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ORIGINAL ARTICLE

Analysis of convective longitudinal fin with temperature-dependent thermal conductivity and internal heat generation

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Longitudinal fin;
Finite difference method;
Temperature-dependent thermal conductivity;
Internal heat generation

Abstract In this study, analysis of heat transfer in a longitudinal rectangular fin with temperature-dependent thermal conductivity and internal heat generation was carried out using finite difference method. The developed systems of non-linear equations that resulted from the discretization using finite difference scheme were solved with the aid of MATLAB using ***fsolve***. The numerical solution was validated with the exact solution for the linear problem. The developed heat transfer models were used to investigate the effects of thermo-geometric parameters, coefficient of heat transfer and thermal conductivity (non-linear) parameters on the temperature distribution, heat transfer and thermal performance of the longitudinal rectangular fin. From the results, it shows that the fin temperature distribution, the total heat transfer, and the fin efficiency are significantly affected by the thermo-geometric parameters of the fin. Also, for the solution to be thermally stable, the fin thermo-geometric parameter must not exceed a specific value. However, it was established that the increase in temperature-dependent properties and internal heat generation values increases the thermal stability range of the thermo-geometric parameter. The results obtained in this analysis serve as basis for comparison of any other method of analysis of the problem.

1. Introduction

The increasing demand and the quest for high-performance heat transfer components with progressively smaller weights, volume, costs or accommodating shapes have greatly increased the use of extended surfaces to enhance heat dissipation from hot primary surfaces. In the design and construction of various types of heat-transfer equipment and components such as air conditioner, refrigerator, superheaters, automobile, power plants, heat exchangers, convectional furnaces, economizers, gas turbines, chemical processing equipment, oil carrying pipelines, computer processors, electrical chips, etc., extended surfaces are used to implement the flow of heat between a source and a sink. In practice, various types of fins with different geometries are used, but due to the simplicity of its design and ease of construction and manufacturing process, the rectangular fins are widely applied in heat-transfer equipment. Also, for ordinary fins problem, the thermal properties of the fin and the surrounding medium (thermal conductivity

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Nomenclature

a_r	aspect ratio or extension factor	P	perimeter of the fin, m
A	cross sectional area of the fins, m^2	T	Temperature, K
Bi	Biot number	T_∞	ambient temperature, K
h	heat transfer coefficient, W m^{-2} k^{-1}	T_b	Temperature at the base of the fin, K
h_b	heat transfer coefficient at the base of the fin, W m^{-2} k^{-1}	x	fin axial distance, m
H	dimensionless heat transfer coefficient at the base of the fin, W m^{-2} k^{-1}	X	dimensionless length of the fin
j	geometric parameter	Q	dimensionless heat transfer
k	thermal conductivity of the fin material, W m^{-1} - k^{-1}	q_i	the uniform internal heat generation in W/m^3
k_b	thermal conductivity of the fin material at the base of the fin, W m^{-1} k^{-1}		*Greek symbols*
K	dimensionless thermal conductivity of the fin material, W m^{-1} k^{-1}	β	thermal conductivity parameter or non-linear parameter
L	Length of the fin, m	δ	thickness of the fin, m
M	dimensionless thermo-geometric fin parameter	δ_b	fin thickness at its base
m^2	thermo-geometric fin parameter m^{-1}	γ	dimensionless internal heat generation parameter
n	convective heat transfer power	θ	dimensionless temperature
		θ_b	dimensionless temperature at the base of the fin
		η	efficiency of the fin
		ε	effectiveness of the fin

and heat transfer coefficient) are assumed to be constant, but if a large temperature difference exists within the fin, typically, between the tip and the base of the fin, the thermal conductivity and the heat transfer coefficient are not constant but temperature-dependent. Therefore, while analyzing the fin, the effects of temperature-dependent thermal properties must be taken into consideration. In carrying out such an analysis, the thermal conductivity may be modeled for such and other many engineering applications by power law and by linear dependency on temperature, while the heat transfer coefficient can be expressed as power law for which the exponents represent different phenomena as reported by Khani and Aziz [1], Ndlovu and Moitsheki [2]. Such dependency of thermal conductivity and heat transfer coefficient on temperature renders the problem highly non-linear and difficult to solve analytically. It is also very realistic to consider the temperature-dependent internal heat generation in the fins as applied in electric-current carrying conductor, nuclear rods or any other heat generating components of thermal systems.

Over the past few decades, the research on the temperature-dependent thermal conductivity and heat transfer coefficient has been on-going in the literature. Also, the solutions of the highly non-linear differential equations have been constructed using different techniques. Aziz and Enamul-Huq [3] and Aziz [4] applied regular perturbation expansion to study a pure convection fin with temperature dependent thermal conductivity. Few years later, Campo and Spaulding [5] predicted the thermal behavior of uniform circumferential fins using method of successive approximation. Chiu and Chen [6] and Arslanturk [7] adopted the Adomian decomposition method (ADM) to obtain the temperature distribution in a pure convective fin with variable thermal conductivity. The same problem was solved by Ganji [8] with the aid of the homotopy perturbation method originally proposed by He [9]. In the same year, Chowdhury and Hashim [10] applied Adomian decomposition method to evaluate the temperature distribution of straight rectangular fins with temperature-dependent surface flux for all possible types of heat transfer while in the following year, Rajabi [11] applied homotopy perturbation method (HPM) to calculate the efficiency of straight fins with temperature-dependent thermal conductivity. Also, a year later, Mustapha [12] adopted homotopy analysis method (HAM) to find the efficiency of straight fins with temperature-dependent thermal conductivity. Meanwhile, Coskun and Atay [13] utilized the variational iteration method (VIM) for the analysis of convective straight and radial fins with temperature-dependent thermal conductivity. Also, Languri et al. [14] applied both the variation iteration and homotopy perturbation methods for the evaluation of the efficiency of straight fins with temperature-dependent thermal conductivity while Coskun and Atay [15] applied variational iteration method to analyze the efficiency of convective straight fins with temperature-dependent thermal conductivity. Besides, Atay and Coskun [16] employed variation iteration and finite element methods to carry out comparative analysis of power-law-fin type problems. Domairry and Fazeli [17] used homotopy analysis method to determine the efficiency of straight fins with temperature-dependent thermal conductivity. Chowdhury et al. [18] investigated a rectangular fin with power law surface heat flux and made a comparative assessment of results predicted by HAM, HPM, and ADM. Khani et al. [19] used Adomian decomposition method (ADM) to provide series solution to fin problems with a temperature-dependent thermal conductivity while Moitheki et al. [20] applied the Lie symmetry analysis to provide exact solutions of the fin problem with a power-law temperature-dependent thermal conductivity and while Hosseini et al. [21] applied homotopy analysis method to generate approximate but accurate solution of heat transfer in fin with temperature-dependent internal heat generation and thermal conductivity. Sadollah et al. [22]

presented metaheuristic algorithms for approximate solution to ordinary differential equations of longitudinal fins having various profiles. The application of differential transform method (DTM) to solve differential equations without linearization, discretization or approximation, linearization restrictive assumptions or perturbation, complexity of expansion of derivatives and computation of derivatives symbolically has made the method popular in recent times. This method was applied by Joneidi et al. [23], Moradi and Ahmadikia [24], Mosayebidorcheh et al. [25], Ghasemi et al. [26], and Ganji and Dogonchi [27] to solve the fin problem. However, the search for the arbitrary value that will satisfy the second boundary condition necessitated the use of Maple or Mathematica software and such could result in additional computational cost in the generation of solution to the problem. This drawback is not only peculiar to DTM, but other approximate analytical methods such as HPM, HAM, ADM and VIM also required additional computational cost and time for the determination of such auxiliary parameters in their procedures of implementation. Also, most of the approximate analytical methods give accurate predictions only when the nonlinearities are weak, and they fail to predict accurately for strong nonlinear models. Also, the methods often involved complex mathematical analysis leading to analytic expression involving a large number terms and when methods such as HPM, HAM, ADM and VIM are routinely implemented, they can sometimes lead to erroneous results as observed by Fernandez [28], Aziz and Bouaziz [29]. In practice, approximate analytical solutions with large number of terms are not convenient for use by designers and engineers. Numerical methods such as Euler and Runge–Kutta methods are limited to solving initial value problems. With the aid of Shooting method, the methods could be carried out iteratively to solve boundary value problems. However, these numerical methods are only useful for solving ordinary differential equations i.e. differential equations with a single independent variable. On the other hand, numerical methods such as finite difference method (FDM), finite element methods (FEM) and finite Volume method (FVM) can be adopted to analyze heat transfer in fins with single and multiple independent variables as they have been used to different linear and non-linear differential equations in literatures. While Han et al. [30] pointed out the accuracy of FDM for the analysis of one-dimensional fin with constant thermal properties and without internal heat generation, the numerical solution of FDM represents an efficient way of obtaining temperature profile for the steady heat transfer processes. The FDM is based on the differential equation of the heat conduction, which is transformed into a difference equation by discretization and the resulting series of recursive or algebraic equations could be solved easily by matrix method. The FDM can be used for solving any complex body by breaking the body into small domains. Also, choice of finer grids which requires high computing capability can remove approximation errors to larger extent. Hence, in this work, finite difference method was applied to analyze heat transfer in a longitudinal rectangular fin with temperature-dependent thermal properties and internal heat generation.

The main objective of this work was to analyze the thermo-geometrical effects on the thermal performance and establish thermal stability of longitudinal fin with temperature-dependent thermal conductivity and temperature-dependent internal heat generation using finite difference method.

2. Problem formulation

The development of the thermal model is based on energy balance analysis in the fin. This analysis is carried out with keen considerations of some simplifying assumptions.

2.1. Model assumptions

The following assumptions were made in the development of the model:

i. The heat flow in the fin and its temperatures remain constant with time.
ii. The temperature of the medium surrounding the fin is uniform.
iii. There is no contact resistance where the base of the fin joins the prime surface.
iv. The temperature of the base of the fin is uniform.
v. The fin thickness is small compared with its width and length, so that temperature gradients across the fin thickness and heat transfer from the edges of the fin is negligible compared with the heat leaving its lateral surface.

According to previous assumptions, one can consider a straight fin of temperature-dependent thermal conductivity $k(T)$, length L and thickness δ and temperature-dependent internal heat generation per unit volume $Q(T)$, upside and downside surfaces are exposed to a convective environment at temperature T_∞ and constant heat transfer coefficient h as shown in Fig. 1. The dimension x pertains to the height coordinate which has its origin at the fin base and has a positive orientation from fin base to fin tip, and it could therefore be stated that

$$\begin{aligned}&\text{Rate of heat conduction into the element at } x\\&\quad = \text{Rate of heat conduction from the element at } x\\&\qquad + dx + \text{Rate of heat convection from the element}\\&\qquad + \text{Rate of heat internal generation in the element}\end{aligned} \tag{1}$$

Mathematically, the thermal energy balance could be expressed as shown in Eq. (2):

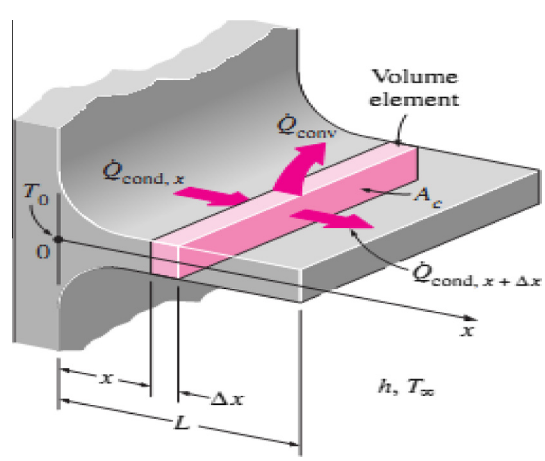

Figure 1 The geometry of straight rectangular convecting fin [31].

$$Q_x = Q_{x+dx} + Q_{conv.} + Q_{\text{int}}. \tag{2}$$

i.e.

$$Q_x - Q_{x+dx} = Q_{conv.} + Q_{\text{int.}} \tag{3}$$

$$Q_x - \left(Q_x + \frac{\delta Q}{\delta x}dx\right) = hP(T - T_c)dx + Q_{\text{int}}.(T)dx \tag{4}$$

As $dx \to 0$, Eq. (4) reduces to Eq. (5)

$$-\frac{dQ}{dx} = hP(T - T_c) + Q_{\text{int.}}(T) \tag{5}$$

From Fourier's law of heat conduction

$$Q = -k(T)A_{cr}\frac{dT}{dx} \tag{6}$$

Substituting Eq. (6) into Eqs. (5), (7) we get

$$\frac{d}{dx}\left(k(T)A_{cr}\frac{dT}{dx}\right) = hP(T - T_c) + q(T) \tag{7}$$

Further simplification of Eq. (7) gives the governing differential equation for the fin as given by

$$\frac{d}{dx}\left[k(T)\frac{dT}{dx}\right] - \frac{h}{A_{cr}}P(T - T_\infty) + Q_{\text{int.}}(T) = 0 \tag{8}$$

where the boundary conditions are

$$\begin{aligned} x &= 0, \quad T = T_o \\ x &= L, \quad \tfrac{dT}{dx} = 0 \end{aligned} \tag{9}$$

For many engineering applications, the thermal conductivity and the coefficient of heat transfer are temperature-dependent. Therefore, the temperature-dependent thermal properties and internal heat generation are given respectively by

$$k(T) = k_\infty[1 + \lambda(T - T_\infty)] \tag{10}$$

and

$$Q_{\text{int}}(T) = Q_\infty[1 + \psi(T - T_\infty)] \tag{11}$$

Substituting Eqs. (10) and (11) into Eq. (1), we arrived at

$$\begin{aligned} &\frac{d}{dx}\left[k_\infty[1 + \lambda(T - T_\infty)]\frac{dT}{dx}\right] - \frac{Ph}{A_{cr}}(T - T_\infty) \\ &\quad + Q_a[1 + \psi(T - T_\infty)] = 0 \end{aligned} \tag{12}$$

Introducing the following dimensionless parameters of Eq. into Eq. (12);

$$\begin{aligned} &X = \tfrac{x}{L}, \quad \theta = \tfrac{T - T_\infty}{T_o - T_\infty}, \quad K = \tfrac{k}{k_\infty}, \quad M^2 = \tfrac{Ph_\infty L^2}{A_{cr}k_\infty}, \\ &Q = \tfrac{Q_\infty A_c}{h_o P(T_o - T_\infty)}, \quad \gamma = \psi(T_o - T_\infty), \quad \beta = \lambda(T_o - T_\infty) \end{aligned} \tag{13}$$

We arrived at the dimensionless governing differential Eq. (14) and the boundary conditions

$$\frac{d^2\theta}{dX^2} + \beta\theta\frac{d^2\theta}{dX^2} + \beta\left(\frac{d\theta}{dX}\right)^2 - M^2\theta + M^2Q(1 + \gamma\theta) = 0 \tag{14}$$

The boundary conditions are

$$\begin{aligned} X &= 0, \quad \theta = 1 \\ X &= 1, \quad \tfrac{d\theta}{dX} = 0 \end{aligned} \tag{15}$$

3. Method of solution

The above non-linear Eq. (14) does not permit the generation of any closed form solution. Therefore, recourse has to be made to either approximation analytical method, semi-numerical method or numerical method of solution. The use and the accuracy of finite difference method for the analysis of heat transfer in fin with constant thermal properties and no internal heat generation has earlier been pointed out by Han et al. [30]. Therefore, in this work, finite difference method is used to discretize the governing Eq. (14) combined with the boundary conditions of Eq. (15).

Consider first a mesh in space formed by points separated by constant spacing Δx as shown in Fig. 2. The mesh points in space are x_1, x_2, x_3 … x_{N-1}, x_N. Note that nodes x_1 and x_N are boundary nodes while all other nodes are interior nodes.

The procedures are shown below:

$$\begin{aligned} &\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{\Delta^2 x} + \beta\theta_i\left(\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{\Delta^2 x}\right) + \beta\left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta x}\right)^2 \\ &\quad - M^2\theta_i + M^2Q(1 + \gamma\theta_i) = 0 \end{aligned} \tag{16}$$

After applying the finite difference approximation and grouping and rearranging the terms, the final algebraic form of the finite difference equation becomes

$$\begin{aligned} &\theta_{i+1} - 2\theta_i + \theta_{i-1} + \beta\theta_i(\theta_{i+1} - 2\theta_i + \theta_{i-1}) + \tfrac{\beta}{4}\theta_{i+1}^2 - \tfrac{\beta}{2}\theta_{i+1}\theta_{i-1} \\ &\quad + \tfrac{\beta}{4}\theta_{i-1}^2 - M^2\theta_i(\Delta^2 x) + M^2Q(\Delta^2 x) + M^2\gamma\theta_i(\Delta^2 x) = 0 \end{aligned} \tag{17}$$

It could be seen that the central difference approximation is used for the differentials, so Eq. (17) is only valid for interior nodes ($i = 2$: $N - 1$) because the central difference approximation cannot be applied at the end points. Thus, for the 48 interior nodes (the remaining 2 nodes of the 50 nodes used in this work are at the boundaries of the fin), we have

$$\begin{aligned} i = 2 \quad &\theta_3 - 2\theta_2 + \theta_1 + \beta\theta_2(\theta_3 - 2\theta_2 + \theta_1) + \frac{\beta}{4}\theta_3^2 - \frac{\beta}{2}\theta_3\theta_1 \\ &+ \frac{\beta}{4}\theta_1^2 - M^2\theta_2(\Delta^2 x) + M^2Q(\Delta^2 x) + M^2\gamma\theta_2(\Delta^2 x) = 0 \end{aligned} \tag{18i}$$

$$\begin{aligned} i = 3 \quad &\theta_4 - 2\theta_3 + \theta_2 + \beta\theta_3(\theta_4 - 2\theta_3 + \theta_2) + \frac{\beta}{4}\theta_4^2 - \frac{\beta}{2}\theta_4\theta_2 \\ &+ \frac{\beta}{4}\theta_2^2 - M^2\theta_3(\Delta^2 x) + M^2Q(\Delta^2 x) + M^2\gamma\theta_3(\Delta^2 x) = 0 \end{aligned} \tag{18ii}$$

$$\begin{aligned} i = 4 \quad &\theta_5 - 2\theta_4 + \theta_3 + \beta\theta_4(\theta_5 - 2\theta_4 + \theta_3) + \frac{\beta}{4}\theta_5^2 - \frac{\beta}{2}\theta_5\theta_3 \\ &+ \frac{\beta}{4}\theta_3^2 - M^2\theta_4(\Delta^2 x) + M^2Q(\Delta^2 x) + M^2\gamma\theta_4(\Delta^2 x) = 0 \end{aligned} \tag{18iii}$$

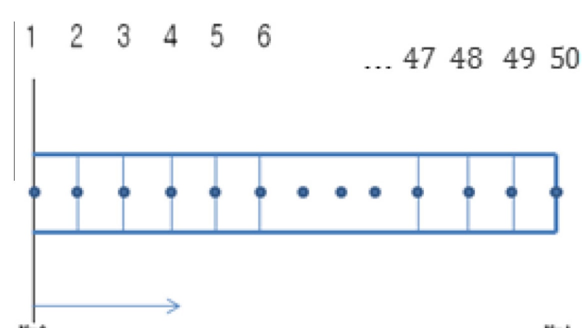

Figure 2 Nodal representation for finite difference method.

$\vdots$

$$i = N-2 \quad \theta_{N-1} - 2\theta_{N-2} + \theta_{N-3} + \beta\theta_{N-2}(\theta_{N-1} - 2\theta_{N-2} + \theta_{N-3}) + \frac{\beta}{4}\theta_{N-1}^2 - \frac{\beta}{2}\theta_{N-1}\theta_{N-3} + \frac{\beta}{4}\theta_{N-3}^2 - M^2\theta_{N-2}(\Delta^2 x) + M^2 Q(\Delta^2 x) + M^2\gamma\theta_{N-2}(\Delta^2 x) = 0 \tag{18xliv}$$

$$i = N-1 \quad \theta_N - 2\theta_{N-1} + \theta_{N-2} + \beta\theta_{N-1}(\theta_N - 2\theta_{N-1} + \theta_{N-2}) + \frac{\beta}{4}\theta_N^2 - \frac{\beta}{2}\theta_N\theta_{N-2} + \frac{\beta}{4}\theta_{N-2}^2 - M^2\theta_{N-1}(\Delta^2 x) + M^2 Q(\Delta^2 x) + M^2\gamma\theta_{N-1}(\Delta^2 x) = 0 \tag{18xlviii}$$

For the first node, $i = 1$, Eq. (17) can be used directly noting that $\theta_{i-1} = \theta_0 = 1$, the fin's base temperature.

$$i = 1 \quad \theta_2 - 2\theta_1 + \theta_0 + \beta\theta_1(\theta_2 - 2\theta_1 + \theta_0) + \tfrac{\beta}{4}\theta_2^2 - \tfrac{\beta}{2}\theta_2\theta_0 + \tfrac{\beta}{4}\theta_0^2 - M^2\theta_1(\Delta^2 x) + M^2 Q(\Delta^2 x) + M^2\gamma\theta_1(\Delta^2 x) = 0 \tag{19}$$

However, the last node at the tip of the fin is treated as a special case i.e. for $i = N$, θ_{N+1} is not defined. Although, this situation can be handled in several ways, possibly the easiest way is to generate a backward approximation to the desired derivatives at $X = 1$. After algebraic manipulations, we arrived at

$$\frac{\theta_{N-2} - \theta_N}{2\Delta^2 x} + \beta\theta_N\left(\frac{\theta_{N-2} - \theta_N}{2\Delta^2 x}\right) + \beta\left(\frac{\theta_N - \theta_{N-1}}{2\Delta x}\right)^2 - M^2\theta_N + M^2 Q(1 + \gamma\theta_N) = 0 \tag{20}$$

which could be simplified as

$$\theta_{N-2} - \theta_N + \beta\theta_N(\theta_{N-2} - \theta_N) + \frac{\beta}{2}(\theta_N - \theta_{N-1})^2 - (2\Delta^2 x)M^2\theta_N + (2\Delta^2 x)M^2 Q(1 + \gamma\theta_N) = 0 \tag{21}$$

The FDM results in a set of 50 non-linear systems of algebraic equations (since 50 nodes are chosen to be used in this work) from Eqs. (18), Eq. (19) and Eq. (21). These equations required are solved simultaneously. The systems of the non-linear equations are solved with the aid of MATLAB using ***fsolve***.

4. Heat flux of the fin

The fin base heat flux is given by

$$q_{bn} = A_c k(T)\frac{dT}{dx} \tag{22}$$

The dimensionless heat transfer rate at the base of the fin ($X = 0$) is obtained as

$$q_b = \frac{q_{bn} L}{k_a A_c (T_b - T_\infty)} = (1 + \beta\theta)\frac{d\theta}{dX} \tag{23}$$

which in finite difference is given by

$$q_b = (1 + \beta\theta_i)\left(\frac{\theta_{i+1} - \theta_i}{\Delta X}\right) \tag{24}$$

The total heat flux of the fin is given by

$$q_T = \frac{q_b}{A_c h(T - T_b)} \tag{25}$$

Substituting Eq. (23) and introducing the dimensionless parameters in Eq. (6) into Eq. (25), we arrived at

$$q_T = \frac{1}{Bi}\frac{k(\theta)}{h}\frac{d\theta}{dX} = \frac{1}{Bi}(1 + \beta\theta)\frac{d\theta}{dX} \tag{26}$$

which in finite difference is given by

$$q_T = \frac{1}{Bi}(1 + \beta\theta_i)\left(\frac{\theta_{i+1} - \theta_i}{\Delta X}\right) \tag{27}$$

5. Fin parameter for thermal performance indication

The performance indication parameter for the fin such as the efficiency of the fin is analyzed.

5.1. Fin efficiency

The amount of heat dissipated from the entire fin is found by using Newton's law of cooling as

$$Q_f = \int_0^1 Ph(T - T_\infty)dX \tag{28}$$

Also, the maximum heat dissipated is obtained if the fin base temperature is kept constant throughout the fin i.e.

$$Q_{\max} = Ph_b L(T_b - T_\infty) \tag{29}$$

Fin efficiency is defined as the ratio of the fin heat transfer rate to the rate that would be if the entire fin were at the base temperature and is given by

$$\eta = \frac{Q_f}{Q_{\max}} = \frac{\int_0^L Ph(T - T_\infty)dx}{Ph_b L(T_b - T_\infty)} \tag{30}$$

Therefore, the fin efficiency in dimensionless variables is given by

$$\eta = \int_0^1 \theta\, dX \tag{31}$$

in which the finite difference is

$$\eta = \sum_{i=1}^{N}\theta_i \tag{32}$$

It is very important to point out that the thermo-geometric parameter or the fin performance factor, M could be written in terms of Biot number, Bi and the aspect ratio, a_r as shown in Eq. (33):

$$M^2 = \frac{Ph_b L^2}{A_c k_a} = \frac{(2L)h_b L^2}{(L\delta)k_a} = \frac{2h_b\delta L^2}{\delta^2 k_a} = \frac{2h_b\delta}{k_a}\left(\frac{L}{\delta}\right)^2 = 2Bia_r^2 \quad Where \quad Bi = \frac{h_b\delta}{k_a},\ a_r = \frac{L}{\delta} \tag{33}$$

From Eq. (33), it implies that $M = a_r\sqrt{2Bi}$.

In order to establish the validity of the solution of the finite difference method, an exact solution was generated to the linear problem of the heat transfer in the straight fin (with constant thermal properties) with and without internal heat generation using an analytical method and also, finite difference method was applied to the same problem.

The exact solution of straight fin with constant thermal properties and no internal heat generation is given in Eq. (34) while Eq. (35) gives the exact solution of straight fin with constant thermal properties with internal heat generation

$$\theta(X) = \frac{Cosh(MX)}{Cosh(M)} \quad (34)$$

$$\theta(X) = \frac{Cosh\ (MX)}{Cosh\ (M)} + Q\left(1 - \frac{Cosh\ (MX)}{Cosh\ (M)}\right) \quad (35)$$

6. Results and discussion

The dimensionless temperature distribution falls monotonically along fin length for all various thermogeometric, thermal conductivity and convective heat transfer parameters as shown in Figs. 3–6. For larger values of the thermogeometric parameter M, the more the heat convected from the fin through its length and the more thermal energy is efficiently transferred into environment through the fin length. In the situation of negligible heat loss from the fin tip (insulated tip) to the environment, the fin temperature decreases along the fin length also, and the temperature decreasing rate is the same around fin base area.

Fig. 3 shows the variation of dimensionless temperature with dimensionless length and also the effect of the thermogeometric parameter on the straight fin with an insulated tip. From the figure, as the thermogeometric parameter increases, the rate of heat transfer (the convective heat transfer) through the fin increases as the temperature in the fin drops faster (becomes steeper which reflects high base heat flow rates) as depicted in the figure. It can be inferred from the results that the ratio of convective heat transfer to conductive heat transfer at the base of the fin (h_b/k_b) has much effect on the temperature distribution, rate of heat transfer at the base of the fin, and efficiency. As h_b increases (or k_b decreases), the ratio h_b/k_b increases at the base of the fin, and consequently the temperature along the fin, especially at the tip of the fin decreases i.e. the tip end temperature decreases as M increases. The profile has the steepest temperature gradient at $M = 1.0$, but its much higher value gotten from the lower value of thermal conductivity than the other values of M in the profiles produces a lower heat-transfer rate. This shows that the thermal performance or efficiency of the fin is favoured at low values of thermogeometric parameter since the aim (high effective use of the fin) is to minimize the temperature decrease along the fin length, where the best possible scenario is when $T = T_b$ everywhere.

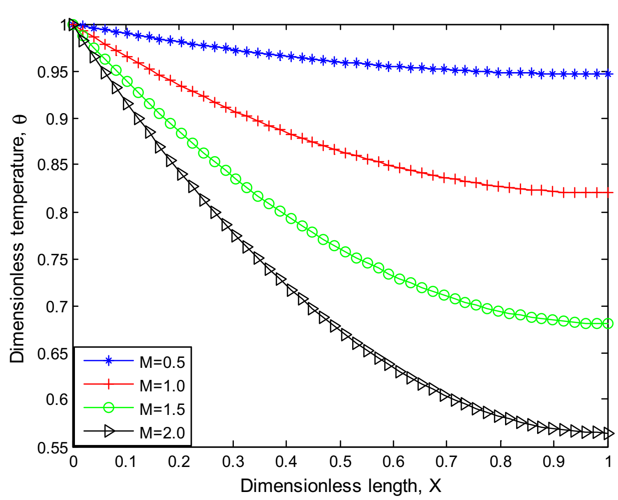

Figure 3 Dimensionless temperature distribution in the fin when $\beta = 0.2$, $Q = 0.3$, $\gamma = 0.5$.

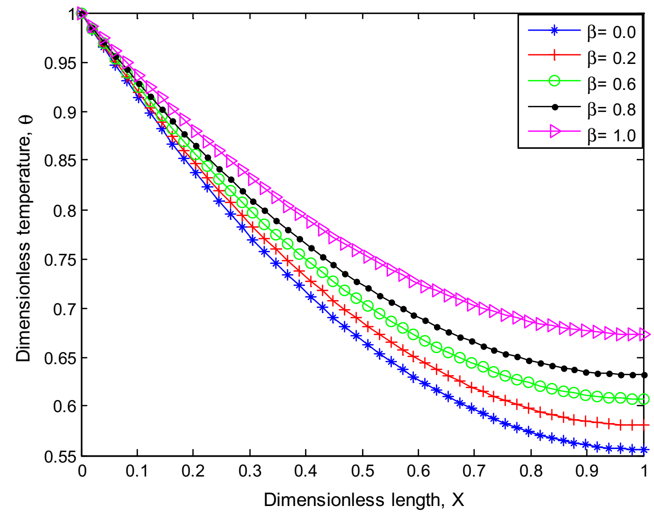

Figure 4 Dimensionless temperature distribution in the fin when $M = 2$, $Q = 0.3$, $\gamma = 0.6$.

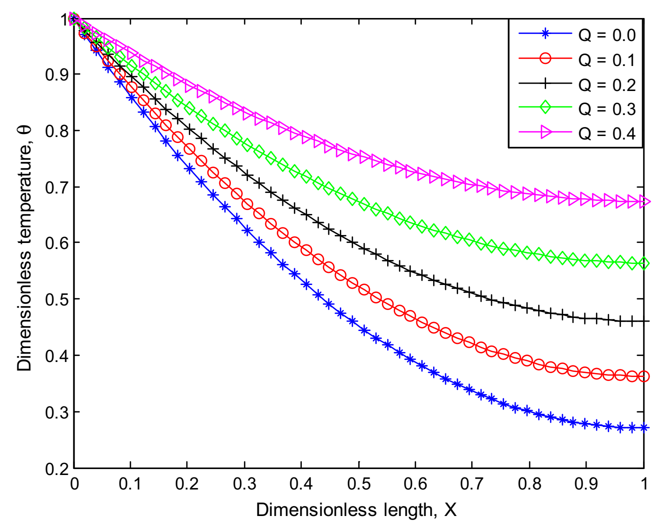

Figure 5 Dimensionless temperature distribution in the fin when $\beta = 0.5$, $M = 2$, $\gamma = 0.2$.

Fig. 4 shows the variation of dimensionless temperature with dimensionless length in longitudinal convective fin with an insulated tip. The effects of thermogeometric and thermal conductivity parameters on the dimensionless temperature distribution and consequently, on the rate of heat transfer are shown. From the figure, it is obvious that as the thermogeometric parameter increases, the rate of heat transfer through the fin increases. This is because as the fin convective heat transfer increases, more heat transferred by conduction through the fin thereby increases the temperature distribution in the fin and consequently the rate of heat transfer.

The effects of internal heat generation parameter on the temperature distribution are depicted in Figs. 5 and 6 while Fig. 7 shows the effects of internal heat generation on the fin

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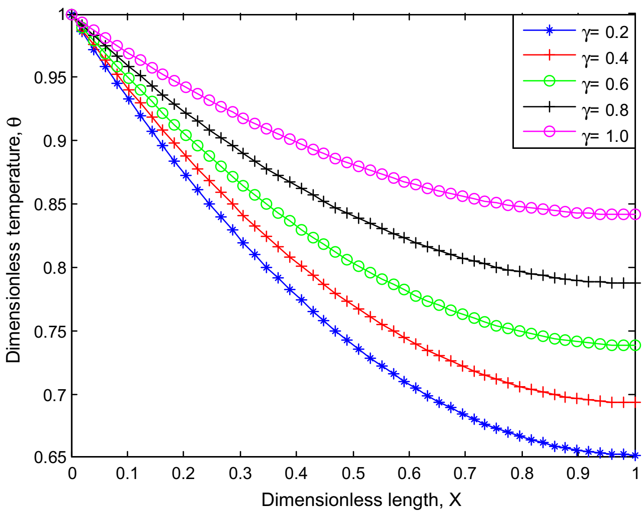

Figure 6 Dimensionless temperature distribution in the fin when $M = 2$, $\beta = 0.6$, $Q = 0.4$.

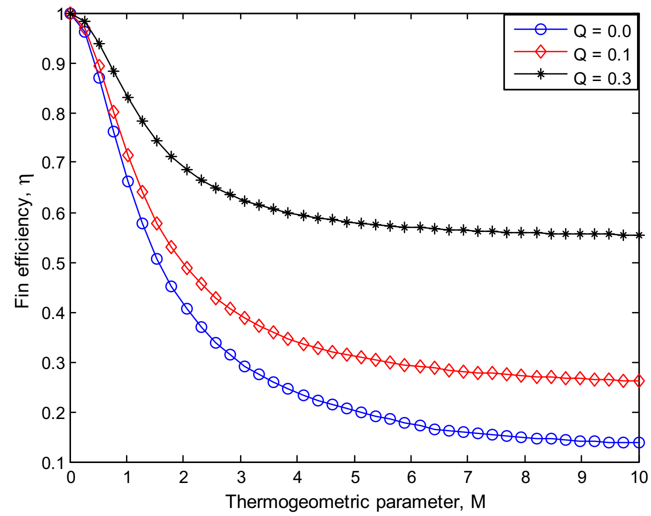

Figure 7 Dimensionless temperature distribution in the fin when $\beta = 0.1$, $\gamma = 0.8$.

thermal performance at different thermogeometric parameters. From the figures, as the internal heat generation parameter increases the temperature gradient of the fins decreases. This is because, as the rate of internal heat generation within the fin increases, the thermal performance of the fin decreases. However, the figures show that the dimensionless temperature gradient of the fin length increases as the thermogeometric parameter increases.

Also, Fig. 8 illustrates the effects of internal heat generation and aspect ratio on the effectiveness of the fin for the temperature dependent thermal conductivity and heat transfer coefficient. From the figures, it could be seen that as the rate of internal heat generation increases and aspect ratio increases (in case of effectiveness of the fin), higher local temperature is produced in the fin, thereby increases the efficiency and the effectiveness of the fin. Also, from the results, it shows that high efficiency and effectiveness of fin could be achieved by using small values of thermogeometric parameter and this could be realized using a fin of small length or by using a material of better thermal conductivity.

The effects of Biot number and aspect ratio on the thermogeometric parameter (the fin performance factor) are shown in Fig. 8. From the results, the fin performance factor increases as the aspect ratio and Biot number increase. However, the thermal performance or efficiency of the fin is favoured at low values of thermogeometric parameter since the aim (high effective use of the fin) is to minimize the temperature decrease along the fin length, where the best possible scenario is when $T = T_b$ everywhere. It must be pointed out that Eq. (40) shows the direct relationship between the thermo-geometric parameter, M and the Biot number, Bi which directly depends on the fin length. A small value of M corresponds to a relatively short and thick fin of poor thermal conductivity and a high value of M implies a long fin or fin with low value of thermal conductivity. Since, the thermal performance or efficiency of the fin is favoured at low values of thermo-geometric fin parameter, very long fins are to be avoided in practice. A compromise is reached for one-dimensional analysis of fins $0 < Bi < 0.1$. When the Biot number is greater than 0.1, two dimensional analysis of the fin is recommended as one-dimensional analysis predicts unreliable results for such limit.

Fig. 9a and b shows the effects of thermal conductivity parameter on the efficiency of the fin when there is no internal heat generation in the fin and at constant heat transfer coefficient while Fig. 10a–d shows that the fin efficiency decreases monotonically (for different thermal conductivity and at a constant heat transfer coefficient) with increasing thermogeometric parameter. Also, it shows the variation of fin efficiency with thermogeometric in longitudinal convecting fin with insulated tip. From the figures, it is shown that as the thermogeometric parameter increases, the efficiency of the fin decreases. When the fin parameter equals to zero, the fin efficiency is 100%, which implies that there is no conduction resistance or no presence of fin at all. As the convective heat transfer coefficient to thermal conductivity ratio approaches zero, the temperature at every point in the fin is equal to the temperature of the base. The figures also depict that there are steep drops in the efficiency for $0 < M < 4$ after which the slopes of the curves decrease and is almost zero for $M > 8$. The inverse variation in the fin efficiency with the thermogeometric parameter is due to the fact that as more material is attached to the prime surface, the resistance to heat flow increases thereby reducing the fin efficiency. Upon further increase in the fin thermo-geometric parameter, the effect of reducing the resistance becomes visible in the sense that the fin efficiency starts to normalize. Therefore, high efficiency of the fin could be achieved by using small values of thermogeometric parameter, which could be realized using a fin of small length or by using a material of better thermal conductivity. Moreover, the results depict that care must be taken in the choice of length of fin used during applications. This is because, thermogeometric parameter (which increases as the fin length increases) tends to infinity, and the fin efficiency tends to zero. The fin to a large extent of its length will remain at ambient. This consequently results in weak conduction limit. The extended area is largely useless in the heat transfer process and hence inefficient. Therefore, very long fins are to be avoided in practice.

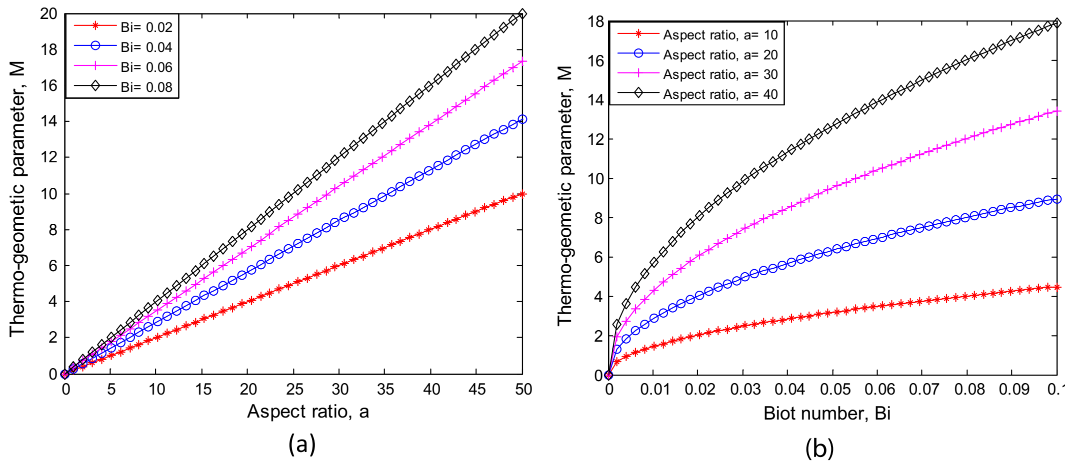

Figure 8 Effects of Biot number on the thermos-geometric parameter of the fin.

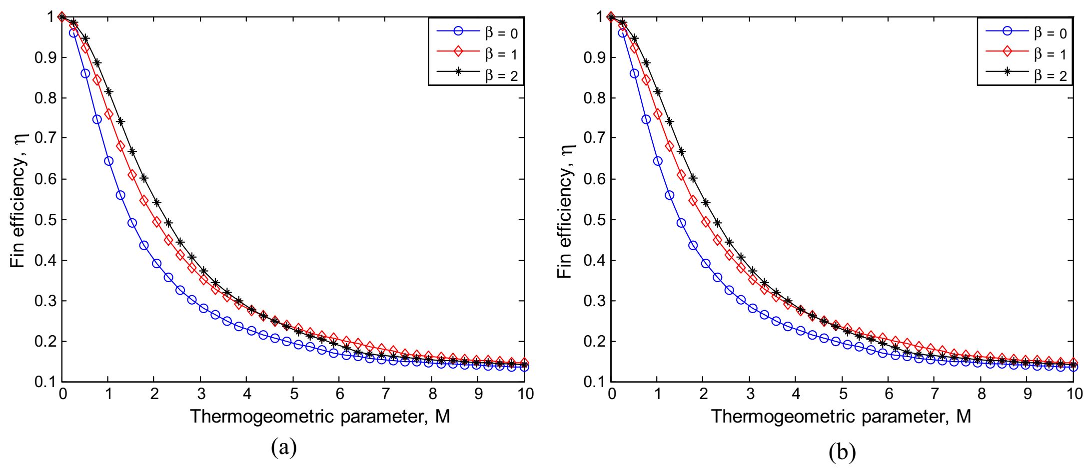

Figure 9 Effects of thermal conductivity parameter on the efficiency of the fin when (a) $Q = 0$, $\gamma = 0.2$ (b) $Q = 0$, $\gamma = 0.8$.

Also, as shown in the figures, the fin efficiency is unity in the limit $M \to 0$. In this limit, the actual heat transfer rate from the fin is zero. This fin parameter (the thermo-geometric parameter) plays a very important role in determining the amount of heat transfer from the fin as it accounts for the effects of decrease in temperature on the heat transfer from the fin. Since, the fin temperature drops along the fin length, the fin efficiency decreases with increase in fin length. Therefore, in practice required fin length should be properly determined because the fin length that causes the fin efficiency to drop below 60% usually cannot be justified economically and should be avoided.

The finite difference method of solution was validated by the exact solution in Fig. 11a and b for the linear thermal model of the fin. This gives the confidence in the predicted results by the finite difference method for the non-linear problems in which no closed-form solution is difficult or impossible to obtain. Fig. 11 shows the comparison of exact solution and finite difference solution for the case of fin with constant thermal properties and internal heat generation. From the results, it could be seen that the finite difference solution agrees with analytical solution. However, during the analysis, it was found that the relative error between the exact solution and the finite difference solution decreases as the numbers of nodes for the finite difference method increase. Also, the relative error increases as the value of the thermo-geometric parameter, M increases for fixed fin length, and increases with the increase of dimensionless fin length for fixed value of the thermo-geometric parameter, M. This fact has also been established by Han et al. [30].

Fig. 12 shows the effects of the thermo-geometric parameter, M and internal heat generation parameter, Q on the thermal behavior of the fin. On maintaining the value of the non-linear term, β while varying the value of M produces physically unsound behavior for larger values of the thermo-geometric parameter. It is shown that for growing values of the thermo-geometric parameter the temperature tends to negative values at the tip of the fin (which shows thermal instability) at $x = 1$ (the tip of the fin), contradicting the assumption of Eq. 15b. It has been observed that in order for a solution to be physically sound the fin thermo-geometric parameter M may not exceed a specific value and this is thought to be related to thermal instability as stated in the literature. This thermal behavior has also been pointed by Harley and Moitsheki [32]. Following the assumptions made regarding the numerical solution of the problem, it was realized that these solutions are

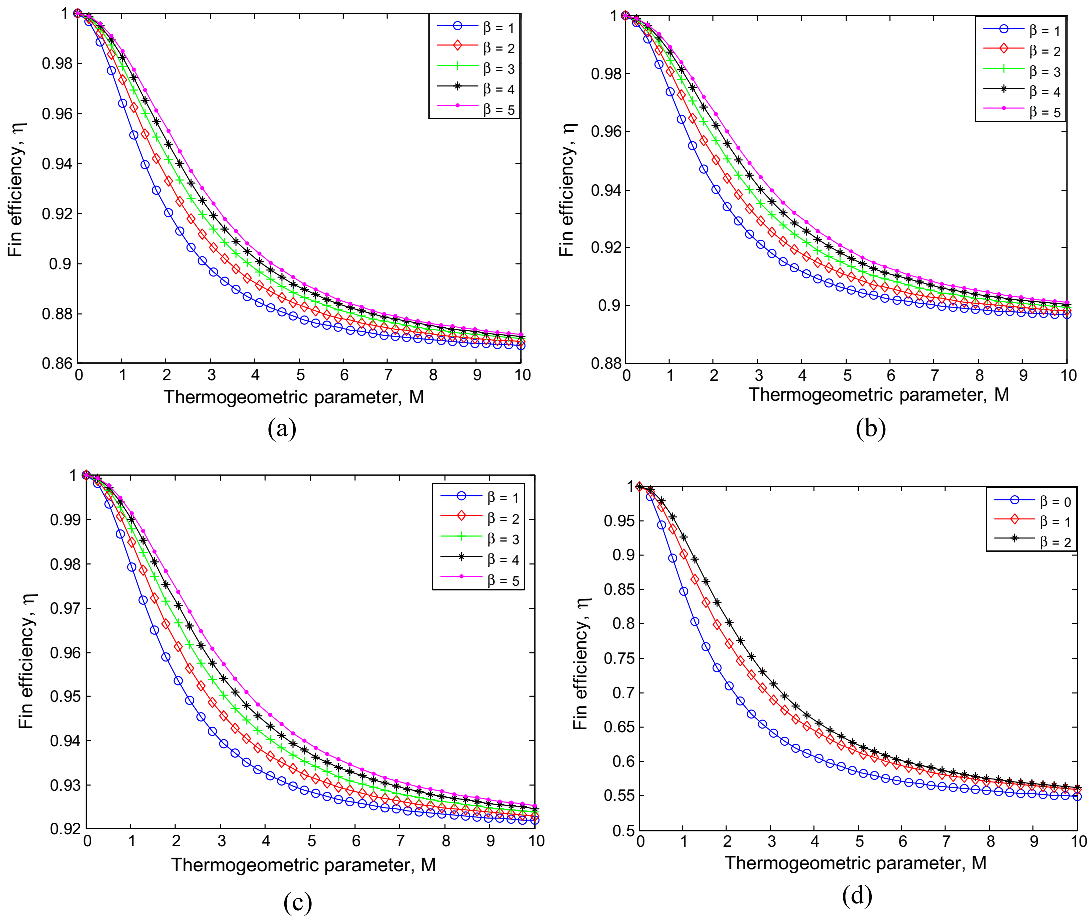

Figure 10 Effects of thermal conductivity parameter on the efficiency of the fin when (a) $Q = 0.6$, $\gamma = 0.2$ (b) $Q = 0.5$, $\gamma = 0.6$ (c) $Q = 0.6$, $\gamma = 0.4$ (d) $Q = 0.2$, $\gamma = 0.2$.

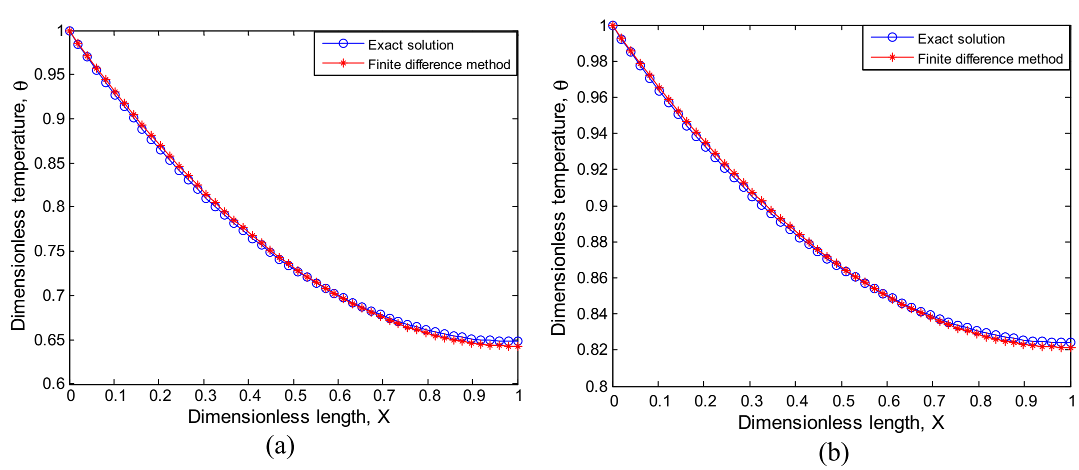

Figure 11 Dimensionless temperature distribution in the fin parameters when (a) $M = 1$, $\beta = 0$, $Q = 0$ (b) $M = 1$, $\beta = 0$, $Q = 0.5$.

not only physically unsound but also point toward thermal instability [33]. Yeh and Liaw [33] observed that a solution could not be found when the fin parameter M exceeded a specific value. They believed this occurrence to be related to thermal instability. Similar situations were found by Harley and Moitsheki [32] for the case of multiboiling heat transfer modes where for each value of n considered there seems to be a $M_{\max}$ such that solutions obtained are only physically sound when the thermo-geometric parameter values are chosen such that $M \leqslant M_{\max}$. Therefore, in order to ensure stability

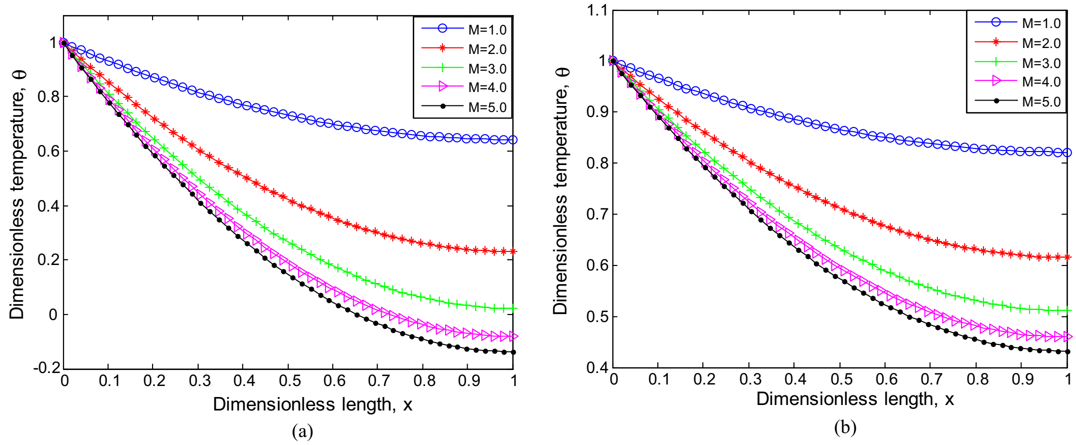

Figure 12 Dimensionless temperature distribution in the fin parameters for different values of M when (a) $\beta = 1$, $\gamma = 0$, $Q = 0$ (b) $\beta = 1$, $\gamma = 0$, $Q = 0.5$.

and avoid numerical diffusion of the solution by the explicit finite difference method, the thermo-geometric parameter, M must not exceed a certain value.

From the analysis of this work, the limiting value of M for thermal instability to the initially in the fin of constant thermal properties without internal heat generation is $\sqrt{2}$. However, when the temperature-dependent properties and internal heat generation in the fin are considered, the value of M for the thermal stability range increases. Furthermore, the thermal stability analysis of the fin (due to the non-linear term, β) with temperature-dependent thermal properties and internal heat generation using finite difference method was analyzed using the approach given by Jain et al. [34]. However, the cubic equation resulting from the stability analysis led to the use of the idea of Yeh and Liaw [33], who observed that a solution could not be found when the fin parameter M exceeded a specific value. They believed this occurrence to be related to thermal instability. So, the determined values of M were used to ensure the thermal stability of the problem.

The finite difference solution was validated with the fourth-Order Runge–Kutta and differential transformation method as shown in Table 1. It is shown that the finite difference method is highly accurate and agrees very well with the fourth-Order Runge–Kutta and differential transform methods.

Table 1 Comparison of the results.

X	4th-R-K Ganji (2014)	DTM Ganji (2014)	FDM (The present study)
0.0	0.9784	0.9785	0.9783
0.1	0.9786	0.9787	0.9785
0.2	0.9792	0.9792	0.9792
0.3	0.9802	0.9803	0.9802
0.4	0.9817	0.9817	0.9817
0.5	0.9835	0.9836	0.9835
0.6	0.9858	0.9859	0.9858
0.7	0.9886	0.9886	0.9886
0.8	0.9919	0.9919	0.9919
0.9	0.9956	0.9957	0.9956
1.0	1.0000	1.0000	1.0000

7. Conclusion

In this work, heat transfer analysis in a longitudinal rectangular fin with temperature-dependent thermal conductivity and internal heat generation was carried out using finite difference method. The results show that the fin temperature distribution, the total heat transfer, and the fin efficiency are significantly affected by the thermo-geometric parameters of the fin. Also, for the solution to be thermally stable, the fin thermo-geometric parameter must not exceed a specific value. However, it was established that the increase in temperature-dependent properties and internal heat generation values increases the thermal stability range of the thermo-geometric parameter. The result was validated with other previous results as presented in the literature. The results obtained in this analysis serve as a basis for comparison of other methods of analysis of the problem.

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