Generalizations of the Hermite–Biehler theorem

Ming-Tzu Ho a,1, Aniruddha Datta b,2, S.P. Bhattacharyya b,*

a Ritek Corporation, No. 42, Kuangfu N. Road, Hsin Chu Industrial Park, Houko 30316, Taiwan
b Department of Electrical Engineering, Texas A & M University, College Station, TX 77843-3128, USA

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Dedicated to Professor Hans Schneider for his many contributions to inertia theory

Abstract

The Hermite–Biehler theorem gives necessary and sufficient conditions for the Hurwitz stability of a polynomial in terms of certain interlacing conditions. In this paper, we generalize the Hermite–Biehler theorem to situations where the test polynomial is not necessarily Hurwitz. The generalization is given in terms of an analytical expression for the difference between the numbers of roots of the polynomial in the open left-half and open right-half planes. The result can be used to solve important stabilization problems in control theory and is, therefore, of both academic as well as practical interest. © 1999 Elsevier Science Inc. All rights reserved.

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* Corresponding author. E-mail: bhatt@ee.tamu.edu
1 E-mail: brutoho0@ms19.hinet.net
2 E-mail: datta@ee.tamu.edu

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1. Introduction

The problem of determining conditions under which all of the roots of a given real polynomial lie in the open left-half complex plane is one of fundamental importance in the study of stability of a dynamic system [1]. This problem has intrigued researchers for more than a hundred years now and one of the earliest solutions, and the most widely known one, is the criterion of Routh–Hurwitz. There are several other equivalent conditions for ascertaining the Hurwitz stability of a given real polynomial (see [1] for a detailed discussion). Of these, the Hermite–Biehler theorem has recently been instrumental in studying the robust parametric stability problem, i.e., the problem of guaranteeing that the roots of a given Hurwitz polynomial continue to lie in the left-half plane under real coefficient perturbations [2,3].

The Hermite–Biehler theorem states that a given real polynomial is Hurwitz iff it satisfies a certain interlacing property [1]. When a given real polynomial is not Hurwitz, the Hermite–Biehler theorem, as currently known, provides absolutely no information about its root distribution. In this paper, we present generalizations of the Hermite–Biehler theorem to real polynomials which are not necessarily Hurwitz stable. These generalizations are not only of academic interest but also have practical implications in control theory.

The paper is organized as follows. In Section 2, we provide a statement of the Hermite–Biehler theorem as well as some equivalent characterizations. In Section 3, we state the relationship between the net phase change of the “frequency response” of a real polynomial as the frequency \( \omega \) varies from 0 to \( \infty \) and the numbers of its roots in the open left-half and open right-half planes. In Section 4, we derive generalizations of the Hermite–Biehler theorem applicable to the case where the test polynomial has no zeros on the imaginary axis. Section 5 shows that the identical theorem statement can accommodate the presence of imaginary axis zeros, provided they are not at the origin. In Section 6, we modify the theorem statement to handle the presence of one or more zeros at the origin. An example verifying the theorem statement is also provided. Finally, Section 7 contains some concluding remarks.

2. The Hermite–Biehler theorem

In this section, we first state the Hermite–Biehler theorem which provides necessary and sufficient conditions for the Hurwitz stability of a given real polynomial. The proof can be found in [1]; see also [4,5] for an alternative proof using the Boundary Crossing Theorem. We also refer the reader to [6] for several results related to the Hermite–Biehler theorem.
Theorem 2.1 (Hermite–Biehler theorem). Let $\delta(s) = \delta_0 + \delta_1 s + \cdots + \delta_n s^n$ be a given real polynomial of degree $n$. Write $\delta(s) = \delta_e(s^2) + s\delta_o(s^2)$ where $\delta_e(s^2)$, $s\delta_o(s^2)$ are the components of $\delta(s)$ made up of even and odd powers of $s$, respectively. Let $\omega_{e_1}, \omega_{e_2}, \ldots$ denote the distinct non-negative real zeros of $\delta_e(-\omega^2)$ and let $\omega_{o_1}, \omega_{o_2}, \ldots$ denote the distinct non-negative real zeros of $\delta_o(-\omega^2)$, both arranged in ascending order of magnitude. Then $\delta(s)$ is Hurwitz stable if and only if all the zeros of $\delta_e(-\omega^2)$, $\delta_o(-\omega^2)$ are real and distinct, $\delta_n$ and $\delta_{n-1}$ are of the same sign, and the non-negative real zeros satisfy the following interlacing property
\[
0 < \omega_{e_1} < \omega_{o_1} < \omega_{e_2} < \omega_{o_2} < \cdots
\] (2.1)

In this paper, our objective is to obtain generalizations of the above theorem for real polynomials that are not necessarily Hurwitz. To clearly understand what it is that we are trying to generalize, we provide below some alternative characterizations and interpretations of the Hermite–Biehler theorem. To do so, we first introduce the standard signum function $\text{sgn} : \mathbb{R} \rightarrow \{-1, 0, 1\}$ defined by
\[
\text{sgn}[x] = \begin{cases} 
-1 & \text{if } x < 0, \\
0 & \text{if } x = 0, \\
1 & \text{if } x > 0.
\end{cases}
\]

Lemma 2.1. Let $\delta(s) = \delta_0 + \delta_1 s + \cdots + \delta_n s^n$ be a given real polynomial of degree $n$. Write $\delta(s) = \delta_e(s^2) + s\delta_o(s^2)$ where $\delta_e(s^2)$, $s\delta_o(s^2)$ are the components of $\delta(s)$ made up of even and odd powers of $s$, respectively. For every frequency $\omega \in \mathbb{R}$, denote $\delta(j\omega) = p(\omega) + jq(\omega)$ where $p(\omega) = \delta_e(-\omega^2)$, $q(\omega) = \omega\delta_o(-\omega^2)$. Let $\omega_{e_1}, \omega_{e_2}, \ldots$ denote the non-negative real zeros of $\delta_e(-\omega^2)$ and let $\omega_{o_1}, \omega_{o_2}, \ldots$ denote the non-negative real zeros of $\delta_o(-\omega^2)$, both arranged in ascending order of magnitude. Then the following conditions are equivalent:

(i) $\delta(s)$ is Hurwitz stable.

(ii) $\delta_n$ and $\delta_{n-1}$ are of the same sign and
\[
n = \begin{cases} 
\text{sgn}[\delta_0] \cdot \{\text{sgn}[p(0)] - 2\text{sgn}[p(\omega_{o_1})] + 2\text{sgn}[p(\omega_{o_2})] + \cdots + (-1)^{m-1} \\
\times 2\text{sgn}[p(\omega_{om-1})] + (-1)^m \cdot \text{sgn}[p(\infty)])\} & \text{for } n = 2m, \\
\text{sgn}[\delta_0] \cdot \{\text{sgn}[p(0)] - 2\text{sgn}[p(\omega_{o_1})] + 2\text{sgn}[p(\omega_{o_2})] + \cdots + (-1)^{m-1} \\
\times 2\text{sgn}[p(\omega_{om-1})] + (-1)^m \cdot 2\text{sgn}[p(\omega_{om})]\} & \text{for } n = 2m + 1.
\end{cases}
\]

(2.2)

(iii) $\delta_n$ and $\delta_{n-1}$ are of the same sign and
Proof. (1) (i) \iff (ii).

We first show that (i) \implies (ii).

By the monotonic phase property of Hurwitz polynomials, we can show that the plot of \( \delta(j\omega) = p(\omega) + jq(\omega) \) must move strictly counterclockwise and goes through \( n \) quadrants in turn as \( \omega \) increases from 0 to \( \infty \) [5]. For Hurwitz \( \delta(s) \), the plots of \( \delta(j\omega) \) are illustrated in Fig. 1.

From Fig. 1, it is clear that

\[
\text{for } n = 2m
\]
\[
\sgn[\delta_0] \cdot \sgn[p(0)] > 0
\]
\[
- \sgn[\delta_0] \cdot \sgn[p(\omega_1)] > 0
\]
\[
\vdots
\]
\[
(-1)^{m-1} \sgn[\delta_0] \cdot \sgn[p(\omega_{m-1})] > 0
\]
\[
(-1)^m \sgn[\delta_0] \cdot \sgn[p(\infty)] > 0
\]

and

Fig. 1. Monotonic phase increase property for Hurwitz polynomials \( \delta(s) \).
for \( n = 2m + 1 \)
\[
\text{sgn}[\delta_0] \cdot \text{sgn}[p(0)] > 0
\]
\[
- \text{sgn}[\delta_0] \cdot \text{sgn}[p(\omega_{o_1})] > 0
\]
\[
\vdots
\]
\[
(-1)^{m-1} \text{sgn}[\delta_0] \cdot \text{sgn}[p(\omega_{o_{m-1}})] > 0
\]
\[
(-1)^m \text{sgn}[\delta_0] \cdot \text{sgn}[p(\omega_{o_m})] > 0
\]

From (2.4) and (2.5), it follows that (2.2) holds.

(ii) \( \Rightarrow \) (i)

Let \( \omega_{o_0} = 0 \) and for \( n = 2m \), denote \( \omega_{o_m} = \infty \). Eq. (2.2) holds if and only if \( \text{sgn}[p(\omega_{o_{l-1}})] \) and \( \text{sgn}[p(\omega_{o_l})] \) are of opposite signs for \( l = 1, 2, \ldots, m \). By the continuity of \( p \), there exists at least one \( \omega_e \in \mathbb{R}, \omega_{o_{l-1}} < \omega_e < \omega_{o_l} \) such that \( p(\omega_e) = 0 \). Moreover, since the maximum possible number of non-negative real roots of \( p(\cdot) \) is \( m \), it follows that there exists one and only one \( \omega_e \in (\omega_{o_{l-1}}, \omega_{o_l}) \) such that \( p(\omega_e) = 0 \), thereby leading us to the interlacing property.

(2) (i) \( \iff \) (iii).

The proof of (2) follows along the same lines as that of (1). \( \square \)

**Remark 2.1.** The interlacing property in Theorem 2.1 gives a graphical interpretation of the Hermite–Biehler theorem while Lemma 2.1 gives an equivalent analytical characterization.

Note that from Lemma 2.1 if \( \delta(s) \) is Hurwitz stable then all zeros of \( p(\omega) \) and \( q(\omega) \) must be real and distinct, otherwise (2.2) and (2.3) will fail.

We now present an example to illustrate the application of Theorem 2.1 and Lemma 2.1 to verify the interlacing property.

**Example 2.1.** Consider the real polynomial \( \delta(s) \) where
\[
\delta(s) = s^7 + 5s^6 + 14s^5 + 25s^4 + 31s^3 + 26s^2 + 14s + 4.
\]
Then
\[
\delta(j\omega) = p(\omega) + jq(\omega),
\]
where
\[
p(\omega) = -5\omega^6 + 25\omega^4 - 26\omega^2 + 4,
\]
\[
q(\omega) = \omega(-\omega^6 + 14\omega^4 - 31\omega^2 + 14).
\]

The plots of \( p(\omega) \) and \( q(\omega) \) are shown in Fig. 2. They show that the polynomial \( \delta(s) \) satisfies the interlacing property.
Also
\[ x e^0: 43106; \]
\[ x e^1: 08950; \]
\[ x e^2: 90452; \]
\[ x o^0: 78411; \]
\[ x o^1: 41421; \]
\[ x o^2: 37419; \]
and
\[ \text{sgn} \delta_0 \cdot [\text{sgn}[p(0)] - 2 \text{sgn}[p(o_1)] + 2 \text{sgn}[p(o_2)] - 2 \text{sgn}[p(o_3)]] = 7, \]
which shows that (2.2) holds.

Now \( \delta(s) \) is of degree \( n = 7 \) which is odd and
\[ \text{sgn}[q(o)] = 1, \quad \text{sgn}[q(o_1)] = -1, \quad \text{sgn}[q(o_2)] = 1, \quad \text{sgn}[q(o_3)] = -1. \]
Also, we have
\[ \text{sgn}[q(o)] = 1, \quad \text{sgn}[q(o_1)] = -1, \quad \text{sgn}[q(o_2)] = 1, \quad \text{sgn}[q(\infty)] = -1 \]
so that
\[ \text{sgn}[\delta_0] \cdot [2 \text{sgn}[q(o_1)] - 2 \text{sgn}[q(o_2)] + 2 \text{sgn}[q(o_3)] - \text{sgn}[q(\infty)]] = 7. \]
Once again, this verifies (2.3).
To verify that $\delta(s)$ is indeed a Hurwitz polynomial, we solve for the roots of $\delta(s)$:

\[-0.5 \pm 1.3229j \quad -0.5 \pm 0.8660j \]
\[-1 \pm j \quad -1\]

We see that all the roots of $\delta(s)$ are in the left-half-plane so that $\delta(s)$ is Hurwitz.

Now consider $\delta(j\omega) = p(\omega) + jq(\omega)$ as illustrated in Fig. 3. From Fig. 3, we know that the polynomial $\delta(s)$ is not a Hurwitz polynomial because it fails to satisfy the interlacing property. However, it is logical to ask: does Fig. 3 provide us with any more information about $\delta(s)$, beyond whether or not it is Hurwitz? As it turns out it is possible to know the number of right-half plane roots of $\delta(s)$ from the above graph. This motivates us to derive generalized versions of the Hermite–Biehler theorem for not necessarily Hurwitz polynomials. This is carried out in Sections 3–6.

3. Signature and net accumulated phase

In this section we develop, as a preliminary step to the generalized Hermite–Biehler theorems, a fundamental relationship between the net accumulated phase of the frequency response of a real polynomial and the difference between the numbers of roots of the polynomial in the open left-half and open right-half planes. To this end, let $C$ denote the complex plane, $C^-$ the open left-half plane and $C^+$ the open right-half plane.
In the beginning, we focus on polynomials without zeros on the imaginary axis. Consider a real polynomial $d_s$ of degree $n$

\[d_s = d_0 + d_1s + d_2s^2 + \cdots + d_ns^n, \quad d_i \in R, \quad i = 0, 1, \ldots, n, \ d_n \neq 0,
\]
such that $d(j\omega) \neq 0 \quad \forall \omega \in (-\infty, \infty)$.

**Definition 3.1.** Let $l$ and $r$ denote the numbers of roots of $d(s)$ in $C^-$ and $C^+$, respectively. Then the signature of $d(s)$ denoted by $\sigma(d)$ is defined as

\[\sigma(d) \equiv l - r.
\]

Since

\[n = l + r\]

it follows that $\sigma(d)$ and $n$ uniquely determine $l$ and $r$, and hence the root distribution of $d(s)$. Now for every frequency $\omega \in R$, $d(j\omega)$ is a point in the complex plane. Let $p(\omega)$ and $q(\omega)$ be two functions defined pointwise by

\[p(\omega) = \text{Re}[d(j\omega)], \quad q(\omega) = \text{Im}[d(j\omega)].\]

With this definition, we have

\[\delta(j\omega) = p(\omega) + jq(\omega) \quad \forall \omega.
\]

Furthermore $\theta(\omega) \equiv \angle d(j\omega) = \arctan[q(\omega)/p(\omega)]$. Let $A^\infty_0 \theta$ denote the net change in the argument $\theta(\omega)$ as $\omega$ increases from $0$ to $\infty$. Then we can state the following lemma [1]:

**Lemma 3.1.** Let $d(s)$ be a real polynomial with no imaginary axis roots. Then

\[A^\infty_0 \theta = \frac{\pi}{2} \sigma(d).
\]

4. Generalizations of the Hermite–Biehler theorem: no imaginary axis roots

In this section, we focus on real polynomials with no imaginary axis roots and derive two generalizations of the Hermite–Biehler theorem by first developing a procedure for systematically determining the net accumulated phase change of the frequency response of a polynomial. We first recall that at any given frequency $\omega$, the phase angle of $\delta(j\omega)$ is given by

\[\theta(\omega) = \tan^{-1} \frac{q(\omega)}{p(\omega)}.
\]

Hence the rate of change of phase with respect to frequency at any given frequency $\omega$ is given by
\[ \frac{d\theta(\omega)}{d\omega} = \frac{1}{1 + q^2(\omega)/p^2(\omega)} \frac{\dot{q}(\omega)p(\omega) - \dot{p}(\omega)q(\omega)}{p^2(\omega) + q^2(\omega)} \]  

(4.1)

If \( p(\omega) \) and \( q(\omega) \) are known for all \( \omega \), we can integrate (4.1) to obtain the net phase accumulation. However, to calculate the net accumulation of phase over all frequencies it is not necessary to know the precise rate of change of phase at each and every frequency. This is because, we know that every time the polar plot \( \delta(j\omega) \) makes a transition from the real axis to the imaginary axis or vice versa, there can be at most a net phase change of \( \pm(\pi/2) \) radians. The actual sign of the phase change can be determined by examining (4.1) at the real or imaginary axis crossing of the \( \delta(j\omega) \) plot. Since at a real or imaginary axis crossing, one of the two terms in the numerator of (4.1) vanishes and the denominator is always positive, the actual determination of sign of the phase change becomes even simpler.

Now, given any polynomial \( \delta(s) \) of degree greater than or equal to one, either the real part or the imaginary part or both of \( \delta(j\omega) \) become infinitely large as \( \omega \to \pm \infty \). However, if we wish to count the total phase accumulation in integral multiples of axis crossings, it is imperative that the frequency response plot used approach either the real or imaginary axis as \( \omega \to \pm \infty \). To accomplish this, one can normalize the plot of \( \delta(j\omega) \) by scaling it with \( 1/f(\omega) \) where \( f(\omega) = (1 + \omega^2)^{n/2} \). Since \( f(\omega) \) does not have any real roots, this scaling will ensure that the normalized frequency response plot \( \delta_f(j\omega) = p_f(\omega) + jq_f(\omega) \) actually intersects either the real axis or the imaginary axis at \( \pm \infty \), while at the same time leaving unchanged the finite frequencies at which \( \delta(j\omega) \) intersects the real and imaginary axes. The subsequent development in this paper makes use of the normalized frequency response plot for determining the net accumulated phase change as we move from \( \omega = 0 \) to \( \omega = +\infty \). Here it should be pointed out that our normalized frequency response plot is analogous to the well-known Mikhailov plot [7] for Hurwitz polynomials.

As in Section 3, we consider a polynomial \( \delta(s) \) of degree \( n \)

\[ \delta(s) = \delta_0 + \delta_1 s + \delta_2 s^2 + \cdots + \delta_n s^n, \quad \delta_i \in \mathbb{R}, \quad i = 0, 1, \ldots, n, \quad \delta_n \neq 0, \]

such that \( \delta(j\omega) \neq 0 \) \( \forall \omega \in (-\infty, \infty) \).

Let \( p(\omega), q(\omega), p_f(\omega), q_f(\omega) \) be as already defined and let

\[ 0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1} \]

be the real, non-negative distinct finite zeros of \( q_f(\omega) \) with odd multiplicities. 

Also define \( \omega_m = +\infty \).

---

3 The function \( q_f(\omega) \) does not change sign while passing through a real zero of even multiplicity; hence such zeros can be skipped while counting the net phase accumulation.
Then we can make the following simple observations:

1. If \( \omega_i, \omega_{i+1} \) are both zeros of \( q_f(\omega) \) then

\[
\Delta_{\omega_i}^{\omega_{i+1}} \theta = \frac{\pi}{2} \left[ \text{sgn}[p_f(\omega_i)] - \text{sgn}[p_f(\omega_{i+1})] \right] \cdot \text{sgn}[q_f(\omega_i^+)].
\] (4.2)

2. If \( \omega_i \) is a zero of \( q_f(\omega) \) while \( \omega_{i+1} \) is not a zero of \( q_f(\omega) \), a situation possible only when \( \omega_{i+1} = \infty \) is a zero of \( p_f(\omega) \) and \( n \) is odd, then

\[
\Delta_{\omega_i}^{\omega_{i+1}} \theta = \frac{\pi}{2} \text{sgn}[p_f(\omega_i)] \cdot \text{sgn}[q_f(\omega_i^+)].
\] (4.3)

3. \( \text{sgn}[q_f(\omega_{i+1}^+)] = - \text{sgn}[q_f(\omega_i^+)], \quad i = 0, 1, 2, \ldots, m-2. \) (4.4)

Eq. (4.2) above is obvious while Eq. (4.4) simply states that \( q_f(\omega) \) changes sign when it passes through a zero of odd multiplicity. Eq. (4.3), on the other hand, can be directly traced to Eq. (4.1).

Using (4.4) repeatedly, we obtain

\[
\text{sgn}[q_f(\omega_i^+)] = (-1)^{m-1-i} \cdot \text{sgn}[q_f(\omega_{m-1}^+)], \quad i = 0, 1, \ldots, m-1. \] (4.5)

Substituting (4.5) into (4.2), we see that if \( \omega_i, \omega_{i+1} \) are both zeros of \( q_f(\omega) \) then

\[
\Delta_{\omega_i}^{\omega_{i+1}} \theta = \frac{\pi}{2} \left[ \text{sgn}[p_f(\omega_i)] - \text{sgn}[p_f(\omega_{i+1})] \right] \cdot (-1)^{m-1-i} \cdot \text{sgn}[q_f(\omega_{m-1}^+)].
\] (4.6)

The above observations enable us to state and prove the following theorem concerning \( \sigma(\delta) \).

**Theorem 4.1.** Let \( \delta(s) \) be a given real polynomial of degree \( n \) with no roots on the \( j\omega \) axis, i.e., the normalized plot \( \delta_f(j\omega) \) does not pass through the origin. Let \( 0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1} \) be the real, non-negative, distinct finite zeros of \( q_f(\omega) \) with odd multiplicities. Also define \( \omega_m = \infty \). Then

\[
\sigma(\delta) = \begin{cases} 
\left\{ \text{sgn}[p_f(\omega_0)] \right. & - 2 \text{sgn}[p_f(\omega_1)] + 2 \text{sgn}[p_f(\omega_2)] + \cdots + (-1)^{m-1} \\
\times 2 \text{sgn}[p_f(\omega_{m-1})] + (-1)^m \text{sgn}[p_f(\omega_m)] \} \cdot (-1)^{m-1} \\
\left. \times \text{sgn}[q(\infty)] \right\} \quad \text{if } n \text{ is even}, \\
\left\{ \text{sgn}[p_f(\omega_0)] \right. & - 2 \text{sgn}[p_f(\omega_1)] + 2 \text{sgn}[p_f(\omega_2)] + \cdots + (-1)^{m-1} \\
\times 2 \text{sgn}[p_f(\omega_{m-1})] \} \cdot (-1)^{m-1} \text{sgn}[q(\infty)] \quad \text{if } n \text{ is odd}. 
\end{cases}
\] (4.7)

**Proof.** First, let us suppose that \( n \) is even. Then \( \omega_m = \infty \) is a zero of \( q_f(\omega) \). The desired expression, i.e. the first one in (4.7), now follows by repeatedly using (4.6) to determine \( \Delta_{\omega_i}^{\infty} \theta \), applying Lemma 3.1, and then using the fact that \( \text{sgn}[q_f(\omega_{m-1}^+) = \text{sgn}[q(\infty)] \right\}


Now let us consider the case in which \( n \) is odd. Then \( \omega_m = \infty \) is not a zero of \( q_f(\omega) \). Hence,

\[
\Delta_0^\infty \theta = \sum_{i=0}^{m-2} \Delta_{\omega_i}^0 \theta + \Delta_{\omega_{m-1}}^\infty \theta \\
= \sum_{i=0}^{m-2} \frac{\pi}{2} \left[ \text{sgn}[p_f(\omega_i)] - \text{sgn}[p_f(\omega_{i+1})] \right] \cdot (-1)^{m-1-i} \text{sgn}[q_f(\omega_{m-1}^+)] \\
+ \frac{\pi}{2} \text{sgn}[p_f(\omega_{m-1})] \cdot \text{sgn}[q_f(\omega_{m-1}^+)] \\
\text{(using (4.6) and (4.3)). (4.8)}
\]

Applying Lemma 3.1, and then using the fact that \( \text{sgn}[q_f(\omega_{m-1}^+)] = \text{sgn}[q(\infty)] \), the desired expression follows. \( \square \)

We now state the result analogous to Theorem 4.1 where the signature \( \sigma(\delta) \) of a real polynomial \( \delta(s) \) is to be determined using the values of the frequencies where \( \delta_f(j\omega) \) crosses the imaginary axis. The proof is omitted since it follows along essentially the same lines as that of Theorem 4.1.

**Theorem 4.2.** Let \( \delta(s) \) be a given real polynomial of degree \( n \) with no roots on the \( j\omega \) axis, i.e., the normalized plot \( \delta_f(j\omega) \) does not pass through the origin. Let \( 0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1} \) be the real, non-negative, distinct finite zeros of \( p_f(\omega) \) with odd multiplicities. Also define \( \omega_m = \infty \). Then

\[
\sigma(\delta) = \begin{cases} 
-\{2 \text{sgn}[q_f(\omega_1)] - 2 \text{sgn}[q_f(\omega_2)] + \cdots + (-1)^{m-2} \\
\times 2 \text{sgn}[q_f(\omega_{m-1})]\} \cdot (-1)^m \text{sgn}[p(\infty)] & \text{if } n \text{ is even,} \\
-\{2 \text{sgn}[q_f(\omega_1)] - 2 \text{sgn}[q_f(\omega_2)] + \cdots + (-1)^{m-2} \\
\times 2 \text{sgn}[q_f(\omega_{m-1})] + (-1)^{m-1} \text{sgn}[q_f(\omega_m)]\} \cdot (-1)^m \\
\times \text{sgn}[p(\infty)] & \text{if } n \text{ is odd.} 
\end{cases} \quad (4.9)
\]

**Remark 4.1.** It is easy to verify that Theorems 4.1 and 4.2 essentially generalize Lemma 2.1, parts (ii) and (iii) to the case of not necessarily Hurwitz polynomials. It is precisely in this sense that Theorems 4.1 and 4.2 are generalizations of the Hermite–Biehler theorem.

5. **The generalized Hermite–Biehler theorem: no roots at the origin**

In this section, we extend Theorems 4.1 and 4.2 so that \( \delta(s) \) is now allowed to have non-zero imaginary axis roots. Theorems 5.1 and 5.2 show that the expressions in the statements of Theorems 4.1 and 4.2 are still valid for this
case. We will present a detailed proof of only Theorem 5.1; the proof of Theorem 5.2 follows essentially along the same lines and is, therefore, omitted.

**Theorem 5.1.** Let \( d_s^\dagger \) be a given real polynomial of degree \( n \) with no roots at the origin. Let \( 0 < x_0 < x_1 < x_2 < \cdots < x_{m-1} \) be the real, non-negative, distinct finite zeros of \( q_f(\omega) \) with odd multiplicities. Also define \( x_m \). Then

\[
\sigma(\delta) = \begin{cases} 
\{\text{sgn}[p_f(\omega_0)] - 2 \text{sgn}[p_f(\omega_2)] + 2 \text{sgn}[p_f(\omega_2)] + \cdots + (-1)^{m-1} \times 2 \text{sgn}[p_f(\omega_{m-1})] \cdot (-1)^{m-1} \text{sgn}[q(\infty)] \} & \text{if } n \text{ is even}, \\
\{\text{sgn}[p_f(\omega_0)] - 2 \text{sgn}[p_f(\omega_1)] + 2 \text{sgn}[p_f(\omega_2)] + \cdots + (-1)^{m-1} \times 2 \text{sgn}[p_f(\omega_{m-1})] \cdot (-1)^{m-1} \text{sgn}[q(\infty)] \} & \text{if } n \text{ is odd}.
\end{cases}
\]

(5.1)

**Proof.** Now, \( \delta(s) \) can be factored as

\[ \delta(s) = \delta_o^*(s)\delta_e^*(s)\delta'(s), \]

where \( \delta_o^*(s) \) contains all the \( j\omega \) axis roots of \( \delta(s) \) with odd multiplicities, \( \delta_e^*(s) \) contains all the \( j\omega \) axis roots of \( \delta(s) \) with even multiplicities, while \( \delta'(s) \) has no \( j\omega \) axis roots. Also \( \delta_o^*(s) \) and \( \delta_e^*(s) \) must necessarily be of the form

\[ \delta_o^*(s) = \prod_{\omega}(\omega^2 + x_\omega^2)^{n_\omega}, \quad \omega > 0, \quad n_\omega \geq 0, \quad n_\omega \text{ is odd, and } x_1 < x_2 < \cdots \]

\[ \delta_e^*(s) = \prod_{\omega}(\omega^2 + \beta_\omega^2)^{n_\omega}, \quad \beta_\omega > 0, \quad n_\omega \geq 0, \quad n_\omega \text{ is even}. \]

The proof is carried out in two steps. First, we show that multiplying \( \delta'(s) \) by \( \delta_e^*(s) \) has no effect on the expression (4.7). Thereafter, we use an inductive argument to show that multiplying \( \delta_e^*(s)\delta'(s) \) by \( \delta_o^*(s) \) also does not affect (4.7).

**Step I:** Define

\[ \delta_0(s) = \delta_e^*(s)\delta'(s) = \prod_{\omega}(\omega^2 + \beta_\omega^2)^{n_\omega} \delta'(s). \]

(5.2)

We want to show that \( \delta_0(s) \) satisfies (5.1).

Now define

\[ \delta'(j\omega) = p'(\omega) + jq'(\omega), \]

\[ \delta_0(j\omega) = p_0(\omega) + jq_0(\omega), \]

so that \( p'(\omega), p_0(\omega), q'(\omega), q_0(\omega) \) are related by
\[ p_0(\omega) = \prod_{i_e} (-\omega^2 + \beta_{i_e}^2)^{m_e} p'(\omega), \quad (5.3) \]
\[ q_0(\omega) = \prod_{i_e} (-\omega^2 + \beta_{i_e}^2)^{m_e} q'(\omega). \quad (5.4) \]

Let \( 0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1} \) be the real, non-negative, distinct finite zeros of \( q'_j(\omega) \) with odd multiplicities. Also define \( \omega_m = \infty \). First let us assume that \( \delta'(s) \) is of even degree. Then, from Theorem 4.1, we have
\[
\sigma(\delta') = \{ \text{sgn}[p'_j(\omega_0)] - 2 \text{sgn}[p'_j(\omega_1)] + 2 \text{sgn}[p'_j(\omega_2)] + \cdots + (-1)^{m-1} \times 2 \text{sgn}[p'_j(\omega_{m-1})] + (-1)^m \text{sgn}[q'_j(\omega_m)] \} \cdot (-1)^{m-1} \text{sgn}[q'(\infty)].
\]

Now, from (5.4), it follows that \( \omega_i, i = 0, 1, \ldots, m-1 \) are also the real, non-negative, distinct finite zeros of \( q_{0_j}(\omega) \) with odd multiplicities. Furthermore, from (5.3) and (5.4), we have
\[
\text{sgn}[p'_j(\omega_i)] = \text{sgn}[p_{0_j}(\omega_i)], \quad i = 0, 1, \ldots, m,
\]
\[
\text{sgn}[q'(\infty)] = \text{sgn}[q_{0}(\infty)].
\]

Since \( \sigma(\delta_0) = \sigma(\delta') \), it follows that the first expression of (5.1) is true for \( \delta_0(s) \) of even degree. The second expression of (5.1), corresponding to \( \delta'(s) \) of odd degree, can be verified by proceeding along exactly the same lines.

**Step II: Proof by Induction:**
Let \( j = 1 \) and consider
\[
\delta_1(s) = (s^2 + \omega_1^2)^{n_1} \prod_{i_e} (s^2 + \beta_{i_e}^2)^{m_e} \delta'(s)
= (s^2 + \omega_1^2)^{n_1} \delta_0(s). \quad (5.5)
\]

Now define
\[
\delta_1(j\omega) = p_1(\omega) + jq_1(\omega)
\]
so that \( p_1(\omega), p_0(\omega), q_1(\omega), q_0(\omega) \) are related by
\[
p_1(\omega) = (-\omega^2 + \omega_1^2)^{n_1} p_0(\omega), \quad (5.6)
\]
\[
q_1(\omega) = (-\omega^2 + \omega_1^2)^{n_1} q_0(\omega). \quad (5.7)
\]

Let \( 0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1} \) be the real, non-negative, distinct finite zeros of \( q_{0_j}(\omega) \) with odd multiplicities. Also define \( \omega_m = \infty \). First let us assume that \( \delta_0(s) \) has even degree. Then, from Step I, we have
\[
\sigma(\delta_0) = \{ \text{sgn}[p_{0_j}(\omega_0)] - 2 \text{sgn}[p_{0_j}(\omega_1)] + 2 \text{sgn}[p_{0_j}(\omega_2)] + \cdots + (-1)^{m-1} \times 2 \text{sgn}[p_{0_j}(\omega_{m-1})] + (-1)^m \text{sgn}[p_{0_j}(\omega_m)] \} \times (-1)^{m-1} \text{sgn}[q_{0}(\infty)]. \quad (5.8)
\]
Now, from (5.7), it follows that $\omega_i, i = 0, 1, \ldots, m - 1$; $z_1$ are the real, non-negative, distinct finite zeros of $q_{l}(\omega)$ with odd multiplicities. Let us assume that $\omega_i < z_1 < \omega_{i+1}$. Then, from (5.6) and (5.7), we have

$$\begin{align*}
\text{sgn}[p_{0t}(\omega_0)] &= \text{sgn}[p_{1l}(\omega_1)], \quad i = 0, 1, \ldots, l, \\
\text{sgn}[p_{lt}(\omega_i)] &= -\text{sgn}[p_{1l}(\omega_1)], \quad i = l + 1, l + 2, \ldots, m, \\
\text{sgn}[p_{1l}(z_1)] &= 0, \\
\text{sgn}[q_{0}(\infty)] &= -\text{sgn}[q_{1}(\infty)].
\end{align*}$$

(5.9)

Since $\sigma(\delta_1) = \sigma(\delta_0)$, using (5.8) and (5.9), we obtain

$$\begin{align*}
\sigma(\delta_1) &= \{\text{sgn}[p_{1l}(\omega_0)] - 2 \text{sgn}[p_{1l}(\omega_1)] + 2 \text{sgn}[p_{1l}(\omega_2)] \\
&\quad + \cdots + (-1)^{l+1}2 \text{sgn}[p_{1l}(\omega_l)] + (-1)^{l+2} \text{sgn}[p_{1l}(z_1)] + (-1)^{n+2} \\
&\quad \times 2 \text{sgn}[p_{1l}(\omega_{l+1})] + \cdots + (-1)^m2 \text{sgn}[p_{1l}(\omega_{m-1})] + (-1)^m \text{sgn}[q_{1}(\infty)],
\end{align*}$$

which shows that the first expression of (5.1) is true for $\delta_1(s)$ of even degree.

The second expression of (5.1), corresponding to $\delta_1(s)$ of odd degree, can be verified by proceeding along exactly the same lines. This completes the first step of the induction argument.

Now let $j = k$ and consider

$$\delta_k(s) = \prod_{i=1}^{k} (s^2 + \chi_n^2)^{n} \prod_{i=1}^{k} (s^2 + \beta_i^2)^{n_e} \delta'(s).$$

(5.10)

Assume that (5.1) is true for $\delta_k(s)$ (inductive assumption). Then

$$\delta_{k+1}(s) = \prod_{i=1}^{k+1} (s^2 + \chi_n^2)^{n} \prod_{i=1}^{k+1} (s^2 + \beta_i^2)^{n_e} \delta'(s)$$

$$= (s^2 + \chi_{k+1}^2)^{n} \delta_k(s).$$

(5.11)

Now, define

$$\begin{align*}
\delta_k(j\omega) &= p_k(\omega) + jq_k(\omega), \\
\delta_{k+1}(j\omega) &= p_{k+1}(\omega) + jq_{k+1}(\omega),
\end{align*}$$

so that $p_{k+1}(\omega), p_k(\omega), q_{k+1}(\omega), q_k(\omega)$ are related by

$$\begin{align*}
p_{k+1}(\omega) &= (-\omega^2 + \chi_{k+1}^2)^{n} p_k(\omega), \quad (5.12) \\
q_{k+1}(\omega) &= (-\omega^2 + \chi_{k+1}^2)^{n} q_k(\omega). \quad (5.13)
\end{align*}$$

Let $0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1}$ be the real, non-negative, distinct finite zeros of $q_{l}(\omega)$ with odd multiplicities. Also define $\omega_m = \infty$. First let us assume that $\delta(s)$ is of even degree. Then from the inductive assumption, we have
\[
\sigma(\delta_k) = \{ \text{sgn}[p_k(\omega_0)] - 2 \text{sgn}[p_k(\omega_1)] + 2 \text{sgn}[p_k(\omega_2)] + \cdots + (-1)^{m-1} \\
\times 2 \text{sgn}[p_k(\omega_{m-1})] + (-1)^m \text{sgn}[p_k(\omega_m)] \}
\times (-1)^{m-1} \text{sgn}[q_k(\infty)].
\] (5.14)

Now from (5.13), it follows that \(\omega_i, i = 0, 1, \ldots, m - 1\); \(\alpha_{k+1}\) are the real, non-negative, distinct finite zeros of \(q_{k+1}(\omega)\) with odd multiplicities. Let us assume that \(\omega_i < \alpha_{k+1} < \omega_{i+1}\). Then from (5.12) and (5.13), we have
\[
\begin{align*}
\text{sgn}[p_k(\omega_i)] &= \text{sgn}[p_{k+1}(\omega_i)], \quad i = 0, 1, \ldots, l, \\
\text{sgn}[p_k(\omega_i)] &= -\text{sgn}[p_{k+1}(\omega_i)], \quad i = l + 1, l + 2, \ldots, m, \\
\text{sgn}[q_{k+1}(\alpha_{k+1})] &= 0, \\
\text{sgn}[q_k(\infty)] &= -\text{sgn}[q_{k+1}(\infty)].
\end{align*}
\] (5.15)

Since \(\sigma(\delta_{k+1}) = \sigma(\delta_k)\), using (5.14) and (5.15), we obtain
\[
\sigma(\delta_{k+1}) = \{ \text{sgn}[p_{k+1}(\omega_0)] - 2 \text{sgn}[p_{k+1}(\omega_1)] + 2 \text{sgn}[p_{k+1}(\omega_2)] \\
+ \cdots + (-1)^{l+1} 2 \text{sgn}[p_{k+1}(\omega_{l+1})] + (-1)^l \\
\times 2 \text{sgn}[p_{k+1}(\alpha_{k+1})] + (-1)^{l+2} 2 \text{sgn}[p_{k+1}(\omega_{l+1})] + \cdots + (-1)^m \\
\times 2 \text{sgn}[p_{k+1}(\omega_{m-1})] + (-1)^{m+1} \\
\times \text{sgn}[p_{k+1}(\omega_m)] \} \cdot (-1)^m \text{sgn}[q_{k+1}(\infty)],
\]
which shows that the first expression of (5.1) is true for \(\delta_{k+1}(s)\) of even degree. The second expression of (5.1), corresponding to \(\delta_{k+1}(s)\) of odd degree, can be verified by proceeding along exactly the same lines. This completes the induction argument and hence the proof. \(\square\)

**Theorem 5.2.** Let \(\delta(s)\) be a given real polynomial of degree \(n\) with no roots at the origin. Let \(0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1}\) be the real, non-negative, distinct finite zeros of \(p_f(\omega)\) with odd multiplicities. Also define \(\omega_m = \infty\). Then
\[
\sigma(\delta) = \begin{cases} 
-\{2 \text{sgn}[q_f(\omega_1)] - 2 \text{sgn}[q_f(\omega_2)] + \cdots + (-1)^{m-2} \\
\times 2 \text{sgn}[q_f(\omega_{m-1})] \} (-1)^m \\
\times \text{sgn}[p(\infty)] \quad \text{if } n \text{ is even}, \\
-\{2 \text{sgn}[q_f(\omega_1)] - 2 \text{sgn}[q_f(\omega_2)] + \cdots + (-1)^{m-2} \\
\times 2 \text{sgn}[q_f(\omega_{m-1})] + (-1)^{m-1} \text{sgn}[q_f(\omega_m)] \} \cdot (-1)^m \\
\times \text{sgn}[p(\infty)] \quad \text{if } n \text{ is odd}.
\end{cases}
\] (5.16)
6. The generalized Hermite–Biehler theorem: no restriction on root locations

Theorems 5.1 and 5.2 presented in the last section require that the polynomial $d_s^+$ have no roots at the origin. In this section, we provide a refinement of Theorem 5.1 whereby the presence of roots of $d_s^+$ at the origin can be handled.

**Theorem 6.1.** Let $d_s^+$ be a given real polynomial of degree $n$ with a root at the origin of multiplicity $k$. Let $0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1}$ be the real, positive, distinct finite zeros of $q_f(\omega)$ with odd multiplicities. Also define $\omega_0 = 0$, $\omega_m = \infty$ and denote $p^{(k)}(\omega_0) = (d^k/d\omega^k)p(\omega)|_{\omega=\omega_0}$. Then

$$
\sigma(\delta) = \begin{cases} 
\{\text{sgn}[p^{(k)}(\omega_0)] - 2 \text{sgn}[p_f(\omega_1)] + 2 \text{sgn}[p_f(\omega_2)] + \cdots + (-1)^{m-1} \times 2 \text{sgn}[p_f(\omega_{m-1})] + (-1)^m \text{sgn}[p_f(\omega_m)]\} \cdot (-1)^{m-1} \times \text{sgn}[q(\infty)] & \text{if } n \text{ is even}, \\
\{\text{sgn}[p^{(k)}(\omega_0)] + 2 \text{sgn}[p_f(\omega_1)] + 2 \text{sgn}[p_f(\omega_2)] + \cdots + (-1)^{m-1} \times 2 \text{sgn}[p_f(\omega_{m-1})]\} \cdot (-1)^{m-1} \times \text{sgn}[q(\infty)] & \text{if } n \text{ is odd}.
\end{cases}
$$

(6.1)

**Proof.** Since $\delta(s)$ has a root of multiplicity $k$ at the origin, we can write

$$
\delta(s) = s^k \delta'(s),
$$

where $\delta'(s)$ is a real polynomial of degree $n'$ with no roots at the origin. Define

$$
\delta'(j\omega) = p'(\omega) + jq'(\omega), \quad \text{and} \quad \delta(j\omega) = p(\omega) + jq(\omega).
$$

The proof can be completed by considering four different cases, namely $k = 4l$, $k = 4l + 1$, $k = 4l + 2$ and $k = 4l + 3$. Due to space limitations, and the fact that each of these cases is handled by proceeding along similar lines, we do not treat all of the cases here. Instead, we focus on a representative case, say $k = 4l + 1$, and provide a detailed treatment for it.

Now, for $k = 4l + 1$, we have

$$
\delta(j\omega) = p(\omega) + jq(\omega),
$$

$$
= -\omega^{4l+1} q'(\omega) + jq^{4l+1} p'(\omega).
$$

First let us assume that $n'$ is even. Then, from Theorem 5.2, we have
\[
\sigma(\delta') = - \{2 \text{sgn}[q'_f(\omega_1)] - 2 \text{sgn}[q'_f(\omega_2)] \\
+ \cdots + (-1)^{m-2} \text{sgn}[q'_f(\omega_{m-1})]\} \cdot (-1)^{m-1} \text{sgn}[p'(\infty)],
\]

(6.2)

where \(0 < \omega_1 < \omega_2 < \cdots < \omega_{m-1}\) are the real, non-negative, distinct finite zeros of \(p'_f(\omega)\) with odd multiplicities.

Define \(\omega_0 := 0\).

Since
\[p(\omega) = -\omega^{4l+1}q'(\omega), \quad p^{(4l+1)}(\omega_0) = -(4l + 1)!q'(\omega_0) = 0\]
and
\[q(\omega) = \omega^{4l+1}p'(\omega),\]
we have
\[
\text{sgn}[p^{(4l+1)}(\omega_0)] = 0, \quad (6.3)
\]
\[
\text{sgn}[q'_f(\omega_i)] = -\text{sgn}[p'_f(\omega_i)], \quad i = 1, 2, \ldots, m, \quad (6.4)
\]
\[
\text{sgn}[p'(\infty)] = \text{sgn}[q(\infty)]. \quad (6.5)
\]

Since \(n'\) is even and \(k = 4l + 1\), it follows that \(n\) is odd. Moreover, since \(\sigma(\delta) = \sigma(\delta')\), using (6.2)–(6.5), we have
\[
\sigma(\delta) = \{\text{sgn}[p^{(k)}(\omega_0)] - 2 \text{sgn}[p'_f(\omega_1)] + 2 \text{sgn}[p'_f(\omega_2)] \\
+ \cdots + (-1)^{m-1} 2 \text{sgn}[p'_f(\omega_{m-1})]\} \cdot (-1)^{m-1} \text{sgn}[q(\infty)],
\]

(6.6)

which shows that the second expression of (6.1) holds for \(\delta(s)\) of odd degree. The first expression of (6.1), corresponding to \(\delta(s)\) of even degree or equivalently \(n'\) odd, can be verified by proceeding along exactly the same lines.

We conclude this section by presenting the following example to verify Theorem 6.1.

**Example 6.1.** Consider the real polynomial
\[
\delta(s) = s^3(s^2 + 1)^2(s^2 + 5)(s - 3)(s^2 + s + 1).
\]
Substituting \(s = j\omega\), we have \(\delta(j\omega) = p(\omega) + jq(\omega)\), where
\[p(\omega) = \omega^{12} - 5\omega^{10} - 3\omega^8 + 17\omega^6 - 10\omega^4\]
and
\[q(\omega) = 2\omega^{11} - 17\omega^9 + 43\omega^7 - 43\omega^5 + 15\omega^3.\]

The real, positive finite zeros of \(q_f(\omega)\) with odd multiplicities are \(\omega_1 = 1.22474\) and \(\omega_2 = \sqrt{5}\). Also define \(\omega_0 = 0\) and \(\omega_3 = \infty\). Hence, \(\text{sgn}[p^{(3)}(\omega_0)] = 0,\)
sgn\[p_f(\omega_1)] = -1, \ sgn[p_f(\omega_2)] = 0, \text{ and } \ sgn[p_f(\omega_3)] = 1. \text{ Since } \delta(s) \text{ is of even degree and with a root at the origin of multiplicity 3, from formula (6.1), it follows that}

\[
\sigma(\delta) = \{\text{sgn}[p^{(3)}(\omega_0)] - 2\text{sgn}[p_f(\omega_1)] + 2\text{sgn}[p_f(\omega_2)] - \text{sgn}[p_f(\omega_3)]\} \cdot (-1)^2\text{sgn}[q(\infty)] = 0 + 2 + 0 - 1 = 1.
\]

This agrees with the value obtained from visual inspection of the factored form of \(\delta(s)\), so that Theorem 6.1 is verified.

7. Concluding remarks

In this paper, we have presented generalizations of the Hermite–Biehler Theorem applicable to not necessarily Hurwitz polynomials. These results are not of mere academic interest and can be used for solving important stabilization problems in control theory. Indeed, a special case of these results has been successfully used in [8,9] to obtain new results on P, PI and PID stabilization.

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References


[9] M.T. Ho, A. Datta, S.P. Bhattacharyya, A new approach to feedback design part II: PI and PID controllers, Department of Electrical Engineering, Texas A & M University, College Station, TX, Tech. Report TAMU-ECE97-001-B.