C(\tau_C^{-s}(\tau_B^{-N}Y), \mathrm{res}(X))$, and $\tau_B^{-N}Y$ is regular in $C-mod$.

If $\mathrm{res}(X)$ is regular in $C-mod$, the result follows from [2]; if $\mathrm{res}(X)$ is preinjective we get the result since only finitely many modules in the τ_c-orbit of $\tau_B^{-N}Y$ are non-sincere.

(b) The proof of (b) is dual to (a).

Let us finally mention the following immediate consequence of the above proof:

COROLLARY 3.3. *If $\mathscr{C}$ is a component in $\mathscr{Y}(T)$ different from the preprojective and the connecting component, then the stable component $\mathscr{C}_{st}$ of $\mathscr{C}$ is of type $\mathbb{Z}A_{\infty}$.*

Remark. Theorem 1 answers a question in [10]. Naturally it also answers the weaker question on growth numbers in [10]: If $\mathscr{D}$ is a regular component in $\mathscr{Y}(T)$ and X is indecomposable in $\mathscr{D}$, then

$$\lim_{n\to\infty} \sqrt[n]{\dim_k(\tau^{-n}X)}=\varrho(C),$$

where $\varrho(C)$ is the growth number (see [3]) of the concealed algebra C.

4. APPLICATIONS

It $T=T_p\oplus T_1$ is a tilting module of a finite dimensional connected wild hereditary algebra without preinjective direct summand (such that $F(T_p)$ is the direct sum of all preprojective projective B-modules), then we have seen in the preceding section that all stable components $\mathscr{C}_{st}$ in $\mathscr{Y}(T)$ are of type $\mathbb{Z}A_{\infty}$ with the following additional properties:

1. There exists a quasi-simple module $X = F(U) \in \mathscr{C}$ (with U a quasi-simple regular A-module) such that $(\to X) = F(\to U)$.

2. There exists a quasi-simple module $Y \in \mathscr{C}$ such that $(Y \to)$ is in C-mod, $C = \operatorname{End}(T_p)$.

We may visualise this situation as shown in Fig. 2 (for a regular component $\mathscr{C}$).

We have already mentioned in the introduction that all regular components of $\Gamma(A)$ occur in this way as a left cone of some stable component of $\mathscr{Y}(T)$. Thus T defines a bijection between the regular components of $\Gamma(A)$ and the stable components of $\Gamma(\mathscr{Y}(T))$, see [8] and therefore by Theorem 1 a bijection between the regular components of $\Gamma(A)$ and the set $\mathscr{R}'$ of components of $\mathscr{Y}(T)$ different from the preprojective and connecting component. Especially we get an injection μ_T from the set of regular components of $\Gamma(A)$ to the set of regular components of $\Gamma(C)$. Let us show that μ_T is a bijection:

THEOREM 3. *Let A be a finite dimensional connected wild hereditary algebra and $T = T_p \oplus T_1$ a tilting module without preinjective direct summands. Let be $B = \operatorname{End}(T)$ and $C = \operatorname{End}(T_p)$ the wild, connected, and concealed algebra defining the preprojective component of B. Then μ_T is bijective.*

Proof. Consider the wing decomposition of T

$$T = T_p \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j).$$

Let be $m = r + s$ and suppose again for simplicity that T is normalised.

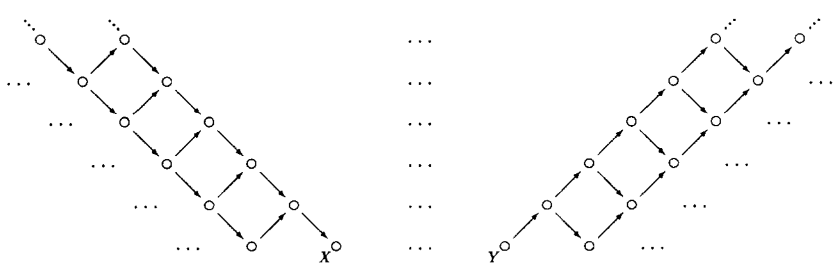

FIGURE 2

Iterating the argument that $T'(W_j)$ is a tilting module in $T(W_j)^{\perp}$, we see that B is an iterated branch-enlargement

$$B = C[Z_1, Q_1] \cdots [Z_m, Q_m]$$

with linear quivers $Q_i = \circ \to \circ \to \cdots \to \circ$, see [13].

Moreover, for all $j = 0, ..., m$ the algebra $C_j = C[Z_1, Q_1] \cdots [Z_j, Q_j]$ is a tilted algebra of some connected wild hereditary algebra A_j with tilting module $T_{(j)}$ without preinjective direct summand and the branch Q_{j+1} is rooted at the quasi-simple C_j-module Z_{j+1}.

The theorem will be proved by induction on $m = r + s$. If $m = 0$, we have $B = C$ and there is nothing to be done. So suppose for some j with $0 < j < m$ Theorem 3 is true. Therefore, by induction, if $\mathscr{D}$ is a regular component in $\Gamma(C)$, then there exists a (quasi-simple) module $Y \in \mathscr{D}$ such that $(Y \to)$ is a full, successor-closed subquiver of some stable component $\mathscr{S}$ in $\Gamma(C_j)$.

By Corollary 3.2 there exists an $N_1 \in \mathbb{N}$ such that $\mathrm{Hom}_{C_j}(Z_{j+1}, \tau_{C_j}^{-l} Y) = 0$ for all $l \geqslant N_1$. Therefore, after identifying $C_j - mod$ with its image in $C_{j+1} - mod$ under the canonical embedding, the Auslander–Reiten sequence $0 \to \tau_{C_j}^{-l} Y \xrightarrow{f} E \xrightarrow{g} \tau_{C_j}^{-l-1} Y \to 0$ in $C_j - mod$ also is an Auslander–Reiten sequence in $C_{j+1} - mod.$

Thus the cone $(\tau_{C_j}^{-N_1} Y \to)$ is a full subquiver of some stable component in the torsion-free part of $C_{j+1} - mod.$

Therefore from the bijectivity of the map $\mu_{T_{(j)}}$ we can deduce that $\mu_{T_{(j+1)}}$ is bijective and induction works.

If A is a wild connected hereditary algebra over some algebraically closed field with at least three simple modules, it was shown in [9] that for all $N \in \mathbb{N}$ there exists a tilting module $T = T_p \oplus T_1$ without preinjective direct summand, such that $\mathrm{End}(T_p)$ is the generalised Kronecker-algebra $K_r = k(\circ \overset{\vdots}{\rightrightarrows} \circ)$ with $r \geqslant N$ arrows. A more constructive proof of the same result was recently given by Unger, see [19]. Her approach has the additional advantage that without any change if works for arbitrary fields as long as we consider only path-algebras of quivers (with trivial valuation). Especially, if A_1' is the wild k-algebra of the quiver $\circ \rightleftarrows \circ \leftarrow \circ$ then for all $r \geqslant 3$ there exists an A_1'-tilting module $T = T_p \oplus T_2$ with $\mathrm{End}(T_p) = K_r$.

If $A = K_r$ and T is a tilting module, then T is either preprojective or preinjective and $\mathrm{End}(T) \cong A$.

Thus we get from Theorem 3:

COROLLARY 4.1. *Let A and B be finite dimensional connected wild hereditary path-algebras of some quivers where k is some field.*

1. *There exists an A-tilting module T, such that μ_T defines a bijection between the regular components of $\Gamma(A)$ and $\Gamma(K_r)$ for some r.*

2. *There exist an A-tilting module T_1, a B-tilting module T_4, and two A'_1-tilting modules T_2 and T_3 such that $\mu_{T_4}^{-1}\mu_{T_3}\mu_{T_2}^{-1}\mu_{T_1}$ is a bijection between the regular components of A and B.*

References

1. I. Assem, "Tilting Theory—An Introduction," Rep. No. 47, Université de Sherbrooke, 1988.
2. D. Baer, Wild hereditary Artin algebras and linear methods, *Manuscripta Math.* **55** (1986), 69–82.
3. V. Dlab and C. M. Ringel, Eigenvalues of Coxeter transformations and the Gelfand–Kirillov dimension of the preprojectiv algebra, *Proc. Amer. Math. Soc.* **83** (1981), 228–232.
4. W. Geigle and H. Lenzing, Perpendicular categories with applications to representations and sheaves, *J. Algebra*, in press.
5. D. Happel and C. M. Ringel, Tilted algebras, *Trans. Amer. Math. Soc.* **274** (1982), 399–443.
6. M. Hoshino, On splitting torsion theories induced by tilding modules, *Comm. Algebra* **11** (1983), 427–441.
7. M. Hoshino, Modules without self-extensions and Nakayama's conjecture, *Arch. Math.* **43** (1984), 493–500.
8. O. Kerner, Tilting wild algebras, *J. London Math. Soc.* **39** (1989), 29–47.
9. O. Kerner, Preprojective components of wild tilted algebras, *Manuscripta Math.* **61** (1988), 429–445.
10. O. Kerner, Universal exact sequences for torsion-theories, *in* "Topics in Algebra—Representations of Algebras," Banach Centre Publications, PWN, Warsaw, **26** (1990), 317–326.
11. C. M. Ringel, Representations of *K*-species and bimodules, *J. Algebra* **41** (1976), 269–302.
12. C. M. Ringel, Finite dimensional algebras of wild representation type, *Math. Z.* **161** (1978), 235–255.
13. C. M. Ringel, Tame algebras and integral quadratic forms, "Lecture Notes in Math.," Vol. 1099, Springer-Verlag, New York/Berlin, 1984.
14. C. M. Ringel, Representation theory of finite-dimensional algebras, *in* "Representation of Algebras, Proc. Durham Symp., 1985," pp. 7–79, London Math. Soc. Lecture Notes, Vol. 116, Cambridge Univ. Press, London/New York, 1986.
15. C. M. Ringel, The regular components of the Auslander–Reiten quiver of a tilted algebra, *Chinese Ann. Math. Ser. B* **9** (1988), 1–18.
16. H. Strauss, A reduction process via tilting theory, *C.R. Math. Rep. Acad. Sci. Canada* **9** (1987), 161–166.
17. H. Strauss, A reduction process with applications to tilts of wild hereditary algebras, preprint, Paderborn, 1988.
18. L. Unger, Schur modules over wild, finite dimensional path algebras with three pairwise non isomorphic simple modules, *J. Pure Appl. Algebra*, in press.
19. L. Unger, On wild tilted algebras which are squids, *Arch. Math.* **55** (1990), 542–550.