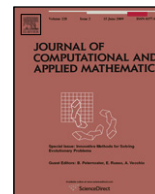




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Obtaining the Quantum Fourier Transform from the classical FFT with QR decomposition

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ABSTRACT

We present the detailed process of converting the classical Fourier Transform algorithm into the quantum one by using QR decomposition. This provides an example of a technique for building quantum algorithms using classical ones. The Quantum Fourier Transform is one of the most important quantum subroutines known at present, used in most algorithms that have exponential speed-up compared to the classical ones. We briefly review Fast Fourier Transform and then make explicit all the steps that led to the quantum formulation of the algorithm, generalizing Coppersmith's work.

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1. Introduction

For many years the spectral analysis of sampled data over a finite range, referred to as the Discrete Fourier Transform (DFT) was performed directly on computers, using $O(N^2)$ operations, where N is the number of data points. A great milestone in Fourier analysis was the paper published in 1965 in [1], where they described the so-called Fast Fourier Transform (FFT) algorithm, which can compute the DFT with only $O(N \log N)$ operations. In the next year, Rudnick presented a computer program which also required $O(N \log N)$ operations [2], being inspired by an earlier method due to [3].

Although the FFT algorithm may seem reasonably fast for classical Computer Science, it turns out that its quantum version can provide exponentially faster algorithms for some problems. In 1994, Shor [4] developed a quantum version of the Fourier Transform when the prime factors of N are not large (smaller than $\log N$). Motivated by Shor's work, Coppersmith [5] developed the quantum version of the FFT when N is a power of two, which required only $O(\log^2 N)$ operations. In the same year [6], using a recursive approach, has also shown how to implement the Fourier Transform in quantum computers.¹ This is, with no doubt, the most important quantum subroutine designed so far. Most quantum algorithms with exponential improvement when compared to the classical counterparts use the Quantum Fourier Transform (QFT) as an essential part. These algorithms solve instances of the Hidden Subgroup Problem (HSP), which has several important applications. A good survey on the HSP can be found in [8].

The quantum algorithm for an approximate QFT was developed in [5], with complexity $O(m \log N)$, where $1 \leq m \leq \log N$ is the parameter that defines the degree of approximation, usually taken as $O(\log \log N)$. Indeed, this approximate algorithm is the one with practical applications. The exact transform yields a level of accuracy that, in most of the cases, cannot even be

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detected by the measuring device. Barenco et al. [9] showed that, in the presence of decoherence, i.e., in realistic situations, the approximate transform may achieve better results than the exact one. They also showed that the accuracy of the exact QFT can be achieved by applying the approximate QFT repeatedly $O\left(\frac{\log^3 N}{m^3}\right)$ times.

In this paper we build the QFT algorithm from the classical FFT, generalizing Coppersmith’s work. In order to accomplish this generalization, we use the QR decomposition. This technique may provide insight for the development of new quantum algorithms.

In Section 2 we review the definition of DFT and the classical FFT algorithm. We also fix the notation that will be used in the generalization of the quantum algorithm. In Section 3 we perform the decomposition of the matrix form of FFT algorithm. QR decomposition plays an essential role in this task. The resulting unitary matrices can be interpreted as quantum logic gates. In Section 4 we summarize the results obtained in the previous sections and show how to build the circuit for the exact QFT.

2. The Discrete Fourier Transform

In this section we address the DFT in a form that will be useful later on. It is important to establish some notations that will be used throughout this paper. Let n be a positive integer, and let a, c be n -bit integers. The binary representations of a and c will be denoted, respectively, by

$$(a_{n-1}a_{n-2} \dots a_0)_2 \quad \text{and} \quad (c_{n-1}c_{n-2} \dots c_0)_2,$$

such that,

$$a = \sum_{j=0}^{n-1} a_j 2^j \quad \text{and} \quad c = \sum_{j=0}^{n-1} c_j 2^j.$$

Binary representations of numbers may have the subscript $(\cdot)_2$ omitted when context is clear. All logarithms are base two. Let X and Y be arrays² of $N = 2^n$ complex numbers. The integers a and c defined above will be useful for indexing X and Y . The notation used in this paper for that indexing is X_a . Let $\omega \equiv \omega_N = \exp(2\pi i/N)$, be the N -root of unity.

Finally, we may define the DFT.

Definition 1 (DFT). The Discrete Fourier Transform takes as input a complex array X , and converts it into a complex array Y , according to the expression

$$Y_c = \frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} X_a \omega^{ac}. \tag{1}$$

This equation may be rewritten by making explicit the binary representation of a and c . We concentrate only on the sub-term ω^{ac} .

$$\omega^{ac} = \exp\left(\frac{2\pi i}{N} \sum_{0 \leq j, k \leq n-1} a_j c_k 2^{j+k}\right). \tag{2}$$

Using $\omega^{(2^n)} = 1$, we can state that

$$\omega^{(2^{j+k})} = \omega^{(2^n)2^{j+k-n}} = 1, \tag{3}$$

if $j + k \geq n$. Therefore, it is possible to eliminate terms from Eq. (2) when $j + k \geq n$, leading us to an equivalent definition of the DFT,

$$Y_c = \frac{1}{\sqrt{N}} \sum_{0 \leq a \leq N-1} X_a \exp\left(\frac{2\pi i}{N} \sum_{\substack{0 \leq j, k \leq n-1 \\ j+k \leq n-1}} a_j c_k 2^{j+k}\right). \tag{4}$$

Coppersmith [5] observed that we may define a more general transform by changing the range of $j + k$ in Eq. (4). If we take $n - m \leq j + k \leq n - 1$, where m is a parameter such that $1 \leq m \leq n$, we obtain the definition of the Approximate Fourier Transform. When $m = n$, we recover the definition of the exact DFT. When $m = 1$, we achieve the a slightly modified definition of the well-known Hadamard transform—with a different indexing of the elements in the array. The argument in the exponent of this new transform differs from that in the exponent of DFT just by

$$i\epsilon = \frac{2\pi i}{N} \sum_{\substack{0 \leq j, k \leq n-1 \\ j+k < n-m}} a_j c_k 2^{j+k}. \tag{5}$$

² Some readers may prefer to see X and Y as functions, respectively $X : \mathbb{Z}_N \rightarrow \mathbb{C}$, and $Y : \mathbb{Z}_N \rightarrow \mathbb{C}$, where $N = 2^n$.

It is not difficult to show that the magnitude $|\epsilon|$ of that error in Eq. (5) is bounded by $2\pi n2^{-m}$. Both the approximate and the exact Fourier Transforms may be expressed by a matrix operation over the array X . The entries of the corresponding matrices only differ by a multiplicative factor $\exp(i\epsilon)$, where $|\epsilon| \leq 2\pi n2^{-m}$. Therefore, the approximate transform is able to provide results very close to the exact Fourier Transform, even if the parameter m is not so close to n , because the error decreases exponentially as m increases. The minimum m required to ensure an error not greater than $|\epsilon_{max}|$ is given by

$$m = \log \frac{2\pi}{|\epsilon_{max}|} + \log \log N. \quad (6)$$

Thus, the Approximate Fourier Transform provides results with a fixed tolerance $|\epsilon_{max}|$ by using a parameter m that barely increases with the size of the input.

We now briefly review the classical FFT [10,11] via Danielson Lanczos lemma. Danielson Lanczos lemma [3] allows the development of a recursive divide-and-conquer scheme to calculate the DFT. According to this lemma, Eq. (1) can be split into two parts. These parts of length $N/2$ can be calculated by employing again Danielson Lanczos lemma, leading to the calculation of four new Fourier Transforms, of length $N/4$ each. This process repeats itself n times until hitting arrays of length one, for which the Fourier Transforms are trivially found—in Eq. (1), when $N = 1$, we have $Y = X$. This is the idea behind Algorithm 1. Here, we denote by $X^{(s)}$ the state of a vector X in a step s of the computation.

Algorithm 1 Classical FFT

Require: This algorithm receives as input a vector $X \in \mathbb{C}^{2^n}$.

Ensure: The output is a vector $Y \in \mathbb{C}^{2^n}$ which is the DFT of vector X .

- 1: **for all** a such that $0 \leq a \leq 2^n - 1$ **do** {Initialization, step n }
- 2: let $X_{(a_{n-1}a_{n-2}\dots a_0)_2}^{(n)} \leftarrow X_{(a_{n-1}a_{n-2}\dots a_0)_2}$.
- 3: **for** s from $n - 1$ to 0, downward **do** {Step s }
- 4: **for all** $0 \leq b_{n-1}, \dots, b_s, a_{s-1}, \dots, a_0 \leq 1$ **do**
- 5:

$$X_{(b_{n-1}\dots b_s a_{s-1}\dots a_0)_2}^{(s)} \leftarrow \frac{1}{\sqrt{2}} X_{(b_{n-1}\dots b_{s+1} 0 a_{s-1}\dots a_0)_2}^{(s+1)} + \frac{1}{\sqrt{2}} \omega^{(b_s b_{s+1}\dots b_{n-1} 0 \dots 0)_2} X_{(b_{n-1}\dots b_{s+1} 1 a_{s-1}\dots a_0)_2}^{(s+1)}. \quad (7)$$

- 6: **for all** b such that $0 \leq b \leq 2^n - 1$ **do** {Re-ordering}
 - 7: set $Y_{(b_{n-1}b_{n-2}\dots b_0)_2} \leftarrow X_{(b_0 b_1 \dots b_{n-1})_2}^{(0)}$.
-

The complexity of the FFT can be easily computed from the above considerations. If we denote by T_{2^n} the approximate number of steps of a DFT on an array of length 2^n , we may write $T_{2^n} = 2T_{2^{n-1}} + 2^n$, for $n \geq 1$, with $T_1 = 0$. Then, $T_{2^n} = n2^n$, which means that the FFT algorithms have complexity $O(n2^n)$ or, equivalently, $O(N \log N)$.

3. Obtaining QFT from FFT

If we treat $X^{(s)}$ as column vectors, it becomes clear that Algorithm 1 may be expressed as a matrix operation, such that

$$X_j^{(s)} = \sum_{0 \leq k \leq N-1} P_{jk}^{(s)} X_k^{(s+1)}, \quad (8)$$

where $P^{(s)}$ is a $N \times N$ matrix for $0 \leq s \leq n - 1$. The indices of vectors, as well as the rows and columns of the matrices will be numbered from 0 to $N - 1$ in this paper.

By observing Algorithm 1, we see in Eq. (7) that each row of $X^{(s)}$ depends only on two rows of $X^{(s+1)}$. Hence, each row of the matrices $P^{(s)}$ have only two nonzero entries. They are located on columns $k = j - j_s 2^s$ and $k = j + (1 - j_s) 2^s$, where j_s is the $(s + 1)$ -th bit of j (counting from the least to the most significant bit). Thus, each nonzero entry of matrices $P^{(s)}$ will be placed only in the main diagonal, or in a subdiagonal, depending of the value of the bit j_s of j .

When $k = j$, we have the entries of the main diagonal of the matrices. If $j_s = 0$, then the first term on the right hand side of Eq. (7) shows us that these entries are $\frac{1}{\sqrt{2}}$. If $j_s = 1$, then the second term on the right hand side of Eq. (7) shows us that the entries are $\frac{1}{\sqrt{2}} \omega^{(j_s j_{s+1} \dots j_{n-1} 0 \dots 0)_2}$.

When $k = j - 2^s$, we have the entries of the lower subdiagonal of the matrices. If $j_s = 0$, then the entries are 0, according to Eq. (7). If $j_s = 1$, then the first term on the right hand side of Eq. (7) shows us that the entries are $\frac{1}{\sqrt{2}}$.

When $k = j + 2^s$, we have the entries of the upper subdiagonal of the matrices. If $j_s = 0$, then the second term on the right hand side of Eq. (7) shows us that the entries are $\frac{1}{\sqrt{2}} \omega^{(j_s j_{s+1} \dots j_{n-1} 0 \dots 0)_2}$. If $j_s = 1$, then the entries are 0, according to Eq. (7).

Before we express these ideas mathematically let us introduce additional notation.

If we define a binary fraction as

$$0.j \equiv 0.j_0 j_1 \dots j_{n-1} = \sum_{0 \leq t \leq n-1} \frac{j_t}{2^{t+1}}, \quad (9)$$

we may rewrite $\omega^{(j_s j_{s+1} \dots j_{n-1} 0 \dots 0)_2}$ as $\omega^{(0.j)2^{n+s}}$.

We note also that

$$j_s = \frac{1 - (-1)^{\lfloor \frac{j}{2^s} \rfloor}}{2}. \tag{10}$$

The generic matrices $P^{(s)}$ can now be written as,

$$P_{jk}^{(s)} = \frac{1}{\sqrt{2}} \begin{cases} \omega^{j_s(0,j)2^{n+s}}, & \text{if } k = j \\ \frac{1 - (-1)^{\lfloor \frac{j}{2^s} \rfloor}}{2}, & \text{if } k = j - 2^s \\ \frac{1 + (-1)^{\lfloor \frac{j}{2^s} \rfloor}}{2} \omega^{(0,j)2^{n+s}}, & \text{if } k = j + 2^s \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

which represent the operations performed by the steps of Algorithm 1.

Proposition 2. *The matrices $P^{(s)}$ are unitary.*

Proof. We may solve

$$\left(P^{(s)} P^{(s)\dagger} \right)_{jk} = \sum_{0 \leq l \leq N-1} P_{jl}^{(s)} P_{kl}^{(s)*}. \tag{12}$$

When $k = j$, we have

$$\begin{aligned} \left(P^{(s)} P^{(s)\dagger} \right)_{jj} &= P_{jj}^{(s)} P_{jj}^{(s)*} + P_{j,j-2^s}^{(s)} P_{j,j-2^s}^{(s)*} + P_{j,j+2^s}^{(s)} P_{j,j+2^s}^{(s)*} \\ &= \frac{1}{2} + \frac{1}{2} \left(\frac{1 - (-1)^{\lfloor \frac{j}{2^s} \rfloor}}{2} \right)^2 + \frac{1}{2} \left(\frac{1 + (-1)^{\lfloor \frac{j}{2^s} \rfloor}}{2} \right)^2 \\ &= 1. \end{aligned} \tag{13}$$

When $k = j - 2^s$ or $k = j + 2^s$ we may perform analogous calculation and obtain $\left(P^{(s)} P^{(s)\dagger} \right)_{jk} = 0$. Therefore, $\left(P^{(s)} P^{(s)\dagger} \right)_{jk} = \delta_{jk}$. \square

Since the matrices $P^{(s)}$ are unitary, they could in principle be implemented on a quantum computer. It is important, when developing a quantum algorithm, to express the unitary operators in terms of universal quantum gates, that is, controlled-NOTs and gates acting on single qubits.

In order to find the universal gates for the quantum version of the FFT algorithm, several decompositions may be applied to matrices $P^{(s)}$. One method that may be insightful in this process is the QR decomposition [12,11], which factors a generic matrix into the product of a unitary matrix M – orthogonal, if the matrix to be decomposed is real – and an upper triangular matrix N . In the case of matrices $P^{(s)}$ we may apply a slightly modified version of QR decomposition which yields orthogonal and diagonal matrices as factors. This can be done because matrices $P^{(s)}$ have the following property: their columns are either real or multiple of real columns. There are at least three well-known methods for computing the QR decomposition: Householder reflections, Givens rotations and Gram–Schmidt decomposition. The last one is particularly interesting in this case because it takes advantage of the property mentioned above.

Observe that any column of matrices $P^{(s)}$ may be obtained by fixing a value of k in Eq. (11) and running j from 0 to $N - 1$. We note that the columns of matrices $P^{(s)}$ are already orthonormal. Multiplying each column k by

$$\alpha_k^{(s)} \equiv (-1)^{\lfloor \frac{k}{2^s} \rfloor} \omega^{2^n - k_s(0,k)2^{n+s}}, \tag{14}$$

for $k = j$, $k = j - 2^s$ and $k = j + 2^s$, and then multiplying the corresponding cases in Eq. (11) by $\alpha_k^{(s)}$, we obtain the orthogonal matrices

$$M_{jk}^{(s)} = \frac{1}{\sqrt{2}} \begin{cases} (-1)^{\lfloor \frac{j}{2^s} \rfloor}, & \text{if } k = j \\ \frac{1 - (-1)^{\lfloor \frac{j}{2^s} \rfloor}}{2}, & \text{if } k = j - 2^s \\ \frac{1 + (-1)^{\lfloor \frac{j}{2^s} \rfloor}}{2}, & \text{if } k = j + 2^s \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

The upper triangular matrices are

$$N_{jk}^{(s)} = \begin{cases} (-1)^{\lfloor \frac{j}{2^s} \rfloor} \omega^{js(0,j)2^{n+s}}, & \text{if } k = j \\ 0, & \text{otherwise.} \end{cases} \quad (16)$$

Now we confirm the decomposition.

Proposition 3. For any step s and for any number of bits n we have

$$P^{(s)} = M^{(s)}N^{(s)}, \quad (17)$$

where the matrices $M^{(s)}$ and $N^{(s)}$ are given by Eqs. (15) and (16), respectively.

Proof. Since matrices $N^{(s)}$ are diagonal we have

$$(M^{(s)}N^{(s)})_{jk} = M_{jk}^{(s)}N_{kk}^{(s)}. \quad (18)$$

When $k = j$,

$$\begin{aligned} M_{jj}^{(s)}N_{jj}^{(s)} &= \frac{(-1)^{\lfloor \frac{j}{2^s} \rfloor}}{\sqrt{2}} (-1)^{\lfloor \frac{j}{2^s} \rfloor} \omega^{js(0,j)2^{n+s}} \\ &= P_{jj}^{(s)}. \end{aligned} \quad (19)$$

When $k = j - 2^s$ or $k = j + 2^s$ we may perform analogous calculations, and easily obtain $(M^{(s)}N^{(s)})_{jk} = P_{jk}^{(s)}$. \square

Let us now analyze the structure of the matrices $M^{(s)}$. Starting with $M^{(0)}$, we note that

$$M_{jk}^{(0)} = \frac{1}{\sqrt{2}} \begin{cases} (-1)^j, & \text{if } j = k \\ \frac{1 - (-1)^j}{2}, & \text{if } k = j - 1 \\ \frac{1 + (-1)^j}{2}, & \text{if } k = j + 1 \\ 0, & \text{otherwise.} \end{cases} \quad (20)$$

It is easy to check that

$$M_{2^n \times 2^n}^{(0)} = I^{\otimes(n-1)} \otimes H, \quad (21)$$

where I is the 2×2 identity matrix, and $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard matrix.

Now, we prove that $M_{2^n \times 2^n}^{(s)} = M_{2^{n-1} \times 2^{n-1}}^{(s-1)} \otimes I$. We use the following formula for a generic matrix A :

$$(A \otimes I)_{jk} = \begin{cases} A_{\lfloor \frac{j}{2} \rfloor \lfloor \frac{k}{2} \rfloor}, & \text{if } k = j \pmod{2} \\ 0, & \text{otherwise,} \end{cases} \quad (22)$$

which can be obtained by analyzing the entries of $A \otimes I$. Replacing A by $M_{2^{n-1} \times 2^{n-1}}^{(s-1)}$ in Eq. (22) and simplifying the result, we get the right-hand side of Eq. (15). Thus

$$M_{2^n \times 2^n}^{(s)} = M_{2^{n-1} \times 2^{n-1}}^{(s-1)} \otimes I. \quad (23)$$

Proposition 4. The matrices $M^{(s)}$ may be decomposed in tensorial products involving only 2×2 identities and a Hadamard matrix, such that

$$M^{(s)} = I^{\otimes(n-s-1)} \otimes H \otimes I^{\otimes s}. \quad (24)$$

Proof. Using (23) recursively s times and replacing n by $n - s$ in Eq. (21) we get Eq. (24). \square

Let us now analyze the structure of the matrices $N^{(s)}$. They are diagonal and the entries are one or $\pm\omega^c$, $0 \leq c < N$. The first attempt in decomposing them is to write $N^{(s)}$ for a fixed s as a product of diagonal matrices, the entries of which are one or $\pm\omega^c$ for a fixed c . We rewrite Eq. (16) using Eq. (9) and (10) to obtain the form

$$N_{jk}^{(s)} = \begin{cases} (-1)^{js} \prod_{t=0}^{n-1} \omega^{js_t t 2^{n+s-t-1}}, & \text{if } k = j, \\ 0, & \text{otherwise.} \end{cases} \quad (25)$$

Since n and s are fixed, the factors we are looking for are obtained by fixing t . So, for given n and s , we define the matrix

$$R^{(s,t,u)} = \begin{cases} \omega^{j_s j_t 2^{n-u}}, & \text{if } k = j \\ 0, & \text{otherwise,} \end{cases} \tag{26}$$

where j_s and j_t are given by Eq. (10). Since matrices $R^{(s,t,u)}$ are diagonal for arbitrary s , t and u , they commute. We can now state the following proposition.

Proposition 5. *The matrices $N^{(s)}$ may be written as*

$$N^{(s)} = \prod_{t=s+1}^{n-1} R^{(s,t,u)}, \tag{27}$$

with $u = t - s + 1$, when $s < n - 1$. When $s = n - 1$, $N^{(n-1)} = I$.

Proof. Using $(-1)^{j_s} = \omega^{j_s j_s 2^{n-1}}$, we see that

$$(-1)^{j_s} \prod_{t=0}^s \omega^{j_s j_t 2^{n+s-t-1}} = 1. \tag{28}$$

Then, using this result in Eq. (25) we get Eq. (27) when $s < n - 1$. Replacing s by $n - 1$ in Eq. (25) we get $N^{(n-1)} = I$. \square

The structure of the matrices $R^{(s,t,u)}$ is the following: they are diagonal; the entries are one when either j_s or j_t is equal to zero; the entries are $\omega^{2^{n-u}} = \exp\left(\frac{2\pi i}{2^u}\right)$ when j_s and j_t are simultaneously equal to one. Therefore, the matrix $R^{(s,t,u)}$ is a controlled operation

$$R^{(u)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(\frac{2\pi i}{2^u}\right) \end{pmatrix}, \tag{29}$$

with control on qubit s and target on qubit t (or vice-versa, in this case). This is a generalization of control gates acting on two qubits in the presence of more qubits.

Summarizing, the QFT can be expressed as

$$F_{2^n} = A^{(n)} \prod_{s=0}^{n-1} M^{(s)} N^{(s)}, \tag{30}$$

where the matrices $M^{(s)}$ and $N^{(s)}$ are given by Eqs. (24) and (27) respectively. $A^{(n)}$ is a $2^n \times 2^n$ matrix implementing the final swaps of the algorithm.

4. Building the QFT from the proposed decomposition

Based on the last section we may finally derive a quantum algorithm to compute the DFT. In the beginning of classical FFT, we have a collection of complex numbers $X_{(a_{n-1} a_{n-2} \dots a_0)}^{(n)}$. These values now correspond to the quantum state

$$|\psi_n\rangle = \sum_{0 \leq a_{n-1}, a_{n-2}, \dots, a_0 \leq 1} X_{(a_{n-1} a_{n-2} \dots a_0)}^{(n)} |a_{n-1} a_{n-2} \dots a_0\rangle. \tag{31}$$

This preparation of the quantum system corresponds to the initialization of the algorithm. Once the state has been prepared, we should apply the matrices $P^{(s)}$ given by Eq. (11) in the following order:

$$|\psi_0\rangle = P^{(0)} P^{(1)} \dots P^{(n-1)} |\psi_n\rangle. \tag{32}$$

The state $|\psi_0\rangle$ is

$$|\psi_0\rangle = \sum_{0 \leq c_{n-1} \dots c_0 \leq 1} X_{(c_0 \dots c_{n-1})}^{(0)} |c_{n-1} \dots c_0\rangle, \tag{33}$$

with the coefficients labelled in the inverse order.

Although the matrices $P^{(s)}$ are, in general, too complex to be directly realized in a physical experiment, it was shown that each of them can be decomposed into simpler matrices $M^{(s)}$ and $N^{(s)}$, which in turn may be decomposed into gates acting on one or two qubits.

In each step s , we must first apply the matrix $N^{(s)}$. In step $n - 1$ we have $N^{(n-1)} = I$. Eq. (27) shows us that each matrix $N^{(s)}$ is a product of other simpler matrices. Hence, we must apply the logical gate $R^{(s,t,t-s+1)}$ – gate $R^{(t-s+1)}$ with control on qubit s and target on qubit t – for each t starting from $t = n - 1$ and going downward until $t = s + 1$. Then, we apply the matrix $M^{(s)}$, which corresponds to a Hadamard gate acting only on qubit s . After running s from $n - 1$ to 0, we must apply swaps to correct the order of the output. We compiled these steps in Algorithm 2

Algorithm 2 QFT

Require: This algorithm must receive as input a vector $X \in \mathbb{C}^{2^n}$.

Ensure: The output is a quantum state $|\psi\rangle$ whose amplitudes correspond to the elements of $Y \in \mathbb{C}^{2^n}$, given by the DFT of X .

- 1: {Initialization, step n }
- 2: prepare the state of the n -qubit quantum register as

$$|\psi_n\rangle = \sum_{k=0}^{N-1} X_k |k\rangle.$$

- 3: **for** s from $n - 1$ to 0 , downward **do** {Step s }
- 4: **for** t from $n - 1$ to $s + 1$, downward **do**
- 5: apply unitary operation $R^{(s,t,t-s+1)}$
- 6: apply a Hadamard gate only on qubit s .
- 7:
- 8: **for** t from 0 to $\lfloor n/2 \rfloor - 1$ **do** {Re-ordering}
- 9: swap qubits t and $n - t - 1$.

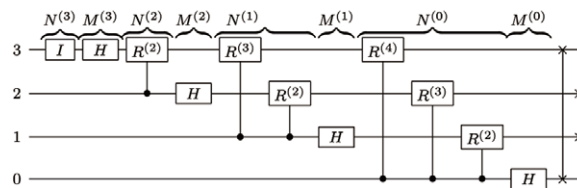


Fig. 1. A circuit for QFT over $n = 4$ qubits. Note that the qubits are numbered from bottom to top, starting from 0 up to $n - 1$.

In Fig. 1 we represent the QFT circuit over four qubits in terms of Hadamard, controlled gates $R^{(u)}$ and swap operations. Note that matrices $M^{(s)}$ and $N^{(s)}$ are also shown on the top. An alternative presentation of the circuit may be obtained by interchanging some of the gates that do not involve operations on the same qubits.

Now we address the computational complexity of Algorithm 2. We assume that the initialization is done in negligible time—which is a reasonable assumption, since in most applications known so far the QFT is applied on a state of the computational basis, or on the output of some earlier step of some algorithm. In the main part of the algorithm (the outer loop) we have $n(n + 1)/2$ steps. Therefore, this part of the algorithm is $O(n^2)$. The last part of the algorithm consists only on $O(n)$ swap operations. We conclude that the QFT has complexity $O(n^2)$ or, equivalently, $O(\log^2 N)$, in terms of one and two qubit operations. It is quite simple to show that this complexity does not change when calculated in terms of universal gates. One just needs to recall that a controlled gate can be decomposed into two CNOTs and three one-qubit gates, and that a swap gate can be decomposed into three CNOTs.

As an intermediate step before deducing the Approximate QFT, we may consider a classical algorithm for the Approximate FFT. This is quite similar to Algorithm 1, the only difference being that instead of $\omega^{(b_s \dots b_{n-1} 0 \dots 0)_2}$ in the second term on the right hand side of Eq. (7), we have $\omega^{(b_s \dots b_{\min(s+m-1, n-1)} 0 \dots 0)_2}$. By repeating all the process of finding the generic matrices $P^{(s)}$ and decomposing them, we find out that this difference reflects only on Eq. (27)—in the approximate algorithm the productory ranges from $t = s + 1$ to $\min(s + m - 1, n - 1)$. Analogously to the exact algorithm, we may check that the Approximate QFT has complexity $O(mn)$ or, equivalently, $O(m \log N)$. In fact, the approximate version of QFT is not only simpler and faster, but also leads to more precise results in the presence of decoherence than its exact counterpart [9].

5. Conclusions

The QFT algorithm represents an important improvement in the complexity of the classical algorithm, from $O(N \log N)$ to $O(\log^2 N)$. The acceleration provided by Approximate QFT algorithm goes even further, as $O(m \log N)$, where m is a parameter that can be taken as $O(\log \log N)$. A practical difference between the classical and the quantum FFT is that the latter provides the result of the calculation as a superposition of quantum states, which cannot be directly read according to the postulates of quantum mechanics. However, a remarkable example showing the advantage of the quantum version of FFT is the quantum algorithm for factorization of large integers [4]. In this algorithm, the quantum FFT subroutine plays an essential role on the exponential speed-up over the best classical algorithms for integer factorization.

In this paper, the building process of the QFT was exposed in detail, generalizing Coppersmith's work. We started from the description of the classical FFT and then obtained generic unitary matrices for each step of the algorithm. In order to get matrices simple enough to represent feasible quantum operations, those generic matrices were factored according to QR decomposition and the formulation in terms of one- and two-qubit gates was obtained. The complexity of this particular algorithm does not change when expressed in terms of universal gates.

We argued that the Approximate QFT is also reobtained by the method proposed here, with analogous calculations. Depending on the chosen parameter, the number of matrices generated by the decomposition of the approximate algorithm can be considerably lower than that of the exact one. This reflects on the complexity of the quantum algorithm.

The deduction of the algorithms addressed in this paper is different from previous works, and we hope it may provide insight for the development of new efficient quantum algorithms. As future directions, we are interested in checking whether other classical algorithms can be analyzed according to this technique, starting from its matrix form and then decomposing it until simpler unitary matrices are obtained.

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