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Small cutsets in quasiminimal Cayley graphs

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Abstract

We continue the recent study carried out by several authors on the cut sets in Cayley graphs with respect to quasiminimal generating sets. We improve the known results on these questions.

The application of our main theorem to symmetric Cayley graphs on minimal generating sets leads to the following result.

Let G be a group containing a minimal generating set M such that $|M| \geqslant 4$. Let $S = M \cup M^{-1}$. Then one of the following conditions holds.

(i) $s^2 = u^2$ and $u^4 = 1$, for all $s, u \in M$

(ii) For all $(d+1)$-subsets A and B of G which are not of the form $\Gamma(x) \cup \{x\}$ for any $x \in G$, there exists $d+1$ disjoint paths from A to B in $Cay(G,S)$.

1. Introduction

Consider a connected regular directed graph X with outdegree d. This means that every vertex dominates d distict vertices. There are sets of vertices having cardinality d whose failure breaks the connectedness of the graph, namely the vertices dominated by a given vertex. If every disconnecting set has cardinality at least d, the graph is said to be maximally connected. This property is important because of its connection with the reliability of networks modeled by graphs. However, it can be improved.

The idea of superconnectivity, introduced first by Boesch and Tindell [4] for undirected graphs and generalized to the directed case by Fàbrega and Fiol [5], selects more efficient models since it minimizes the disconnecting d-subsets. More precisely a graph is superconnected if every disconnecting set has cardinality at least $d+1$, unless it consists of the vertices dominating or dominated by some vertex.

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The Vosper's property considered first in connection with graphs by Hamidoune and Tindell is even more selective. The original definition requires the notion of fragments and will be given in the next section.

By Menger's Theorem a graph is maximally connected if and only if for any two d-subsets $A, B \subset V$, there are d vertex-disjoint (directed) paths from A to B. Vosper's property can also be stated in terms of the following stronger version of Menger's theorem which shows its importance for graph reliability [10].

A maximally connected graph has the Vosper's property if and only if for any two $(d+1)$-subsets $A, B \subset V$ one of the following conditions holds.

(i) There is a vertex x such that either A or B consists of $\{x\}$ and the set of vertices dominating x or dominated by x respectively.

(ii) There are $d+1$ vertex-disjoint (directed) paths from A to B.

Notice that conditions (i) and (ii) cannot hold together.

The connectivity problems on Cayley graphs have been widely studied in Combinatorics. Some results obtained by number theorists may also be translated to give connectivity results [10]. Although we cannot review all the results, let us mention few of them connected to the present work.

In [14], Imrich proved a conjecture of Balinski and Russakov relative to the connectivity of some Cayley graphs on the symmetric groups. In [6], Godsil proved that the connectivity of an undirected Cayley graph defined by a minimal generating set is maximal and this result is generalized to directed graphs in [8].

A subset S of a graph G will be called *quasiminimal* if there is a total ordering '$<$' on S such that for all $x \in S$, $x \notin \langle y;\ y < x \rangle$. Such an ordering is called *hierarchical.*

Akers and Krishnamurty proved in [1] that Cayley graphs on quasiminimal generating sets of transpositions in the symmetric group are also maximally connected.

Independently Alspach [2] and the authors [13] obtained the following result, proved also by Baumslag in a slightly less general form [3].

Theorem 1.1 Alspach ([2]; Hamidoune et al. [12]). *Let G be a group with a generating set $M \subset G \backslash \{1\}$ admiting a hierarchical ordering with u as a first element. Let $X = Cay(G, S)$, where $S \subset M \cup M^{-1}$. Then exactly one of the following conditions holds.*

(i) $\kappa(X) = |S|$.

(ii) $|S| \geqslant 3$, $S = M \cup M^{-1}$, $u^2 = 1$ *and* $s^2 = u$ *for all* $s \in S \backslash \{u\}$.

The superconnectivity problems in Cayley graphs are more recent. Boesch and Tindell mentioned in [4] the difficulty of the characterization of superconnected loop graphs.

The first break on this problem came from Additive group theory. In an unpublished report, Hamidoune and Tindell [10] observed that an additive theorem of Vosper [15] implies that loop graphs of a prime order not defined by an arithmetic progression are Vosperian.

The application of an additive theorem of Kempermann allowed a characterization of Vosperian and superconnected Cayley graphs on abelian groups, obtained by the authors in [12].

More recently, a theory of superatoms introduced in [11] can be used, among other applications, to calculate all the fragments and minimum cutsets in abelian Cayley graphs. It is also used to give easier characterizations for superconnected and Vosperian Cayley graphs.

In this paper we study the superconnectivity and the Vosperianity of Cayley graphs with respect to quasiminimal generating sets. We show here that these graphs are superconnected and verify the stronger Vosper property with few exceptions.

We shall use in our proof several known results summarized in the next section.

2. Vulnerability theory

All graphs we consider are assumed to be directed and without loops and multiple arcs. We identify undirected with symmetric graphs.

For the definitions given briefly, the reader may refer to [9, 12].

Let $X = (V, E)$, be a graph and let $F \subset V$. The *inverse* graph of X is $X^{-1} = (V, E^{-1})$, where $E^{-1} = \{(x,y) | (y,x) \in E\}$. The subgraph induced by F is $X[F] = (F, E \cap (F \times F))$. We write $\Gamma F = (y \in V | (x,y) \in E$ for some $x \in F\}$, $\partial F = \Gamma F \setminus F$ and $\delta F = V \setminus (F \cup \Gamma F)$. The last three sets calculated with respect to X^{-1} will be denoted, respectively, by $\Gamma^{-1}F$, $\partial^{-}F$ and $\delta^{-}F$. We write $\partial_X F$ when the reference to the graph X has to be made explicit. The *degree* of a vertex x is by definition $d(x) = |\Gamma\{x\}|$. If this quantity is independent of x, the graph is said to be regular. The common degree will be denoted by $d(X)$.

The *connectivity* of X is

$$\kappa(X) = \min\{|\partial F| \; 1 \leqslant |F \cup \partial F| \leqslant |V| - 1 \text{ or } |F| = 1\}.$$

The graph X is said to be *connected* if $\kappa(X) > 0$ and maximally connected if $\kappa(X) = \min\{d(x)|, x \in V\}$. If X is connected and $X[V \setminus T]$ is not, then T is a *disconnecting* set of X.

If X is a d-regular graph we have clearly $0 \leqslant \kappa(X) \leqslant d$.

A subset F of V such that $F \cup \partial F \neq V$ and $|\partial F| = \kappa(X)$ is called a *fragment* of X. A fragment of X^{-1} is called a *negative* fragment of X. A fragment of X with minimum cardinality is called an *atom* of X. The cardinality of the atoms of X is denoted by $\mu(X)$.

The following lemma contains a useful duality between fragments. It is implicit in [7].

Lemma 2.1 (Hamidoune [7]). (i) $\kappa(X) = \kappa(X^{-1})$.

(ii) *If F is a fragment then δF is a negative fragment and $\delta^{-}(\delta F) = F$.*

Let us introduce the other basic definitions. A regular graph X with degree d is said to be *superconnected* if every disconnecting set T has cardinality at least $d+1$ unless $T = \partial(x)$ or $T = \partial^-(x)$ for some $x \in V$.

A regular graph X with degree d is said to be *Vosperian* if every fragment has cardinality either 1 or $|V| - d - 1$.

More details on these notions can be found in [10–12]. Notice that a vosperian graph is superconnected but the converse is not always true.

Let us give the characterization of vosperian graphs mentioned in the introduction. We include a proof of it for the convenience of the reader.

Lemma 2.2 (Hamidoune [11]). *Let X be a regular graph such that $\kappa(X) = d(X)$. The following conditions are equivalent.*

(i) *X is a vosperian.*

(ii) *For all $(d+1)$-subsets A and B of V such that for all $x \in V$, $A \neq \partial(x) \cup \{x\}$ and $B \neq \partial^-(x) \cup \{x\}$, there exist $d+1$ disjoint paths from A to B.*

Proof. The implication (ii) ⇒ (i) is straightforward. We only prove the (i) ⇒ (ii) part.

Add two different vertices a, b to X and connect a to all the vertices in A and connect all the vertices of B to b. Denote the resulting graph by Y. The result will follow using Menger's Theorem after proving the following statement.

Let $F \subset V \cup \{a\}$ such that $a \in F$ and $b \notin F \cup \partial_Y F$. Then $|\partial_Y F| \geqslant d+1$.

Assume the contrary. Clearly, $F_0 = F \setminus \{a\} \neq \phi$. and $F_0 \cup \partial F_0 \neq V$. By the definition of the connectivity we have $|\partial F_0| \geqslant d$. Moreover, $|\partial F_0| \leqslant |\partial_Y F| \leqslant d$. Therefore, F_0 is a fragment. Since X is vosperian, we have either $|F_0| = 1$ or $|F_0| = |V| - d - 1$. Consider the first case, the other one follows by duality. Put $F_0 = \{c\}$. We have $|\partial F| = |\partial(c) \cup (A \setminus \{c\})| \geqslant d+1$ since otherwise $A \subset \{c\} \cup \partial(c)$, a contradiction. □

A fragment F is said to be *proper* if $|F| \leqslant |\delta F|$. The following Lemma is straightforward.

Lemma 2.3. *Let X be a regular graph which is nonsuperconnected. Then either X or X^{-1} contain a connected proper fragment F with $|F| > 1$.*

Note that a noncomplete graph X is maximally connected if and only if $|A| = 1$ for an atom A of X. Details from these definitions can be found in [7]. A graph is *vertex transitive* when its automorphism group acts transitively on the set of vertices.

We use the usual notions of elementary group theory. All groups are assumed to be finite. Let A, B be two subsets of a group G. We write

$$AB = \{xy \colon x \in A \text{ and } y \in B\}.$$

Let G be a group containing a subset S. The Cayley graph on G associated to S is by definition

$$Cay(G,S) = (G, \{(x, xs) \mid x \in G \text{ and } s \in S\}).$$

Observe that, for every subset $F \subset G$, $\Gamma F = FS$. We denote by $\langle S \rangle$ the subgroup of G generated by S. $Cay(G,S)$ is connected whenever $\langle S \rangle = G$. Cayley graphs are vertex transitive, the set of left translations being a transitive automorphism group of the graph.

The atoms of Cayley graphs have the following property.

Theorem 2.4 (Hamidoune [9]). *Let G be a group, $S \subset G$ and $X = Cay(G,S)$. Assume that $\mu(X) \leqslant \mu(X^{-1})$. Then there is a unique subgroup H generated by $H \cap S$ which is an atom of $Cay(G,S)$. Moreover, every fragment of X is a union of atoms.*

We use also the following corollary to this result contained in [2, 13]. We include a short proof, based on the above theorem, for the convenience of the reader.

Lemma 2.5 (Alspach [2]; Hamidoune et al. [13]). *Let G be a group with a generating set $M \subset G \setminus \{1\}$ with $|M| \leqslant 3$ admiting a hierarchical ordering with u as a first element. Let $X = Cay(G,S)$, where $S \subset M \cup M^{-1}$.*

If $\kappa(X) < |S|$, then $|M| = 3$ and $S = M \cup M^{-1}$ and $u^2 = 1$ and $s^2 = u$ for all $s \in M \setminus \{u\}$. Moreover $\{1, u\}$ is the unique atom of $X = X^{-1}$ containing 1.

Proof. Since M^{-1} is also quasiminimal, we may assume without loss of generality that $\mu(X) \leqslant \mu(X^{-1})$. By Theorem 2.4, there is an atom A which is the subgroup generated by $A \cap S \neq \emptyset$.

If $Av = Aw$ for $v, w \in S \setminus A$, by the quasiminimality of M we must have $v = w^{-1}$ and then $v^2 \in A$. In particular, $\kappa(X) = |M \setminus A| \cdot |A|$. Since $\kappa(X) \leqslant 5$, then $|M \setminus A| \leqslant 2$.

If $|M \setminus A| = 1$. then $G = A \cup Av$, $\{v\} = M \setminus A$, and A is not a fragment.

Therefore $|M \setminus A| = 2$ and $|A| = 2$ and $|S| = 5$. Let $A = \{1, x\}$, where $x^2 = 1$, and let $S = \{x, v, v^{-1}, w, w^{-1}\}$. We must have $Av = Av^{-1}$ and $Aw = Aw^{-1}$, which implies $v^2 = w^2 = x$. In particular, $x = u$, the first element of the hierarchical ordering. □

A basic tool for our method is the following result proved independently by Hamidoune [7] in the Cayley graphs language and Olson [19] in the additive language.

Theorem 2.6 (Hamidoune [8]; Olson [19]). *Let G be a group with a generating set $S \subset G \setminus \{1\}$. Let $X = Cay(G,S)$. Then $\kappa(X) > |S|/2$.*

We use only the following easy consequence of Theorem 2.6.

Corollary 2.7. *Let G be a group with a subset F and a subgroup H such that $FH \neq G$. Let $s \in G \backslash H$ be such that $H \cup Hs$ generates G.*
Then $|FHs \backslash FH| \geqslant |H|$.

Proof. If $FHs \cup FH = G$ then $|FHs \backslash FH| = |G| - |FH| \geqslant |H|$, since $|H|$ divides both $|G|$ and $|FH|$. Otherwise apply Theorem 2.6 to $Cay(G,(H \backslash \{1\} \cup Hs))$ □

3. A reduction method

Let G be a group and let H be a subgroup and let $s \in G \backslash H$ such that $G = \langle H \cup \{s\} \rangle$. Let S_0 be a generating subset of H and let $S \subset S_0 \cup \{s, s^{-1}\}$. Set $X = Cay(G,S)$ and $X_0 = Cay(H, S_0)$.

We shall say that X_0 is a *faithful* factor of X if the following conditions are satisfied:

(i) The smaller atom of X_0 and X_0^{-1} has cardinality at most 2.
(ii) $\kappa(X_0) = |S_0|$ if $|S_0| \leqslant 4$.
(iii) $|H| \geqslant 2|S| - \kappa(X_0) + 1$.

The following proposition characterizes the proper fragments of Cayley graphs with faithful factors.

Proposition 3.1. *Let G be a group, H a subgroup and $s \in G \backslash H$ such that $G = \langle H \cup \{s\} \rangle$. Let S_0 be a generating subset of H with $|S_0| \geqslant 3$ and let $S \subset S_0 \cup \{s, s^{-1}\}$. Set $X = Cay(G,S)$ and $X_0 = Cay(H, S_0)$.*

Assume that X_0 is faithful factor of X. Let Q be a proper fragment of X containing 1 and $q \in Q \backslash \{1\}$. Then $Q = \{1, q\} \subset H$ and $|\partial Q| = |\partial Q \cap H| + 2$.

Moreover if $|S \backslash S_0| = 2$ then $s^2 = q$ and $q^2 = 1$.

Proof. Let F be a fragment of X and set $T = \partial F$, so that G is the disjoint union $G = F \cup T \cup \delta F$.

We shall prove the following statement.

$$\textit{There is a unique left coset } xH \textit{ such that } xH \cap F \neq \emptyset \textit{ and } xH \cap \delta(F) \neq \emptyset. \tag{1}$$

We first prove that $FH \cap \delta F \neq \emptyset$. Suppose the contrary. Therefore we would have $FH = F \cup (T \cap FH) \neq G$ By Corollary 2.7, we have

$$|H| \leqslant |FHs \backslash FH| = |Fs \backslash FH| + |(T \cap FH)s \backslash FH| \leqslant |T \backslash FH| + |FH \cap T|$$
$$= |T| \leqslant |S|.$$

This inequality contradicts the hypothesis of the proposition. This contradiction proves the existence in (1).

Since the graph induced on xH is isomorphic to $Cay(H, S_0)$, we have

$$|xH \cap \partial F| \geqslant \kappa(X_0). \tag{2}$$

If there was another coset $yH \neq xH$ satisfying (2), by adding the inequalities in (2) we would have

$$2 + |S_0| \geqslant |S| \geqslant |\partial F| \geqslant 2\kappa(X_0) \geqslant 2|S_0| - 2.$$

Therefore, $|S_0| \leqslant 4$, but in this case $\kappa(X_0) = |S_0|$. By substituting the inequality in the above one, we get $|S_0| \leqslant 2$, against the hypothesis.

Therefore

$$\textit{for all } y \notin xH \textit{ either } yH \subset F \cup \partial F \textit{ or } yH \subset \delta(F) \cup \partial(F) \tag{3}$$

This proves the uniqueness part of (1). We prove now the following stronger statement.

$$F \subset xH \textit{ or } \delta(F) \subset xH \tag{4}$$

Assume that $F \not\subset xH$ and $\delta F \not\subset xH$. Set $F_1 = F \cap xH$ and $F_2 = F \setminus xH$. Set $\delta F_1 = \delta F \cap xH$ and $\delta F_2 = \delta F \setminus xH$. Put $T_1 = T \cap xH$ and $T_2 = T \cap F_2 H$ and $T_2' = T \cap \delta F_2 H$.

$$(F_2 \cup T_2)H = F_2 \cup T_2 \textit{ and } (\delta F_2 \cup T_2')H = \delta F_2 \cup T_2' \tag{5}$$

Since $F_2 = \emptyset$, we have using(4) and Corollary 2.7,

$$|(F_2 \cup T_2)s \setminus (F_2 \cup T_2)| \geqslant |H|.$$

Therefore $|F_2 s \setminus (F_2 \cup T_2)| \geqslant |H| - |T_2|$. As $F_2 s \setminus (F_2 \cup T_2) \subset F_1 \cup T_1 \cup T_2'$, then $|F_1 \cup T_1 \cup T_2'| \geqslant |H| - |T_2|$. Hence, $|F_1| + |T| \geqslant |H|$. Similarly, $|\delta F_1| + |T| \geqslant |H|$. By adding these two equations we get $|T| + |T_2| + |T_3| \geqslant |H|$. By (2), $|T_2| + |T_3| \leqslant |S| - \kappa(X_0)$. It follows that $2|S| - \kappa(X_0) \geqslant |H|$, contradicting the definition of a faithful factor. This contradiction proves (4).

Let us now prove the proposition. By (1), there is a unique coset xH such that $xH \cap Q \neq \emptyset$ and $xH \cap \delta Q \neq \emptyset$. By (4), either $Q \subset xH$ or $\delta Q \subset xH$.

Assume that $Q \not\subset xH$. We then have $\delta Q \subset xH$. By (2),

$$|\delta(Q)| \leqslant |H| - (\kappa(X_0) + 1) \leqslant |H| - (\kappa(X) - \kappa(X_0) \leqslant |Q \setminus xH| < |Q|,$$

contradicting the assumption that Q is proper. Therefore, $Q \subset xH$. Since $1 \in Q$, we have $xH = H$. In particular $q \in H$.

Since the atoms of X_0 have cardinality at most 2, we have $\kappa(X_0) \geqslant |S_0| - 1$. Therefore, using (2),

$$|Q(S \setminus S_0)| = |\partial Q \setminus H| \leqslant |S| - (|S_0| - 1) \leqslant 3, \tag{6}$$

and so $|Q(S \setminus S_0)| \leqslant 3$. If the equality holds, $|\partial Q \cap H| = |S_0| - 1$ and Q is a fragment of X_0. By the Theorem 2.4, Q must be a disjoint union of atoms which have then cardinality 2. In any case we have $Q = \{1, q\}$. In particular,

$$2 \leqslant |Q(S \setminus S_0)| = |S| - |\partial Q \cap H|. \tag{7}$$

If $|S \backslash S_0| = 1$, (7) implies that

$$|\partial Q| = |\partial Q \cap H| + 2. \tag{8}$$

Assume $|S \backslash S_0| = 2$. In particular $s^2 \neq 1$. We have $Q\{s, s^{-1}\} \supset \{s, s^{-1}, qs, qs^{-1}\}$. The assumption $s^2 \neq q$ shows that the four elements are distinct, contradicting (6). Similarly for $(s^{-1})^2 \neq q$ and hence $q^2 = 1$. Now we have $Qs = Qs^{-1}$ and $|Q\{s, s^{-1}\}| = 2$. The relation (8) follows by (7). □

We are now able to deduce the result about the connectivity of Cayley graphs with respect to a quasiminimal generating set. We give the proof in order to illustrate the general method in a simpler case and to keep the paper self-contained, as much as possible. We begin with the following lemma.

Lemma 3.2. *Let M be a quasiminimal generating set of G with $|M| \geqslant 4$. Let m be the maximal element of M under a hierarchical order and let $H = \langle M \backslash m \rangle$. Let $S \subset M \cup M^{-1}$ and $S_0 = S \backslash \{m, m^{-1}\}$. Set $X_0 = Cay(H, S_0)$.*

If $\kappa(X_0) \geqslant |S_0| - 1$, then $|H| \geqslant 2|S| - \kappa(X_0) + 1$.

Proof. Set $b = |M|$. We have clearly $2^{b-1} \leqslant |H|$ and $|S| \leqslant 2b$. Therefore, the stronger inequality

$$|H| \geqslant |S| + 4$$

is satisfied for $b \geqslant 5$.

Consider the case $b = 4$. If there is some $s \in S_0$ such that $s^2 \neq 1$, then $|H| \geqslant 12 \geqslant |S| + 4$. Otherwise $s^2 = 1$, for all $s \in S_0$. It follows that $|S| \leqslant 5$. By Lemma 2.5, we have $\kappa(X_0) = 3$. Hence $|H| \geqslant 8 \geqslant 2|S| - 3 + 1$. □

This was the first step to prove that large factors in quasiminimal Cayley graphs are faithful.

Corollary 3.3 (Alspach [2]; Hamidoune et al. [13]). *Let G be a group with a generating set $M \subset G \backslash \{1\}$ admitting a hierarchical ordering with u as a first element. Let $X = Cay(G, S)$, where $S \subset M \cup M^{-1}$*

Then $\kappa(X) \geqslant |S| - 1$. Moreover, the inequality is strict unless $|S| = 2|M| - 1 \geqslant 5$ and $u^2 = 1$ and for all $m \in M \backslash \{u\}$, $m^2 = u$. In this case $\{1, u\}$ is an atom of X.

Proof. The proof is by induction on $|M|$, by Lemma 2.5 the result holds for $|M| \leqslant 3$. Suppose now that $\kappa(X) \leqslant |S| - 1$. Clearly, one of the atoms of$X$ and X^{-1} is a proper fragment. We may assume without loss of generality that X has a proper fragment Q. Choose Q with $1 \in Q$ and let $q \in Q \backslash \{1\}$.

Let m be the maximal element of M and let $S_0 = S \backslash \{m, m^{-1}\}$ and let $H = \langle M \backslash m \rangle$. Set $X_0 = Cay(H, S_0)$.

By the induction hypothesis $\kappa(X_0) \geqslant |S_0| - 1$. By Lemma 3.2, X_0 is faithful. By Proposition 3.1 we have $Q = \{1, q\} \subset H$ and $|S \backslash S_0| = 2$ and $m^2 = q$.

It follows also that Q is a fragment and hence an atom of X. The proof follows now by induction and the uniqueness of the atom. □

Corollary 3.4. *Let M be a quasi minimal generating set of G with $|M| \geqslant 4$. Let m be the maximal element of M under a hierarchical ordering and let $H = \langle M \backslash m \rangle$. Let $S \subset M \cup M^{-1}$ and $S_0 = S \backslash \{m, m^{-1}\}$. Set $X_0 = Cay(H, S_0)$.*

Then X_0 is a faithful factor of X.

Proof. The result follows clearly from Lemma 3.2 and Corollary 3.3. □

4. The superconnectivity

The following lemma contains an argument which will appear in the proofs of Theorems 4.2 and 5.1.

Lemma 4.1. *Let G be a group containing a quasiminimal generating set M such that $|M| \geqslant 4$. Let $M \subset S \subset M \cup M^{-1}$.*

Assume that $\kappa(Cay(G,S)) = |S|$. If $Cay(G,S)$ contains a connected proper fragment Q with cardinality $\geqslant 2$, then there is a hierarchical ordering of M with a minimal element u and $v \in M \backslash \{u\}$ such that the following conditions holds

(i) *$Q = \{1, u\}$ and $u^2 = 1$.*
(ii) *$|S \backslash \{v\}| = 2(|M| - 1)$ and $s^2 = u$ for all $s \in S \backslash \{u, v\}$.*

Proof. Let $m \in M$ *be such that* $H = \langle M \backslash \{m\} \rangle$ is a proper subgroup of G and let $S_0 = S \backslash \{m, m^{-1}\}$. Set $X_0 = Cay(H, S_0)$. By Corollary 3.4, X_0 is faithful.

Assume first that $|S \cap \{m, m^{-1}\}| = 1$. By Proposition 3.1, $|\partial Q \cap H| = |S| - 2 = |S_0| - 1$. It follows that $\kappa(X_0) \leqslant |S_0| - 1$. Then, by Corollary 3.3, $S_0 = S_0^{-1}$ and $s^2 = u$ for all $s \in S \backslash \{u, m\}$ and $u^2 = 1$ and the theorem holds with $v = m$.

Assume now that $|S \cap \{m, m^{-1}\}| = 2$. By Proposition 3.1, $Q = \{1, q\} \subset H$ and $m^2 = q$ and $q^2 = 1$. Moreover, Q is a proper connected fragment of X_0 and $\kappa(X_0) = |S| - 2 = |S_0|$. The result follows now by induction for $|M| > 4$. Let us now consider the case $|M| = 4$. Let $u < v < w < m$ be the 4 elements of M in a hierarchical order. Since Q is connected we have $q \in \{u, v, w\}$ and $|S_0| \leqslant 5$. Since $\partial Q \cap H = QS_0 \backslash Q$, which is a union of Q-cosets, we have $|S_0| = 4$. It follows that there is $x \in S_0 \backslash Q$ such that $|S_0 \cap \{x, x^{-1}\}| = 1$.

We can assume, by reordering if necessary, that $q = u$ and $x = v$. Now $QS_0 \backslash Q = Qv \cup Qw \cup Qw^{-1}$. It follows $Qw = Qw^{-1}$ and $w^2 = u$. This completes the proof. □

Theorem 4.2. *Let G be a group containing a quasiminimal generating set M such that $|M| \geqslant 4$. Let $S \subset M \cup M^{-1}$.*

Assume that $\kappa(Cay(G,S)) = |S|$. Then $Cay(G,S)$ is superconnected unless there is a hierarchical reordering of M with a minimal element u and $v \in M\backslash\{u\}$ such that $|S\backslash\{v\}| = 2(|M|-1)$ and $s^2 = u$ for all $s \in S\backslash\{u,v\}$ and $u^2 = 1$.

Proof. Suppose that X is not superconnected. We may assume without loss of generality using Lemma 2.4 that X contains a connected proper fragment Q with cardinality >1. The result follows now by Lemma 4.1.

On the other hand, if the relations of the theorem hold, then clearly $Q = \{1,u\}$ is a proper connected fragment and X is non superconnected. □

Let us express now this result in the symmetric case (undirected graphs) which is the most studied.

Corollary 4.3. *Let G be a group containing a quasiminimal generating set M such that $|M| \geqslant 4$. Let $S = M \cup M^{-1}$ be such that $\kappa(Cay(G,S)) = |S|$. Then either $Cay(G,S)$ is superconnected or there is a hierarchical reordering of M with a minimal element u and $v \in M\backslash\{u\}$ such that $s^2 = u$ for all $s \in S\backslash\{u,v\}$ and $v^2 = u^2 = 1$.* □

5. The Vosper's property

Proposition 3.1 implies clearly that a quasiminimal Cayley graph with odd order is Vosperian since there cannot be a self-inverse element. It implies also a result proved by Lladó [16] saying that any anti-symmetric quasiminimal Cayley graph is Vosperian. One can use it to calculate the relators that avoid the Vosper's property.

For simplicity we shall solve the problem only in the symmetric case, equivalent to the undirected case.

Theorem 5.1. *Let G be a group containing a quasiminimal generatings set M such that $|M| \geqslant 4$. Let $S = M \cup M^{-1}$ be such that $\kappa(Cay(G,S)) = |S|$. Then $Cay(G,S)$ is Vosperian unless there is a hierarchical ordering of M with three smallest elements $u < v < w$ such that one of the following conditions is satisfied.*

(i) *There is $v > u$ with $v^2 = u^2 = 1$ and $s^2 = u$ for all $s \in S\backslash\{u,v\}$.*
(ii) *$s^2 = u^2$ for all $s \in M$ and $u^4 = 1$.*
(iii) *$s^2 = uv^{-1} = v^{-1}u$, for all $s \in M\backslash\{u,v\}$*
(iv) *$s^2 = uv = vu$ for all $s > v$ and $u^2 = v^2 = 1$.*
(v) *$s^2 = wu = w^{-1}v$ for all $s > w$ and $u^2 = v^2 = 1$.*

Proof. Suppose that X is nonvosperian.

We may assume without loss of generality that X contains a proper fragment Q with cardinality >1. We may assume that $1 \in Q$ and that $q \in Q\backslash\{1\}$. If $Q \cap S \neq \emptyset$, then X contains a connected proper fragment. By Lemma 4.1 the condition (i) holds.

Suppose that $Q \cap S = \emptyset$.

Assume first that there is $m_i \geqslant w$ such that $m_i^2 = 1$. By Proposition 3.1 applied to $X_i = Cay(\langle S_i \rangle, S_i)$, where $S_i = \{s \in S \mid s \leqslant m_i\}$, $Q = \{1, q\}$ is a (connected) atom of X_{i-1} contradicting our assumption that $Q \cup S = \emptyset$. We may therefore assume that $m_i^2 \neq 1$ *for all* $i \geqslant 4$.

By applying iteratively the Proposition 3.1, we have

$$m_i^2 = q, \text{for } i > 3. \tag{1}$$

Moreover, $Q = \{1, q\} \in H_3$ and Q is a fragment of X_3 which, by Lemma 2.5 is maximally connected. Since $|QS_3| = |S_3|$ and $S_3 \subset QS_3$ we have $QS_3 = S_3$. In particular, $|S_3|$ is even. We consider two cases

Case 1: $|S_3| = 6$.

We then have $|H_2| \geqslant 9$. By Proposition 3.1, $q \in H_2$ and $w^2 = q$. Moreover, $QS_2 = S_2$. No three of the cosets Qu, Qu^{-1}, Qv, Qv^{-1} coincide. Therefore, we must have, except for the replacement of an element by its inverse, either $Qu = Qu^{-1}$ and $Qv = Qv^{-1}$ or $Qu = Qv$ and $Qu^{-1} = Qv^{-1}$.

The first equality leads to the relation $u^2 = v^2 = q$ and (ii) holds. The second inequality leads to the relations $vu^{-1} = v^{-1}u = q$ and then (iii) holds.

Case 2: $|S_3| = 4$.

In this case there are two self-inverse elements in S_3. If $w^2 = 1$, then 3.1 implies that $|H_2| < 2|S_3| - \kappa(X_2) + 1$. Hence, $|H_2| = 4$ and $u^2 = v^2 = 1$, a contradiction. Therefore, $u^2 = v^2 = 1$ and $w^2 \neq 1$. No three of the cosets Qw, Qw^{-1}, Qv, Qu can coincide. Therefore, exactly two of them are equal. Except for the replacement of an element by its inverse we have either $Qw = Qw^{-1}$ and $Qu = Qv$ or $Qw = Qu$ and $Qw^{-1} = Qv$. The first equality leads to (iv) and the second one to (v).

It can easily be checked that in any of the cases (i)–(v) the subset $Q = \{1, q\}$ is a nontrivial fragment. This completes the proof. □

The above proof can easily be adapted to give the result in the nonsymmetric case. In particular, when M is a minimal generating set we get the following result.

Corollary 5.2. *Let G be a group containing a minimal generating set M such that $|M| \geqslant 4$. Let $S = M \cup M^{-1}$. Then one of the following conditions holds.*

(i) *$Cay(G, S)$ is Vosperian.*

(ii) *$s^2 = u^2$ and $u^4 = 1$, for all $s, u \in M$.*

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