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Snakes: Oriented Families of Periodic Orbits, Their Sources, Sinks, and Continuation

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Poincaré observed that for a differential equation $x' = f(x, \alpha)$ depending on a parameter α, each periodic orbit generally lies in a connected family of orbits in (x, α)-space. In order to investigate certain large connected sets (denoted Q) of orbits containing a given orbit, we introduce two indices: an orbit index ϕ and a "center" index ⌭ defined at certain stationary points. We show that generically there are two types of Hopf bifurcation, those we call "sources" (⌭ $= 1$) and "sinks" (⌭ $= -1$). Generically if the set Q is bounded in (x, α)-space, and if there is an upper bound for periods of the orbits in Q, then Q must have as many source Hopf bifurcations as sink Hopf bifurcations and each source is connected to a sink by an oriented one-parameter "snake" of orbits. A "snake" is a maximal path of orbits that contains no orbits whose orbit index is 0. See Fig. 1.1.

1. INTRODUCTION

We investigate the maximal continuation of a family of periodic solutions of a generic differential equation that depends on a parameter. We assume

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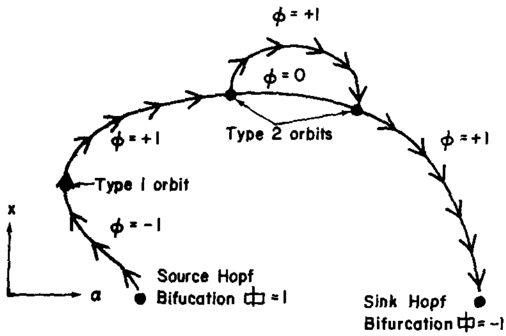

FIG. 1.1. A snake of periodic orbits: hypothetical example illustrating terminology. Each point of the graph except the ends represents a periodic orbit. Arrows point to the left when the orbit index ϕ is -1 and right when $\phi = +1$ and no arrows are shown when $\phi = 0$.

throughout that $f: R^n \times R \to R^n$ is infinitely differentiable in both coordinates. Suppose now that the equation

$$dx/dt = f(x, \alpha), \qquad (x, \alpha) \in R^n \times R, \tag{1.1}$$

has a nonconstant periodic solution

$$x = p_0(t) \qquad \text{at} \quad \alpha = \alpha_0.$$

The simplest continuation question asks if there is still a periodic solution if we change α slightly. The main tool for answering this type of question is the Poincaré map. We define the Poincaré map $G(x, \alpha)$ in terms of a point (x_0, α_0) on the orbit and an n-dimensional disc B through (x_0, α_0) perpendicular to $(f(x_0, \alpha_0), 0)$. For $(x_1, \alpha_1) \in B$ sufficiently close to (x_0, α_0) we may define $G(x_1, \alpha_1)$ to be the x coordinate of the point where the trajectory through (x_1, α_1) next hits the disc. (The α coordinate is α_1.) We say μ is a *multiplier* of the orbit if it is an eigenvalue of the $(n-1) \times (n-1)$ matrix of partial derivatives $D_x G(x_0, \alpha_0)$. If none of the multipliers of this solution equals one, it follows from the implicit function theorem that there is unique family (p_α, α) of periodic solutions (for $|\alpha - \alpha_0|$ sufficiently small) with the following properties: first, $p_{\alpha_0}(t) = p_0(t)$; and second, p_α is a continuous function of α and the (minimum) period $T(p_\alpha)$ varies smoothly. We call this the *Poincaré continuation* of p_0 and we call the set $\{(p_0(t), \alpha_0): 0 \leqslant t \leqslant T(\alpha_0)\}$ an *orbit*, that is, a periodic orbit.

When extending this family p_α as far as possible, we may reach some α_1 and an orbit $p_1(t)$ for which there is a multiplier μ equal to $+1$. Generally the family may be followed further, but not as a function of α. If α has been increasing along the family, it now decreases, and vice versa. Hence, we introduce a new parameter β and write the family of orbits interchangeably as $(p(t,\beta),\alpha(\beta))$ or $(p_\beta(t),\alpha_\beta)$. We let T denote the (minimum) period of an orbit so we write $T(\beta)$ or $T(p_\beta)$ or $T(p_\alpha(0),\alpha_\beta)$ for the (minimum) period. The α_1 orbit is sometimes called a *jug handle bifurcation orbit.*

Generically there are two more important phenomena that cannot be avoided. The first important phenomena is called a *period-doubling bifurcation,* (that is, subharmonic bifurcation) (see Fig. 1.2). For example, it is possible to reach a value β_1 for which this (minimum) period suddenly drops by a factor of 2, that is $\lim_{\beta\uparrow\beta_1} T(\beta)=2T(\beta_1)$. In this situation any continuation beyond β_1 passing through new orbits, rather than backtracking will generally have $\lim_{\beta\downarrow\beta_1} T(\beta)=T(\beta_1)$, so this factor-of-2 discontinuity is essential and not just transitory. The orbit $p(t,\beta_1)$ will have a multiplier $\mu=-1$, and generically this will be algebraically simple and $+1$ will not be a multiplier. This orbit will therefore lie on a second family of orbits $\bar{p}_\alpha$, α whose period varies continuously. This second family is the Poincaré continuation of the original orbit (p_0,α_0). For a period-doubling bifurcation to occur, x must be at least three dimensional. An example with a large number of period-doubling bifurcations can be found in [1].

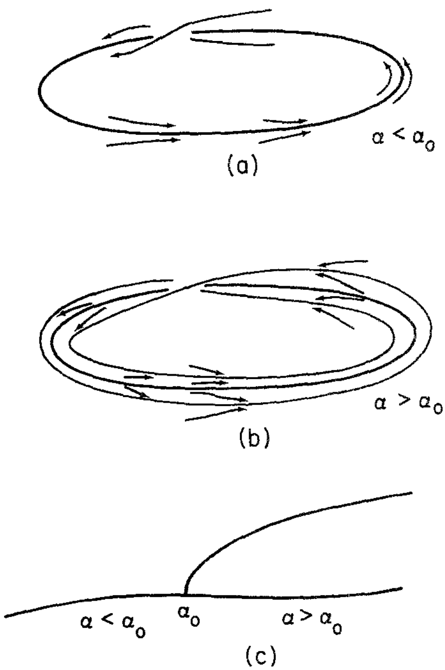

FIG. 1.2. In (a) an attracting orbit in R^3 is shown. In (b) the orbit has become unstable as α increases beyond α_0, and a period-doubling bifurcation occurs (at α_0). In (b) the low-period orbit is contained in a Möbius strip with the long-period orbit on the boundary. The sequence shown might be called a supercritical period-doubling bifurcation. In (c) a schematic diagram is shown representing (x,α) space with each point representing a single orbit.

The second phenomenon concerns *stationary points*, that is points (x, α) at which $f = 0$. The family may be continued to some value β_2 at which $T_2 = \lim_{\beta \to \beta_2} T(\beta)$ exists but the diameter of the orbit drops to 0. This is, there is a stationary point $(x_2, \alpha(\beta_2))$ for which $p(t, \beta) \to x_2$ as $\beta \to \beta_2$, for all t. At such a point $D_x f(x_2, \alpha(\beta_2))$ will have a pair of imaginary eigenvalues $\pm i\beta$ for some $\beta \neq 0$. (See Fig. 1.3.) We say a stationary point (x, α) is a *center* if $D_x f(x, \alpha)$ has non-zero imaginary eigenvalues, and it is *isolated* if furthermore 0 is not an eigenvalue and there is a neighborhood of (x, α) in $R^n \times R$ in which (x, α) is the only center. Generically there are two types of such stationary points and we will call these "source" and "sink" Hopf

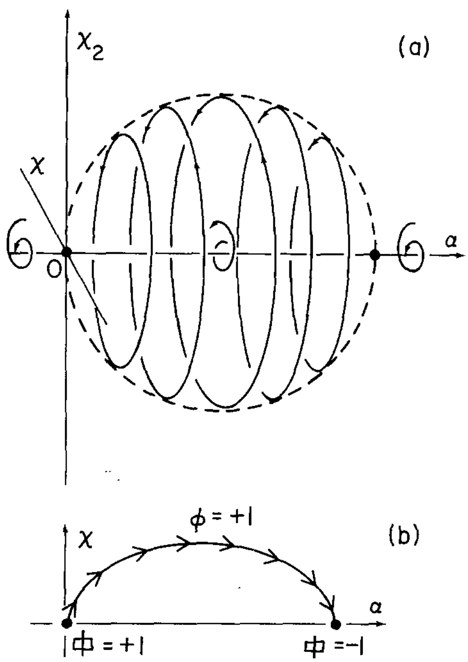

FIG. 1.3. The difficulty of representing families of orbits in high dimensions is considerable. Both (a) and (b) represent the differential equation

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} g(x, \alpha),$$

where $g(x, \alpha) = \alpha(1 - \alpha) - x_1^2 - x_2^2$. In polar coordinates it is $d\theta/dt = 1$, $d\rho/dt = \rho[\alpha(1 - \alpha) - \rho^2]$. The orbits have $\rho^2 = \alpha(1 - \alpha)$. In the terminology developed in this paper the stationary points $x = 0$, $\alpha = 0$ and $x = 0$, $\alpha = 1$ are "centers" and in fact are "generic Hopf points," though $\oplus(0, 0) = +1$, and so $(0, 0)$ is called a source of the family of orbits, while $\oplus(0, 1) = -1$, and so is called a sink. The orbits are all stable, and this implies their "orbit index" has value $\phi = +1$. The arrows are drawn pointing toward the right in (b) because $\phi = +1$.

points. The distinction between these two is *not* based on whether the bifurcation is subcritical or supercritical, but is more fundamental in that it depends only on $(n+1) \times (n)$ matrix $D_{x,\alpha} f(x_0, \alpha_0)$.

We denote by Q_0 a set which is maximal amongst arc-wise connected sets of orbits and centers containing a given (p_0, α_0). The objective of this paper is to investigate the properties of this set, and certain of its subsets first in a simple generic case. Brunovsky [2] who emphasized maps and Sotomyer [3] showed that generically, systems that depend on a parameter have only three types of orbits, and our versions of these are called type 0, 1, and 2. For another proof see [17, Appendix]. To investigate generic Hamiltonian systems, we also introduce what we call type m orbits, $m \geqslant 3$, and Meyer [4] shows that these types together (defined slightly differently) are sufficient for generic Hamiltonians. For Hamiltonians, the total energy plays the role of α and equilibria (Liapunov centers) play the role of Hopf bifurcations.

Readers not interested in Hamiltonian systems should assume throughout that all orbits are of types 0, 1, *or* 2. The paper also will be substantially easier to read if the reader assumes *all centers are generic Hopf points* (see Section 3 for definitions). Most applications of our results will probably be covered by these special cases.

We have not restricted ourselves to the simplest generic class of stationary points since cases of interest often involve nongeneric cases, such as the rest state of a frictionless multiple pendulum. A separate paper will discuss both the non-generic theory and the extension to Hamiltonian systems.

An example in [18] (see also an example in [19]) is given (see Fig. 1.4) in

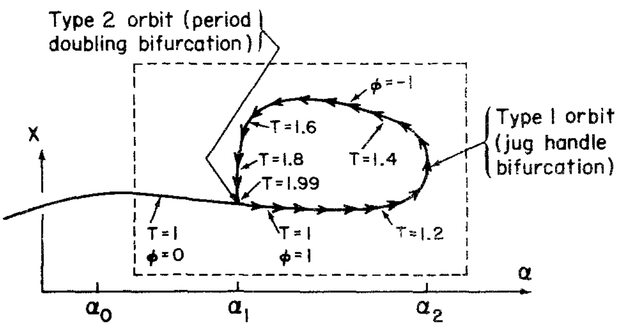

FIG. 1.4. When the orbit index ϕ is zero, global continuation is sometimes not possible. Each orbit is represented by one point and the arrows along the family indicate the value of the orbit index as in Fig. 1.1. This figure describes the continuation of an orbit at α_0. The continuation family enters and then remains inside a compact set (delimited by a dashed boundary), even though the period T remains bounded ($1 \leqslant T < 2$) and the compact set contains no centers. We say the α_0 obit is not "globally continuable." This behavior is possible only because the orbit at α_0 has orbit index $\phi = 0$. If ϕ was not 0 it would have to be globally continuable. This example is only realizable when $n \geqslant 4$. See [18].

which the jug handle and the period-doubling bifurcations are combined in a striking way. In this example an orbit at $\alpha = \alpha_0$ is followed by Poincaré continuation to a value $\alpha = \alpha_2 > \alpha_0$, passing an intermediate value $\alpha = \alpha_1$. The α_1 orbit has a multiplier -1 and is a period-doubling bifurcation orbit. The α_2 orbit is a jug handle bifurcation orbit and the family has α decreasing after we pass through the α_2 orbit. The period gradually increases (as α now decreases) to a value approaching twice the period of the α_1 orbit. Then the family closes up onto the orbit previously passed at α_1, thus effectively terminating the family. A further "extension" would retrace orbits already traversed. This example is stable in the sense that the behavior persists when the differential equation is perturbed slightly. The equation can even be chosen to be real analytic. The point of the example is that the maximal continuation for $\alpha > \alpha_0$ lies in a compact subset of R^{n+1} and the periods remain bounded and in particular the continuation contains no centers. For any orbit such as this α_0 orbit, the orbit index ϕ introduced here must be 0 since a non-zero orbit index guarantees that a more satisfactory global continuation must exist.

While it is not obvious from the text, this paper owes a great debt to the work of Fuller [5], and our orbit index arose from our failure to adapt the Fuller index to the problems considered here. Our results make no direct use of Fuller's index.

2. A Continuum of Orbits

Here we deal with families of orbits of Eq. (1.1). The term *orbit* refers exclusively to the trajectory of a non-constant *periodic* solution as a subset of (x, α)-space. For any orbit p, α we write $T(p, \alpha)$ for its (minimum) period, and the term "period" always means "minimum period." We sometimes write $T(x, \alpha)$ when (x, α) is on an orbit.

We emphasize several types of orbits in our discussion of generic behavior:

Type 0. An orbit having no multipliers that are roots of unity will be called type 0; that is, if $e^{i2\pi\theta}$ is a multiplier, then θ must be irrational.

When 1 is not a multiplier of p_0, α_0, we know it lies on a unique one-parameter family whose periods vary continuously. This family (the Poincaré continuation) can be parameterized by α for α near α_0, and we will denote the family p_α, α. For this main family we will write A_α for the $(n-1) \times (n-1)$ matrix $D_x G(p_\alpha, \alpha)$. When p_0 has one or more multipliers that are roots of unity, it is possible for additional families with longer periods to bifurcate from p_0, α_0.

Let p_0, α_0 be an orbit. We will say γ is an *arc* (of orbits) *emanating from*

p_0, α_0 if $\gamma: [0, 1) \to R^n \times R$ is continuous and is piecewise differentiable and satisfies the following conditions.

(A1) $\gamma(\beta)$ is on an orbit for each β, distinct β's giving distinct orbits, and $\gamma(0)$ is on the orbit p_0, α_0;

(A2) $T(\gamma(\beta))$ is continuous for $\beta \in (0, 1)$ but not necessarily at $\beta = 0$; and $\lim_{\beta \downarrow 0} T(\gamma(\beta))$ exists and is finite;

(A3) for any k the set $B(k, \gamma) = \{\beta \in [0, 1)$: the $\gamma(\beta)$ orbit has multipliers that are kth roots of unity$\}$ is at most countable and 1 is the only allowed limit point; and $\beta \in (0, 1)$ implies that neither $+1$ nor -1 are multipliers of the $\gamma(\beta)$ orbit.

It follows that for a dense set of $\beta \in (0, 1)$, the $\gamma(\beta)$ orbit is type 0. In addition, for each $\beta \in (0, 1)$, the fact that 1 is not a multiplier implies there is a unique continuation for which T is continuous. But (A2) assumes T is continuous along the arc, so the arc is in fact the Poincaré continuation of each of its orbits for $\beta \in (0, 1)$. Since the Poincaré continuation is parametrized by α, we have the following corollary property of the above assumptions.

(A4) The α coordinate α_β of an arc is a strictly monotonic function of β.

We will say an arc γ is an *m-arc* if $m = \lim_{\beta \downarrow 0} T(\gamma(\beta))/T(\gamma(0))$. Since f satisfies a Lipschitz condition in a neighborhood of the $\gamma(0)$ orbit, it follows that $m > 0$; see [6] or [7] or our proof of Proposition 3.1. In addition, it can be argued that m must be an integer, and in fact if $m > 1$ then p_0, α_0 must have a multiplier (other than 1) that is a root of unity. The question of what type of m-arcs can bifurcate from an orbit for non-generic orbits is analyzed in detail in [16].

We also say the family p_β, α_β for $\beta \in [0, 1)$ is an arc (or m-arc) if when we define $\gamma(\beta) = (p_\beta(0), \alpha_\beta)$, then γ is an arc or m-arc, respectively.

We say p_0, α_0 is a *regular bifurcation orbit* if it is not a type-0 orbit and there are a finite number of arcs γ_i, $i = 1, ..., k$, emanating from p_0, α_0, and the following conditions hold.

(B1) $\gamma_i(0)$ is on the p_0, α_0 orbit and the arcs are otherwise distinct; that is, $\gamma_i(\beta_1)$ is on the same orbit as $\gamma_j(\beta_2)$ for $i \neq j$ only if $\beta_1 = \beta_2 = 0$.

(B2) If p_j, α_j is any sequence of orbits with $p_j, \alpha_j \to p_0, \alpha_0$ and $T(p_j, \alpha_j)$ bounded, then for all but a finite number of j, there is some i and some $\beta \in [0, 1)$ such that $\gamma_i(\beta)$ is on the p_j, α_j obit.

We will say p_0, α_0 is a *type*-1 orbit if it is a regular bifurcation orbit and $+1$ is a multiplier and no other root of unity is a multiplier and there are precisely two arcs γ_1, γ_2 emanating from p_0, α_0 and the α coordinates $\alpha_1(\beta)$, $\alpha_2(\beta)$ both are strictly less than α_0 for all $\beta \in (0, 1)$ or both are strictly greater. Since 1 is the only multiplier that is a root of unity, $T(\gamma_i(\beta))$ will be continuous at $\beta = 0$ for both $i = 1$ and 2.

We say an orbit p_0, α_0 is *type* 2 (that is a period-doubling bifurcation) if it is a regular bifurcation orbit with three arcs emanating from it and -1 is a multiplier and there are no other multipliers that are roots of unity. Again writing A_α for $D_x G(p_\alpha, \alpha)$, we require in addition that

$$\det[A_\alpha^2 - I] \text{ changes sign at } \alpha = \alpha_0, \tag{2.1}$$

and, in particular, has an isolated 0 at $\alpha = \alpha_0$. As was shown in studies of zeroes of maps by Krasnoselski [8] using local degree arguments and by Rabinowitz [9] using the global degree theoretic arguments, condition (2.1) implies a connected family of fixed points of $G \circ G$ bifurcates from p_0, α_0. (They show that if the Jacobian of a map—$G \circ G$ (I in our case)—changes sign, when evaluated on a path of zeroes, then additional zeroes must bifurcate from that point. If the rate of change in (2.1) is non-zero at α_0, this result can be derived from the Implicit Function Theorem. Since orbits of the Poincaré continuation p_α, α correspond to all the fixed points of G near $p_0(0)$, α_0, it follows that the fixed points of $G \circ G$ represent orbits whose (minimum) period is approximately $2 \cdot T(p_0, \alpha_0)$. Two of the arcs are the Poincaré continuation (one for $\alpha > \alpha_0$ and one for $\alpha < \alpha_0$). The third must be a 2-arc.

When discussing Hamiltonian systems, another type of bifurcation frequently occurs. We say an orbit p_0, α_0 is *type m*, when m is an integer, $m \geqslant 3$, if it is a regular bifurcation orbit with either two or four arcs and the following conditions are satisfied.

(M1) For some integer k that is relatively prime to m, $e^{\pm i2\pi k/m}$ are simple multipliers and no other multipliers are roots of unity;

(M2) $\det[A_\alpha^m - I]$ does *not* change sign at α_0 and α_0 is an isolated zero;

(M3) Either there are two distinct monotonic m-arcs or there are none.

3. Stationary Points

Let $x_0 \in R^n$ be a stationary point for the equation $dx/dt = g(x)$. Let A_0 denote $D_x g(x_0)$. We will say T is a *virtual period of* x_0 (for g) if $T > 0$ and T is the period of some (non-constant) periodic orbit of the equation

$$dy/dt = A_0 y. \tag{3.1}$$

If the differential equation depends on parameter α, then we use the phrase

"T is a virtual period for (x_0, α_0)." Notice that if a stationary point has a virtual period, it must be a center.

The following result plays a key role in our theory.

3.1. Virtual Period Proposition. *Let $g_i: R^n \to R^n$ be C^2 functions for $i = 0, 1, \ldots$ with $g_i(x)$ and its first partial derivatives converging uniformly to $g_0(x)$ and its first partials as $i \to \infty$. For $i = 1, 2, \ldots$, let p_i be periodic solution with period T_i of the differential equation*

$$dy/dt = g_i(y).$$

Assume there is some x_0 such that $p_i(t) \to x_0$ as $i \to \infty$, uniformly in t, and assume $T_\infty = \lim T_i$ exists and is finite. Then $g(x_0) = 0$ and T_∞ is a virtual period of x_0 for g.

This result is proved in Appendix 2. The following variant is more obviously related to our situation.

3.2. Corollary. *Let $\{p_i, \alpha_i\}$ be a sequence of orbits of (1.1) converging to a single point (x_0, α_0) with the diameters of the orbits tending to 0. Assume $T(p_i, \alpha_i)$ is convergent to some number $T_\infty < \infty$. Then (x_0, α_0) is a center and T_∞ is a virtual period of (x_0, α_0).*

This improves a result in [10] which stated T_∞ was a multiple of the period of some periodic solution of Eq. (3.1). The corollary follows immediately from the proposition by letting $g_i(x) = f(x, \alpha_i)$.

Since 0 is not an eigenvalue of $D_x f$ for any isolated centers (x_0, α_0), the Implicit Function Theorem implies (x_0, α_0) lies on a (unique) continuous family of stationary points $(x(\alpha), \alpha)$ for α near α_0. Let $E(\alpha)$ denote the sum of the multiplicities of the eigenvalues of $D_x f(x(\alpha), \alpha)$ having strictly positive real parts. Let $E(\alpha_0+)$ and $E(\alpha_0-)$ denote right- and left-hand limits of E at α_0. Define the *crossing number* χ, the net number of pairs of eigenvalues crossing the imaginary axis at α_0, by

$$\chi = \tfrac{1}{2}(E(\alpha_0+) - E(\alpha_0-)).$$

The Chinese symbol for center is 中 (pronounced "tzong"). We define the *center index* of an isolated center (x_0, α_0) to be the product

$$\text{中}(x_0, \alpha_0) = \chi \cdot (-1)^{E(\alpha_0)}. \tag{3.2}$$

3.3. Example. Let the dimension n be even and let $k = n/2$. Let A be an $n \times n$ matrix with eigenvalues $\alpha_j \pm i\beta_j$ with $\beta_j \neq 0$ for $j = 1, \ldots, k$. Assume f in Eq. (1.1) satisfies $f(0, \alpha) \equiv 0$ and writing $A_\alpha = D_x f(0, \alpha)$, assume

$$A_\alpha = A + \alpha I.$$

Then $(0, \alpha_j)$ is an isolated center for Eq. (1.1). Since A_α has eigenvalues $\{\alpha + \alpha_j \pm i\beta_j\}$, the center index of each $(0, \alpha_j)$ is a positive integer, and if the α_j's are distinct, the center indices of these points are all $+1$.

While it is sometimes important in applications to investigate non-generic cases, the most common centers have $⏀ = \pm 1$ and a single family of orbits emanates from the center. In this case we say an isolated center is a *source* if $⏀ = +1$ and a *sink* if $⏀ = -1$. Hopf proved periodic orbits bifurcate from an isolated center, with the restriction of analyticity and $|\chi| = 1$ and some additional hypotheses. The case where $⏀$ is odd was dealt with in [10], and [11] showed that if $\chi \neq 0$ then orbits bifurcate from (x_0, α_0). The extra factor using $E(\alpha_0)$ is needed to understand how different isolated centers can be connected by families of orbits. Examples are easy to construct in which there is an isolated center having $\chi = 0$ and having no orbits nearby.

We will say x_0, α_0 is a *Hopf point* if the following conditions are satisfied.

(H1) The point (x_0, α_0) is a stationary point and $D_x f(x_0, \alpha_0)$ is non-singular. Hence there is a differentiable function $x(\alpha)$ for α near α_0 such that $(x(\alpha), \alpha)$ is a stationary point for each α. Let $L(\alpha) = D_x f(x(\alpha), \alpha)$.

(H2) The matrix $L(\alpha_0)$ has exactly one pair of eigenvalues $\pm i\theta$ that are pure imaginary, and these are algebraically simple. (Hence $2\pi/\theta$ is the virtual period of (x_0, α_0)).

(H3) Their continuous extension $r(\alpha) \pm i\theta(\alpha)$ (as eigenvalues of $L(\alpha)$) satisfies

$$\frac{d}{d\alpha} r(\alpha) \neq 0.$$

Under the additional unnecessary assumption that f is real analytic, Hopf proved that there exists a smooth family of non-constant periodic solutions $p(t, \beta)$, $\alpha(\beta)$ for $\beta \in (0, 1]$ with the properties that $\alpha(0) = \alpha_0$, $p(t, \beta) \to x_0$ as $\beta \downarrow 0$ and $T(p(t, \beta), \alpha(\beta)) \to 2\pi/\theta$ as $\beta \downarrow 0$. We will say $p(t, \beta)$, $\alpha(\beta)$ is the family *associated* with the Hopf point. He also showed that for any sequence of orbits converging to x_0, α_0 with a bound on the set of their periods, all but finitely many must lie on the associated family.

We will say a Hopf point is *generic* if $p(t, \beta)$, $\alpha(\beta)$ is type 0 for all $\beta \in (0, 1)$ and if for all sufficiently small $\beta > 0$

$$\frac{d}{d\beta} \alpha(\beta) \neq 0.$$

4. Generic Results

In this section we assume that every center is isolated and that each orbit is either a regular bifurcation orbit or is type 0. To simplify the exposition we prove the results in detail only in the case where each orbit is type m $(m \geqslant 0)$. Section 8 describes the modification necessary in the more general case. Let $P \subset R^n \times R$ be the union of the orbits and the centers of (1.1).

While it is possible to establish properties of the isolated centers of P (i.e., the sum of the center indices must be 0, provided P is bounded and T is bounded on P) it is more enlightening to examine subsets of P that are as small as possible so that the property is still valid. The subset of P might be bounded and have T bounded on it while P itself is unbounded. We will say $Q \subset P$ is *2-arcwise connected* if for any two points (x_0, α_0), (x_1, α_1) in Q there is a continuous γ: $[0, 1] \to Q$ such that

$$\gamma(0) = (x_0, \alpha_0) \qquad \text{and} \qquad \gamma(1) = (x_1, \alpha_1)$$

and $T(\gamma(\cdot))$ is bounded and if $\gamma(s)$ is on a type m orbit with $m \geqslant 3$ then we require

$$\lim_{\tau \uparrow s} T(\gamma(\tau)) = \lim_{\tau \downarrow s} T(\gamma(\tau)),$$

and, in particular, we require that these limits exist. These limits need not equal $T(\gamma(s))$. Notice that $T(\gamma(\cdot))$ can also be discontinuous when $\gamma(s)$ is a center or lies on a type-2 orbit. In the latter case the left and right hand limits of $T(\gamma(\cdot))$ can differ by a factor of 2, and thus we justify the "2" in the name. Write $p, \alpha \subset Q$ when the orbit $\{(p(t), \alpha)\}$ is a subset of Q.

We will say Q is a *2-component* if Q is 2-arc-wise connected and there are no strictly larger 2-arc-wise connected sets of which it is a subset. Each point of Q_0 lies in a 2-component. The intersection of two distinct 2-components can be non-empty but then it contains only type m orbits, $m \geqslant 3$.

4.1. Theorem. *Assume Q is a 2-component of P that is a bounded set in $R^n \times R$. Assume T is bounded on Q. Then Q contains a finite set (possibly empty) of (isolated) centers $c_i = (x_i, \alpha_i)$, $i = 1,..., k$, and*

$$\sum_{i=1}^{k} \text{⊕}(c_i) = 0.$$

A major objective in beginning research of this paper was to answer the following question. Suppose a 2-component Q contains a single isolated center with $\chi = \pm 1$ and suppose Q is bounded. Is T unbounded on Q? The above theorem gives the answer "yes." Motivated strongly by investigations of the Fuller Index, we hoped to be able to define an orbit index to be able to resolve this question, and we were amazed at how simple this index turned out to be in the generic case under investigation here. We now define this orbit index on type-0 orbits.

DEFINITION. Let $p(t)$ be a periodic solution of $\dot{x} = f(x)$. Suppose no roots of unity are multipliers, and there are σ^+ multipliers in $(1, \infty)$ and σ^- in $(-\infty, -1)$. That is, σ^+ and σ^- are the sums of the multiplicities of multipliers. Define the *orbit index* $\phi(p)$ to be

$$\begin{aligned} \phi(p) &= (-1)^{\sigma^+} \quad \text{if } \sigma^- \text{ is even,} \\ &= 0 \quad \text{if } \sigma^- \text{ is odd.} \end{aligned} \tag{4.1}$$

Since the Poincaré fixed point index of G is defined to be $(-1)^{\sigma^+}$, it follows that $\phi(p)$ is the Poincaré fixed index for G if σ^- is even. It is easy to check that the following formulation is equivalent to (4.1): ϕ is the average of the Poincaré fixed point indices of G and $G \circ G$.

The index ϕ will allow us to impose a natural orientation on families of orbits having non-zero orbit index at orbit p_0, α_0 with $\phi(p_0, \alpha_0) \neq 0$. The *positive direction along the family* is the direction in which α is increasing (decreasing) provided $\phi(p_0, \alpha_0) > 0$ (or, <0 respectively). In Fig. 1.1 we ilustrate a possible configuration representing each orbit by a single point and using arrows along the family to illustrate the orientation of the family. In Appendix 1 we describe how to define the index ϕ for a larger class of orbits.

In the next theorem we restrict consideration to orbits whose α coordinate lies in an interval I. To define a *2-component of* $P \cap (R^n \times I)$, substitute $P \cap (R^n \times I)$ for P in the above definitions. In particular, a 2-component of $P \cap (R^n \times I)$ is a connected subset of $P \cap (R^n \times I)$.

4.2. THEOREM. *Let $I = [a_1, a_2]$, where $a_1 < a_2$. Let Q be a 2-component of $P \cap (R^n \times I)$ such that if $(p, \alpha) \subset Q$ and $\alpha = a_1$ or a_2, then p, α is type 0, and if $(x, \alpha) \in Q$ is a center then $\alpha \neq a_1, a_2$. Then*

$$\sum_{(p,\alpha_2) \subset Q} \phi(p, a_2) - \sum_{(p,\alpha_1) \subset Q} \phi(p, a_1) = \sum \text{⊕}(c_i), \tag{4.2}$$

where the first two sums are taken over the orbits at a_2 and a_1 respectively and the last sum is taken over all isolated centers $c_i = (x_i, \alpha_i)$ in Q, $i = 1, \ldots, k$.

Notice that boundedness of T and Q implies the three sums are taken over finite sets. When the first two sums are taken over empty sets, this theorem reduces precisely to Theorem 4.1. (See Fig. 4.1.)

The results here are stated for finite dimensional situations, but in several

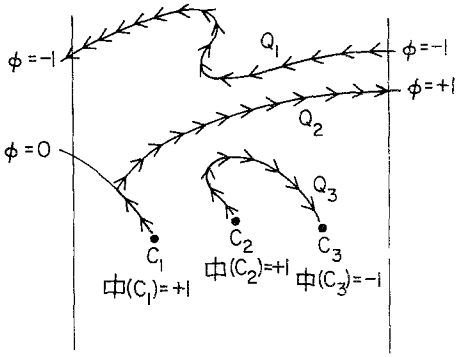

FIG. 4.1. Three examples of what Q might look like in Theorem 4.2. The arrows along families of orbits point to the right where $\phi=+1$ and to the left when $\phi=-1$.

infinite dimensional cases, there is sufficient compactness to guarantee that if Q is bounded and T is bounded on Q, then Q is compact, and to guarantee σ^-, σ^+, and $E(\alpha)$ are all finite. The results here would appear to extend naturally to such cases. See [12] and [13] for infinite-dimensional problems to which the results in this paper might extend.

We introduced our orbit index and it immediately appeared in Theorem 4.2 without explanation. We now wish to motivate more clearly its role in studying families of orbits. The index is in fact an invariant of an unusual sort. It is designed to be useful even if orbits are dense for each parameter value. In such a situation it would be useless to add the indices of all orbits in the x space and conclude that this is invariant under changes in α or invariant under perturbations. An alternative would be to restrict attention to all orbits whose period is less than some number. This is the approach in the Fuller index, and while we find the Fuller index an excellent tool for studying generalized Hopf bifurcation, it fails in practice to enable us to distinguish between the (minimum) period of an orbit and multiples of the period. Instead we restrict attention to connected sets of orbits, and the next proposition may be interpreted as saying ϕ is a *bifurcation invariant*.

Let $\gamma(\beta)=(x(\beta),\alpha(\beta))$ be a monotonic arc. The multiplier numbers σ^+ and σ^- are not necessarily constant, but as β is varied, these integers must change by even numbers because of the arc property (A3). Hence two type-0 orbits that are hit by the arc must have the same orbit index, and we write $\phi(\gamma)$ for the value. Recall that property (A4) guarantees each $\alpha_i(\beta)$ is strictly monotonic so $\operatorname{sign}(\alpha_i(\beta)-\alpha_0)$ is well defined for $\beta>0$.

4.3. *Bifurcation Invariance Proposition*

Let p_0, α_0 be a regular bifurcation orbit with monotonic arcs $\gamma_i(\beta)=(x_i(\beta),\alpha_i(\beta))$ emanating from it, $i=1,\ldots,k$, then

$$\sum_{i=1}^{k} \phi(\gamma_i) \operatorname{sign}(\alpha_i(\beta) - \alpha_0) = 0, \qquad \beta > 0. \tag{4.3}$$

This proposition can be reduced to a question of periodic orbits of a map by investigating the Poincaré map for p_0, α_0, and so we choose to put the proof in [16]. We now state some simple corollaries of Proposition 4.3.

The Bifurcation Invariance Proposition is of interest to us here when p_0, α_0 is of type m, $m \geqslant 1$. Let $b_i = \phi(\gamma_i)$, $i = 1,..., k$. When $m = 1$, there are two arcs and both are on the same side of α_0. Either $b_1 = b_2 = 0$ or $b_1 + b_2 = 0$, one being $+1$ and the other -1.

If p_0, α_0 is type 2, there are three arcs. Since 1 is not a multiplier there are two arcs that can be obtained from Poincaré continuation, whose indices we denote by b_1 and b_2. Since the definition of type 2 requires $\det[A_\alpha^2 - I]$ to change sign at α_0, the number of real eigenvalues λ of A_α^2, such that $\lambda > 1$ changes by an odd number as α crosses α_0. It follows that the number of eigenvalues λ of A_λ such that $\lambda < -1$ changes by an odd number. From Proposition 4.3 either b_1 or b_2 is 0, but not both. Bifurcation invariance then implies that the monotonic 2-arc has index $b_3 \neq 0$. If the two arcs with non-zero indices are on the same side of α_0, the indices must total 0 while if these arcs are on opposite sides, their indices must be equal.

If p_0, α_0 is type m with $m \geqslant 3$, there are two 1-arcs and either zero or two m-arcs. The two 1-arcs represent the Poincaré continuation and lie on opposite sides of α_0 and their indices b_1 and b_2 must be equal since the numbers σ^- and σ^+ are equal for the two arcs. If there are monotonic m-arcs, it follows that either they are on the same side of α_0 and their indices b_3, b_4 satisfy $b_3 = -b_4$, or they are on opposite sides and $b_3 = b_4$.

5. Paths of Orbits

In this section, we assume all centers are isolated and all orbits are of type m for some $m \in \{0, 1, 2,...\}$.

In order to prove the results in Section 4, it is necessary to develop some results on paths of orbits, results which are interesting on their own. We will say γ is a *path* if the domain J of γ is an interval and the following properties are satisfied.

(P1) The function $\gamma: J \to R^n \times R$ is continuous.

(P2) For all $\beta \in J$, $\gamma(\beta)$ is on an orbit. (When non-type m orbits are permitted we require that $\gamma(\beta)$ is on an orbit of type m $(m \geqslant 0)$.)

(P3) If β_1 is in the interior of J and $\gamma(\beta_1)$ is on an orbit of type m, $m \geqslant 3$, then for $\beta \neq \beta_1$ we require

$$\lim_{\beta \uparrow \beta_1} T(\gamma(\beta)) = \lim_{\beta \downarrow \beta_1} T(\gamma(\beta)).$$

(P4) If $\gamma(\beta_1)$ and $\gamma(\beta_2)$ are on the same orbit where β_1 and β_2 are distinct points in the interior of J, then the orbit is type m with $m \geqslant 3$.

The last condition rules out the situation in which a path runs back and forth over the same set of orbits. Notice that $T(\gamma(\beta))$ can have essential discontinuities (at type-2 orbits) and at such a discontinuity β_1, the left- and right-hand limits of $T(\gamma(\beta))$ at β_1 must differ by a factor of 2.

If γ is a path, we write dom(γ) for the interval on which it is defined. We say p_β, α_β or $p(t, \beta)$, $\alpha(\beta)$ is a path if the function $\gamma(\beta) =^{\text{df}} (p(0, \beta), \alpha(\beta))$ is a path. For a path γ define $T^{\text{ess}}(\gamma, \beta_0) = T(\gamma(\beta_0))$ unless $\gamma(\beta_0)$ ison an orbit of type m with $m \geqslant 3$. In that case define

$$T^{\text{ess}}_{\beta_0} = T^{\text{ess}}(\gamma, \beta_0) = \lim_{\substack{\beta \to \beta_0 \\ \beta \neq \beta_0}} T(\gamma(\beta)).$$

We have $T^{\text{ess}}_{\beta_0} = T(\gamma(\beta_0))$ except possibly in the case that $\gamma(\beta_0)$ is type m for $m \geqslant 3$, in which case we can alternatively have $T^{\text{ess}}(\gamma, \beta_0) = mT(\gamma(\beta_0))$.

Let the *modulus* of γ at β be

$$M(\gamma, \beta) = T^{\text{ess}}_{\beta} + |\alpha_\beta| + \max_t \|\gamma(\beta)(t)\|,$$

where $\gamma(\beta)(t)$ is x coordinate at time t of the trajectory starting from $\gamma(\beta)$ at time 0. Let $\gamma(\beta) = (p_\beta, \alpha_\beta)$. If β_0 is an end point of dom(γ) where we permit $-\infty \leqslant \beta_0 \leqslant \infty$, define the *limit set at* β_0 to be $\Lambda = \Lambda(\beta_0) = \{(x_1, \alpha_1)$: there is a sequence $\beta_i \to \beta_0$ such that $T(\gamma(\beta_i))$ is bounded and $\alpha(\beta_i) \to \alpha_1$, and there is a sequence $\{t_i\}$ such that $p(\beta_i, t_i) \to x_1\}$.

5.1. Proposition. *Let* $\gamma(\beta) = p_\beta$, α_β *be a path whose domain is a bounded interval. Let* β_0 *be an end point of* $J = \text{dom}(\gamma)$. *Assume*

$$\liminf_{\beta \to \beta_0} M(\gamma, \beta) < \infty. \tag{5.1}$$

Then the limit set Λ *at* β_0 *is either an orbit or* $\Lambda = \{(x_1, \alpha_1)\}$, *where* (x_1, α_1) *is an isolated center. Furthermore*

$$d(\gamma(\beta), \Lambda) \to 0 \qquad \textit{as} \quad \beta \to \beta_0 \tag{5.2}$$

and

$$\limsup_{\beta \to \beta_0} T(\gamma(\beta)) < \infty. \tag{5.3}$$

Proof of Proposition 5.1. Assume (5.1) is satisfied. Then there is a sequence $\{\beta_i\}$ in J and a point (x_1, α_1) and a $T_1 > 0$ such that $\beta_i \to \beta_0$ and

$$\gamma(\beta_i) \to (x_1, \alpha_1) \qquad \text{and} \qquad T(\gamma(\beta_i)) \to T_1.$$

In fact $T_1 < 0$. This follows easily if $f(x_1, \alpha_1) \neq 0$, and if (x_1, α_1) is a stationary point $T_1 > 0$ follows from arguments similar to those about $T_\infty > 0$ in Appendix 2. Since $p(\beta_i, 0) \to x_1$ and $\alpha(\beta_i) \to \alpha_1$ and the solutions $p(\beta_i, t)$ are periodic with periods T_i tending to T_1, the solution through (x_1, α_1) must "return" to (x_1, α_1) at time T_1. Possibly (x_1, α_1) is a stationary point (and then by Corollary 3.2, it is a center and all centers are isolated), but suppose now that it is not. Then it lies on an orbit p_1, α_1 of type $m \geqslant 0$. Since p_1, α_1 is a bifurcation orbit or is type 0, all but a finite number of the $\gamma(\beta_i)$ must lie on the arcs emanating from p_1, α_1, and it follows that for all β near β_0, $\gamma(\beta)$ must lie on the orbits in the arcs emanating from p_1, α_1. Since those arcs are distinct and $\gamma(\beta)$ can hit the orbit p_1, α_1 at most once, $\gamma(\beta)$ must lie on a single arc emanating from p_1, α_1, and it follows easily that Λ is the p_1, α_1 orbit. Therefore if Λ contains any point (x_1, α_1) that is not a center, Λ is an orbit, and clearly $d(\gamma(\beta), \Lambda) \to 0$ as $\beta \to \beta_0$.

We may assume now Λ is a union of centers. Suppose that (5.2) fails so that there is a sequence $\beta_i^* \to \beta_0$ for which

$$\varepsilon_0 = \inf_i d(\gamma(\beta_i^*), (x_1, \alpha_1)) > 0,$$

where d is the metric on $R^n \times R$. Let $\Pi \subset R$ be the set of virtual periods for (x_1, α_1). By Corollary 3.2, Π is non-empty. Let $m_0 = \max \Pi$. Let $B(\varepsilon)$ denote the closed ball in $R^n \times R$ with center (x_1, α_1) and radius ε. We say *an orbit p, α is in $B(\varepsilon)$* if $(p(t), \alpha) \in B(\varepsilon)$ for all t. Corollary 3.2 implies we may choose $\varepsilon > 0$ sufficiently small that no orbit in $B(\varepsilon)$ has its period in $[3m, 9m]$. In addition, choose $\varepsilon \in (0, \varepsilon_0)$ sufficiently small that (x_1, α_1) is the only center in $B(\varepsilon)$. We may assume the sequences are chosen so that $\gamma(\beta_i)$ is on an orbit in $B(\varepsilon)$ and $T(\gamma(\beta_i)) < 3m_0$ and so that

$$\beta_i < \beta_i^* < \beta_{i+1} \qquad \text{for all } i.$$

Since $\gamma(\beta_i^*) \notin B(\varepsilon)$ and since $T^{\text{ess}}(\beta)$ is continuous except for discontinuities at which it changes by a factor of 2, there must be some $\beta_i^{**} \in (\beta_i, \beta_{i+1})$ such that $\gamma(\beta_i^{**})$ lies on an orbit in B and that orbit intersects the boundary of B and $T(\gamma(\beta_i^{**})) < 3m_0$. Let (x_3, α_3) be a limit point of $\gamma(\beta_i^*)$. Then $(x_3, \alpha_3) \in \Lambda$, and since it is the limit of points whose orbits touch boundary $B(\varepsilon)$ and have periods less than $3\beta_0$, it must be a center but cannot be (x_1, α_1), a contradiction. Hence $\Lambda = \{(x_1, \alpha_1)\}$, and $\gamma(\beta) \to (x_1, \alpha_1)$ as $\beta \to \beta_0$. In addition, lim sup $T(\gamma(\beta)) < 3m_0$ so (5.3) holds. ∎

Let γ and γ_0 be paths. We will say γ is an *extension* of γ_0 if for each $\beta_0 \in \text{dom}(\gamma_0)$ there is a $\beta \in \text{dom}(\gamma)$ such that $\gamma_0(\beta_0)$ and $\gamma(\beta)$ are on the same orbit. Notice that a path is an extension of itself. We will say an extension γ of γ_0 is *strict* if γ_0 is not an extension of γ, that is, if γ hits at least one orbit that is not hit by γ_0. If γ_0 has no strict extensions, we will say it is *maximal* path.

Suppose a family of orbits bifurcate from a center. It is naive to expect a simple answer when asking where it goes since in continuing the family bifurcations can occur, and choices must be made as to which branch to follow (unless we can formulate a general answer involving the entire "tree"). Condition (P3) tells how to continue at type-m orbits $m \geqslant 3$. At type type-2 orbits some choice is left. Imagine a hypothetical situation in which the family passes through a type-2 orbit, approaching along the remaining arc, that is, the path "doubles back." It is easy to make choices in Fig. 1.1 so that this situation might arise. The resulting path is maximal and terminates at a type-2 orbit.

Let γ be a path whose domain J is bounded. We say γ is *open* (*closed*) *at an endpoint* β_0 if β_0 is an endpoint of J and $\beta_0 \notin J$ (or, $\beta_0 \in J$, respectively). When γ is an open at an endpoint β_0, one possible behavior is that the γ family of orbits is *convergent to a center at* β_0. We mean by this phrase that $\gamma(\beta)$ converges to a center as $\beta \to \beta_0$ and diameter of the $\gamma(\beta)$ orbit tends to 0 as $\beta \to \beta_0$ and $\limsup_{\beta \to \beta_0} T(\gamma(\beta)) < \infty$. In particular, $\Lambda(\beta_0)$ contains exactly one point.

5.2. PROPOSITION. *Let γ be a maximal path that is open at an endpoint β_0. Then either*

$$M(\gamma, \beta) \to \infty \qquad \text{as} \quad \beta \to \beta_0 \tag{5.4}$$

or the γ family of orbits is convergent to a center at β_0.

Proof. Let $\gamma(\beta)$ be a path and assume β_0 is an end of its domain J and $\beta_0 \notin J$. Assume $\liminf_{\beta \to \beta_0} M(\beta) < \infty$. Then by Proposition 5.1 there is a limit point (x_0, α_0) of $\gamma(\beta)$ as $\beta \to \beta_0$. Suppose (x_0, α_0) is not a center. Then for β near β_0 the points $\gamma(\beta)$ lie on the orbits of an arc emanating from the (x_0, α_0) orbit so $T(\gamma(\beta))$ has a limit as $\beta \to \beta_0$. Redefining $\gamma(\beta)$ if necessary without changing the orbit that $\gamma(\beta)$ is on, we can assume $\gamma(\beta)$ has a limit as $\beta \to \beta_0$. Therefore γ can be extended to a path that is defined at β_0, contradicting our assumption that γ is maximal. Hence (x_0, α_0) must be a center. It is easy to show that γ is convergent to this center. ∎

Let γ be a path with domain J. We say γ is a *cyclic path* if J is a closed interval $[\beta_0, \beta_1]$ of positive length, and $\gamma(\beta_0)$ and $\gamma(\beta_1)$ are on the same orbit.

5.3. PROPOSITION. *Let γ be a path which is maximal and cyclic. Then*

$$\lim_{\substack{\beta \to \beta_1 \\ \beta \neq \beta_1}} T(\gamma(\beta)) = \lim_{\substack{\beta \to \beta_0 \\ \beta \neq \beta_0}} T(\gamma(\beta)) \tag{5.5}$$

and the $\gamma(\beta_0)$ orbit is not type 2.

Proof. A type-2 orbit has three arcs emanating from it. If the $\gamma(\beta_0)$ orbit was type 2, then one of the arcs would be traversed for β near β_0 and other when β was near β_1, leaving the third arc untraversed. Hence, the path would not be maximal. Hence the maximality of a cyclic path implies the $\gamma(\beta_0)$ orbit is not type 2. We may assume then the $\gamma(\beta_0)$ orbit is type m, $m \neq 2$, and therefore has an even number of arcs emanating from it and these come in pairs with similar T values. It is possible that there are points β interior to J for which $\gamma(\beta)$ is also on the orbit, but condition (P3) guarantees that if (5.5) was not satisfied, γ would not be maximal. ∎

5.4. Double-Back Proposition. *Let γ be a maximal path that is not a cyclic path. Let $J = \text{dom}(\gamma)$. Let β_0 be an endpoint and assume γ is closed at β_0. Then there exists β_1 in the interior of J such that $\gamma(\beta_1)$ and $\gamma(\beta_2)$ are on the same orbit and that orbit is type 2.*

The proof follows from arguments like those of Proposition 5.3.

In the next section we describe how it is possible to avoid this situation in which a maximal path that terminates by doubling back to a type-2 orbit.

6. Snakes

We will say a path γ with domain J_0 is *oriented on the interval* $J \subset J_0$ if it satisfies the following conditions.

(S1) For all but countably many $\beta \in J$, the orbit index $\phi(\gamma(\beta))$ is defined and is non-zero;

(S2) If $\phi(\gamma(\beta_0)) > 0$ for some β_0, then the α coordinate $\alpha(\beta)$ of $\gamma(\beta)$ is strictly monotonically increasing near β_0 and if $\phi(\gamma(\beta)) < 0$, then $\alpha(\beta)$ is strictly decreasing locally.

We say γ is *oriented* if it is oriented on the interval J_0. We say an oriented path γ is a *maximal oriented path* if it has no strict extension that is oriented. For a path γ define $\text{Orb}(\gamma)$ to be the set of orbits for which there is a β such that $\gamma(\beta)$ is on that orbit.

Let S be a set of orbits. We say S is a *snake* if there is a maximal oriented path γ such that $\text{Orb}(\gamma) = S$.

We say path γ_1 is a path *through* the snake S if γ_1 is a maximal oriented path with $S = \text{Orb}(\gamma_1)$. If p_0, α_0 is a type-0 orbit in a snake S, then $\phi(p_0, \alpha_0) \neq 0$. Conversely if p_0, α_0 is an orbit whose orbit index is non-zero, then it lies in some snake. This follows since there is a path γ with $\gamma(0)$ on p_0, α_0, and either γ or the path $\gamma(-\beta)$ is oriented on some neighborhood J of 0. Suppose γ is the one that is oriented near 0. Then $\gamma \mid J$ is an oriented path and so there must be an extension γ_1 of $\gamma \mid J$ that is a maximal oriented path, so p_0, α_0 is in the snake $\text{Orb}(\gamma_1)$.

6.1. Uniqueness Proposition. *Let S_1 and S_2 be snakes. Assume*

$$S_1 \cap S_2 \tag{6.1}$$

contains an orbit p_0, α_0 of type 0, 1, *or* 2. *Then*

$$S_1 = S_2. \tag{6.2}$$

If γ_1 and γ_2 are maximal oriented paths and $\gamma_1(\beta_1)$ and $\gamma_2(\beta_2)$ are on the same orbit from some β_1, β_2 and if (6.2) does not hold, then it follows that orbit must be type m, $m \geqslant 3$, and

$$T^{\text{ess}}(\gamma_1, \beta_1) \neq T^{\text{ess}}(\gamma_2, \beta_2).$$

The Uniqueness Proposition is proved by studying the relationship a snake S and an arbitrary path γ that is only oriented on an interval. Let γ be a path with domain J such that for some $\beta_0 \in J$, $\gamma(\beta_0)$ is on an orbit in S having type 0, 1, or 2. Let $J_S \subset J$ be the largest interval containing β_0 for which $\text{Orb}(\gamma \mid J_1) \subset S$. Assume γ is oriented on an interval containing β_0. Let $J_0 \subset J$ be the largest interval containing β_0 on which γ is oriented.

Claim 1. $J_0 = J_S$.

Claim 2. *If $\beta_1 \in \text{int}\, J$ is an endpoint of J_0 then the $\gamma(\beta_1)$ orbit is type* 2.

Claim 1 is more general than Proposition 6.1 since if $S_1 = S$ and γ is a curve through S_2 (with $\gamma(\beta_0)$ on an orbit in $S_1 \cap S_2$), Claim 1 implies $J_0 = J$ and so $S_2 = \text{Orb}(\gamma) \subset S_1$. By symmetry we also must have $S_1 \subset S_2$, so (6.2) holds.

Proof of claims. Let $\beta_1 \in \text{int}(J)$ be such that $\gamma(\beta_1)$ is on an orbit p_1, α_1 of type m where $m \neq 2$. Assume $\beta_1 \in J_0$ and $\beta_1 \in J_S$. We first prove Claim 2. For some sufficiently small $\varepsilon > 0$ the restricted paths $\gamma \mid (\beta_1, \beta_1 + \varepsilon)$ and $\gamma \mid (\beta_1 - \varepsilon, \beta_1)$ lie on two different arcs emanating from p_1, α_1. Since one of these intervals lies in J_0 (and in J_S), ϕ is non-zero on these arcs (that is, on the type 0 orbits of that arc). If $m = 0$ or 1, there are two arcs emanating from p_1, α_1 and from Proposition 4.3 and the discussion after it, ϕ is non-zero on both arcs since it is non-zero on one of them. For a type-0 orbit, ϕ has the same value on both arcs and for type 1, ϕ has opposite sign on the two arcs. It follows that since γ is oriented on one of these arcs, it is oriented on both. Hence $m = 0$ or 1 implies $\beta_1 \in \text{int}(J_0)$. If $m \geqslant 3$, then the two arcs of p_1, α_1 that γ passes through must either be 1-arcs or both must be m-arcs from condition (P3). Since the number σ^+ (and also σ^-) is the same on both 1-arcs of a type m orbit and the α coordinate is increasing for both or is decreasing for both the path would be oriented in the neighborhood of β_1 if it traversed the 1-arcs. Knowing this about 1-arcs, it is easy to show from (4.3) that γ would also be oriented in a neighborhood of β_1 if it traversed the m-arcs. Hence Claim 2 is true. To prove Claim 1, notice similarly that if $\beta_1 \in J_S$ and the $\gamma(\beta_1)$ orbit is not type 2, we have $\beta_1 \in \operatorname{int} J_S$.

We have then proved: *if* $\beta_1 \in \operatorname{int} J$ *and* b_1 *is an endpoint of either* J_0 *or* J_S, *then* $\gamma(\beta_1)$ *i on a type-2 orbit.*

Each type-2 orbit p_2, α_2 has three arcs emanating from it, and ϕ is non-zero on two of these and is zero on the third, and from (4.3) there is an oriented path passing through it. Hence p_2, α_2 lies in a snake and the two arcs on which ϕ is non-zero must also lie in that snake (and in any snake containing p_2, α_2), while the third arc cannot lie in any snake since ϕ is 0 on it. Hence if β_1 is in $J_0 \cap J_S$, then it is an endpoint of J_0 if and only if it is an endpoint of J_S. ∎

We say a path γ is *orientable* if whenever $\gamma(\beta)$ is on an orbit of type 0, $\phi(\gamma(\beta)) \neq 0$. In the above arguments we describe what a path must be like if it is oriented on some interval but not on its entire domain. There is a maximal interval J_0 on which it is oriented and immediately outside J_0 but in dom γ, the path travels along an arc on which ϕ is 0. The following corollary is therefore immediate.

6.2. COROLLARY. *Let* γ *be an orientable path. Define* γ^* *by* $\gamma^*(\beta) \equiv \gamma(-\beta)$. *Then either* γ *is oriented or* γ^* *is oriented.*

If γ_1 and γ_2 are two paths through a snake S, then γ_1 and γ_2 are both closed paths or neither is, and both or neither are open. We therefore say S is *open* or *cyclic* if the paths through it are open or cyclic, respectively.

6.3. THEOREM: SNAKE TERMINATION PRINCIPLE. *Let* S *be a snake. Then* S *is either open or it is cyclic.*

A perusal of the proofs in Section 5 shows the arguments for Propositions 5.2, 5.3, and 5.4 remain valid if instead of considering all paths, only orientable paths γ are considered. That is, the conclusions remain valid in the class of orientable paths, so "maximal path" would be replaced by "maximal" in the class of orientable paths. Proposition 5.4, however, is satisfied vacuously in this class. If a maximal path doubles back, Proposition 5.4 in effect says that all three arcs emanating from some type-2 orbit must be traversed, which is impossible for an orientable path. Hence a maximal orientable path is either open or cyclic. Corollary 6.2 says that a maximal orient*ed* path is a maximal orient*able* path. ∎

If a center (x_0, α_0) is the source or sink for some path through a snake S, then it is the source or sink respectively for all paths through S. Hence we say (x_0, α_0) is the *source* (or *sink*) *of a snake* S if it is the source or sink respectively of the paths through S. Similarly if

$$M(\gamma(\beta)) \to \infty \qquad \text{as} \quad \beta \to \sup \operatorname{dom}(\gamma)$$

for one path γ through S, it is true for all its paths and we say ∞ *is the sink of* S. If

$$M(\gamma(\beta)) \to \infty \qquad \text{as} \quad \beta \to \inf \operatorname{dom}(\gamma)$$

for some path γ through S, we say ∞ *is the source of* S. Using

PROPOSITION 5.2 (we may now restate the Snake Termination Principle). *The source of an open snake S is either ∞ or a center, and its sink is either ∞ or a center. If S is cyclic, it has no ends. In fact S is cyclic if and only if the union of the orbits in S is compact.*

Proof. For a Hopf point (x_0, α_0) the crossing number χ equals sign

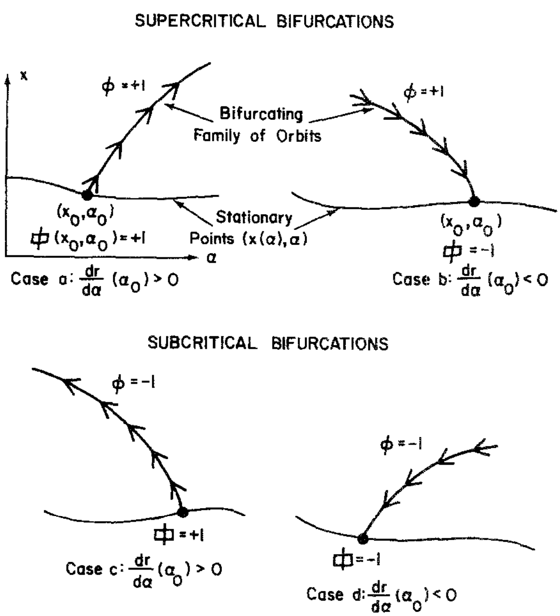

FIG. 7.1. Generic Hopf point (x_0, α_0) in $R^2 \times R$. The oriented families of orbits are indicated by thick curves whose arrows are directed to the right (increasing α) when $\phi = +1$ and to the left when $\phi = -1$. The real part of the eigenvalues of $D_x f(x(\alpha), \alpha)$ is $r(\alpha)$. To two supercritical cases (see text) and the two subcritical cases are shown. It is important to notice that supercritical bifurcation can have ⌽ $= +1$ or $= -1$ as can subcritical. Hopf points with ⌽ $= +1$ are called sources and the arrows along the curve on the orbits lead away from the centers that are sources while they lead to the sinks (⌽ $= -1$).

$[(dr/d\alpha)(\alpha_0)]$ (using r as in the definition of Hopf points). Special insight is provided when the dimension n is 2. Then $E(\alpha_0) = 0$ so

$$\boxplus(x_0, \alpha_0) = \chi$$

and near a generic Hopf point every periodic orbit has exactly one multiplier and it is positive. Hence $\sigma^- = 0$ and

$$\phi(p_\beta, \alpha_\beta) = (-1)^{\sigma^+}$$

In Fig. 7.1 cases (a) and (b) show the *supercritical* (also called normal) bifurcations, that is

$$\text{sign}(dr/d\alpha)(\alpha_0) = \text{sign}[\alpha(\beta) - \alpha_0] \qquad \text{for} \quad \beta > 0.$$

When $n = 2$, supercritical bifurcation implies the orbits near (x_0, α_0) are attractors so $\sigma^+ = 0$ and $\phi(p_\beta, \alpha_\beta) = +1$. Therefore Eq. (7.1) holds in these cases. In the *subcritical* cases (c) and (d) (inverted bifurcations) characterized by

$$\text{sign } r'(\alpha_0) = -\text{sign}[\alpha(\beta) - \alpha_0] \qquad \text{for} \quad \beta > 0,$$

the orbits near the center must be unstable with $\sigma^+ = +1$. Hence $\phi(p_\beta, \alpha_\beta) = -1$. Hence we have shown Eq. (7.1) is true when $n = 2$.

The Center Manifold Theorem (CMT) is used to prove the Hopf Theorem in [14]. Their restriction $E(\alpha_0) = 0$ is unnecessary in finite dimensions. The CMT shows that there is a three dimensional manifold M containing the curve of stationary points $(x(\alpha), \alpha)$ for α near α_0 and in particular (x_0, α_0). The manifold M is invariant under trajectories near (x_0, α_0).

The real two dimensional subspace of R^n corresponding to the imaginary eigenvalues $\pm i\theta$ is tangent to the manifold at (x_0, α_0). When $n = 2$, M is an open neighborhood of (x_0, α_0). See [14] for a more complete description. It follows that the cases in Fig. 7.1 display the possible behaviors on M, where $\boxplus$ and ϕ are calculated for the flow in M. (In particular, the bifurcating family of orbits must lie in M.) We will write $\boxplus^M$ and ϕ^M when these numbers are computed for the differential equation on M and we write $\boxplus^{(n+1)}$ and $\phi^{(n+1)}$ when calculation is made using the full differential equation on $R^n \times R$. Calculating $E(\alpha_0)$ for the full system we obtain immediately

$$\boxplus^M = \boxplus^{(n+1)}(-1)^{E(\alpha_0)}. \tag{7.2}$$

For each orbit near (x_0, α_0) the period is approximately $2\pi/\theta$ (that is, $T(p(t, \beta), \alpha(\beta)) \to 2\pi/\theta$ as $\beta \to 0$, as mentioned in Section 3). If λ is an eigenvalue of $D_x f(x_0, \alpha_0)$, then each orbit near (x_0, α_0) has a multiplier approximately $\exp(\lambda 2\pi/\theta)$. Complex conjugate pairs λ, λ^* of eigenvalues yield multipliers in pairs in the sense that for each orbit sufficiently near (x_0, α_0), there is a pair of multipliers near $\exp(\lambda 2\pi\theta)$ and $\exp(\lambda^* 2\pi\theta)$ and both multipliers are real or both are off the real axis and for each of the eigenvalues λ other than $\pm i\theta$, we have $\mathrm{Re}(\lambda) \neq 0$. Hence the pair of multipliers are both strictly outside the unit circle or both are inside. If real, both are positive or both negative. From these facts it follows that $(\sigma^-)^{(n+1)}$ is even (while $(\sigma^-)^M$ is 0). Hence ϕ^M and $\phi^{(n+1)}$ are both non-zero for orbits near the center. Further

$$(\sigma^+)^M + E(\alpha_0) = (\sigma^+)^{(n+1)} \bmod 2$$

that is, both sides are odd or both are even, and therefore

$$\phi^M(p_\beta, \alpha_\beta)(-1)^{E(\alpha_0)} = \phi^{(n+1)}(p_\beta, \alpha_\beta).$$

Together with Eq. (7.2), this proves Eq. (7.1) holds. Notice $\mathrm{sign}[\alpha(\beta) - \alpha_0]$ does not depend on the choice of M vs R^{n+1}. ∎

We will say a snake is a *tail-biter* if it is open and its source and sink are the same isolated center. It is possible for an isolated center to have an infinite number of tail-biter snakes, though most would be rather small. In fact for any sufficiently small closed neighborhood N of an isolated center c_0, only finitely many snakes that have c_0 as source or sink can intersect the boundary of N. If there were infinitely many there would be infinitely many orbits p_i, α_i (on distinct snakes) lying wholly in N, but touching the boundary of N. The orbits could furthermore be chosen so that $T(p_i, \alpha_i) \leqslant 2T_0$, where T_0 is the maximum of the virtual periods of (x_0, α_0). There would then be a cluster point (x_1, α_1) on the boundary of N, and (x_1, α_1) could not be on an orbit, since that would be a bifurcation orbit with a finite number of arcs emanating from it. The nearby orbits (p_i, α_i) would be on infinitely many distinct arcs emanating from the (x_1, α_1) orbit since they are distinct snakes. Therefore (x_1, α_1) would be a stationary point and it would have to have a virtual period $t_1 \leqslant 2T_0$. But then every small neighborhood N of c_0 would contain a center on its boundary contradicting our assumption that centers are isolated.

For any center c_0 let $s^-(c_0)$ and $s^+(c_0)$ be the numbers of snakes, excluding tail-biters, that have c_0 as their source or sink, respectively. By the above remark, s^+ and s^- are finite.

7.2. PROPOSITION. *Let c_0 be an isolated center. Then*

$$⊕(x_0, \alpha_0) = s^-(c_0) - s^+(c_0). \tag{7.3}$$

In our terminology, Hopf proved there is a unique snake emanating from a Hopf point c_0. Lemma 7.1 says that if c_0 is a generic Hopf point and

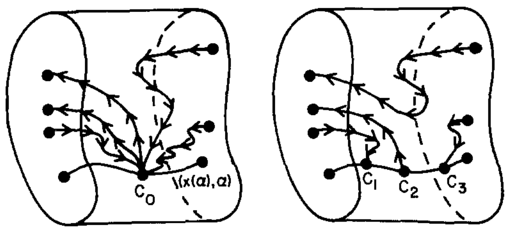

FIG. 7.2. Perturbing a non-generic center C_0 to a generic situation.

$\oplus(c_0)=+1$, then c_0 is the source of the snake (and $s^-(c_0)=+1$ and $s^+(c_0)=0$) while if $\oplus=-1$, c_0 is the snakes sink and $s^-=0$ and $s^+=1$. Hence Eq. (7.3) is now proved for generic Hopf points.

The idea of the proof is to perturb f slightly in a neighborhood of c_0 so that c_0 is replaced by a collection of generic Hopf points. Hence we investigate centers and snakes of

$$dx/dt = f(x,\alpha) + g(x,\alpha). \tag{7.4}$$

Write $C(f+g,Q)$ for the centers of Eq. (7.4) in $Q \subset R^{n+1}$, while $C(f,Q)$ will denote the centers in Q for Eq. (1.1). Let T_0 be the maximum virtual period of c_0 for f. We consider only perturbations $g\colon R^n \times R \to R^n$ which are identically 0 outside a specified neighborhood Q of c_0 in R^{n+1}, perturbations g for which $f(x,\alpha)=0$ if and only if $f(x,\alpha)+g(x,\alpha)=0$. We let $F(\varepsilon,Q)$ denote the set of such g whose C^1 norm satisfies $\|g\|_{C^1}<\varepsilon$. Proposition 3.1 plays a key role. If S is snake that enters Q and terminates at c_0, we must argue that the corresponding snake for Eq. (7.4) either leaves Q or terminates at a Hopf point, that is the periods of the orbits of the snake do not tend to ∞ while the snake remains in Q.

Proof. Let $B(\varepsilon)$ be the ball of radius ε in R^{n+1} with center c_0.

Let $J(\varepsilon)=[\alpha_0-\varepsilon, \alpha_0+\varepsilon]$.

Choose $\varepsilon_0>0$ so that the stationary points in $B(\varepsilon_0)$ lie on a curve that may be written $(x(\alpha),\alpha)$, and so that c_0 is the only center in $B(\varepsilon_0)$ and is also the only center in the family $(x(\alpha),\alpha)$ for $\alpha\in J(\varepsilon_0)$, and so that $D_x f(x(\alpha),\alpha)$ does not have 0 as an eigenvalue when $\alpha \in J(\varepsilon_0)$. We will consider the set of perturbations $F(\varepsilon,Q)$ only when $Q\subset B(\varepsilon_0)$. We further require ε_0 be chosen sufficiently small that no periodic orbit of (7.4) lying wholly in $B(\varepsilon_0)$ has period in $[2T_0, 5T_0]$ when $g\in F(\varepsilon_0,\beta(\varepsilon_0))$. That this will be true for some $\varepsilon_0>0$ follows from Proposition 3.1. If there were no such ε_0, there would be a sequence $g_i\to 0$ in the C^1 norm and a convergent sequence of periodic solution p_i, α_i of $x'=f+g_i$ such that p_i, $\alpha_i\to(x_0,\alpha_0)$ and $T(p_i,\alpha_i)\in [2T_0, 5T_0]$. By Proposition 3.1, c_0 would have a virtual period for f in that interval, contradicting the maximality of T_0. When $g \in F(\varepsilon_0, \beta(\varepsilon_0))$ is chosen so that the centers in $C(f + g, R^{n+1})$ are isolated, we have

$$\sum_{c \in C(f+g, B(\varepsilon_0))} ⊕_{f+g}(c) = ⊕_f(c_0), \tag{7.5}$$

where $⊕_{f+g}$ and $⊕_f$ denote the center index calculated for Eqs. (7.4) and (1.1), respectively. This equality holds because the sum of the crossing indices $\chi_{f+g}(c)$ must equal the crossing index $\chi_f(c_0)$ and because the number e of real positive eigenvalues of $Df(x(\alpha), \alpha)$ is independent of α for $\alpha \in J(\varepsilon_0)$ and equals the number of real positive eigenvalues of $D_x[(f + g)(x, \alpha)]$ for all $g \in F(\varepsilon_0, \beta(\varepsilon_0))$. Hence in the formula [Eq. (3.2)] defining $⊕_f(c_0)$, the factor $(-1)^{E(\alpha_0)}$ equals $(-1)^e$ and also equals the corresponding factor for $⊕_{f+g}(c)$. We further assume ε_0 is small enough that for every $g \in F(\varepsilon_0, \beta(\varepsilon_0))$, every center in $C(f + g)$ has its virtual periods less than $2T_0$.

Let $N \subset R^n$ be a closed neighborhood of x_0 such that $|x_0 - y| < \varepsilon_0$ for all $y \in N$ and such that if $x_1 \in \operatorname{bnd} N$, then (x_1, α_0) is not on an orbit with $T(x_1, \alpha_0) \leqslant 5T_0$. Such an N does exist: let V_0 be the union of x_0 and the points x for which (x, α_0) is on an orbit with period $T(x, \alpha_0) \leqslant 5T_0$; let $V_\delta = \{x \mid |x - y| \leqslant \delta$ for some $y \in V_0\}$: then the connected component of V_δ that contains x_0 satisfies the conditions on N for δ sufficiently small.

Let $P(\varepsilon)$ be the set of orbits for Eq. (1.1) that lie wholly in $N \times \{\alpha_0 - \varepsilon, \alpha_0 + \varepsilon\}$ and have period $T \leqslant 5T_0$.

Let $\varepsilon_1 \in (0, \varepsilon_0)$ be a number such that

$$N \times J(\varepsilon_1) \subset B(\varepsilon_0) \tag{7.6}$$

and if $x \in \operatorname{bnd} N$ and $\alpha \in J(\varepsilon_1)$, then (x, α) is not a point of an orbit of Eq. (1.1) that has period $T \leqslant 5T_0$, and such that every orbit of (1.1) in $P(\varepsilon_1)$ is type 0.

Let $g \in F(\varepsilon_1, N \times J(\varepsilon_1))$ be chosen so that all points in $C(f + g) \cap B(\varepsilon_1)$ are generic Hopf points and all orbits of (7.4) are type m, $m \geqslant 0$. Standard arguments show in fact that almost every $g \in F(\varepsilon_1, N \times J(\varepsilon_1))$ (in the sense of Baire Category) satisfies this condition.

Figure 7.2 illustrates some of the possible differences in the snakes for Eqs. (1.1) and (7.4). Neither equation has any orbits in $B(\varepsilon_1)$ with periods between $2T_0$ and $5T_0$. Let $\gamma(\beta)$ be an oriented path with $\gamma(\beta) \in N \times J(\varepsilon_1)$ for $\beta \in [\beta_1, \beta_0)$ or $(\beta_0, \beta_1]$ and with $\gamma(\beta_1) \in N \times \{\alpha_1 - \varepsilon_1, \alpha_0 + \varepsilon_1\}$. Assume further that $T(\gamma(\beta)) \to \infty$ as $\beta \to \beta_0$. Then $T(\gamma(\beta_1)) > 5T_0$. The perturbation g is 0 on $N \times \{\alpha_0 - \varepsilon_1, \alpha_0 + \varepsilon_1\}$ so any orbit in that set is also an orbit of Eq. (7.4) and for such orbits $\phi_{f+g} = \phi_f$. The "net number" of snakes terminating at c_0 may be defined to be $s_f^+(c_0) - s_f^-(c_0)$ and this number equals the "net number" entering $N \times J(\varepsilon_1)$ with period $T < 2T_0$; that is

$$s_f^+(c_0) - s_f^-(c_0) = \sum_{\substack{p, \alpha \in P(\varepsilon_1) \\ \alpha = \alpha_0 - \varepsilon_1}} \phi(p, \alpha) - \sum_{\substack{p, \alpha \in P(\varepsilon_1) \\ \alpha = \alpha_0 + \varepsilon_1}} \phi(p, \alpha).$$

We have shown the right-hand side equals $\sum \boxplus_{f+p}(c)$, summing over $c \in C(f+g, N \times J(\varepsilon_1))$. By Eq. (7.5), that sum equals $\boxplus_f(c_0)$. Hence Eq. (7.3) is proved. ∎

Proof of Theorem 4.1. From Proposition 7.2,

$$\boxplus(c_i) = s^-(c_i) - s^+(c_i).$$

Hence

$$\sum_{i=1}^{k} \boxplus(c_i) = \sum_{i=1}^{k} s^-(c_i) - \sum_{i=1}^{k} s^+(c_i). \tag{7.7}$$

If a source c is in a 2-component, then all orbits in all snakes emanating from c are also. Furthermore if an orbit of type 0 with non-zero orbit index is in Q, then so is its snake, and if that snake is not cyclic, its source and sink will also be in Q. Since Q is bounded and T is bounded on Q, each snake S has a source in $\{c_i\}$ if and only if it has a sink there. Hence the number of snakes having sources in $\{c_i\}$, namely, $\sum_i s^-(c_i)$ equals the number that have sinks in $\{c_i\}$, namely, $\sum_i s^+(c_i)$. Hence from Eq. (7.7)

$$\sum \boxplus(c_i) = 0. \quad ∎$$

Proof of Theorem 4.2. We may assume without loss of generality that $f(x, \alpha) = f(x, a_1)$ for all $\alpha \leqslant a_1$ and $f(x, \alpha) = f(x, a_2)$ for all $\alpha \geqslant a_2$. If this were not the case we could substitute $f^*(x, \alpha) \equiv f(x, h(\alpha))$ for $f(x, \alpha)$, where h is a differentiable function such that $h(\alpha) = a_1$ for all $\alpha \leqslant a_1$, and $h(\alpha) = a_2$ for all $\alpha \geqslant a_2$, and $h(\alpha_i) = \alpha_i$ for $i = 1,..., k$, and h is strictly monotonically increasing on (a_1, a_2). Such a substitution would leave unchanged the values in Eq. (4.2) of the orbit index and center index. Let Q^+ be the points in Q plus the points (x, α), where $(x, a_1) \in Q$ and $\alpha \leqslant a_1$, plus the points (x, α), where $(x, a_2) \in Q$ and $\alpha \geqslant a_1$.

A snake has ∞ as its source if and only if it either contains an orbit p, a_1 having $\phi(p, a_1) = +1$ or contains an orbit p, a_2 with $\phi(p, a_2) = -1$. Similarly a snake has ∞ as its sink if there is such an orbit with $\phi(p, a_1) = -1$ or $\phi(p, a_2) = +1$. Let $s^+(\infty, Q^+)$ be the number of snakes in Q^+ that have ∞ as their sink and $s^-(\infty, Q^+)$, the number with ∞ as their source. The left-hand side of Eq. (4.2) therefore equals

$$s^+(\infty, Q^+) - s^-(\infty, Q^+). \tag{7.8}$$

Since (7.7) is also true for our situation, and each snake has a source if and only if it has a sink, Eq. (7.8) equals the right-hand side of Eq. (7.7). ∎

8. When There Are Orbits That Are Not Type m

Despite our emphasis so far, situations abound in which non-generic behavior is seen. Consider for the moment the studies of zeroes of maps $\phi(x, \alpha) = 0$ rather than periodic solutions. Generically 0 is a "regular value," that is $M = D_{x,\alpha}(x, \alpha)$ always has full rank whenever $\phi(x, \alpha) = 0$. Yet the bifurcation theory of Krasnoselski and Rabinowitz for example concentrates on the interesting cases where there are zeroes at which M does not have full rank. More specifically the one dimensional map $\phi(x, \alpha) = \alpha x(1 - x) + \beta$ is frequently studied for the case $\beta = 0$, yet this is precisely the value of β for which the map $(x, \alpha) \mapsto \phi(x, \alpha)$ does not have 0 as a regular value.

Since one of the objectives of this paper is to indicate to the reader the kinds of orbits he is likely to observe in practice, we now describe some mildly non-generic cases. In fact our theory can be extended to general systems by approximating them by the generic systems whose theory we develop here. The non-generic extension is being carried out in separate papers [17, 20] with Alligood.

In the planar restricted three-body problem (e.g., Earth, Moon, pebble), there are orbits which are symmetric about the Earth–Moon axis, in rotating coordinates for which this axis remains fixed. From one of these orbits, there bifurcates a family whose orbits are not symmetric about their axis. The bifurcation orbit is observed to have all multipliers equal to $+1$, and so is not analogous to any of our orbits of type m. Of course this system is Hamiltonian, so it is not quite of the form of (1.1). However, it would be possible to choose coordinates in a neighborhood of the bifurcation orbit, with α corresponding to energy and x corresponding to the remaining three dimensions. In this framework, the bifurcation orbit is a regular bifurcation orbit, but is not of type m for any m. Figure 8.1 displays this type of bifurcation.

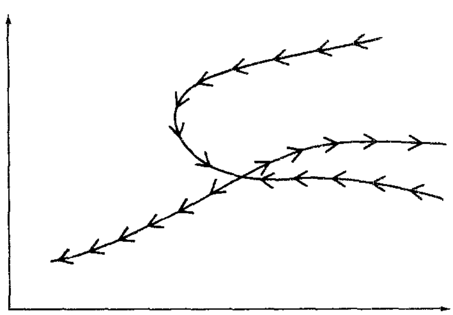

FIG. 8.1. A common configuration when there are symmetries that is easily incorporated into the theory even though such a bifurcation is nongeneric, in that most small symmetry breaking perturbations would completely change the picture into a situation in which two snakes pass by each other without touching.

To give another example, we again depart from the form of (1.1) for the sake of simplicity. Consider the non-autonomous equation

$$\ddot{x} + \alpha\dot{x}(1 - x^2) + \beta x^{2k+1} = \gamma \cos t. \tag{8.1}$$

If $x(t)$ is a solution then

$$y(t) \stackrel{\text{def}}{=} -x(t + \pi) \tag{8.2}$$

is also a solution. For some choices of α, β, γ there are periodic solutions $x(t)$ which satisfy the symmetry

$$x(t) = -x(t + \pi). \tag{8.3}$$

Sometimes families of such orbits are observed in $(\alpha, x, \dot{x})$ space, all with period 2π. An orbit in such a family can be a bifurcation orbit with a second family splitting off. The orbits of this family can also have period 2π. The second family can fail to satisfy (8.3). If $x(t)$ is a periodic solution in the second family the periodic solution $y(t)$ given by (8.2) will be a different orbit in the family. The bifurcation orbit will have $+1$ as a multiplier and does not correspond to a type m orbit for any m. It is not difficult to embed this system in an autonomous equation of the type in (1.1). Let $n = 4$. Introduce polar coordinates (r, θ) into the (x_1, x_2) plane, $r^2 = x_1^2 + x_2^2$. Then we write

$$\begin{aligned}
&\theta' = 1 \qquad \text{(where } ' \text{ denotes } d/dt\text{)},\\
&r' = -r(1 - r),\\
&x_3' = x_4,\\
&x_4' = \alpha x_3(1 - x_4^2) - \beta x_4^{2k+1} + \gamma \cos \theta.
\end{aligned}$$

Notice the cylinder $r = 1$ is stable and on this cylinder the flow corresponds to the previous equation. The resulting bifurcation orbit that corresponds to the one in (8.1) is a regular bifurcation orbit but is not type m.

We assume now that *each orbit of Eq.* (1.1) *is either type* 0 *or is a regular bifurcation orbit.* For any compact set $A \subset R^n \times R$ and any $T_0 \leqslant 0$, it follows from the definition of regular bifurcation orbit that there exists at most a finite number of regular bifurcation orbits in A having period $T \leqslant T_0$. An orbit that is not type m for any m will be called a *special* orbit. Figure 8.1 displays a special orbit with a configuration as occurs in the above two examples. The statements of Theorems 4.1 and 4.2 permit the existence of special orbits, but we have given the proofs in the case where no orbits are "special."

Sketch of proofs of Theorems 4.1 *and* 4.2 *allowing special orbits.* We now modify the definition of path by changing (P2) to read "For all $\beta \in J$, $\gamma(\beta)$ is on an orbit of type m ($m \geqslant 0$)." The wording of the other definitions remains unchanged. Now some of our various propositions must be modified in simple ways. For example, in addition to the possibilities mentioned in Proposition 5.2, it is possible for a maximal path to terminate at a regular bifurcation orbit. The Bifurcation Invariance Proposition (Proposition 4.3) assures that this causes no difficulties in the proofs because it implies each special orbit must be the source and sink of equal numbers of orbits. We can then regular assign each snake that has a source at such an orbit to one that has its sink there, and thereby create *chains* of snakes, and then *maximal chains* of snakes. Results that were true for snakes are now true of the maximal chains of snakes. ∎

Appendix 1: The Orbit Index in More General Circumstances

Even when the connected set of orbits Q is quite degenerate, it is sometimes still possible to define the orbit index ϕ. If an orbit p_0, α_0 has multipliers that are roots of unity, it may still be "*relatively isolated* in $R^n \times \{\alpha_0\}$" in the sense that for any sequence of orbits p_i, α_0 converging to p_0, α_0, we have $T(p_i, \alpha_0) \to \infty$.

We first fix some notation. If $F: R^m \to R^m$ is continuous, $\Omega \subseteq R^m$ open and bounded, and $F(x) \neq 0$ on $\partial\Omega$ lot

$$d(F, \Omega) = \text{the degree of } F \text{ on } \Omega;$$

if $a \in R^m$ is an isolated fixed point of F let

$$i(F, a) = \text{the fixed point index of } F \text{ at } a.$$

Thus if F has finitely many fixed points in $\bar{\Omega}$, none of them on $\partial\Omega$, then

$$d(F - \text{id}, \Omega) = \sum_{\substack{f(a) = a \\ a \in \Omega}} i(F, a) \tag{A.1}$$

where id represents the identity map on R^m.

Suppose the differential equation

$$\dot{x} = f(x) \tag{E}$$

has a periodic solution $p(t)$ with least period $T > 0$; let $\sum$ be a transverse hyperplane in R^m at $p(0) = a_0$ and $a \to P(a)$ the Poincaré map on $\sum$, so $P(a_0) = a_0$. Set

$$\nu_k(P) = i(P^k, a_0).$$

the index of the kth iterate of P; this is well defined whenever the solution $p(t)$ is isolated from solutions of (E) with periods (or multiples thereof) near kT.

The basic idea in defining the orbit index is quite simple. Let

$$\phi(p) = \text{average value of } \{\nu_k(p)\}.$$

If none of the multipliers of p are roots of unity, then an easy calculation yields $\nu_k = \nu_1$ for k odd and $\nu_k = \nu_2$ for k even. If σ^- (defined in Section 5) is even, then $\nu_1 = \nu_2 = \cdots$ and the average is ν_1, while if σ^- is odd, $\nu_1 = -\nu_2$ and the average is 0. More generally however the question of whether this average even exists arises. Shub and Sullivan [15] proved that the sequence $\{\nu_k(p)\}$ is bounded, assuming all $\nu_k(p)$ are defined. We prove in [16] that the average exists and is an integer.

Appendix 2

The purpose of this appendix is to prove Proposition 3.1, a partial converse to the Hopf Bifurcation Theorem, which given conditions at a stationary point guaranteeing that a family of periodic orbits will bifurcate from it. Here we ask what can be concluded if it is known that a family of periodic orbits bifurcates from a stationary point.

Proof of Proposition 3.1. The assertion "$T_\infty > 0$" follows from the fact that in any bounded neighborhood of x_0, g_i are uniformly Lipshitzean in x, so that there is an $L > 0$ (independent of i) such that $\| g_i(x_1) - g_i(x_2)\| \leqslant L \|x_1 - x_2\|$ for all x_1, x_2 in that neighborhood. If the norm $\| \cdot \|$ used is the Euclidean norm, it follows that the period T of any periodic orbit in that set satisfies $T \geqslant 2\pi/L > 0$; see [6]. For any norm we always have $T \geqslant 4/L$ [7]. In particular, $T_\infty > 0$.

To prove the main result we first argue that (3.1) has a periodic solution. Choose $\tau \in (0, 1)$ and let

$$\varepsilon_i = \max_t |p_i(t) - p_i(t + \tau T_i)|, \tag{A.2}$$

$$y_i(t) = (p_i(t) - p_i(t + \tau T_i))/\varepsilon_i. \tag{A.3}$$

Since the orbits are converging to a single point, $\varepsilon_i \to 0$. For $i = 1, 2,...,$ write A_i for the matrix $D_x g_i(x_0)$ so that $A_i \to A_0$, and write

$$N_i(x) = g_i(x) - g_i(x_0) - D_x g_i(x_0) \cdot (x - x_0)$$

so that $N_i(x) = O(|x - x_0|^2)$ since g_i is C^2. Writing $p_i^*(t)$ for the shifted function $p_i(t + \tau T_i)$, the functions in the sequence $\{y_i\}$ satisfy

$$\frac{d}{dt} y_i = A_i y_i + [N_i(p_i) - N_i(p_i^*)]/\varepsilon_i.$$

The fact that $D_x N_i(x_0) = 0$ implies the existence of constants $L_i \downarrow 0$ such that

$$\varepsilon_i^{-1} |N_i(p_i(t)) - N_i(p_i^*(t))| \leqslant L_i |p_i(t) - p_i^*(t)| \varepsilon_i^{-1}$$
$$\leqslant L_i |y_i(t)| \to 0 \qquad \text{as} \quad i \to \infty.$$

Some subsequence converges to some function $z_\tau(t)$ which is a solution of (3.1). Since the maximum value of $|y_i|$ is one and the average value of each y_i is 0, the same is true of z_τ, that is, z_τ is a non-constant periodic solution of (3.1) and in fact $z_\tau(T_\infty + t) = z_\tau(t)$. Hence, T_∞ is an integer multiple of the period T_τ of z_τ; i.e., T_τ equals T_∞/k for some integer $k > 0$. If $k = 1$ we are done. If not, we wish to find a collection of orbits of (3.1) having T_∞ as the least common multiple of their periods.

CLAIM. *τT_∞ is not an integer multiple of T_τ. If τ is irrational, then τT_∞ cannot an integer multiple of T_τ since T_∞ is an integer multiple of T_τ, so we may assume τ is a rational number m/k, where m and k are relatively prime integers. Discarding terms from the sequence $\{y_i\}$ if necessary, we may assume*

$$[y_i(t + \tau T_i) - y_i(t)] \varepsilon_i^{-1} - z_\tau(t) \to 0 \qquad \textit{as} \quad i \to \infty,$$

where ε_i depends on τ. Since $y_i(t + k\tau T_i) = y_i(t)$, we have

$$0 = [y_i(t + \tau T_i) - y_i(t)] \varepsilon_i^{-1} + [y_i(t + 2\tau T_i) - y_i(t + \tau T_i)] \varepsilon_i^{-1} + \cdots$$
$$+ [y_i(t + k\tau T_i) - y_i(t + (k-1) \tau T_i)] \varepsilon_i^{-1}$$
$$\approx z_\tau(t) + \cdots + z_\tau(t + (k-1) \tau T_\infty).$$

If τT_∞ is an integer multiple of T_τ, this final sum would be $k z_\tau(t)$. But z_τ is not zero, so the claim is proved.

Hence for every rational τ in (0, 1), there is a periodic solution z_τ whose period divides T_∞ but does not divide τT_∞. Hence we may choose some collection $\{z_j\}_0^{k-1}$ of solutions of (3.1) such that T_∞ is the smallest number which is a multiple of all their periods. It follows that for almost any choice of real numbers $c_0, \ldots, c_{k-1}$,

$$\sum_{j=1}^{k-1} c_j z_j(t)$$

is a periodic solution of (3.1) having (least) period T_∞. ∎

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