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Prime Characters and Primitivity

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INTRODUCTION

A natural question, in the character theory of finite groups, is which quasi-primitive characters are actually primitive. A character is said to be quasi-primitive if it is irreducible and its restriction to every normal subgroup is homogeneous. If the group is solvable, T. Berger [9, Th. 11.33] has shown that every quasi-primitive character is actually primitive. If the group is simple, however, every irreducible character is quasi-primitive and may or may not be primitive. The character of degree 5 of Alt(5), for example, is not primitive. It follows that a generalization of Berger's Theorem to non-solvable groups must include some extra hypotheses which become vacuous when the group is solvable. In this paper we prove the following such theorem, with hypotheses on the chief factors and the character degrees.

THEOREM A. *Let χ be a quasi-primitive irreducible character of a finite group G. Assume that if C is a chief factor of G the following hold:*

(1) *If $C \simeq S \times \cdots \times S$ (n factors) where S is some non-Abelian simple group, $\hat{S}$ is the universal cover of S, $\psi \in \mathrm{Irr}(\hat{S})$, $\psi \neq 1$, and $\psi(1)^n \mid \chi(1)$, then $(\chi(1)/\psi(1))\psi$ is not induced from a character of a proper subgroup of $\hat{S}$.*

(2) *If C is an Abelian non-central p-group, $|C| \mid \chi(1)^2$, and $O_{p'}(G/C_G(C)) = C_G(C)$ then p is odd and if, furthermore, S is a subnormal simple subgroup of $G/C_G(C)$, $\hat{S}$ is the universal covering group of S, and*

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$\psi \neq 1$ is an irreducible character of $\hat{S}$ of degree a power of p then $p \nmid |Z(\hat{S})/\ker(\rho)|$.

Then χ is primitive.

Theorem A is proved by applying the theory of primitive characters developed in [3]. When the degree of χ is suitably restricted, hypotheses on the composition factors of G suffice, as demonstrated by the following corollaries.

COROLLARY B. *Assume that χ is a quasi-primitive irreducible character and $\chi(1) = p^a$ where p is a prime and $p > 7$. Suppose that the following groups are not isomorphic to composition factors of G:*

(1) *$L_n(q)$ when $(q^n - 1)/(q - 1) = p^b$,*
(2) *$L_n(q)$ when $p \mid (q - 1, n)$,*
(3) *$U_n(q)$ when $p \mid (q + 1, n)$,*
(4) *A_{p^b},*
(5) *M_{11} if $p = 11$.*

Then χ is primitive.

COROLLARY C. *Assume that χ is a quasi-primitive character of G of odd degree and that if S_0 is a non-Abelian composition factor of G then $S_0 \in \{L_2(q)\, q$ odd $q > 9$. $L_2(2^k)$ if $2^k \geqslant 8$ and $2^k + 1 \nmid \chi(1)$, $Sz(2^k)$ if $2^k \geqslant 32$ and $(2^{2k} + 1) \nmid \chi(1)$, M_{12}, J_1, J_2, M_{23}, HS, He, Ru, Co_3, Co_2, HN, Ly, Co_1, J_3, Suz, ON, $Fi_{22}\}$. Then χ is primitive.*

If the product of an admissible set of prime characters is primitive, then each prime is primitive. However, the converse is not true as shown by the following example. Let G be the semidirect product of $\hat{A}_6$ with the central product E of an extra special group of order 2^5 with Z_4. There is an admissible set of primes $\{\rho_1, \rho_2\}$ of G, with $\rho_1(1) = \rho_2(1) = 4$, $\ker(\rho_2) = E$, ρ_1 and ρ_2 primitive, but $\rho_1 \rho_2$ is induced from a linear character of $Z(E)\, \hat{A}_6$.

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Most of the notation is standard and is taken from [5, 1]. If X is a group and p a prime, then X_p denotes a Sylow p subgroup of X. If $g \in X$, then g^X denotes the set of conjugates of g.

If φ is a quasi-primitive irreducible character of a group X, then $F^*(\varphi)$, $Z(\varphi)$, and $M^*(\varphi)$ are defined by

$$\frac{F^*(\varphi)}{\ker(\varphi)} = F^*\left[\frac{X}{\ker(\varphi)}\right],$$

$$\frac{Z(\varphi)}{\ker(\varphi)} = Z\left[\frac{X}{\ker(\varphi)}\right],$$

$$M^*(\varphi) = \frac{F^*(\varphi)}{Z(\varphi)}$$

($F^*(X)$ denotes the generalized Fitting subgroup of a group X.) A non-linear irreducible quasi-primitive character is said to be a *prime character* if $M^*(\varphi)$ is homogeneous and $\varphi_{F^*(\varphi)}$ is irreducible.

A set $\{\varphi_1, ..., \varphi_n\}$ of prime characters of a group X is said to be *admissible* if $M^*(\varphi_i)$ and $M^*(\varphi_j)$ contain no X-isomorphic X-chief factors for $i \neq j$.

A character χ is *standard* if the determinantal order of χ divides some power of $\chi(1)$.

LEMMA 1. *Let G be a finite group and ρ a prime character of G with $M^*(\rho)$ an Abelian p-group. Suppose that H is a maximal subgroup of G which avoids $M^*(\rho)$. If $O_{p'}(G/F^*(\rho)) = F^*(\rho)$ assume that p is odd and that for every S a covering group of a non-Abelian composition factor of $F^*(G/F^*(\rho))/F^*(\rho)$ and every $\delta \in \mathrm{Irr}(S)$ with $\delta(1) \mid \rho(1)$, we have $p \nmid |Z(S)/\ker(\delta)|$. Then ρ_H is not a quasi-primitive irreducible character of H.*

Proof. By replacing ρ by a $\lambda\rho$ for an appropriate linear character λ, we may assume that ρ is a standard character of G, and we may further assume that $\ker(\rho) = 1$. We set $K = F^*(\rho) = F^*(G)$, and by [4, Th. 1.11], $C_G(K/Z(G)) = K$. By [4, Prop. 1.9], $\rho(1) = |K/Z(K)|^{1/2} = p^a$, and since ρ is standard, K is a p-group. By the maximality, $H \cap K = Z(G)$ and $HK = G$. We assume that ρ_H is a quasi-primitive irreducible character of H.

$H \cap O_{p'}(G/K)$ is a normal subgroup of H such that $H \cap O_{p'}(G/K)/Z(G) \simeq O_{p'}(G/K)/K$, a p'-group. It follows that there exists a subgroup Q such that $H \cap O_{p'}(G/K) = Q \times Z(G)$ and therefore $Q \lhd H$. Since ρ_H is quasi-primitive, $Q \subseteq Z(G) \subset K$, a p-group, and $Q = 1$. Hence $H \cap O_{p'}(G/K) = Z(G)$ and $O_{p'}(G/K) = K$. In particular, by hypothesis, p is odd.

Since H is a maximal subgroup, $G \neq K$. Let H_1 be a subgroup of H of minimal order subject to $H_1 \lhd H$ and $H \not\subseteq Z(G)$. Set $R = H_1 K$. Since G is completely reducible on $M^*(\rho)$, and $C_G(M^*(\rho)) = K$, $O_p(G/K) = K$, whence R/K is not solvable. Hence, the minimality of H_1 yields $H_1 = H'_1$,

$H_1 = A_1 \cdots A_t$ where $[A_i, A_j] = 1$ if $i \neq j$, and the A_i are isomorphic quasi-simple groups. It follows that the A_i are homomorphic images of S where S is a universal covering group of $A_i/Z(A_i)$ and $A_i/Z(A_i)$ is isomorphic to a non-Abelian composition factor of $F^*(G/F^*(\rho))/F^*(\rho)$. Now $\rho_{H_1} = e\delta$ where $\delta \in \mathrm{Irr}(H_1)$, $e \in Z^+$, and $\delta(1) = p^b \geqslant p$. Since ρ is faithful, $Z(H_1)$ is a p-group, If $p \mid |Z(H_1)|$, then ρ faithful and H a central product imply that there is a $\delta_1 \in \mathrm{Irr}(S)$ where $\delta_1(1) \mid \delta(1)$ and $p \mid |Z(S)/\ker(\delta_1)|$. This contradicts the hypothesis of the lemma so that H_1 is a direct product of simple groups.

By [10, Cor. 4.4], there is a subgroup U of G such that $G = UK$, $U \cap K = L = \langle \zeta \rangle = Z(K)$, $U \supseteq C_G(K)$, and there is a $\sigma \in \mathrm{Aut}(G)$ such that $\sigma^2 = 1$, σ inverts K/L and $U = C_G(\sigma)$.

By [10, Th. 7.1], $\rho_U = \psi\xi$ where ψ is a character of $U/Z(G)$, $|\psi(u)|^2 = |C_{K/L}(u)|$ for all $u \in U$, and $\xi \in \mathrm{Irr}(U)$. In particular, $\psi(1) = p^a$ whence ξ is linear. Since $G = UK$, and H_1 is perfect, $R = U_1 K$ where $U_1 = (R \cap U)'$. Furthermore $R \cap U = U_1 Z(G)$ implies $U_1 = U_1'$. If $\zeta^j \in Z(G) \cap U_1$, then $U_1 = U_1'$ and ξ linear imply that $\zeta^j \in \ker(\psi) \cap \ker(\xi) \subseteq \ker(\rho) = \{1\}$. Thus, $U_1 \cap K = U_1 \cap (U \cap K) \subseteq U_1 \cap Z(G) = 1$.

$H_1 K = R = U_1 K$ yields

$$H_1 \simeq \frac{H_1}{H_1 \cap K} \simeq \frac{H_1 K}{K} = \frac{U_1 K}{K} \simeq \frac{U_1}{U_1 \cap K} \simeq U_1.$$

Let h be a p'-element of H_1; then $h \to u$ under the isomorphism of H_1 and U_1 where u is the p'-element of U_1 such that $hK = uK$. Now $hK = uK$ and $\langle h \rangle K = \langle u \rangle K$, with h and u p'-elements imply that h is conjugate to u by an element of K. Thus, $\rho(h) = \rho(u)$. Let $(\rho_{H_1})^*$ and $(\rho_{U_1})^*$ denote the Brauer characters for the prime p corresponding to ρ_{H_1} and ρ_{U_1}. The previous discussion implies that $(\rho_{U_1})^*$ corresponds to $(\rho_{H_1})^*$ under the isomorphism from U_1 to H_1. Since ρ_H is quasi-primitive, $\rho_{H_1} = e\delta$ where $\delta \in \mathrm{Irr}(H_1)$ and $e \in Z^+$ implies that all the irreducible Brauer constituents of $(\rho_{U_1})^*$ lie in the same p-block B. Since ρ is a faithful character of degree p^a, δ corresponds to a faithful character of U_1 of prime power degree and $Z(U_1) = 1$ [3, Lemma 4.24] implies that B is not of full defect. U is isomorphic to a subgroup of $\tilde{G} = Sp(2a, p)$. Moreover, it follows from [10, Th. 4.8] that ψ is the restriction to U_1 of a corresponding character $\tilde{\psi}$ of $\tilde{G}$ where $\tilde{\psi} = \tilde{\psi}_1 + \tilde{\psi}_2$ with $\tilde{\psi}_1(1) = (p^a - 1)/2$ and $\tilde{\psi}_2(1) = (p^a + 1)/2$. Since $((p^a - 1)/2, (p^a + 1)/2) = 1$, there is $\gamma \in \mathrm{Irr}(U_1)$ such that $0 \neq (\gamma, \psi_{U_1}) = (\gamma, \rho_{U_1})$ and $(\gamma(1), p) = 1$. Then $(\rho_{U_1})^*$ a sum of irreducible Brauer characters in the same p block B implies that $\gamma \in B$. Thus, $(p, \gamma(1)) = 1$ yields that B is a block of full defect. This is a contradiction.

Proof of Theorem A. Assume χ is a character of a group G of minimal degree such that the hypothesis but not the conclusion of the theorem is satisfied. By [4, Th. A] there is a central extension (G_1, π) of G such that $\chi = \prod_{i=1}^{n} \rho_i$ where $\{\rho_1, ..., \rho_n\}$ is an admissible set of prime characters of G_1. Since G_1 is a central extension of G, G_1 and χ also satisfy the hypothesis of the theorem, and it is sufficient to prove that χ is a primitive character of G_1. Hence, we may replace G by $G_1/\bigcap_{i=1}^{n} \ker(\rho_i)$ and assume $\chi = \prod_{i=1}^{n} \rho_i$ where $\rho_i \in \operatorname{Irr}(G)$ and $Z(G) = Z(\chi)$. We may also assume χ and each ρ_i are standard characters of G.

$\chi = \zeta^G$ where $\zeta \in \operatorname{Irr}(H)$ and H is a maximal subgroup of G. Thus, $H \supseteq Z(G)$. Suppose there is some K with $K \triangle G$, $K \not\subseteq Z(G)$, and $K \subseteq H$. Choose K of minimal order such that $K \lhd G$, $K \subseteq H$, and $K \not\subseteq Z(G)$; then $\chi_K = e\varphi$ where $e \in Z^+$ and $\varphi \in \operatorname{Irr}(K)$. $\varphi(1) > 1$ since $Z(\chi) = Z(G)$ and $K \not\subseteq Z(G)$. By [9, Th. 11.17], there is a central extension (Γ, π) of G; and viewing χ as an irreducible quasi-primitive character of Γ, $\chi = \gamma\theta$ where γ, $\theta \in \operatorname{Irr}(\Gamma)$, $\theta(1) = e$, and $\theta_{\pi^{-1}(K)}$ is a multiple of a linear character. Thus, $\chi_K = e\varphi$ where $\varphi \in \operatorname{Irr}(K)$ yields $\gamma_{\pi^{-1}(K)} \in \operatorname{Irr}(\pi^{-1}(K))$. Let $N = \pi^{-1}(K) \cap \ker(\theta)$; then $\chi_N = e\gamma_N$ and χ quasi-primitive imply that γ_N is homogeneous. Since $\pi^{-1}(K)/N$ is cyclic, [9, Th. 11.22] yields that γ_N is irreducible. $N \subseteq \ker(\theta)$ and χ is irreducible and quasi-primitive, whence [4, Lemma 1.1] implies that θ is a quasi-primitive character of Γ. Since $\theta(1)$ is a proper divisor of m and Γ satisfies the hypothesis of Theorem A, θ is primitive.

$\chi = \zeta^\Gamma$ where ζ may be viewed as a character of $\pi^{-1}(H)$, $\chi_N = e\gamma_N$, and $\gamma_N \in \operatorname{Irr}(N)$ with $N \subseteq \pi^{-1}(H)$ now imply that $\zeta_N = b\gamma_N$ where $b \in Z^+$. Since γ_N is irreducible, $\gamma_{\pi^{-1}(H)} \in \operatorname{Irr}(\pi^{-1}(H))$. Thus, [8, Th. 6.17] yields $\zeta = \lambda\gamma_{\pi^{-1}(H)}$ where $\lambda \in \operatorname{Irr}(\pi^{-1}(H)/N)$. It follows that $\theta\gamma = \chi = \zeta^\Gamma = (\lambda\gamma_{\pi^{-1}(H)})^\Gamma = \lambda^\Gamma\gamma$. However, $N \subseteq \ker(\lambda)$ and $N \triangle \Gamma$ imply that $N \subseteq \ker(\lambda^\Gamma)$. Now $N \subseteq \ker(\theta)$ and [9, Th. 6.17] yield $\theta = \lambda^\Gamma$. This is a contradiction since θ is primitive, and such K cannot exist.

Now let K be a normal subgroup of minimal order such that $K \not\leq Z(G)$; then $G = HK$. Since $\{\rho_1, ..., \rho_n\}$ is an admissible set of primes [4, Theorem 1.11], the minimality of K and [9, Th. 11.22] imply that we may choose notation so that $K \subseteq \ker(\chi')$ where $\chi' = \prod_{i=2}^{n} \rho_i$ and $K \not\subseteq \ker(\rho_1)$. Let $\rho = \rho_1$. Suppose $K/K \cap Z(G)$ is not an Abelian p-group; then by the minimality of K, $K = A_1 ... A_t$ where $[A_i, A_j] = 1$ for $i \neq j$, and the A_i are isomorphic quasi-simple groups. $\chi_K = \chi'(1)\, \rho_K$ so that $\zeta^G = \chi$, $G = HK$, and [9, Problem 5.2] yield $\chi'(1)\, \rho_K = (\zeta^G)_K = (\zeta_{H \cap K})^K$. Hence, there is a maximal subgroup K_1 of K such that $H \cap K \subseteq K_1$ and $\chi'(1)\, \rho_K = \zeta_1^K$ where $\zeta_1 = (\zeta_{H \cap K})^{K_1}$. Since $A_i \not\subseteq K_1$ for at least one i, we may assume $A_1 \not\subseteq K_1$. Thus $A_1 K_1 = K$ and $\chi'(1)\, \rho_{A_1} = (\zeta_1^K)_{A_1} = (\zeta_1^K)_{A_1} = ((\zeta_1)_{A_1 \cap K_1})^{A_1}$. Since K is a central product of the A_i and χ_K is homogeneous, $\rho_{A_1} = f\delta$ where $\delta \in \operatorname{Irr}(A_1)$ and $\delta(1)^{t-1} \mid f$. Further, $\delta(1) > 1$ since $Z(\chi) = Z(G)$. Thus, $\chi_{A_1} = \chi'(1) f\delta = ((\zeta_1)_{A_1 \cap K_1})^{A_1}$. This contradicts the hypothesis of the theorem since $K/K \cap Z(G)$ is a chief factor of G. Thus, the minimality of K implies that K is p-group for some odd prime p and $K \cap Z(\rho) = K \cap Z(G)$.

Since $KZ(\rho)/Z(\rho)$ is a normal p-subgroup of $G/Z(\rho)$, $K \subseteq F^*(\rho)$. Thus, $F^*(\rho)/Z(\rho)$ is Abelian and $[F^*(\rho), F^*(\rho)] \subseteq Z(\rho)$. Now $G = HK$ yields $F^*(\rho) = (H \cap F^*(\rho))K$ and $[F^*(\rho) \cap H, K] \subseteq [F^*(\rho), F^*(\rho)] \cap K \subseteq Z(\rho) \cap K \subseteq Z(G) \subseteq H \cap F^*(\rho)$. Therefore, $F^*(\rho) \cap H$ is a normal subgroup of G so that $F^*(\rho) \cap H = Z(G)$. It follows that $F^*(\rho) = Z(G)K$ and that H avoids $M^*(\rho)$. By [4, Prop. 1.9], if $|M^*(\rho)| = p^{2a+1}$, then $\rho(1) = p^a$, so that $M^*(\rho)$ is a chief factor with $|M^*(\rho)| \mid \chi(1)^2$, and, by [4, Th. 1.11], $C_G(M^*(\rho)) = F^*(\rho)$. Further, $[G : H] = p^{2a}$.

Now, $\rho\chi' = \chi = \zeta^G$ yields $p^a\chi'(1) = \zeta(1)[G : H] = \zeta(1)\, p^{2a}$ so that $\rho(1)\zeta(1) = \chi'(1)$. $\chi' \in \operatorname{Irr}(G/K)$, Gallagher's Theorem [9, (6.17)], and $\chi'(1) < \chi(1)$ imply that χ' is a primitive irreducible character of G/K. Thus, $G = HK$ yields χ'_H a primitive irreducible character of H. By Frobenius Reciprocity, $\zeta^G = \rho\chi'$ yields $1 = (\zeta, \rho_H\chi'_H) = (\zeta\bar{\rho}_H, \chi'_H)$. Now $\zeta(1)\,\bar{\rho}_H(1) = \zeta(1)\,\rho(1) = \chi'(1)$ and $\chi'_H \in \operatorname{Irr}(H)$ imply that $\zeta\bar{\rho}_H = \chi'_H$. Since χ'_H is primitive and irreducible so is $\bar{\rho}_H$. This contradicts Lemma 1.

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In this section, Corollaries B and C are proved by verifying the hypothesis of Theorem A. Heavy use of [1] is involved. We will frequently adopt the notation of [1].

We will consider the following situation. χ denotes a quasi-primitive irreducible character of odd degree of a group G and S_0 is a non-Abelian composition factor of G. Suppose $(\chi(1)/\delta(1))\,\delta = \mu^S$ where S is a homomorphic image of a universal covering group of S_0, δ is a faithful irreducible character of S, and μ is a character of a maximal subgroup R of S. If $A \subseteq S$, let A_0 denote the image of A in $S/Z(S)$. The conditions $\chi(1)$ odd and $R \supseteq Z(S)$ yield $[S_0 : R_0] = [S : R]$ is odd. Further, there is a $\phi \in \operatorname{Irr}(R)$ such that $(\phi, \mu) \neq 0$, $\phi^S = a\delta$ where a and $\phi(1)$ are odd. We note that

$$a\delta(1) = [S : R]\,\varphi(1) = [S_0 : R_0]\,\varphi(1). \tag{2.1}$$

We recall that if $x^S \cap R = \bigcup_{i=1}^n x_i^R$, then

$$a\delta(x) = \varphi^S(x) = |C_S(x)| \left(\sum_{i=1}^n \frac{\varphi(x_i)}{|C_R(x_i)|} \right). \tag{2.2}$$

Suppose $F \vartriangle R$ and $(\varphi_F, \lambda) \neq 0$ for $\lambda \in \operatorname{Irr}(F)$; then $\varphi = \theta^R$ where $\theta \in \operatorname{Irr}(I(\lambda))$ and $\theta|_F$ is a multiple of λ by [9, Th. 6.11]. If $x^S \cap I(\lambda) =$

$\bigcup_{i=1}^{m} y_i^{I(\lambda)}$ where $y_1 = x$ and $y_2 \ldots y_m$ are representatives of the distinct $I(\lambda)$-conjugacy classes of elements which are S-conjugate to x, then

$$a\delta(x) = \varphi^S(x) = \theta^S(x) = |C_S(x)| \left(\sum_{i=1}^{m} \frac{\theta(y_i)}{|C_{I(\lambda)}(y_1)|} \right). \tag{2.3}$$

Suppose $F \subseteq K$ are subgroups of R, $F \triangle R$, T is a set of representatives of some different K-conjugacy classes where all elements in T have order t, and there is an element k in K such that $wF = kF$ for all $w \in T$. The number of K-conjugates of w in $wF = kF$ is $|C_{K/F}(kF)|\ |F|/|C_K(w)|$. Thus we obtain,

$$\sum_T \frac{1}{|C_K(w)|} \leqslant \frac{\alpha}{|C_{K/F}(kF)|\ |F|}, \tag{2.4}$$

where α is the number of elements of order t in kF.

The following inequality is also useful:

$$a|R| = |R|\ (\delta_R, \varphi) \leqslant \varphi(1)\,\delta(1) + \beta(|R| - 1)\,\varphi(1), \tag{2.5}$$

where $\beta = \max\{|\delta(g)|\mid g \in R^{\#}\}$.

Proof of Corollary B. We first show $(\chi(1)/\delta(1))\,\delta$ is not induced from a proper maximal subgroup R of S if S is a covering group of a non-Abelian composition factor S_0 of G with $\delta(1) \mid \chi(1)$. Assume the contrary; then as in the previous paragraphs $a\delta = \varphi^S$ where a and $\varphi(1)$ are odd and $\varphi \in \mathrm{Irr}(R)$, R a maximal subgroup of S. Since $S_0 \not\simeq L_n(q)$ where $(q^n - 1)/(q - 1) = p^b$, or A_{p^b}, or M_{11} if $p = 11$, and $p \geqslant 11$, [7, Theorem 2.3] implies that $S_0 = L_2(11)$, $R/Z(S) \simeq A_5$, and $p = 11$, or $S_0 = M_{23}$, $R/Z(S) = M_{22}$, and $p = 23$. In each case, $\delta(1) = p$. However, by [1] S has no irreducible character of degree 23, if $S_0 = M_{23}$. Thus, $S_0 = L_2(11)$, $p = 11$, and $R/Z(S) \simeq A_5$. Hence, we may assume $S = S_0$ and $\delta = \chi_6$ of [1, p. 7]. Thus, [1, p. 2] implies $(\delta_{A_5}, \varphi) = 1$ and $\varphi(1) = 3$ or 5. However, (2.1) now yields $11 = a\delta(1) = [S : R]\ \varphi(1) = 33$ or 55. This is a contradiction. Thus $(\chi(1)/\delta(1))\,\delta$ is not induced from a proper subgroup of S.

If $p \mid |Z(S)|$ where S is a universal covering group of S_0 a non-Abelian composition factor of G, then [6, pp. 302–303] and $p \geqslant 11$ imply that $S_0 \simeq L_n(q)$ and $p \mid (q - 1, n)$ or $S_0 \simeq U_n(q)$ and $p \mid (q + 1, n)$. Both these cases are not possible by hypothesis so the corollary follows from Theorem A.

Proof of Corollary C. In order to ease notation and verify the hypothesis of Theorem A, we may assume S_0 is a non-Abelian composition factor of G, δ is a faithful irreducible character of S where S is a homomorphic image of a universal covering group of S_0, and $\delta(1) \mid \chi(1)$. The values for $\delta(1)$ appear in [8] if $S_0 = L_2(q)$ or $Sz(q)$ and in [1] otherwise. Direct inspection implies there is no δ with $\delta(1)=p^a$, p an odd prime, and $p \mid |Z(S)/\ker(\delta)|$. Thus, by the discussion preceding the proof of Corollary B, it is sufficient to show that $a\delta \neq \varphi^S$ where a and $\varphi(1)$ are odd and $\varphi \in \mathrm{Irr}(R)$ for R a maximal subgroup of S. We assume the contrary. We repeatedly use the trivial fact that if $\delta(x) \neq 0$ then $a\delta(x)=\varphi^S(x)$ implies that we may assume by conjugation that $x \in R$.

The proof now proceeds as a case by case analysis using the previous remarks.

(i) Suppose $S_0 = L_2(q)$ where q is odd and $q > 9$. $\delta(1)$ odd implies that we may assume $S = S_0$. By [8, Th. XI 5.6, 5.7] $\delta(1) \in \{(q \pm 1)/2, q\}$. If $\delta(1)=(q\pm 1)/2$, then δ does not vanish on any elements of S_p where $q=p^m$, p a prime. Thus, R contains a Sylow p-subgroup of S. By [12, Th. 6.25], $[S:R]$ is even which is a contradiction. If $\delta(1)=q$, then $\delta(x) \neq 0$ if x is not conjugate to an element of $S_p^{\#}$. Thus, $a\delta = \varphi^S$ implies that R contains an S-conjugate of any element whose order divides $(q-1)(q+1)$. Since S contains elements of order $(q+1)/2$ and $(q-1)/2$ and $q>9$, $R/Z \not\simeq S_4$, A_5, or A_4. Now $(q^2-1)/4 \mid |R|$ together with $[S:R]$ odd and [12, Theorem 6.25] now imply that no subgroup R exists.

(ii) Suppose that $S_0 = L_2(2^k)$ where $2^k+1 \nmid \chi(1)$ and $k \geqslant 3$; then $S=S_0$. By [8, Th. XI 5.5] δ is an exceptional character belonging to the Singer cycle of $L_2(2^k)$ and $\delta(1)=2^k-1$. Since δ does not vanish on involutions or elements of order dividing 2^k+1, R must contain an element of order 2^k+1 and an element of order 2. Now [12, Theorem 6.25] implies that R is a dihedral group of order $2\cdot(2^k+1)$. This contradicts $[S:R]$ odd.

(iii) Suppose $S_0 = Sz(2^k)$ where $(2^{2k}+1) \nmid \chi(1)$ and $k \geqslant 5$. Set $k=2m+1$. Then $S_0=S$ and by [8, Theorem XI 5.10], δ is an exceptional character associated with a cyclic group U where $(|U|, \delta(1)) \in \{(2^{2m+1}+2^{m+1}+1,\ (2^{2m+1}-1)(2^{2m+1}-2^{m+1}+1)),\ (2^{2m+1}-2^{m+1}+1,\ (2^{2m+1}-1)(2^{2m+1}+2^{m+1}+1))\}$. δ does not vanish on any elements of U. By [13], $R=N_S(U)$ (up to conjugation). This contradicts $[S:R]$ odd.

If $S_0 \in \{M_{12}, J_1, J_2, M_{23}, HS, He, Ru, Co_3, Co_2, HN, Ly, Co_1\}$, then the Schur multiplier of S_0 is 1 or 2. Thus, we may assume $S=S_0$ for these groups. We adopt the notation of [1]. In particular, we let $\delta=\chi_n$ where χ_n is one of the characters given in [1].

(iv) Suppose $S_0 = M_{12}$; then $R = 2_+^{1+4} : S_3$ or $4^2 : D_{12}$. In particular, R contains no elements of order 5, 10, or 11. R contains one conjugacy class of 3-elements. Now [1, p. 33] implies no δ exists where $\delta(1)$ is odd and $a\delta=\varphi^S$.

(v) Assume that $S=J_1$; then $R=2^3:7:3$ or $2\times A_5$ by [1, p. 36].

Thus, R contains no elements of order 11, 15, or 19. Direct inspection of [1, p. 36] yields a contradiction.

(vi) Suppose that $S=J_2$; then by [1, pp. 42–43], $[S:R]=315$ and $R=2^{1+4}:A_5$ or $[S:R]=525$ and $R=2^{2+4}:(3\times S_3)$. Neither R contains representatives of all four S-conjugacy classes of 5-elements, or any element of order 15, or any element of order 7. Hence, it follows from [1, p. 43] that $\delta=\chi_{13}$. χ_{13} does not vanish on $3A$ or $3B$ of [1, pp. 424]. Since $\langle 3A\rangle$ and $\langle 3B\rangle$ are not S-conjugate, $R\neq 2^{1+4}:A_5$. $R=2^{2+4}:(3\times S_3)$ and (2.1) imply that $a=3\varphi(1)$. There is a normal subgroup F in R where $F=2^{2+4}$. If $(8A)^S\cap R=\bigcup_{i=1}^{n} x_i^R$, then xF is an involution and we may choose notation so that $x_iF=(8A)F$ for $i=2, ..., t$. Thus, $|F|\;|C_{R/F}((8A)F)|=6|F|$ and applying (2.4) with $K=R$ and $T=\{x_i\mid i=1, ..., n\}$ yields $\sum_{i=1}^{n} 1/|C_R(x_i)|\leqslant \frac{1}{6}$. Now (2.2) implies that

$$3\varphi(1)=|a\chi_{13}(8A)|=|C_S(8A)|\left(\sum_{i=1}^{n}\frac{\varphi(x_i)}{|C_R(x_i)|}\right)$$
$$\leqslant 8\varphi(1)\sum_{i=1}^{n}\frac{1}{|C_R(x_i)|}\leqslant\frac{8\varphi(1)}{6}.$$

This is a contradiction.

(vii) Suppose $S=M_{23}$. Then $[S:R]=23$ and $R=M_{22}$, $[S:R]=253$ and $R=L_3(4):2_2$, and $[S:R]=253$ and $R=2^4:A_7$ or $[S:R]=1771$ and $R=2^4:(3\times A_5):2$. Hence, R contains no elements of order 23 and no R contains elements of order 15 and 7. Thus, $\delta=\chi_{16}$ of [1, p. 71] and $\delta(1)=1035$. Since χ_{16} does not vanish on 11-elements, $R=M_{22}$. Thus, (2.1) implies that $a=\varphi(1)/45$. In particular, $45\mid\varphi(1)$ whence [1, p. 40] implies that $\varphi(1)=45$ and $\varphi(x)$ is not real if $|\langle x\rangle|=7$. However, M_{22} and M_{23} both have exactly two conjugacy classes of 7-elements. Therefore, (2.2) yields $-a=a\chi_{16}(7A)=|C_S(7A)|(\varphi(7A)/|C_R(7A)|)=2\varphi(7A)$. This contradicts $\varphi(7A)$ non-real.

(viii) Suppose that $S=HS$; then $R=4^3:L_3(2)$ or $4\cdot 2^4:S_5$ by [1, p. 80]. In particular, R contains no elements of order 15 and at most one conjugacy class of 5-elements. Examination of character values of 5-elements and 15-elements for irreducible characters of HS of odd degree yields a contradiction in each case.

(ix) Suppose $S=He$; then $R=2_+^{1+6}\cdot L_3(2)$ and $[S:R]=187{,}425$ or $R=2^6:3^{\cdot}S_6$ and $[S:R]=29{,}155$. $2_+^{1+6}\cdot L_3(2)$ contains no elements of order 5, 21, 15, or 17 and only two conjugacy classes of 7-elements. $2^6:3^{\cdot}S_6$ contains no elements of order 7, 21, or 17. Thus, [1, p. 105] implies that $\delta=\chi_{28}$ and $\delta(1)=17{,}493=17\cdot 7^3\cdot 3$. Since χ_{28} does not vanish on 5-elements, $R=2^6:3^{\cdot}S_6$. Thus, R and S both have exactly one conjugacy class of 5-elements. We choose notation so that $5A \in R$. It is direct to compute that $|C_R(5A)| = 4 \cdot 5 \cdot 3$. Thus, (2.2) yields $a(-7) = a\delta(5A) = 5\varphi(5A)$. However, by (2.1), $a = 5\varphi(1)/3$. Hence, $\varphi(5A) = \varphi(1)(-7)/3$ which is a contradiction.

(x) Suppose $S = Ru$; then $R = 2^{3+8} : L_3(2)$ and $[S : R] = 424{,}125$ or $R = 2^{\cdot}2^{4+6} : S_5$ and $[S : R] = 593{,}775$ by [1, p. 126]. Neither R contains conjugates of both $5A$ and $5B$, and R contains no elements of order 15, 29, or 13. Now direct inspection using [1, p. 127] implies $a\delta \neq \varphi^S$ for $\varphi \in \mathrm{Irr}(R)$ and $\varphi(1)$ odd.

(xi) Suppose $S = Co_3$; then $R \in \{2^{\cdot}S_6(2),\ 2^{4\cdot}A_8,\ 2^2 \cdot [2^7 \cdot 3^2] \cdot S_3\}$ by [1, p. 134]. Hence, R does not contain representatives of both $5A$ and $5B$ or any elements of order 11, 23, or 21. Thus, [1, p. 135] implies that $\delta \in \{\chi_{15}, \chi_{37}\}$. If $\delta = \chi_{15}$, then $\chi_{15}(5A) \neq 0$ yields $R = 2^{\cdot}S_6(2)$ or $2^4 \cdot A_8$. However, δ also does not vanish on $9A$ and $9B$, but neither $2^{\cdot}S_6(2)$ nor $2^4 \cdot A_8$ contain two conjugacy classes of 9-elements. If $\delta = \chi_{37}$ then δ does not vanish on $9A$ or $9B$. Thus $R = 2^2 \cdot [2^7 \cdot 3^2] \cdot S_3$. Let $x_1 = 9A$ and $(9B)^S \cap R = \bigcup_{i=1}^n x_i^R$. If $F = 2^2 \cdot [2^7 \cdot 3^2]$, then notation may be chosen so that $x_i \in (9A)F$ for $i = 1, ..., n$ since $R/F = S_3$. Since $|C_{R/F}((9B)F)|\ |F| = 3|F|$, (2.4) with $K = R$ and $T = \{x_i \mid i = 1, ..., n\}$ yield $\sum_{i=1}^n 1/|C_R(x_i)| \leqslant \frac{1}{3}$. By (2.1) $a = 3^4\varphi(1)$ so that (2.2) implies $3^4\varphi(1) \cdot 2 = a\delta(9A) = |C_S(9A)|\ (\sum_{i=1}^n \varphi(x_i)/|C_R(x)|) \leqslant 81 \cdot \varphi(1)/3$. This is a contradiction.

(xii) Suppose $S = Co_2$; then $R = 2^{10} : M_{22} : 2$ and $[S : R] = 46{,}575$, $R = 2_+^{1+8} : S_6(2)$ and $[S : R] = 56{,}925$, or $R = 2^{4+10} : (S_5 \times S_3)$ and $[S : R] = 3{,}586{,}275$ by [1, p. 154].

R contains no elements of order 23. Also R does not contain representatives of both $5A$ and $5B$, or representatives of both $15A$ and $15B$. Hence, $\delta \in \{\chi_8, \chi_{16}, \chi_{17}, \chi_{20}, \chi_{29}, \chi_{35}, \chi_{38}, \chi_{39}, \chi_{46}, \chi_{47}, \chi_{48}, \chi_{49}, \chi_{56}, \chi_{60}\}$. (See [1, p. 155].)

Suppose $\delta = \chi_8$ or χ_{60}; then $\delta(11A) \neq 0$ so $R = 2^{10} \cdot M_{22} : 2$. Since $\chi_8(15A) \neq 0$, $\delta = \chi_{60}$. It follows from (2.2), [1, p. 155], and [1, p. 39] that $a = a\delta(11A) = \varphi(x_1) + \varphi(x_2)$ where x_1 and x_2 denote representatives of the R-conjugacy classes of 11-elements. Since $|C_S(11A)|$ and $\varphi(1)$ are both odd $\varphi_{M_{22}} \in \mathrm{Irr}(M_{22})$. By (2.1), $a \cdot 5 \cdot 3^2 = \varphi(1)$. Hence $\varphi_{|M_{22}} = \chi_3$ or χ_4 of [1, p. 40]. Thus $1 = a\delta(11A) = \varphi(x_1) + \varphi(x_2) = 2$ which is another contradiction.

Suppose $R = 2_+^{1+8} : S_6(2)$ and $|\langle x \rangle| = 16$, where $x \in R$; then $|C_S(x)| = 32$ by [1, pp. 154]. Let $F = 2^{1+8}$; then [1, p. 47] implies that if $x^S \cap R = \bigcup_{i=1}^n x_i^R$ we may choose notation so that $Fx_i \in \{(4E)F, (8A)F, (8B)F\}$ where $4E$, $8A$, $8B$ are given on [1, p. 47]. The number of R-conjugates of x_i in x_iF is

$$|F| \frac{|C_{R/F}(x_iF)|}{|C_R(x_i)|} = \frac{|C_{R/F}(x_iF)|\ F|}{32}.$$

If $x_iF=(4E)F$, then $|C_{S_6(2)}(4E)|=32$ implies that $(4E)F$ contains only elements of order 16. Since R is a split extension, $(4E)F$ contains an element of order 4. Thus, $x_iF\in(8A)F$ or $(8B)F$ and each of these cosets contains elements of order 8. The number of R conjugates of x_i in $(8A)F$ or $(8B)F$ is $16|F|/|C_R(x_i)|=|F|/2$ by [1, p. 47]. Hence $(8A)F$ and $(8B)F$ contain representatives of only one conjugacy class of 16 elements. Therefore, if x is either of the two conjugacy classes of elements of order 16 in S, $x^S\cap R=x^R$, if $x=16A$ or $16B$ of [1, p. 154]. Thus by (2.2)

$$a\delta(x)=\varphi(x) \qquad \text{if} \quad x=16A \text{ or } 16B. \tag{xii, 1}$$

Suppose $(\varphi_F,\lambda)\neq 0$ where $\lambda\in\mathrm{Irr}(F)$ and $I(\lambda)\neq R$. $[R:I(\lambda)]\,|\,\varphi(1)$ and [1, p. 46] yield $d\,|\,\varphi(1)$ where $d\in\{63, 135, 315\}$. Since a is an integer, [1, pp. 154–155] and (2.1) now yield $a\geqslant 9$. However, (xii, 1) yields $9=a|\delta(16A)|\leqslant|\varphi(16A)|$ which implies that $|C_R(16A)|\geqslant 81$. Thus, $I(\lambda)=R$. Since $\varphi(1)$ is odd and $|F/\ker(\lambda)|\leqslant 2$, $\varphi_{S_6(2)}\in\mathrm{Irr}(S_6(2))$ and $|\varphi(16A)|=|\varphi_{S_6(2)}(g)|$ where $(16A)F=gF$ and $g\in S_6(2)$. Now $|\delta(16A)|=1$ by [1, pp. 155–156] and $|\varphi_{S_6(2)}(g)|=1$ by [1, p. 47] since $|\langle g\rangle|=8$. Hence, by (xii, 1), $a=|a\delta(16A)|=|\varphi(16A)|=1$. Thus, (2.1) now yields $\delta=\chi_{38}$ or χ_{39} and $\varphi_{S_6(2)}=\chi_2$ of [1, p. 47], $\delta=\chi_{56}$ and $\varphi_{S_6(2)}=\chi_7$ or χ_8 of [1, p. 47], or $\delta=\chi_{49}$ and $\varphi_{S_6(2)}=\chi_3$ of [1, p. 47].

Suppose $\delta=\chi_{49}$ and $\varphi_{S_6(2)}=\chi_3$. There is one class of 7-elements in $S_6(2)$ and no elements of order 14 or 28. Therefore, we may choose notation so $g\in S_6(2)$ has order 7 and if $x\in\{7A, 14A, 14B, 14C, 28A\}$ of [1, p. 154], and $x^S\cap R=\bigcup_{i=1}^n x_i^R$, then all x_i lie in gF. Since $|C_{S_6(2)}(g)|=7$, it is direct to compute that $(7A^R\cap gF)\cup(14A^R\cap gF)\cup(14B^R\cap gF)\cup(14C^R\cap gF)\cup(28A^R\cap gF)=gF$. It follows that $x^S\cap R=x^R$. If $F\not\subseteq\ker(\varphi)$, then gF contains an element gf where $\varphi(gf)$ is negative. Since gf is S-conjugate to x, $\varphi(x)$ is negative. However, (2.2), (2.1), $x^S\cap R=x^R$, and [1, p. 155] yield $1=a\delta(x)=\varphi(x)$ which is a contradiction. Therefore, $F\subseteq\ker(\varphi)$. Hence, $\delta(16A)=1=\delta(16B)$ and (xii, 1) imply that $\varphi(8A)$ and $\varphi(8B)=1$ where $8A$ and $8B$ represent the conjugacy classes of elements of order 8 in $S_6(2)$. This contradicts $\varphi_{S_6(2)}=\chi_3$.

If $\varphi_{S_6(2)}=\chi_2$, χ_7, or χ_8, then $\varphi(y)\neq 0$ if y has order 9. However, S and R both have exactly one conjugacy class of elements of order 9. Thus, by (2.2), $\delta(y)\neq 0$ if $|\langle y\rangle|=9$. This contradicts $\delta\in\{\chi_{38}, \chi_{39}, \chi_{56}\}$. Hence, $R\neq 2_+^{1+8}:S_6(2)$.

If $R=2^{10}:M_{22}:2$, then δ must vanish on at least one conjugacy class of 3-elements. Thus, $\delta=\chi_{49}$ and $\delta(7A)=1$. Now $(7A)^S\cap R=x_1^R\cup x_2^R$ by [1, p. 40]. Hence, (2.2) yields $a=a\delta(7A)=|C_S(7A)|(\sum_{i=1}^2\varphi(x_i)/|C_R(x_i)|)=(56/7\cdot 2^\alpha)(\varphi(x_1)+\varphi(x_2))$. Since φ does not vanish on 7-elements, [1, p. 39] implies that $I(\lambda)=R$ if $(\varphi_{|F},\lambda)\neq 0$ for $\lambda\in\mathrm{Irr}(F)$ where $F=2^{10}$. Now $\varphi(1)$ odd yields $\varphi_{M_{22}}\in\mathrm{Irr}(M_{22})$. By (2.1), $a=3\varphi(1)/55$. Since φ does not vanish on 7-elements, [1, p. 40] yields $\varphi_{M_{22}}=\chi_5$ of [1, p. 40]. Thus, $\varphi(x_1)+\varphi(x_2)=-2$ which contradicts $a\delta(7A)=a>0$.

Thus $R=2^{4+10}:(S_5\times S_3)$. Since R contains no element of order 9 or 7, we may assume $\delta\in\{\chi_{38},\chi_{39},\chi_{46},\chi_{47},\chi_{48},\chi_{56}\}$. Let $F=2^{4+10}$, $K\in\{2^{4+10}:(S_4\times S_3), R\}$, and let $\bar{A}$ denote the image of the set A in K/F. K/F has classes bF and cF of elements of order 4, and we may choose notation so that $|C_{\bar{K}}(\bar{b})|=2^3\cdot 3$, $|C_{\bar{K}}(\bar{c})|=8$. Let T_1 be a set of representatives of the K-conjugacy classes of 16 elements such that $wF=bF$ for $w\in T_1$. By (2.4), $\sum_{T_1} 1/|C_K(w)|\leqslant \frac{1}{24}$. Let T_2 be a set of representatives of the K-conjugacy classes of 16 elements such that $wF=cF$ for $w\in T_2$. By (2.4), $\sum_{T_2} 1/|C_K(w)|\leqslant\frac{1}{8}$. If g has order 16 in K, then $\bar{g}$ has order 4. Thus, $T_1\cup T_2$ is a set of representatives of the K-conjugacy classes of elements of order 16. Hence, we obtain

$$\sum_{T_1\cup T_2}\frac{1}{|C_K(w)|}\leqslant\frac{1}{6}. \tag{xii, 2}$$

Suppose that $a\delta=\gamma^S$ where $\gamma\in\operatorname{Irr}(K)$. We may choose notation so that $16A$, $16B\in K$, $(16A)^S\cap K=\bigcup_{S_1} w^K$, $(16B)^S\cap K=\bigcup_{S_2} w^K$ where $S_1\cup S_2=T_1\cup T_2$. Thus, $a\delta(16A)=\gamma^S(16A)=|C_S(16A)|(\sum_{S_1}\gamma(w)/|C_K(w)|)$, $a\delta(16B)=\gamma^S(16B)=|C_S(16B)|(\sum_S\gamma(w)/|C_K(w)|)$. Reference [1, p. 155] now yields $2a=|a\delta(16A)|+|a\delta(16B)| = |\gamma^S(16A)|+|\gamma^S(16B)| \leqslant 32(\sum_{T_1\cup T_2}|\gamma(w)|/|C_K(w)|)$. Using (xii, 2) we obtain

$$2a\leqslant\tfrac{36}{6}\alpha,\qquad \text{where } \alpha=\max\{|\gamma(w)|\mid w\in T_1\cup T_2\}. \tag{xii, 3}$$

Since $\alpha^2\leqslant\max\{|C_S(16A)|-1,\ |C_S(16B)|-1\}$, $\alpha\leqslant\sqrt{31}$. Now letting $K=R$ yields $a\leqslant 14$. By (2.1) $a=9\varphi(1)$ if $\delta=x_{38}$ or χ_{39}. Thus, $\delta\in\{x_{46},\chi_{47},\chi_{48}\}$ and $a=27\varphi(1)/5$ or $\delta=\chi_{56}$ and $a=9\varphi(1)/5$. Since a is an integer, $5\mid\varphi(1)$. Thus, $a\leqslant 14$ yields $\delta=\chi_{56}$ and $\varphi(1)=5$.

If $(\varphi_F,\lambda)\neq 0$ where λ is not invariant in R and $\lambda\in\operatorname{Irr}(F)$, then $\varphi(1)=5$ implies that $K=I(\lambda)=FS_4\times S_3$ and $\varphi=\gamma^R$ where $\gamma(1)=1$. Now (xii, 3) yields $18=2\cdot 9\varphi(1)/5\leqslant 32/6\cdot 1$ which is a contradiction. Hence, $I(\lambda)=R$. Since $\varphi(1)$ is odd and $\lambda(1)=1$, $|\varphi(x)|=|\hat{\varphi}(g)|$ where $\hat{\varphi}$ is an irreducible character of $S_5\times S_3$ with $\hat{\varphi}(1)=5$, $x\in\{16A, 16B\}$, and g is an element of order 4 in $S_5\times S_3$. By [1, p. 2], $|\varphi(x)|=1$ which again contradicts (xii, 3).

(xiii) Assume that $S=HN$; then $R=2_+^{1+8}:(A_5\times A_5)\cdot 2$ or $R=2^3\cdot 2^2\cdot 2^6\cdot(3\times L_3(2))$ by [1, p. 166]. Reference [1, p. 164] implies that $\delta(5A)\neq 0$. Thus, $R=2^{1+8}:(A_5+A_5)\cdot 2$ and we may choose notation so $5A\in R$. By [1, p. 166], $R=N(2B)$. But $5A\in R$ implies there is an element of order 10 with 2-part conjugate to $2B$ and 5-part conjugate to $5A$. This contradicts [1, p. 164].

(xiv) Suppose $S=Ly$; then $R=3\cdot McL:2=N(3A)$ or $R=2^{\cdot}A_{11}=$

$N(2A)$ by [1, p. 174]. Neither R contains elements of order 31, 37, 67, or 25. $2^{\cdot}A_{11}$ contains no elements of order 33. $3^{\cdot}McL:2=N(3A)$ contains no elements of order 40 since R contains no elements of order 60. Thus, $\delta \in \{\chi_{14}, \chi_{34}\}$.

If $\delta=\chi_{14}$, then $\delta(7A)=4$. Both R's have one class of 7-elements so that by (2.2) $a4=(7\cdot 3\cdot 8/|C_R(7A)|)\,\varphi(7A)$. It is direct to check for both R that $|C_R(7A)|=7\cdot 3\cdot 2^{\alpha}$ where $\alpha \geqslant 0$. By (2.1) $a=\varphi(1)\,5^2/11$ if $R=3^{\cdot}McL:2$ and $a=\varphi(1)\,5^3\cdot 3^3/11$ if $R=2^{\cdot}A_{11}$. Thus, $|\varphi(7A)|\,8 \geqslant 4\varphi(1)5^2/11$ which is a contradiction.

If $\delta=\chi_{34}$, then $\delta(5A)\neq 0$, $\delta(5B)\neq 0$, and $\delta(40A)\neq 0$. Thus, $R=2^{\cdot}A_{11}$ and $a=\varphi(1)\,5^3\cdot 3^3/7\cdot 11$. Both R and S have two conjugacy classes of 5-elements. Thus, by (2.2) $(\varphi(1)\,5^3\cdot 3^3/77)(-10) = a\delta(5B) = (|C_S(5B)|/|C_R(5B)|\,\varphi(5B)$. Now $|C_R(5B)|=5^2\cdot 2$ and $|C_S(5B)|=5^4\cdot 3\cdot 2$. Thus, $\varphi(1)\,5^3\cdot 3^3\cdot 10/77 \leqslant (5^4\cdot 3\cdot 2/5^2\cdot 2)\,\varphi(1)=5^2\cdot 3\varphi(1)$ which is another contradiction.

(xv) Assume that $S=Co_1$; then $R=2^{11}:M_{24}$, $R=2^{1+8}_{+}{}^{\cdot}0_8^{+}(2)$, $R=2^{2+12}:(A_8\times S_3)$, or $R=2^{4+12}{}^{\cdot}(S_3\times 3S_6)$. None of the R contain elements of orders 33, 35, or 39. No R contains both elements of order 9 and elements of order 21. No R contains two non-conjugate 7 subgroups. If R contains representatives of at least four different S-conjugacy classes of 15-elements, then R contains at most one conjugacy S-classes of elements of order 5. Only $2^{1+8}_{+}{}^{\cdot}0_8^{+}(2)$ contains elements of order 9. Direct inspection using [1, pp. 184–186] yields $\delta \in \{\chi_{56}, \chi_{63}, \chi_{80}, \chi_{87}\}$.

χ_{63}, χ_{87}, and χ_{56} do not vanish on some 9-element. χ_{80} does not vanish on $7A$, $15A$, $15B$, and $15D$. Thus, $R=2^{1+8}_{+}{}^{\cdot}0_8^{+}(2)=N(2A)$. If $\delta(15A)\neq 0$ then there is an element of order 30 with 2-part conjugate to $2A$ and 15-part conjugate to $15A$. This is a contradiction. Thus, $\delta \in \{\chi_{63}, \chi_{87}\}$.

We assume that $9A$, $9B$, $9C \in R$. Suppose that $(\varphi|_{2^{1+8}_{+}}, \lambda)\neq 0$ for $\lambda \in \mathrm{Irr}(F)$ for $F=2^{1+8}_{+}$. Since $I(\lambda)$ contains an element of order 9 and a Sylow 2 subgroup of R [1, p. 85] implies that $I(\lambda)=R$. Since $\varphi(1)$ is odd, it follows that there is a $\hat{\varphi} \in \mathrm{Irr}(0_8^{+}(2))$ such that $\hat{\varphi}(r)=\varphi(r)$ if r has odd order. By (2.1), $5^2\cdot 7\cdot a=\varphi(1)\,3^4$ if $\delta=\chi_{63}$ and $5^2\cdot 7\cdot a=3^3\varphi(1)$ if $\delta=\chi_{87}$. Hence, $175\,|\,\hat{\varphi}(1)$. Since $\delta(9A)\neq 0$, $\delta(9B)\neq 0$, and $\delta(9C)\neq 0$, [1, p. 86] yields $(9A)^S\cap R=(9A)^R$, $(9B)^S\cap R=(9B)^R$, and $(9C)^S\cap R=(9C)^R$. Hence, by (2.2) $a\delta(x)=(|C_S(x)|/|C_R(x)|)\,\hat{\varphi}(x)$ for $x\in\{9A, 9B, 9C\}$. Thus, $\hat{\varphi}\in \mathrm{Irr}(0_8^{+}(2))$, $\hat{\varphi}$ does not vanish on any 9-element, and $175\,|\,\hat{\varphi}(1)$. By [1, p. 86], $\hat{\varphi}=\chi_{10}$ of [1, p. 86]. Thus, $27|\delta(9C)| \leqslant a|\delta(9C)|=|C_S(9C)|/|C_R(9C)|=2\cdot 3^5/2\cdot 3^3=9$ by (2.2). This is a contradiction.

Thus, we may assume $S_0\in\{J_3, Suz, 0'N, Fi_{22}\}$. Since $\delta(1)$ is odd, $S/Z(S)$ is odd. Thus, for ease of notation we may assume $S=S_0$ or $3S_0$, the triple cover of S_0. If $A\subseteq S$, then A_0 denotes the image of A in $S/Z(S)$.

(xvi) Assume $S_0=J_3$; then $R_0=2^{1+4}:A_5$ or $R_0=2^{2+4}:(3\times S_3)$. If

$S=S_0$, then no R contains elements of order 9, 19, or 17. Hence, by [1, p. 83], we may assume $S=3J_3$. No R contains elements of order 19 or 17. Thus, we may assume $\delta=\chi_{36}$ of [1, p. 83]. Let x and y, respectively, be cohort representatives of $5A$ and $5B$ with $|\langle x\rangle|=|\langle y\rangle|=5$. Since $\delta(x)\neq 0$ and $\delta(y)\neq 0$, $R_0=2_-^{1+4}:A_5$. Hence, (2.1) implies that $a=3^2\varphi(1)$. Since R and S both have 2-classes of 5-elements, (2.2) yields $9\varphi(1)2=a\delta(x)=(|C_S(x)|/|C_R(x)|)\,\varphi(x)=(3\cdot 30/3\cdot 2\cdot 5)\,\varphi(x)$. This contradicts $\varphi(x)\leqslant\varphi(1)$.

(xvii) Suppose that $S_0=Suz$; then by [1, p. 131], $R_0=2_-^{1+6}\cdot U_4(2)$, $2^{4+6}:3A_6$, or $2^{2+8}:(A_5\times S_3)$. $S=Suz$ or $3Suz$ and S has no element g of order 9 with $|\langle gZ(S)\rangle|=3$. It now follows from the structure of R_0 that $R=R_1\times Z(S)$ where $R_1\simeq R_0$. ($|Z(S)|=1$ is possible.) Hence, R contains no elements of order 7, 21, 11, 13, or 33. R does not contain two non-conjugate subgroups of order 5. If R contains an element of order 9, then $R_0=2_-^{1+6}\cdot U_4(2)$. Thus, by [1, pp. 128–130], we may assume $\delta\in\{\chi_i\mid i=7$, 8, 13, 14, 17, 18, 19, 28, 100, 101, 106, 113$\}$.

Let x be an element of order 8 in S which is a cohort representative of $8C$ in Suz (see [1, pp. 128–130]). $\delta(x)\in\{\pm 1\}$ and $|C_S(x)|=32|Z(S)|$. We may assume $x\in R_1$. Let u_i be cohort representatives of $3A$, $3B$, and $3C$, respectively, in S_0 where $|\langle u_i\rangle|=3$ and $\delta(u_i)$ is real for $i=1,2,3$. (See [1, pp. 128–130].) Then $\langle u_1\rangle_0$ and $\langle u_2\rangle_0$ are non-conjugate. Each δ is non-vanishing and real on $\langle u_1\rangle$ and $\langle u_2\rangle$. We may assume $u_i\in R_1$ for $i=1,2$.

Let v be an element of order 15 which is a cohort representative of $15C$ in S_0 (see [1, pp. 128–130]) and let $\delta(v)$ be real.

Suppose $R_0=2_-^{1+6}\cdot U_4(2)$; then R has a normal subgroup $F=F_2\times Z(S)$ where $F_2\subseteq R_1$ and $F_2=2_-^{1+6}$. By [1, p. 131], $R_0=N_{S_0}(2A)$. Since Suz contains no element of order 6 with 2-part conjugate to $2A$ and 3-part conjugate to $3C$, $(3C)^{S_0}\cap R_0=\phi$. Thus, $\delta\notin\{\chi_7,\chi_8,\chi_{13}\chi_{14},\chi_{17},\chi_{18},\chi_{19},\chi_{28}\}$. $\chi_{106}(v)\neq 0$ and $v^S\cap R=\phi$ by [1, p. 26]. Hence $\delta\in\{\chi_{100},\chi_{101},\chi_{113}\}$. Therefore, $|C_S(x)|=|Z(S)|\cdot 32=3\cdot 32$, and $R=R_1\times Z(S)$. Now $x^S\cap R=\bigcup_{i=1}^n x_i^R$ where notation may be chosen so that $x_iF=(4B)F$ where $(4B)F$ is an element of $R/F\simeq U_4(2)$ corresponding to $4B$ of [1, p. 27]. $|C_{U_4(2)}(4B)|=8$ and at most $|F|/3$ of the elements in $(4B)F$ have order 8. Hence, by (2.2) and (2.4) $a=a|\delta(x)|\leqslant|C_S(x)|\,(\sum_{i=1}^n|\varphi(x_i)|/|C_R(x_i)|)\leqslant 96\alpha/24=4\alpha$ where $\alpha=\max\{|\varphi(x_i)|,\ i=1,\dots,n\}$, i.e.,

$$a\leqslant 4\alpha. \qquad \text{(xvii, 1)}$$

$|C_S(x)|=96$ and $R=R_1\times Z(S)$ yield $|\varphi(x_i)|^2\leqslant|C_{R_1}(x_i)|-1\leqslant 31$ for $i=1,\dots,n$. Hence, $\alpha\leqslant\sqrt{31}$ and $a\leqslant 22$. By (2.1) $a=9\varphi(1)/5$ if $\delta=\chi_{100}$ or χ_{101} and $a=3\varphi(1)/5$ if $\delta=\chi_{113}$. In particular, (xvii, 1) now yields $\varphi(1)=5$ if $\delta=\chi_{100}$ or χ_{101} and $\varphi(1)\leqslant 35$ if $\delta=\chi_{113}$. Suppose $(\varphi_F,\lambda)\neq 0$ for $\lambda\in\mathrm{Irr}(F)$. Since $[R:I(\lambda)]$ is odd and $\varphi(1)\leqslant 35$, [1, p. 26] yields $I(\lambda)=R$ or $I(\lambda)/F=2^4:A_5$. However, $\langle u_1\rangle_0$ and $\langle u_2\rangle_0$ are non-conjugate which implies that $I(\lambda)=R$. Since $R_1/F_2 \simeq U_4(2)$, it follows that there is a homomorphism σ of $R \to U_4(2)$ such that $|\varphi(r)|=|\hat{\varphi}(\sigma(r))|$ where $\hat{\varphi} \in \mathrm{Irr}(U_4(2))$. Since $\sigma(x_i)$ is conjugate to $4B$ of [1, p. 27] for $i=1, ..., n$, [1, p. 27] implies that $\alpha=1$. Now $a \leqslant 4\alpha=4$ and a an integer yield $\delta=\chi_{113}$. Since $a=3(\varphi(1)/5)$ and $\varphi(1)$ is odd, $\hat{\varphi}=\chi_2$ or χ_3 or [1, p. 27]. In particular, $\hat{\varphi}$ does not vanish on 9-elements. R and S have the same number of conjugacy classes of 9-elements so (2.2) yields that χ_{113} does not vanish on 9-elements. This is a contradiction. Therefore, $R \neq 2^{1+6}\cdot U_4(2)$.

χ_i does not vanish on some 9-element if $i=7$, 8, 17, 18, 19, 100, and 101, and R contains elements of order 9 only if $R_0=2_-^{1+6}\cdot U_4(2)$; thus, $\delta \in \{\chi_{13}, \chi_{14}, \chi_{28}, \chi_{106}, \chi_{113}\}$.

Suppose that $R_0=2^{4+6}:3A_6$; then R contains a normal subgroup $F=F_2 \times Z(S)$ where $F_2=2^{2+8}$ and $F_2 \subseteq R_1$. $\langle u_1\rangle_0$ and $\langle u_2\rangle_0$ non-conjugate together with $\delta(1)$ odd and [1, p. 4] imply that $I(\lambda)=R$ if $\lambda \in \mathrm{Irr}(F)$ and $(\varphi_F, \lambda) \neq 0$. $|F_2/\ker(\lambda)| \leqslant 2$ and $\varphi(1)$ odd yield $\varphi|_{3A_6} \in \mathrm{Irr}(3A_6)$. Since δ is non-zero and integer-valued on $\langle u_1\rangle$ and $\langle u_2\rangle$, $\varphi|_{3A_6}=\chi_2$ or χ_3 of [1, p. 5]. Hence, $\delta(g)=0$ if $|\langle gZ(S)\rangle|=15$. Therefore, $\delta \in \{\chi_{28}, \chi_{113}\}$.

Let x' be an element of orders 8 in R_1 which is a cohort representative of $8B$ [1, pp. 129–130]. Then $|C_S(x_1)|=|Z(S)|\,2^6$. Hence $x_j'F \cup x_iF \subseteq (2A)F \cup (4A)F$ where $(2A)F$ and $(4A)F$ are elements of order 2 and 4 in R/F described on [1, p. 5]. $x^S \cap R=\bigcup_{i=1}^n (x_i)^R$ and $(x')^S \cap R=\bigcup_{i=1}^{n'} (x_i')R$. Let $T_1=\{x_i, x_j' \mid x_iF=(2A)F$ or $x_j'F=(2A)F\}$ and $T_2=\{x_i, x_j' \mid x_iF=(4A)F$ or $x_j'F=(4A)F\}$. $|C_{R/F}((2A)F)|=8$ and $|C_{R/F}((4A)F)|=4$. Since $F=F_2 \times Z(S)$, (2.4) yields $\sum_{T_1} 1/|C_{R(g)}| \leqslant 1/8|Z(S)|$ and $\sum_{T_1'} 1/|C_R(g)| \leqslant 1/4|Z(S)|$.

Since $\varphi \in \{\chi_2, \chi_3\}$ of [1, p. 5] $|\varphi(x_i)|=1=|\varphi(x_j')|$ for all i, j. Further, $a=3^3\varphi(1)/5=3^3$ if $\delta=\chi_{28}$ and $a=3^2\varphi(1)/5=3^2$ if $\delta=\chi_{113}$ by (2.1). Thus, (2.2) and $|\delta(x)|=|\delta(x')|=1$ now imply that $2a=2a|\delta(x)|=2|C_S(x)|\,|\sum_{i=1}^n \varphi(x_i)/|C_R(x_i)|| \leqslant 2^6\cdot|Z(S)|(\sum_{i=1}^n 1/|C_R(x_i)|)$ and $a=a|\delta(x')|=|C_S(x')|\ \ |\sum_{j=1}^{n'} \varphi(x_j')/|C_R(x_j')|| \leqslant 2^6|Z(S)|(\sum_{j=1}^{n'} 1/|C_R(x_j')|)$. Hence, $3^3 \leqslant 3a=2a|\delta(x)|+a|\delta(x')| \leqslant 2^6|Z(S)|(\sum_{T_1 \cup T_2} 1/|C_R(g)|) \leqslant 2^6|Z(S)|(3/8|Z(S)|)$. This is a contradiction.

Hence, $R_0=2^{2+8}:(A_5 \times S_3)$, and R has a normal subgroup $F=F_2 \times Z(S)$ where $F_2=2^{2+8}$ and $F_2 \subseteq R_1$. If $x^S \cap R=\bigcup_{i=1}^n x_i^S$, then Fx_i has order 2 and $|C_{R_1}(x_i)| \leqslant 32$. Hence, there are 2-elements g_1, g_2 in $A_5 \times S_3$ with $|C_{R/F}(g_1F)|=24$, $|C_{R/F}(g_2F)|=8$, and we may assume $\bigcup_{i=1}^n x_iF \subseteq g_1F \cup g_2F$. $|\delta(x)|=1$ by [1, pp. 128–130]. Hence, (2.4), (2.2), and $F=F_2 \times Z(S)$ now yield $a=a|\delta(x)| \leqslant |C_S(x)|(\sum_{i=1}^n |\varphi(x_i)|/|C_R(x_i)|) \leqslant (|Z(S)|\,32/|Z(S)|\,6)\alpha$ where $\alpha=\max\{|\varphi(x_i)| \mid i=1, ..., n\}$. Since $R=R_1 \times Z(S)$, $|\varphi_i(x)|^2 \leqslant |C_S(x)|/|Z(S)|-1=31$ and $a \leqslant 29$. By (2.1), $a=3^4\varphi(1)$ if $\delta=\chi_{13}$ or χ_{14}, $a=\varphi(1)3^4/5 \geqslant 3^4$ if $\delta=\chi_{28}$, $a=\varphi(1)3^2$ if $\delta=\chi_{106}$, and $a=(\varphi(1)/5)\,3^3$ if $\delta=\chi_{113}$. Thus, $\delta=\chi_{106}$ or χ_{113}.

If $(\varphi_F, \lambda) \neq 0$ for $\lambda \in \mathrm{Irr}(F)$, then $\langle u_1\rangle_0$ and $\langle u_2\rangle_0$ non-conjugate imply that $I(\lambda)=R$ or $I(\lambda)/F=A_4\times S_3$. If $I(\lambda)=R$, then $\varphi(1)$ odd implies that $\varphi|_{A_5\times S_3}\in \operatorname{Irr}(A_5\times S_3)$. Further, $|\varphi(x_i)|=|\varphi(g_j)|$ if $x_i\in g_jF$. Thus, $|\varphi(x_i)|=1$ and $\alpha=1$. However, $9\leqslant a\leqslant 16/3\cdot\alpha=16/3$ is a contradiction. Hence, $I(\lambda)/F=A_4\times S_3$. Further, $\theta(u_i)\neq 0$ for $i=1,2$ implies that $\theta(1)=1$ where $\varphi=\theta^R$. Since $\chi_{106}(v)\neq 0$ and $I(\lambda)$ contains no elements of order 15, $\delta=\chi_{113}$. Notation may be chosen so that g_1F, $g_2F\in I(\lambda)$. By using $(1_{A_4})^{A_5}$ on [1, p. 2], it is direct to see that $(g_iF)^{R/F}\cap I(\lambda)/F=(g_iF)^{I(\lambda)/F}$. It follows that if $x^S\cap I(\lambda)=\bigcup_{i=1}^m y_i^{I(\lambda)}$, then notation may be chosen so that $\bigcup_{i=1}^m y_iF\subseteq g_1F\cup g_2F$. Since $|C_{I(\lambda)/F}(g_iF)|=|C_{R/F}(g_iF)|$, the argument used in previous paragraphs yields $\sum_{i=1}^m 1/|C_{I(\lambda)}(y_i)|\leqslant 1/|Z(S)|\,6=\frac{1}{18}$. By (2.4), (2.2), and $a=\varphi(1)\,3^3/5$, $\varphi(1)3^3/5=a|\delta(x)|\leqslant|C_S(x)|\,(\sum_{i=1}^m|\theta(y_i)|/|C_{I(\lambda)}(y_i)|)\leqslant 32\cdot 3(\frac{1}{18})\theta(1)=16/3$. This contradicts $5=[R:I(\lambda)]\theta(1)=\varphi(1)$.

(xviii) Suppose that $S_0=O'N$. $R_0=4_2{}^{\cdot}L_3(4):2_1$, or $R_0=4^{3\cdot}L_3(2)$ by [1, p. 132]. Thus, R does not contain two non-conjugate subgroups of order 7. In particular, either $x^S\cap R=\phi$ or $y^S\cap R=\phi$ if x and y have order 7, x is a cohort representative of $7A$, and y is a representative of $7B$ in S_0. (See [1, p. 132].) Further, R contains no elements of order 11, 19, or 31. Now direct inspection using [1, p. 132] implies that $a\delta\neq\varphi^S$ for $\delta\in\operatorname{Irr}(S)$.

(xix) Suppose $S_0=Fi_{22}$; then $R_0=2^{10}:M_{22}$, $R_0=(2\times 2_+^{1+8}:U_4(2)):2$, or $R_0=2^{5+8}:(S_3\times A_6)$. Thus, R contains no elements g such that $gZ(S)$ has order 21, 13, or 30 in R_0. If $|\langle gZ(S)\rangle|=22$, where $gZ(S)\in(22A)^{S_0}\cup(22B)^{S_0}$ of [1, p. 156], and $gZ(S)\in R_0$, then $R_0=2^{10}:M_{22}$. Hence, $(1_{R_0})^{S_0}$ may be deduced from [1, p. 163], and it is direct to compute $(1_{R_0})^{S_0}(gZ(S))=0$. Thus, no R contains an element of order 22. Hence, $\delta\in\{\chi_{17},\chi_{18},\chi_{19},\chi_{20},\chi_{45},\chi_{59},\chi_{121},\chi_{141},\chi_{142}\}$ of [1, pp. 156–162].

Suppose $R_0=2^{10}:M_{22}$; then R_0 contains only one class of 3-elements, and no elements of order 9 or 15. Direct inspection using [1, pp. 156–162] now yields that for none of the δ is $a\delta=\varphi^S$ where $R_0=2^{10}:M_{22}$.

Suppose $R_0=(2\times 2_+^{1+8}:U_4(2)):2$, then $R_0=N_{R_0}(2B)$. Thus $3D^{S_0}\cap R_0=\phi$, since Fi_{22} does not contain an element of order 6 with 3 part $3D$ and 2 part $2B$. Thus, $\delta\notin\{\chi_{45},\chi_{17},\chi_{18},\chi_{19}\}$. R_0 contains no element of order 15 so that $\delta\neq\chi_{121}$. Hence, $\delta\in\{\chi_{20},\chi_{59},\chi_{141},\chi_{142}\}$. $R=R_1\times Z(S)$ where $R_1\simeq R_0$. R has a normal subgroup $F=2\times 2_+^{1+8}\times Z(S)$. Let x and x' denote elements of order 16 which are cohort representatives of $16A$ and $16B$ of [1, p. 156]; then $|C_S(x)|=|Z(S)|\,32$. Hence, if $x^S\cap R=\bigcup_{i=1}^n x_i^R$, and $x'^S\cap R=\bigcup_{i=1}^{n'}(x_i')^R$, [1, p. 27] implies that there are elements g_1, g_2 in $U_4(2):2$ corresponding to $4B$ and $8A$ of [1, p. 27] such that $\bigcup_{i=1}^n x_iF\subseteq g_1F\cup g_2F$, where $|C_{R/F}(g_1F)|=16$ and $|C_{R/F}(g_2F)|=8$. (Since R is a split extension, if $g_3\in U_4(2):2$ where g_3 has order 4 and corresponds to $4D$ of [1, p. 27], then $|C_{R/F}(g_3F)|=2^5$. If x_i' or $x_i\in g_3F$, then the number of R-conjugates of x_i' or x_i in g_3F would be

$|C_{R/F}(g_3F)|/|C_R(x_i')| = |C_{R/F}(g_3F)|/|C_R(x_i)| = 2^5|F|/(2^5|Z(S)|)$. However, g_iF contains at most $|F|/|Z(S)|$ elements of order a power of 2 and some of these elements have order 4. This is a contradiction.) By standard arguments $\sum_{i=1}^{n} 1/|C_R(x_i)| + \sum_{i=1}^{n'} 1/|C_R(x_i')| \leqslant 3/(|Z(S)|\,16)$. Thus, (2.2) and [1, pp. 156–158] yield $a = a|\delta(x)| \leqslant (C_S(x)|\,(\sum_{i=1}^{n} |\varphi(x_i)|/|C_R(x_i)|)$ and $a = a|\delta(x')| \leqslant |C_S(x')|\,(\sum_{i=1}^{n'} |\varphi(x_i')|/|C_R(x_i')|)$. Thus $2a \leqslant 32 \cdot |Z(S)|$ $\alpha(3/|Z(S)|\,16)$ or $a \leqslant 3\alpha$ where $\alpha = \max\{|\varphi(x_i)|, \ |\varphi(x_j')|\ |\ i = 1, ..., n, j = 1, ..., n'\}$. Since $|C_S(x)| = |C_S(x')| = 32|Z(S)|$ and $R = R_1 \times Z(S)$, $\alpha \leqslant \sqrt{31}$. Hence, $a \leqslant 16$. By (2.1), $a = 15\varphi(1)$ if $\delta = \chi_{20}$, $a = \varphi(1)3/5$ if $\delta = \chi_{59}$, and $a = (\varphi(1)/5)3^2$ if $\delta = \chi_{141}$ or χ_{142}. Since a is an odd integer, $\delta \neq \chi_{20}$ and $\varphi(1) = 5$ or 15. It now follows from [1, p. 26] that $I(\lambda) = R$ if $(\varphi_F, \lambda) \neq 0$ and $\lambda \in \mathrm{Irr}(F)$. Now $\varphi(1)$ odd implies that $\varphi_{(U_4(2):2)} \in \mathrm{Irr}(U_4(2):2)$ and $|\varphi(x_i)| = |\varphi(g_j)|$ if $x_iF = g_jF$. Also $|\varphi(x_i')| = |\varphi(g_j)|$ if $x_i'F = g_jF$. By [1, p. 27], $|\varphi(x_i)| = |\varphi(x_i')| = 1$, whence $\varphi(1)3/5 = a \leqslant 3$. But $U_4(2):2$ has no irreducible character of degree 5.

Thus, $R_0 = 2^{5+8} : (S_3 \times A_6)$. Since R_0 contains no elements of order 9, $\delta \notin \{\chi_{17}, \chi_{18}, \chi_{19}, \chi_{121}, \chi_{141}, \chi_{142}\}$. Hence, $\delta \in \{\chi_{20}, \chi_{45}, \chi_{59}\}$ and $S = S_0$. Let $x = 16A$ and $x' = 16B$ of [1, p. 156]; then $|C_S(x)| = |C_S(x')| = 32$, and $R_0 = R$ has a normal subgroup $F = 2^{5+8}$.

There are 2-elements g_1, $g_2 \in R$ such that R/F has two conjugacy classes g_1F and g_2F of elements of order 4 where $|C_{R/F}(g_1F)| = 24$ and $|C_{R/F}(g_2F)| = 8$. If $x^S \cap R = \bigcup_{i=1}^{n} x_i^R$ then x_iF has order 4, and $|C_{R/F}(x_iF)| \leqslant 32$ so we may assume $x_i \in g_1F \cup g_2F$ for $i = 1, ..., n$. A similar argument yields if $x'^S \cap R = \bigcup_{i=1}^{n'} x_i'R$, then we may assume $\bigcup_{i=1}^{n'} x_i'F \subseteq g_1F \cup g_2F$. It follows from (2.4) that $(\sum_{i=1}^{n} 1/|C_R(x_i)| + \sum_{i=1}^{n'} 1/|C_R(x_i')|) \leqslant \frac{1}{24} + \frac{1}{8} = \frac{1}{6}$. Now by (2.2) and [1, pp. 156–158], $a = a|\delta(x)| \leqslant |C_S(x)|\,(\sum_{i=1}^{n} |\varphi(x_i)|/|C_R(x_i)|)$ and $a = a|\delta(x')| \leqslant |C_S(x)|\,(\sum_{i=1}^{n'} |\varphi(x_i')|/|C_R(x_i)|)$. Adding yields $2a \leqslant 32(\sum_{i=1}^{n} |\varphi(x_i)|/|C_R(x_i)| + \sum_{i=1}^{n'} |\varphi(x_i')|/|C_R(x_i)|) \leqslant (32/6)\alpha$ where $\alpha = \max\{|\varphi(x_i)|, |\varphi(x_i')|\ |i = 1, ..., n, i = 1, ..., n'\}$.

By (2.1) $a = 3^2 \cdot 5\varphi(1)$ if $\delta = \chi_{20}$, $a = 3^3\varphi(1)/5$ if $\delta = \chi_{45}$, and $a = 3^2\varphi(1)/5$ if $\delta = \chi_{59}$. Since $\alpha \leqslant \min\{\sqrt{31}, \varphi(1)\}$, $a \leqslant \frac{8}{3}\alpha$ which yields $\delta = \chi_{59}$ and $\varphi(1) = 5$. Hence, if $(\varphi_F, \lambda) \neq 0$ for $\lambda \in \mathrm{Irr}(F)$, then $I(\lambda) = R$. In particular, $(\varphi)_{S_3 \times A_6} \in \mathrm{Irr}(S_3 \times A_6)$ and $|\varphi_j(x_j')| = |\varphi(x_i)| = |\varphi(g_k)|$ if g_kF contains x_i or x_j'. Thus, $\alpha = 1$ whence $9 \leqslant 9\varphi(1)/5 \leqslant \frac{8}{3}$ a final contradiction.

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