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Local Box Adjacency Algorithms for Cylindrical Algebraic Decompositions

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We describe new algorithms for determining the adjacencies between zero-dimensional cells and those one-dimensional cells that are sections (not sectors) in cylindrical algebraic decompositions (cad). Such adjacencies constitute a basis for determining all other cell adjacencies. Our new algorithms are local, being applicable to a specified 0D cell and the 1D cells described by specified polynomials. Particularly efficient algorithms are given for the 0D cells in spaces of dimensions two, three and four. Then an algorithm is given for a space of arbitrary dimension. This algorithm may on occasion report failure, but it can then be repeated with a modified isolating interval and a likelihood of success.

1. Introduction

The basic theory of, and terminology about, cylindrical algebraic decomposition (cad) can be found in Arnon *et al.* (1984a). The cad algorithm in its original formulation did not produce information concerning the adjacency relation on the set of cells comprising a cad. However it was realized that such cell adjacency information would be essential for certain applications of cads. A particularly important potential application is that of path planning, discussed in an early paper (Schwartz and Sharir, 1983). A more recent attempt at realization of path planning using cad in a simple case is discussed in McCallum (1997). Another possible application, that of clustering cells into maximal sign-invariant connected sets for the purpose of aiding in the cad computation process itself, was pursued in Arnon (1988).

Let A be a set of bivariate irreducible integral polynomials. Arnon *et al.* (1984b) presented an algorithm which, given such a set A of bivariate polynomials, constructs an A-invariant cad of the plane, and determines all pairs of adjacent cells in that cad. The algorithm of Arnon *et al.* (1984b) determines which sections are adjacent to a given 0-cell c^0 in the plane by analyzing the sides of a suitable box constructed around c^0. For this reason we call the adjacency algorithm of Arnon *et al.* (1984b) the “box algorithm” in the plane.

Now let A be a set of trivariate irreducible integral polynomials. Arnon *et al.* (1988) presented an algorithm which, given such a set A of trivariate polynomials, constructs an

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A-invariant cad of three-space having the so-called boundary property, and determines all pairs of adjacent cells in that cad.

Collins and McCallum (1995) described an algorithm for determining in a cad of $\mathbf{R}^3$ the section–section interstack adjacencies over a given adjacency in the induced cad of the plane. The algorithm is called a "box algorithm" for three-space. A somewhat informal extension to four-dimensional space of the box algorithm for three-space was also presented. The box algorithm for four-space uses a subalgorithm called the "real projection algorithm".

We were motivated by some concerns about the efficiency of the real projection subalgorithm of Collins and McCallum (1995) to seek another approach to adjacency determination which might avoid the need for this subalgorithm. The present paper contains descriptions of, and correctness proofs for, new box algorithms for two-, three- and four-dimensional spaces which do not require the real projection algorithm. These new algorithms are local, being applicable to a specified 0-cell and the 1D sections determined by specified polynomials. This could be computationally advantageous, as we shall indicate in the next section.

In the final section of this paper we present a local box adjacency algorithm for n-space, where n is arbitrary. Moreover, this algorithm permits the input polynomials to have coefficients from an algebraic number field, unlike the three- and four-space adjacency algorithms, which require the input polynomials to be irreducible integral polynomials. In Section 3.2 we summarize how to find all cell adjacencies in a cad of 3-space using our box algorithms. This summary reveals that to find all cell adjacencies in n-space for $n \geq 3$ will require algorithms allowing algebraic polynomials as inputs. Our n-space algorithm is fallible in the sense that it may occasionally report failure, but it can then be reapplied with a high probability of success.

2. A Local Box Adjacency Algorithm for the Plane

The box adjacency algorithm for the plane presented by Arnon *et al.* (1984b) is satisfactory, but is not an explicitly local algorithm. In this section we formulate a local version of the box algorithm of Arnon *et al.* (1984b). Our motivation for doing so is to provide a foundation upon which our presentation of the local box adjacency algorithms for higher dimensional spaces will be based. This section could also serve as a helpful review for the reader of the basic idea of cell adjacency computation in the plane.

The following theorem will be applied repeatedly in this paper.

Theorem 2.1. *Let $\mathbf{R}^*$ denote the two-point compactification of $\mathbf{R}$. Let c be a cell in a cad of $\mathbf{R}^n$ and let p be a point in $(\mathbf{R}^*)^n$ that does not belong to c. Then p is a limit point of c if and only if p is adjacent to c.*

Proof. Recall that by definition p is adjacent to c if and only if $\{p\} \cup c$ is connected, and that all cells in a cad are connected sets. Let $C = p \cup c$. Assume that p is a limit point of c. If p is not adjacent to c then C is not connected, so there exist open sets, O_1 and O_2 such that $O_1 \cap C$ and $O_1 \cap C$ are non-empty disjoint sets whose union is C. We may assume that $p \in O_1 \cap C$. But since p is a limit point of c, O_1 also contains points of c. Therefore $O_1 \cap C$ also contains points of c. Now $O_1 - \{p\}$ is an open set and

$(O_1 - \{p\}) \cap c$ is non-empty, so $(O_1 - \{p\}) \cap c$ and $O_2 \cap c = O_2 \cap C$ are non-empty disjoint sets whose union is c, contradicting the connectedness of c.

Next assume that p is not a limit point of c. Let ϵ be a positive real number such that the ball of radius ϵ centered at p contains no point of c. Let O_1 be the ball of radius $\epsilon/2$ centered at p, and let O_2 be the set of all points less than $\epsilon/2$ from some point of c. Then O_1 and O_2 are non-empty disjoint open sets containing p and c, respectively, so $c \cup \{p\}$ is not connected. □

Now we are ready to state a theorem, which is strongly suggestive of our method for cell adjacency determination in the plane. We will use the notion of a *strong* isolating interval for a root α of a real polynomial $A(x)$: this is an isolating interval for α whose closure is also an isolating interval for α.

THEOREM 2.2. *Let $A(x)$ and $B(x, y)$ be real polynomials of positive degrees in x and y, respectively. Let (a_1, a_2) and (b_1, b_2) be strong open isolating intervals for roots α of $A(x)$ and β of $B(\alpha, y) \neq 0$, respectively, such that B is delineable over $(\alpha, a_2]$ and B has no zeros in $[\alpha, a_2] \times \{b_1, b_2\}$. Let σ be a section of B over $(\alpha, a_2]$. Then σ is adjacent to (α, β) if and only if σ and $\{a_2\} \times [b_1, b_2]$ have non-empty intersection.*

PROOF. The section σ determines a continuous function $y = f(x)$ on $(\alpha, a_2]$. First, assume that σ is adjacent to (α, β). By Theorem 2.1, (α, β) is a limit point of σ. Therefore there is a point (α', β') of σ in $(\alpha, a_2) \times (b_1, b_2)$ (with $\beta' = f(\alpha')$). We claim that $f(a_2) \in (b_1, b_2)$. Suppose that this is not the case. Then, by continuity of f, there exists $x_0 \in (\alpha, a_2]$ such that $f(x_0) = b_1$ or $f(x_0) = b_2$. Either possibility is contrary to an hypothesis. This proves the claim. We have shown that σ and $\{a_2\} \times [b_1, b_2]$ have non-empty intersection.

Conversely assume that σ and $\{a_2\} \times [b_1, b_2]$ have non-empty intersection. That is, $f(a_2) \in [b_1, b_2]$. This implies $f(a_2) \in (b_1, b_2)$, by an hypothesis. We claim that $f(x) \in (b_1, b_2)$, for every $x \in (\alpha, a_2]$. Suppose that this is not the case. Then by continuity of f there exists $x_0 \in (\alpha, a_2)$ such that $f(x_0) = b_1$ or $f(x_0) = b_2$. This proves the claim. It follows that $\sigma \subset (\alpha, a_2] \times [b_1, b_2]$.

By the property $\sigma \subset (\alpha, a_2] \times [b_1, b_2]$ and the compactness of the subset $\{\alpha\} \times [b_1, b_2]$ of the plane (Munkres, 1975), σ has a limit point (α, β') in $\{\alpha\} \times [b_1, b_2]$. By the closure of the real variety of B, $B(\alpha, \beta') = 0$. But β is the unique real root of $B(\alpha, y)$ in $[b_1, b_2]$, since (b_1, b_2) is a strong isolating interval for β as a root of $B(\alpha, y)$, an hypothesis. Hence $\beta' = \beta$. By Theorem 2.1, σ is adjacent to (α, β). □

Here now is our algorithm based on the above theorem. The inputs and outputs of the algorithm make sense in the context of cad computation in the plane. The inputs to the theorem, $A(x)$ and $B(x, y)$, are required to be algebraic polynomials, by which we mean polynomials with coefficients in some algebraic number field. If we have computed some cad of $\mathbf{R}^n$ and wish only to compute cell adjacencies in the induced cad of $\mathbf{R}^2$, we need to use the algorithm only with irreducible integral polynomials as inputs. But, as we shall see in the next section, to compute all cell adjacencies in $\mathbf{R}^3$, we will need to allow algebraic polynomials as inputs. With the notation and hypotheses of the theorem, the algorithm computes the number of sections of B over $(\alpha, a_2]$ that are adjacent to (α, β).

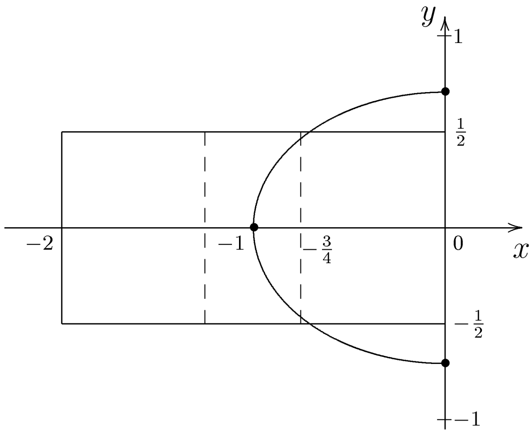

Figure 1. An application of the local box adjacency algorithm.

That is, the algorithm determines the number of sections of B that are *right-adjacent* to (α, β).

Local box adjacency algorithm in $\mathbf{R}^2$

Inputs:

1. $A(x), B(x, y)$: algebraic polynomials of positive degrees in x and y, respectively.
2. $(a_1, a_2), (b_1, b_2)$: strong open isolating intervals with rational endpoints for roots α of A and β of $B(\alpha, y) \neq 0$ such that B is delineable over $(\alpha, a_2]$.

Outputs:

1. (a'_1, a'_2): a subinterval of (a_1, a_2) with rational number endpoints and containing α such that B has no zeros in $[a'_1, a'_2] \times \{b_1, b_2\}$.
2. n: the number of sections of B over $(\alpha, a_2]$ that are adjacent to (α, β).

Steps:

1. Refine the interval (a_1, a_2) to an interval $(a'_1, a'_2]$ such that B has no zeros in $[a'_1, a'_2] \times \{b_1, b_2\}$. To accomplish this, proceed as follows. Refine (a_1, a_2) to an interval $(a'_1, a'_2]$ such that $[a'_1, a'_2]$ contains no root of $B(x, b_1)$ or of $B(x, b_2)$.
2. Set $n \leftarrow$ the number of real roots of $B(a'_2, y)$ in (b_1, b_2).
3. Return (a'_1, a'_2) and n. □

This algorithm performs the portion of the work of steps 2 and 4 of algorithm SSADJ2 from Arnon *et al.* (1984b) that pertains to the root β of $B(\alpha, y)$. Figure 1 illustrates an application of the algorithm to an example in which $(\alpha, \beta) = (-1, 0)$, $A(x) = x + 1$, $B(x, y) = x^2 + 2y^2 - 1$, $(a_1, a_2) = (-2, 0)$ and $(b_1, b_2) = (-1/2, 1/2)$. The dashed vertical line segments represent the subinterval $(a'_1, a'_2) = (-5/4, -3/4)$ computed by the algorithm. The number of sections of B that are right-adjacent to (α, β) is, of course, two. We now provide a correctness argument for the above algorithm.

THEOREM 2.3. *The local box adjacency algorithm for the plane is correct.*

PROOF. We first note that both $B(x, b_1)$ and $B(x, b_2)$ are non-zero polynomials. The reason is that $B(\alpha, b_1) \neq 0$ and $B(\alpha, b_2) \neq 0$, since (b_1, b_2) is a strong isolating interval for β as a root of $B(\alpha, y)$ (input condition 2). Hence step 1 can achieve its goal in finite time.

We shall prove that B has n sections over $(\alpha, a_2]$ that are adjacent to (α, β) by exhibiting a bijection ϕ between the sections of B over $(\alpha, a_2]$ adjacent to (α, β) and the real roots of $B(a_2', y)$ in (b_1, b_2). Let σ be a section of B over $(\alpha, a_2]$ which is adjacent to (α, β). By Theorem 2.2, in which we take $a_2 = a_2'$, σ and $\{a_2'\} \times [b_1, b_2]$ have non-empty intersection. That is, there is a point (a_2', β') of σ such that $\beta' \in [b_1, b_2]$. It follows that $\beta' \in (b_1, b_2)$. We define $\phi(\sigma) = \beta'$. Now ϕ is a one-to-one mapping since B is delineable over $(\alpha, a_2]$, by hypothesis.

Let β' be a real root of $B(a_2', y)$ in (b_1, b_2). Then (a_2', β') lies in some section σ of B over $(\alpha, a_2]$. By Theorem 2.2, in which we take $a_2 = a_2'$, σ is adjacent to (α, β). Clearly $\phi(\sigma) = \beta'$. We have shown that ϕ is also an onto mapping, which completes the correctness proof. □

In an actual implementation of this algorithm we may wish to also have it return the number of left adjacencies, namely the number of real roots of $B(a_1', y)$ in (b_1, b_2). To keep the exposition simple we have omitted throughout this paper the treatment of left adjacencies.

Knowing how many sections of $B(x, y)$ are adjacent to every (α, β_i) does not, in general, suffice to conclude which sections these are, because if the leading coefficient of $B(x, y)$ vanishes at α, some sections of $B(x, y)$ may be adjacent to $(\alpha, -\infty)$. That is, they may approach $(\alpha, -\infty)$ as x approaches α from the right. To determine how many sections are adjacent to $(\alpha, -\infty)$, we need a slightly modified version of the above algorithm to determine the number of sections adjacent to $(\alpha, -\infty)$. In this modification, the input (b_1, b_2) is replaced by a rational number b which is less than all roots of $B(\alpha, y)$. Step 1 then refines $(\alpha, a_2]$ to an interval $(\alpha, a_2']$ such that B contains no root of $B(x, b)$, Step 2 sets n to the number of roots of $B(a_2', y)$ in $(-\infty, b)$, the number of sections of B over $(\alpha, a_2]$ which are adjacent to $(\alpha, -\infty)$. The local box adjacency algorithms of the following sections of this paper also have similar modifications.

As suggested earlier, the algorithm of this section is a localized version of algorithm SSADJ2 of Arnon *et al.* (1984b), so to speak. The availability of such a local adjacency algorithm would leave open the possibility that one could make use of root multiplicity information to determine some of the adjacencies in the plane. Use of root multiplicity information for this purpose would be expected to speed up considerably the computation of some of the cell adjacencies in the plane.

The local box adjacency of this section and its variations that we have discussed enable us to determine all adjacencies between two sections in any cad of $\mathbf{R}^2$. From these all other adjacencies can be inferred as discussed in Arnon *et al.* (1984b). For the reader's convenience we summarize here how this is accomplished. There are, first, the obvious intrastack adjacencies, those between two cells in the same stack. Namely, any sector in a stack is adjacent to any section immediately above or below it. Then there are the interstack adjacencies. Let c_0 be any section in the induced cad of $\mathbf{R}$ and c_1 be an adjacent sector. Let S_0 and S_1 be the stacks over c_0 and c_1, respectively. Let s be any sector of S_1, and let s_1 and s_2 be the sections in S_1 that are immediately below and above s. Here we imagine that each stack contains sections at $-\infty$ and ∞. Let t_1 and t_2 be the sections in S_0 that are adjacent to s_1 and s_2. Then, by Corollary 2.5 of Arnon *et al.* (1984b), the cells in S_0 that are adjacent to s are t_1, t_2 and all cells between t_1 and t_2.

3. Adjacencies in Three-space

In this section we first present in Section 3.1 a local box adjacency algorithm for 3-space. Following that in Section 3.2 we shall, for the reader's convenience, outline how, based on theorems from Arnon *et al.* (1988), all other cell adjacencies in three-space can be deduced from the adjacencies between 0D cells and 1D cells.

3.1. A local box adjacency algorithm for three-space

Let $A(x)$, $B(x, y)$ and $C(x, y, z)$ be irreducible integral polynomials of positive degrees in their main variables and let (α, β, γ) be a point of $\mathbf{R}^3$ for which

$$A(\alpha) = B(\alpha, \beta) = C(\alpha, \beta, \gamma) = 0$$

and $C(\alpha, \beta, z) \neq 0$. We present a local box adjacency algorithm for determining the number of sections of C adjacent to (α, β, γ) over each of the sections of B that are adjacent to (α, β).

The following theorem is an analogue of Theorem 2.2, and is strongly suggestive of our algorithm for cell adjacency determination in three-space. The theorem gives a criterion for a section τ of C over a section of B to be adjacent to (α, β, γ): namely, τ is adjacent to (α, β, γ) if and only if τ intersects a certain side (face) of a box in $\mathbf{R}^3$ satisfying certain hypotheses. The theorem provides the basis for our local box adjacency algorithm for $\mathbf{R}^3$.

Theorem 3.1. *Let $A(x)$, $B(x, y)$ and $C(x, y, z)$ be real polynomials of positive degrees in their main variables. Let (a_1, a_2), (b_1, b_2) and (c_1, c_2) be strong open isolating intervals for roots α of $A(x)$, β of $B(\alpha, y) \neq 0$ and γ of $C(\alpha, \beta, z) \neq 0$, respectively, such that B is delineable over $(\alpha, a_2]$, B has no zeros in $[\alpha, a_2] \times \{b_1, b_2\}$, C is delineable over each section of B over $(\alpha, a_2]$, and B and C have no common zeros in $[\alpha, a_2] \times [b_1, b_2] \times \{c_1, c_2\}$. Let σ be a section of B over $(\alpha, a_2]$ and let τ be a section of C over σ. Then τ is adjacent to (α, β, γ) if and only if τ and $\{a_2\} \times [b_1, b_2] \times [c_1, c_2]$ have non-empty intersection.*

Proof. The section σ determines a continuous function $y = f(x)$ on $(\alpha, a_2]$. Likewise τ determines a continuous function $z = g(x, y)$ on σ. We define $h(x) = g(x, f(x))$, for $x \in (\alpha, a_2]$. Then h is continuous. First, assume that τ is adjacent to (α, β, γ). By Theorem 2.1, (α, β, γ) is a limit point of τ. Therefore (α, β) is a limit point of σ; hence σ is adjacent to (α, β), by Theorem 2.1. Now by Theorem 2.2 σ and $\{a_2\} \times [b_1, b_2]$ have non-empty intersection, that is, $f(a_2) \in [b_1, b_2]$. Since (α, β, γ) is a limit point of τ there is a point $(\alpha', \beta', \gamma')$ of τ in the interior of the box $[\alpha, a_2] \times [b_1, b_2] \times [c_1, c_2]$ (with $\beta' = f(\alpha')$ and $\gamma' = h(\alpha')$). We claim that $h(a_2) \in (c_1, c_2)$. Suppose that this is not the case. Then, by continuity of h, there exists $x_0 \in (\alpha, a_2]$ such that $h(x_0) = c_1$ or $h(x_0) = c_2$. Now by Theorem 2.2, in which we take $a_2 = x_0$, σ and $\{x_0\} \times [b_1, b_2]$ have non-empty intersection, that is, $f(x_0) \in [b_1, b_2]$. Thus $(x_0, f(x_0), h(x_0))$ is a common zero of B and C in $[\alpha, a_2] \times [b_1, b_2] \times \{c_1, c_2\}$, contrary to hypothesis. The claim is proved. We have shown that $(a_2, f(a_2), h(a_2)) \in \{a_2\} \times [b_1, b_2] \times [c_1, c_2]$, so τ and $\{a_2\} \times [b_1, b_2] \times [c_1, c_2]$ have non-empty intersection.

To prove the converse, assume that τ and $\{a_2\} \times [b_1, b_2] \times [c_1, c_2]$ have non-empty intersection, that is, $f(a_2) \in [b_1, b_2]$ and $h(a_2) \in [c_1, c_2]$. Then σ and $\{a_2\} \times [b_1, b_2]$ have non-empty intersection, so σ is adjacent to (α, β) (Theorem 2.2). Let $x \in (\alpha, a_2]$. By Theorem 2.2, taking $a_2 = x$, σ and $\{x\} \times [b_1, b_2]$ have non-empty intersection, that is, $f(x) \in [b_1, b_2]$. We claim that $h(x) \in (c_1, c_2)$. For if this were not the case then by continuity of h there would exist $x_0 \in (\alpha, a_2]$ such that either $h(x_0) = c_1$ or $h(x_0) = c_2$; so $(x_0, f(x_0), h(x_0))$ would be a common zero of B and C in $[\alpha, a_2] \times [b_1, b_2] \times \{c_1, c_2\}$, contrary to hypothesis. It follows from what we have established that $\tau \subset (\alpha, a_2] \times [b_1, b_2] \times [c_1, c_2]$.

Since $\tau \subset (\alpha, a_2] \times [b_1, b_2] \times [c_1, c_2]$, σ is adjacent to (α, β) and the subset $\{(\alpha, \beta)\} \times [c_1, c_2]$ of $\mathbf{R}^3$ is compact (Munkres, 1975), τ has a limit point (α, β, γ') in $\{(\alpha, \beta)\} \times [c_1, c_2]$. By the closure of the real variety of C, $C(\alpha, \beta, \gamma') = 0$. But γ is the unique root of $C(\alpha, \beta, z)$ in $[c_1, c_2]$, by an hypothesis. Hence $\gamma' = \gamma$. By Theorem 2.1 τ is adjacent to (α, β, γ). □

We now describe a new local adjacency algorithm for three-space based upon Theorem 3.1. The inputs and outputs for the algorithm make sense in the context of the computation of a cad of $\mathbf{R}^3$. All of the requirements on the inputs with the exception of requirement 2b are fulfilled by the theory of cad construction. The requirement 2b could be fulfilled by application of the local box adjacency algorithm described in Section 2. As for the algorithm of Section 2, the following algorithm determines what we might call the *right adjacencies* for (α, β, γ) (and the polynomials A, B, C). Unlike the algorithm of Section 2, this algorithm requires irreducible integral polynomials as inputs; its correctness proof depends on the irreducibility of B. A more general three-space algorithm, applicable to algebraic polynomials, is needed for some adjacency computations in four-space. In Section 5 we describe an n-space algorithm for algebraic polynomials which can be applied for $n = 3$.

In the following algorithm description, and elsewhere in this paper, we shall use the convention that the resultant of two polynomials with respect to any variable such that one of the polynomials, but not both, is of degree zero in that variable, is equal to the polynomial of degree zero (regardless of the degree of the other polynomial in that variable).

Local box adjacency algorithm in $\mathbf{R}^3$

Inputs:

1. $A(x), B(x, y), C(x, y, z)$: irreducible integral polynomials of positive degrees in their main variables.
2. $(a_1, a_2), (b_1, b_2), (c_1, c_2)$: strong open isolating intervals with rational endpoints for roots α of A, β of $B(\alpha, y)$ and γ of $C(\alpha, \beta, z) \neq 0$ such that
   (a) B is delineable over $(\alpha, a_2]$,
   (b) neither $B(x, b_1)$ nor $B(x, b_2)$ has any root in $(\alpha, a_2]$, and
   (c) C is delineable over each section of B over $(\alpha, a_2]$.

Outputs:

1. (a_1', a_2'): a subinterval of (a_1, a_2) with rational endpoints and containing α such that B and C have no common zeros in $[\alpha, a_2'] \times [b_1, b_2] \times \{c_1, c_2\}$.

2. L: the list of all lists (i, N_i) such that among the sections of B over $(\alpha, a_2]$ that are adjacent to (α, β), for the ith such section, σ_i, there are $N_i > 0$ sections of C that are adjacent to (α, β, γ).

Steps:

1. Refine the isolating interval (a_1, a_2) for α to a subinterval (a'_1, a'_2) such that B and C have no common zeros in $[\alpha, a'_2] \times [b_1, b_2] \times \{c_1, c_2\}$. To accomplish this, proceed as follows. For $k = 1, 2$ do: set $R_k(x) \leftarrow \mathrm{res}_y(B(x, y), C(x, y, c_k))$, the resultant with respect to y of $B(x, y)$ and $C(x, y, c_k)$. Refine $[\alpha, a_2]$ to an interval $[\alpha, a'_2]$ such that $(\alpha, a'_2]$ contains no root of $R_1(x)$ or $R_2(x)$. [The correctness proof for this algorithm will show that both $R_1(x)$ and $R_2(x)$ are non-zero polynomials, and that step 1 in fact guarantees that B and C have no common zeros in $[\alpha, a'_2] \times [b_1, b_2] \times \{c_1, c_2\}$.]
2. Compute isolating intervals for the real roots $\beta_1 < \beta_2 < \cdots < \beta_N$ of $B(a'_2, y)$ which are contained in (b_1, b_2).
3. Set $L \leftarrow ()$, the null list. For $i = 1, \ldots, N$ do compute the number N_i of real roots of $C(a'_2, \beta_i, z)$ in (c_1, c_2), and if $N_i > 0$, append (i, N_i) to L.
4. Return (a'_1, a'_2) and L. □

Figure 2 illustrates an example to which the three-space adjacency algorithm could be applied. In this example, $(\alpha, \beta, \gamma) = (-1, 0, 0)$, $A(x) = x + 1$, $B(x, y) = x^2 + 2y^2 - 1$, $C(x, y, z) = x^2 + y^2 + z^2 - 1$, $(a_1, a_2) = (-5/4, -3/4)$, $(b_1, b_2) = (-1/2, 1/2)$ and $(c_1, c_2) = (-1/4, 1/4)$. Notice that this example is closely related to that illustrated in Figure 1. In particular, requirement 2b is satisfied because the values of a_1 and a_2 used for this example are the values of a'_1 and a'_2 constructed by the two-space adjacency algorithm for the example of Figure 1. Figure 3 illustrates the three-space adjacency algorithm's output for this example: $(a'_1, a'_2) = (-17/16, -15/16)$ and $L = ((1, 2), (2, 2))$.

Theorem 3.2. *The above algorithm is correct.*

Proof. We shall show that B and C have no common zeros in $[\alpha, a'_2] \times [b_1, b_2] \times \{c_1\}$. A symmetric argument will show that B and C have no common zeros in $[\alpha, a'_2] \times [b_1, b_2] \times \{c_2\}$. We first show that the polynomial $R_1(x)$ computed in step 1 is non-zero. Suppose not. In case $\deg_y C(x, y, c_1) = 0$, $R_1(x) = C(x, y, c_1)$, by definition. So $C(x, y, c_1) = 0$. In particular $C(\alpha, \beta, c_1) = 0$, contradicting the hypothesis that (c_1, c_2) is a strong isolating interval for γ. In case $\deg_y C(x, y, c_1) > 0$, by Theorem 2 of Collins (1971), $B(x, y)$ and $C(x, y, c_1)$ would have a common divisor of positive degree in $\mathbf{Q}[x, y]$. Since $B(x, y)$ is irreducible in $\mathbf{Z}[x, y]$, and hence in $\mathbf{Q}[x, y]$, $B(x, y)$ itself would therefore be a divisor of $C(x, y, c_1)$. Therefore $C(\alpha, \beta, c_1)$ would vanish, contrary to the hypothesis that (c_1, c_2) is a strong isolating interval for γ. The proof that $R_1(x) \neq 0$ is complete.

Suppose first that $\deg_y C(x, y, c_1) = 0$, in which case $R_1(x) = C(x, y, c_1)$, by definition. Then $(\alpha, a'_2]$ contains no root of $C(x, y, c_1)$. Therefore B and C have no common zeros in $(\alpha, a'_2] \times \mathbf{R} \times \{c_1\}$. Suppose on the other hand that $\deg_y C(x, y, c_1) > 0$. Then $(\alpha, a'_2]$ contains no root of $R_1(x)$. Therefore there is no common zero of $B(x, y)$ and $C(x, y, z)$ in $(\alpha, a'_2] \times \mathbf{R} \times \{c_1\}$, by Theorem 5 of Collins (1971). It remains to show that $B(\alpha, y)$ and $C(\alpha, y, c_1)$, as polynomials in y, have no common zeros in $[b_1, b_2]$. This is so because $B(\alpha, \beta') \neq 0$ for every $\beta' \in [b_1, b_2]$ with $\beta' \neq \beta$, since (b_1, b_2) is a strong isolating interval

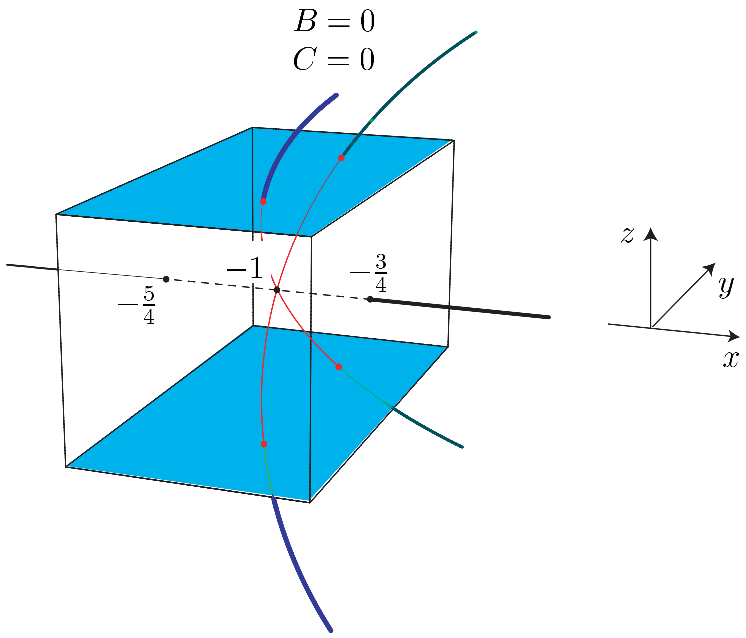

Figure 2. The curves enter and leave through the bottom and top of the box.

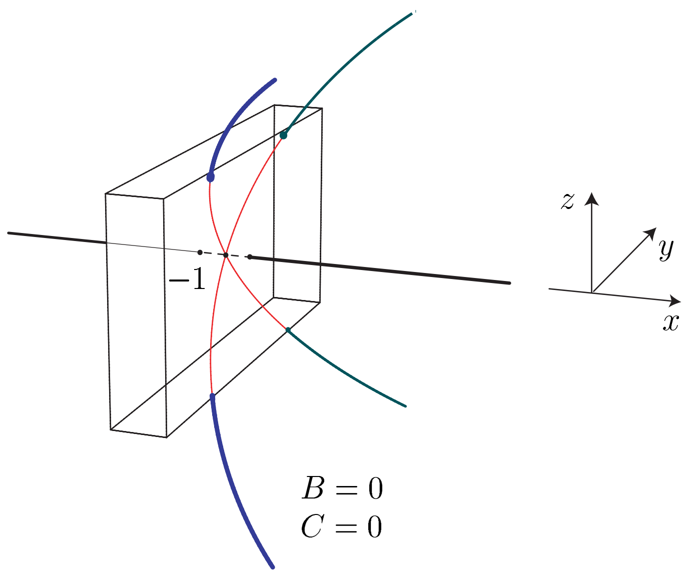

Figure 3. The curves enter through the face $x = -\frac{15}{16}$, very close to the edges where this meets the top and bottom faces.

for β as a root of $B(\alpha, y)$, and $C(\alpha, \beta, c_1) \neq 0$, since (c_1, c_2) is a strong isolating interval for γ as a root of $C(\alpha, \beta, z) \neq 0$. This completes the proof that B and C have no common zeros in $[\alpha, a_2'] \times [b_1, b_2] \times \{c_1\}$.

By Theorem 2.2, in which we take $a_2 = a'_2$, there is a one-to-one correspondence between the sections of B over $(\alpha, a_2]$ adjacent to (α, β) and the β_i that are isolated in Step 2. Hence there are N sections $\sigma_1 < \cdots < \sigma_N$ of B over $(\alpha, a_2]$ adjacent to (α, β), and $(a'_2, \beta_i) \in \sigma_i$. Let $1 \leq i \leq N$. We can prove that C has N_i sections over σ_i adjacent to (α, β, γ) by exhibiting a bijection between the real roots of $C(a'_2, \beta_i, z)$ in (c_1, c_2) and the sections of C over σ_i adjacent to (α, β, γ). The bijection is defined in the same manner as in the proof of Theorem 2.3. The mapping definition and the proof of bijectivity use Theorem 3.1. This completes the proof of the theorem. □

Algorithm SSADJ3 from Arnon *et al.* (1988) is the key subalgorithm of Arnon *et al.* (1988) for $\mathbf{R}^3$ adjacency determination. The algorithm of this section differs from SSADJ3 in at least two respects. First, the algorithm presented here is an algorithm *localized* at a particular point (α, β, γ) of $\mathbf{R}^3$, in contrast with SSADJ3 which treats the entire cylinder over a point (α, β) of the plane. Second, the strategy of SSADJ3 is to extract cell adjacencies in three-space from adjacency information for a suitable plane projection of three-space. The algorithm presented here, on the other hand, makes use of a suitable three-dimensional box about (α, β, γ) for adjacency determination.

In principle an analogue of SSADJ3 could be obtained by applying the local box algorithm for every root γ of $C(\alpha, \beta, z)$ and for the adjacencies at $(\alpha, \beta, -\infty)$. As for the algorithm of Section 2, the use of a local adjacency algorithm might permit the use of root multiplicity information to determine some of the adjacencies in three-space.

Our local box adjacency algorithm for three-space determines adjacencies between a point (α, β, γ) and sections of C over sections of B over a sector that is adjacent to α. We also need to determine adjacencies between (α, β, γ) and sections of C over a sector in $\mathbf{R}^2$ that is adjacent to (α, β). Substituting α for x reduces this to an adjacency problem in the y, z-plane, determining the number of sections of $C(\alpha, y, z)$ that are adjacent to (β, γ). We can apply the local box adjacency algorithm for $\mathbf{R}^2$ with input polynomials $B(\alpha, y)$ and $C(\alpha, y, z)$ and strong isolating intervals (b_1, b_2) and (c_1, c_2).

We have proved the correctness of our local box adjacency algorithm for $\mathbf{R}^3$ under the assumption that $C(\alpha, \beta, z) \neq 0$. But if $C(\alpha, \beta, z) = 0$ we may nevertheless wish to determine points (α, β, γ) that are adjacent to sections of C over 1D cells in $\mathbf{R}^2$ that are adjacent to (α, β). To do this, we may employ a technique used in Arnon *et al.* (1988), Theorems 4.2 and 4.3. Let $R(x, z) = \mathrm{res}_y(B(x, y), C(x, y, z))$ and $P(x, z) = \mathrm{pp}_z(R(x, z))$, the primitive part of $R(x, z)$ with respect to z. If σ is a section of C over a section of B that is adjacent to (α, β) over a sector adjacent to α then σ is adjacent to (α, β, γ) where γ is either a real root of $P(\alpha, z)$, ∞ or $-\infty$. Let γ be a real root of $P(\alpha, z)$ and let (c_1, c_2) be a strong isolation interval for γ as a root of $P(\alpha, z)$. Slight modification to the proofs of Theorems 3.1 and 3.2 shows that the local box adjacency algorithm for $\mathbf{R}^3$ is applicable and will determine the number of sections of C that are adjacent to (α, β, γ). In case σ is a section of C over a sector adjacent to (α, β), a similar technique applies. We set $R(y, z) = \mathrm{res}_x(A(x), C(x, y, z))$ and $P(y, z) = \mathrm{pp}_z(R(x, z))$. Then we let (c_1, c_2) be an isolating interval for a root γ of $P(\beta, z)$.

3.2. Determining all other adjacencies in three-space

As in $\mathbf{R}^2$, the intrastack adjacencies are obvious. The interstack adjacencies are those between the two adjacent stacks over two adjacent cells in the induced cad of $\mathbf{R}^2$. We classify these by the dimensions of the cells in the induced cad, D', which may be {0,1}, {0,2}, or {1,2}.

We begin with the case $\{0, 1\}$. Let c_i be the cell in D' of dimension i. We assume that every section in the stack S_1 over c_1 is adjacent to a unique section in the stack S_0 over c_0, including possibly sections at ∞ and $-\infty$. This requires that if any polynomial $C(x, y, z)$ is nullified by c_0 that S_0 has been refined by including the needed sections, using the technique described in the last paragraph of Section 3.1. Then, as in $\mathbf{R}^2$, if s is a sector in S_1 with sections σ_1 and σ_2 immediately above and below, and if σ_i is adjacent to section τ_i in S_0, $i = 1, 2$, then s is adjacent to both τ_1 and τ_2 and all cells in S_0 between τ_1 and τ_2.

Next consider the case $\{1, 2\}$. Let c_i be the cell in D' of dimension i. Suppose first that c_1 is a section. If a section of $C(x, y, z)$ over c_2 is adjacent to a section over c_1, that section over c_1 must also be a section of $C(x, y, z)$. Suppose that the section of C over c_2 is a section of $C(x, y, z)$ over a section of $B(x, y)$ over a sector s. Let a be a rational number in s. The adjacencies among sections of C over sections of B over s are in one-to-one correspondence with the adjacencies of sections of $C(a, x, y)$ over sections of $B(a, y)$ in $\mathbf{R}^2$. We can determine the latter adjacencies by applying the local box adjacency algorithm for $\mathbf{R}^2$ with input polynomials $B(a, y)$ and $C(a, x, y)$ and strong isolating intervals for the roots β_i of $B(a, y)$ and the roots $\gamma_{i,j}$ of $C(a, \beta_i, z)$.

Now suppose that c_1 is a sector. Consider any section σ_2 of $C(x, y, z)$ over c_2. Again, any section σ_1 over c_1 that is adjacent to σ_2 must be a section of C. Let (α, b) be a point in c_1 with b rational. Let b_1 and b_2 be rational numbers such that $b_1 < b < b_2$, and such that (α, b_1) and (α, b_2) are in c_1. The adjacencies among sections of C over c_2 and adjacencies of C over c_1 are in one-to-one correspondence with sections of $C(\alpha, y, z)$ over an interval $(b, , b_2]$ or an interval $[b_1, b)$. Let $B(y) = y - b$ be the minimal polynomial of b. Apply the local box adjacency algorithm for $\mathbf{R}^2$ with input polynomials $y - b$ and $C(\alpha, y, z)$, the strong isolating interval (b_1, b_2) for b as the root of $B(y)$ and strong isolating intervals for roots of $C(\alpha, b, z)$.

Finally we consider the case $\{0, 2\}$. We now suppose that adjacencies for the cases $\{0, 1\}$ and $\{1, 2\}$ have already been determined. Let c_0 and c_2 be cells in $\mathbf{R}^2$ of dimensions 0 and 2, respectively. Let s_2 be a section of $C(x, y, z)$ over c_2. By Theorem 6.1 of Arnon *et al.* (1988), if p and q are limit points of s_2 in $c_0 \times \mathbf{R}^*$ then so is every point between p and q. Suppose first that c_0 does not nullify C. Then by the theorem just cited, s_2 has a unique limit point in $c_0 \times \mathbf{R}^*$. Let c_1 be a section or sector of dimension 1 in $\mathbf{R}^2$ that is adjacent to c_2. Let s_1 be the unique section of C over c_1 that is adjacent to s_2. Let s_0 be the unique section of C over c_0 that is adjacent to s_1. Then s_0 is the unique section of C over c_0 that is adjacent to s_2.

Now assume that c_0 nullifies C. In this case we may need to insert more sections into the stack over c_0. Let C_x denote the partial derivative of C with respect to x and let $B(x, y) = \text{res}_z(C(x, y, z), C_x(x, y, z))$ if C_x has positive degree in z, otherwise $B(x, y) = C_x(x, y, o)$. We need to augment our set of projection factors by including B as a new bivariate projection polynomial. This will result in adding the irreducible factors of B as new bivariate projection factors, and will also produce new univariate projection factors. If $c_0 = \{(\alpha, \beta)\}$ and (a_1, a_2) is the isolating interval for α, we need to refine (a_1, a_2) to an isolating interval (a'_1, a'_2) whose closure contains no roots of the new univariate projection factors. Consider the example $C(x, y, z) = y^3z + xy^2 - x^3$. The original projection factors are $y^3z + xy^2 - x^3$, $y + x$, $y - x$, y and x and $(0, 0)$ nullifies

C. $C_x(x,y,z) = y^2 - 3x^2$ so $B(x,y) = y^2 - 3x^2$, which is irreducible. Adding B to the projection factor set does not introduce any new univariate projection factors.

Let $c_2' = c_2 \cap ((a_1', a_2') \times \mathbf{R})$. We need to identify sections of B that are interior to c_2' and adjacent to (α, β). This can be done by isolating the roots of $B(a_2', y)$ (assuming that c_2 is to the right of c_0) and the roots of $B'(a_2', y)$ where $B'(x,y)$ is any original projection factor having a section adjacent to c_2. We omit details of this process, which depend on whether c_2 is bounded both below and above. Applications of the local box adjacency algorithm for $\mathbf{R}^2$ at (α, β) with isolating interval (a_1', a_2') and polynomials B_i will be required. Suppose that a section σ of B over c_2' that is adjacent to (α, β) has been identified. We now need to find the limit point (α, β, γ) in the stack over c_0 of the section of C over σ. We cannot do this directly, but if we compute $G(x,z) = \mathrm{res}_y(B(x,y), C(x,y,z))$, then γ will be among the roots of $G(\alpha, z)$ if γ is finite. For our example let $c_0 = \{(0,0)\}$ and let c_2 be the sector bounded on the left by the y-axis and bounded below by the line $y = x$. There is a unique section of $B(x,y)$ that is interior to $c_2' = c_2$ and adjacent to c_0, namely a portion of the line $y = \sqrt{3}x$. $G(x,z) = x^6(27z^2 - 4)$.

Let (b_1, b_2) be an isolating interval for the root β' of $B(a_2', y)$ such that (a_2', β') is in σ. Let γ_j be one of the roots of $G(\alpha, z)$ and let (c_1^*, c_2^*) be an isolating interval for γ_j. With (c_1^*, c_2^*) in place of an isolating interval for a root of $C(\alpha, \beta, z)$, we can then use the local box adjacency algorithm in $\mathbf{R}^3$ to determine the number of sections of C over σ that are adjacent to $(\alpha, \beta, \gamma_j)$. We can also determine the number adjacent to $(\alpha, \beta, -\infty)$. If we repeat this procedure for each γ_j, we can determine which sections of C over σ are adjacent to which $(\alpha, \beta, \gamma_j)$s. Now suppose that we have found all points (α, β, γ) that are adjacent to some section of C over some section of some B_i that is adjacent to (α, β) and interior to c_2'. Call these $\gamma_1, \ldots, \gamma_n$. For each section or sector in $\mathbf{R}^2$ that is adjacent to both c_2 and c_0, add an additional γ_i such that $(\alpha, \beta, \gamma_i)$ is the limit of the section of C over that section or sector. For every such γ_i make $(\alpha, \beta, \gamma_1)$ a section of the stack over c_0. Now let τ be any particular section of C over c_2. Let γ_1 (possibly $-\infty$) be least such that the section of τ over some section of some B_i has $(\alpha, \beta, \gamma_1)$ as limit point; let γ_2 (possibly ∞) be greatest. Then by Theorem 6.1 of Arnon *et al.* (1988), τ is adjacent to $(\alpha, \beta, \gamma_1)$, $(\alpha, \beta, \gamma_2)$ and all sections and sectors between in the stack over c_0. The process we have outlined for finding the γ_is follows from Theorems 6.2, 6.3 and 6.4 of Arnon *et al.* (1988). In our example, the limit of the section of C over the sector that bounds c_2 on the left is 0. Also the limit of the section of C over the section of $y - x$ that bounds c_2 below is 0. There are two γ_is, namely $2/3\sqrt{3}$ and $-2/3\sqrt{3}$. By application of the modified local box adjacency algorithm for 3-space we find that the section of C over the section of $y^2 - 3x^2$ that is interior to c_2 is adjacent to $(0, 0, -2/3\sqrt{3})$. Therefore c_2 is adjacent to the section $(0,0,0)$, the section $(0, 0, -2/3\sqrt{3})$, and the sector between these two sections.

4. A Local Box Adjacency Algorithm for Four-space

Let $A(x)$, $B(x,y)$, $C(x,y,z)$ and $D(x,y,z,w)$ be irreducible integral polynomials of positive degrees in their main variables. Let $(\alpha, \beta, \gamma, \delta)$ be a point of $\mathbf{R}^4$ for which

$$A(\alpha) = B(\alpha, \beta) = C(\alpha, \beta, \gamma) = D(\alpha, \beta, \gamma, \delta) = 0,$$

$C(\alpha, \beta, z) \neq 0$ and $D(\alpha, \beta, \gamma, w) \neq 0$. We present a local box adjacency algorithm for determining the number of sections of D adjacent to $(\alpha, \beta, \gamma, \delta)$ over each of the sections of C adjacent to (α, β, γ) over each of the sections of B adjacent to (α, β).

We begin with a four-dimensional analogue of Theorem 3.1. This theorem provides the basic idea for the algorithm for cell adjacency determination in four-space which is described subsequently.

THEOREM 4.1. *Let $A(x)$, $B(x,y)$, $C(x,y,z)$ and $D(x,y,z,w)$ be real polynomials of positive degrees in their main variables. Let (a_1,a_2), (b_1,b_2), (c_1,c_2) and (d_1,d_2) be strong open isolating intervals for roots α of $A(x)$, β of $B(\alpha,y) \neq 0$, γ of $C(\alpha,\beta,z) \neq 0$ and δ of $D(\alpha,\beta,\gamma,w) \neq 0$ such that B is delineable over $(\alpha,a_2]$, B has no zeros in $[\alpha,a_2]\times\{b_1,b_2\}$, C is delineable over each section of B over $(\alpha,a_2]$, B and C have no common zeros in $[\alpha,a_2]\times[b_1,b_2]\times\{c_1,c_2\}$, D is delineable over each section of C over each section of B over $(\alpha,a_2]$ and B, C and D have no common zeros in $[\alpha,a_2]\times[b_1,b_2]\times[c_1,c_2]\times\{d_1,d_2\}$. Let σ be a section of B over $(\alpha,a_2]$, let τ be a section of C over σ and let ρ be a section of D over τ. Then ρ is adjacent to $(\alpha,\beta,\gamma,\delta)$ if and only if ρ and $\{a_2\}\times[b_1,b_2]\times[c_1,c_2]\times[d_1,d_2]$ have non-empty intersection.*

We omit the proof of this theorem since it follows the same pattern as the proofs of Theorems 2.2 and 3.1, and because the theorem is a special case of Theorem 5.1 in the following section.

While this theorem suggests the strategy we might wish to employ for cell adjacency determination in four-space, we have found that the simplest method of achieving one of the key hypotheses of the theorem—namely, that "B, C and D have no common zeros in $[\alpha,a_2]\times[b_1,b_2]\times[c_1,c_2]\times\{d_1,d_2\}$"—is by no means obvious. We have discovered what we think is a relatively simple and elegant method of ensuring this hypothesis. The method itself is described in steps 1 and 2 of the adjacency algorithm which will shortly follow. The complete validity of the method, which will be established as part of the correctness proof for the adjacency algorithm, depends upon the following result which is stated and proved by McCallum (1999). Recall from McCallum (1988) or McCallum (1998) that an r-variate real polynomial (or analytic function) f is said to be *order-invariant* in a subset S of r-space if the order of f is the same at every point of S.

THEOREM 4.2. *Let $r \geq 2$, let $f(x_1,\ldots,x_r)$ and $g(x_1,\ldots,x_r)$ be real polynomials of positive degrees in the main variable x_r, let $R(x_1,\ldots,x_{r-1})$ be the resultant of f and g with respect to x_r, and suppose that $R \neq 0$. Let S be a connected subset of $\mathbf{R}^{r-1}$ on which f is delineable and in which R is order-invariant. Then g is sign-invariant in each section of f over S.*

We now describe a new local adjacency algorithm for four-space based upon Theorem 4.1. The inputs and outputs for the algorithm make sense in the context of cad computation in $\mathbf{R}^4$. All of the requirements on the inputs except for 2c and 2e are fulfilled by the theory of cad construction with improved projection, for which the reader can consult either McCallum (1988) or McCallum (1998). Requirements 2c and 2e can be fulfilled by application of the adjacency algorithms from Sections 2 and 3.

Local box adjacency algorithm in $\mathbf{R}^4$

Inputs:

1. $A(x), B(x,y), C(x,y,z), D(x,y,z,w)$: irreducible integral polynomials of positive degrees in their main variables.

2. $(a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2)$: strong open isolating intervals with rational endpoints for roots α of A, β of $B(\alpha, y)$, γ of $C(\alpha, \beta, z) \neq 0$ and δ of $D(\alpha, \beta, \gamma, w) \neq 0$ such that
   (a) B is delineable over $(\alpha, a_2]$,
   (b) B is order-invariant in each of its sections over $(\alpha, a_2]$,
   (c) neither $B(x, b_1)$ nor $B(x, b_2)$ has any root in $(\alpha, a_2]$,
   (d) C is delineable over each section of B over $(\alpha, a_2]$,
   (e) B and C have no common zeros in $[\alpha, a_2] \times [b_1, b_2] \times \{c_1, c_2\}$ and
   (f) D is delineable over each section of C over each section of B over $(\alpha, a_2]$.

Outputs:

1. (a'_1, a'_2): a subinterval of (a_1, a_2) with rational endpoints and containing α such that B, C and D have no common zeros in $[\alpha, a'_2] \times [b_1, b_2] \times [c_1, c_2] \times \{d_1, d_2\}$.
2. L: a list of all lists $(i, j, N_{i,j})$ such that
   (a) there are at least i sections of B over $(\alpha, a_2]$ that are adjacent to (α, β); let σ_i be the ith such section,
   (b) there are at least j sections of C over σ_i that are adjacent to (α, β, γ); let $\tau_{i,j}$ be the jth such section,
   (c) there are $N_{i,j} > 0$ sections of D over $\tau_{i,j}$ that are adjacent to $(\alpha, \beta, \gamma, \delta)$.

Steps:

1. The goal of the first two steps is to refine the interval (a_1, a_2) to an interval (a'_1, a'_2) such that B, C and D have no common zeros in $[\alpha, a'_2] \times [b_1, b_2] \times [c_1, c_2] \times \{d_1, d_2\}$. The first step is to perform the following simple loop:
   for $k = 1, 2$ carry out the following sequence of operations:
   (a) if $\deg_z D(x, y, z, d_k) = 0$ then

   set $Q_k(x, y) \leftarrow D(x, y, z, d_k)$

   else

   set $Q_k(x, y) \leftarrow \mathrm{res}_z(C(x, y, z), D(x, y, z, d_k))$

   [end "if-then-else"];
   (b) $\bar{Q}_k \leftarrow Q_k$;
   while $B \mid \bar{Q}_k$ do

   set $\bar{Q}_k \leftarrow \bar{Q}_k / B$

   [end "while"];
   if $\deg_y \bar{Q}_k = 0$ then

   set $R_k(x) \leftarrow \bar{Q}_k(x, y)$

   else

   set $R_k(x) \leftarrow \mathrm{res}_y(B, \bar{Q}_k)$

   [end "if-then-else"].

   [end "for"].

2. Refine (a_1, a_2) to an interval (a_1', a_2') such that $(\alpha, a_2']$ contains no root of $R_1(x)$ or $R_2(x)$. [The correctness proof for this algorithm will show that steps 1 and 2 in fact guarantee that B, C and D have no common zeros in $[\alpha, a_2'] \times [b_1, b_2] \times [c_1, c_2] \times \{d_1, d_2\}$.]
3. Compute isolating intervals for the real roots $\beta_1 < \beta_2 < \cdots < \beta_N$ of $B(a_2', y)$ that are contained in (b_1, b_2).
4. For $i = 1, \ldots, N$ do compute isolating intervals for the N_i real roots $\gamma_{i,1} < \gamma_{i,2} < \cdots < \gamma_{i,N_i}$ of $C(a_2', \beta_i, z)$ in (c_1, c_2).
5. Set $L \leftarrow$ the null list. For $i = 1, \ldots, N$ do for $j = 1, \ldots, N_i$ do compute the number $N_{i,j}$ of real roots of $D(a_2', \beta_i, \gamma_{i,j}, w)$ in (d_1, d_2) and, if $N_{i,j} > 0$, append the list $(i, j, N_{i,j})$ to L. □

Theorem 4.3. *The above algorithm is correct.*

Proof. We shall show that B, C and D have no common zeros in $[\alpha, a_2'] \times [b_1, b_2] \times [c_1, c_2] \times \{d_1\}$. A symmetric argument will show that B, C and D have no common zeros in $[\alpha, a_2'] \times [b_1, b_2] \times [c_1, c_2] \times \{d_2\}$. Let $D_1(x, y, z) = D(x, y, z, d_1)$. First, we show that $Q_1(x, y) \neq 0$. Suppose not. In case $\deg_z D_1 = 0$, we would have $D_1 = 0$. So, in particular, $D(\alpha, \beta, \gamma, d_1) = 0$, contrary to the hypothesis that (d_1, d_2) is a strong isolating interval for δ as a root of $D(\alpha, \beta, \gamma, w) \neq 0$. In case $\deg_z D_1 > 0$, we would have $\mathrm{res}_z(C, D_1) = 0$, which would imply $C \mid D_1$, since C is irreducible. This would imply $D_1(\alpha, \beta, \gamma) = 0$, since $C(\alpha, \beta, \gamma) = 0$, by hypothesis. Again we have reached a contradiction. The proof that $Q_1(x, y) \neq 0$ is complete. We remark that the property $Q_1 \neq 0$ ensures the termination of the "while" loop in step 1b.

Second, we show that $R_1(x) \neq 0$. Suppose not. In case $\deg_y \bar{Q}_1 = 0$, we would have $\bar{Q}_1 = 0$, contrary to the property proved above. In case $\deg_y \bar{Q}_1 > 0$, we would have $\mathrm{res}_y(B, \bar{Q}_1) = 0$. This would imply that B is a divisor of $\bar{Q}_1$, since B is irreducible. But this contradicts the property that B is not a divisor of $\bar{Q}_1$, which is the condition for the termination of the "while" loop in step 1b. The proof that $R_1(x) \neq 0$ is complete.

Now $Q_1 = B^n \bar{Q}_1$, for some $n \geq 0$, by construction of $\bar{Q}_1$. We observe that $\bar{Q}_1 \neq 0$ throughout each section of B over $(\alpha, a_2']$, by construction of a_2'. So $\bar{Q}_1$ is (rather trivially) order-invariant in each such section of B. Moreover B is order-invariant in each of its sections over $(\alpha, a_2']$ by input hypothesis 2b. Therefore, by Lemma A.3 of McCallum (1988), Q_1 is order-invariant in each section of B over $(\alpha, a_2']$.

Suppose that B, C and D_1 have some common zero $(\alpha', \beta', \gamma')$ in $(\alpha, a_2'] \times [b_1, b_2] \times [c_1, c_2]$. Then $(\alpha', \beta', \gamma')$ lies in some section τ of C over some section σ of B over $(\alpha, a_2']$, by input hypotheses 2a and 2d. As shown previously, Q_1 is order-invariant in σ. Therefore, by Theorem 4.2, D_1 is sign-invariant in τ. Hence, since D_1 vanishes at the point $(\alpha', \beta', \gamma')$ of τ, D_1 vanishes throughout τ. By input hypotheses 2(a,c,d,e), and Theorem 3.1, in which we take $a_2 = \alpha'$, τ is adjacent to (α, β, γ). Therefore, by the closure of the real variety of D_1, $D_1(\alpha, \beta, \gamma) = 0$. This contradicts the hypothesis that (d_1, d_2) is a strong isolating interval for δ as a root of $D(\alpha, \beta, \gamma, w) \neq 0$. The proof that B, C and D have no common zeros in $(\alpha, a_2'] \times [b_1, b_2] \times [c_1, c_2] \times \{d_1\}$ is complete.

Output condition 1 has almost been completely established. It remains to show that $B(\alpha, y)$, $C(\alpha, y, z)$ and $D_1(\alpha, y, z)$ have no common zeros in $[b_1, b_2] \times [c_1, c_2]$. This is so because $B(\alpha, \beta') \neq 0$ for every $\beta' \in [b_1, b_2]$ with $\beta' \neq \beta$, $C(\alpha, \beta, \gamma') \neq 0$ for every

$\gamma' \in [c_1, c_2]$ with $\gamma' \neq \gamma$, and $D_1(\alpha, \beta, \gamma) \neq 0$ (since (b_1, b_2), (c_1, c_2) and (d_1, d_2) are strong isolating intervals for β, γ and δ, respectively).

Output condition 2 can be proved by exhibiting, for each i in range $1 \leq i \leq N$ and each j in range $1 \leq j \leq N_i$, a one-to-one correspondence between the real roots of $D(a'_2, \beta_i, \gamma_{i,j}, w)$ in (d_1, d_2) and the sections of D over $\tau_{i,j}$ that are adjacent to $(\alpha, \beta, \gamma, \delta)$. The proof is a straightforward application of Theorem 4.1, and is analogous to the proof of output condition 2 of the algorithm of Section 3. □

As already mentioned in the previous section, the following section will present an n-space algorithm for algebraic polynomials. The algorithm above, however, will be much more efficient for the numerous adjacency computations in 4-space involving only irreducible integral polynomials.

We observed in Section 3 the considerable complexity of computing all cell adjacencies in three-space. A comparable treatment of all cell adjacencies in four-space is therefore beyond the scope of this paper, as the complexity of this task in four-space would certainly be considerably greater. It is reasonable to expect, however, that consideration of all the required cases, would show how to reduce this to applications of the algorithms of Sections 2 and 3, the above algorithm, and the n-space algorithm of the following section for $n = 3$.

5. A Local Box Adjacency Algorithm for n-Space

Let $A_1(x_1), A_2(x_1, x_2), \ldots, A_n(x_1, \ldots, x_n)$ be irreducible integral polynomials of positive degrees in their main variables. Let $(\alpha_1, \ldots, \alpha_n)$ be a point of $\mathbf{R}^n$ for which

$$A_1(\alpha_1) = A_2(\alpha_1, \alpha_2) = \cdots = A_n(\alpha_1, \ldots, \alpha_n) = 0,$$

and suppose that $A_2(\alpha_1, x_2), \ldots, A_n(\alpha_1, \ldots, \alpha_{n-1}, x_n)$ are all non-zero polynomials. We shall present in this section a local box adjacency algorithm whose output can be described as follows. Let σ_2 be a section of A_2 over $(\alpha_1, b_1]$ adjacent to (α_1, α_2) and, for $3 \leq i < n$, let σ_i be a section of A_i over σ_{i-1} which is adjacent to $(\alpha_1, \ldots, \alpha_i)$. The algorithm's output comprises, for every such sequence of sections $\sigma_2, \ldots, \sigma_{n-1}$, the number of sections of A_n over σ_{n-1} which are adjacent to $(\alpha_1, \ldots, \alpha_n)$.

As in the previous sections, we begin with a theorem, which provides the basis for our cell adjacency algorithm.

Theorem 5.1. *Let $n \geq 2$. Let $A_1(x_1), A_2(x_1, x_2), \ldots, A_n(x_1, \ldots, x_n)$ be real polynomials of positive degrees in their main variables. Let (a_1, b_1), $(a_2, b_2), \ldots, (a_n, b_n)$ be strong open isolating intervals for roots α_1 of $A_1(x_1)$, α_2 of $A_2(\alpha_1, x_2) \neq 0, \ldots, \alpha_n$ of $A_n(\alpha_1, \ldots, \alpha_{n-1}, x_n) \neq 0$, respectively. Suppose that, for $2 \leq i \leq n$, the polynomials $A_2, A_3, \ldots, A_i$ have no common zeros in $[\alpha_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{i-1}, b_{i-1}] \times \{a_i, b_i\}$. Suppose that A_2 is delineable over $(\alpha_1, b_1]$ and let σ_2 be a section of A_2 over $(\alpha_1, b_1]$. For $3 \leq i \leq n$, suppose that A_i is delineable over σ_{i-1} and let σ_i be a section of A_i over σ_{i-1}. Then σ_n is adjacent to $(\alpha_1, \ldots, \alpha_n)$ if and only if σ_n and $\{b_1\} \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ have non-empty intersection.*

Proof. The theorem is proved by induction on n. The induction base (that is, the case $n = 2$) is Theorem 2.2, proved in Section 2. Let $n > 2$. Assume as the induction hypothesis that the assertion to be proved holds with n replaced by $n-1$. We must now establish the truth of the assertion, making use of the induction hypothesis. The section σ_2 determines a continuous function $x_2 = f_2(x_1)$ on $(\alpha_1, b_1]$. Likewise, for $3 \leq i \leq n$, σ_i determines a continuous function $f_i(x_1, \ldots, x_{i-1})$ on σ_{i-1}. For $2 \leq i \leq n$ we define $h_i(x_1) = f_i(x_1, h_2(x_1), \ldots, h_{i-1}(x_1))$, for $x_1 \in (\alpha_1, b_1]$. Then, for $2 \leq i \leq n$, h_i is continuous.

First, assume that σ_n is adjacent to $(\alpha_1, \ldots, \alpha_n)$. By Theorem 2.1, $(\alpha_1, \ldots, \alpha_n)$ is a limit point of σ_n. Therefore $(\alpha_1, \ldots, \alpha_{n-1})$ is a limit point of σ_{n-1}, hence adjacent to σ_{n-1} by Theorem 2.1. By the induction hypothesis, σ_{n-1} and $\{b_1\} \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}]$ have non-empty intersection. That is, for $2 \leq i \leq n-1$, $h_i(b_1) \in [a_i, b_i]$. Since $(\alpha_1, \ldots, \alpha_n)$ is a limit point of σ_n, there is a point $(\alpha'_1, \ldots, \alpha'_n)$ of σ_n in the interior of the box $[\alpha_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ (with $\alpha'_i = h_i(\alpha'_1)$, for $2 \leq i \leq n$). We claim that $h_n(b_1) \in (a_n, b_n)$. Suppose that this is not the case. Then, by continuity of h_n, there exists $x \in (\alpha_1, b_1]$ such that $h_n(x) = a_n$ or $h_n(x) = b_n$. Now by the induction hypothesis, in which we take $b_1 = x$, σ_{n-1} and $\{x\} \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}]$ have non-empty intersection. That is, for $2 \leq i \leq n-1$, $h_i(x) \in [a_i, b_i]$. Thus $(x, h_2(x), \ldots, h_n(x))$ is a common zero of $A_2, \ldots, A_n$ in $[\alpha_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n, b_n\}$, contrary to an hypothesis. The claim is proved. We have shown that $(b_1, h_2(b_1), \ldots, h_n(b_1))$ is an element of $\{b_1\} \times [a_2, b_2] \times \cdots [a_n, b_n]$, so σ_n and $\{b_1\} \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ have non-empty intersection.

To prove the converse, assume that σ_n and $\{b_1\} \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ have non-empty intersection. That is, for $2 \leq i \leq n$, $h_i(b_1) \in [a_i, b_i]$. Then σ_{n-1} and $\{b_1\} \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}]$ have non-empty intersection, so σ_{n-1} is adjacent to $(\alpha_1, \ldots, \alpha_{n-1})$, by the induction hypothesis. Let $x' \in (\alpha_1, b_1]$. By the induction hypothesis, in which we take $b_1 = x'$, σ_{n-1} and $\{x'\} \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}]$ have non-empty intersection. That is, for $2 \leq i \leq n-1$, $h_i(x') \in [a_i, b_i]$. We claim that $h_n(x') \in (a_n, b_n)$. For if this were not the case then by continuity of h_n there would exist $x \in (\alpha_1, b_1]$ such that either $h_n(x) = a_n$ of $h_n(x) = b_n$. So $(x, h_2(x), \ldots, h_n(x))$ would be a common zero of $A_2, \ldots, A_n$ in $[\alpha_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n, b_n\}$, contrary to an hypothesis. It follows from what we have established that

$$\sigma_n \subset (\alpha_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

By this relation, the observation noted above that σ_{n-1} is adjacent to $(\alpha_1, \ldots, \alpha_{n-1})$ and the compactness of the subset $\{(\alpha_1, \ldots, \alpha_{n-1})\} \times [a_n, b_n]$ of $\mathbf{R}^n$ (Munkres, 1975), σ_n has a limit point $(\alpha_1, \ldots, \alpha_{n-1}, \alpha'_n)$ in $\{(\alpha_1, \ldots, \alpha_{n-1})\} \times [a_n, b_n]$. By the closure of the real variety of A_n, $A_n(\alpha_1, \ldots, \alpha_{n-1}, \alpha'_n) = 0$. But α_n is the unique root of $A_n(\alpha_1, \ldots, \alpha_{n-1}, x_n)$ in $[a_n, b_n]$, by an hypothesis. Hence $\alpha'_n = \alpha_n$. By Theorem 2.1, σ_n is adjacent to $(\alpha_1, \ldots, \alpha_n)$. □

We now describe our n-space local adjacency algorithm based upon Theorem 5.1. The inputs and outputs for the algorithm make sense in the context of cad computation in $\mathbf{R}^n$. The algorithm inputs $A_1(x_1), \ldots, A_n(x_1, \ldots, x_n)$ are required to be algebraic polynomials, by which we mean polynomials with coefficients in some algebraic number field. The requirement on inputs 2b is fulfilled by the theory of cad construction. The requirement 2a could be fulfilled by successive application of the algorithm for dimensions $n = 2, 3, \ldots, n-1$.

Local box adjacency algorithm in $\mathbf{R}^n$

Inputs:

1. $A_1(x_1), A_2(x_1, x_2), \ldots, A_n(x_1, \ldots, x_n)$: algebraic polynomials of positive degrees in $x_1, x_2, \ldots, x_n$, respectively, where $n \geq 2$.
2. $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$: strong open isolating intervals with rational endpoints for roots α_1 of $A_1(x_1)$, α_2 of $A_2(\alpha_1, x_2) \neq 0, \ldots, \alpha_n$ of $A_n(\alpha_1, \ldots, \alpha_{n-1}, x_n) \neq 0$, respectively, such that:
   (a) for $2 \leq i \leq n-1$, $A_2, \ldots, A_i$ have no common zeros in $[\alpha_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{i-1}, b_{i-1}] \times \{a_i, b_i\}$, and
   (b) A_2 is delineable over $(\alpha_1, b_1]$ and, for $3 \leq i \leq n$, A_i is delineable over each section of A_{i-1} over ... over each section of A_2 over $(\alpha_1, b_1]$.

Outputs:
either "failure" or both 1 and 2 described as follows:

1. (a_1', b_1'): a subinterval of (a_1, b_1) with rational endpoints and containing α such that $A_2, \ldots, A_n$ have no common zeros in $[\alpha_1, b_1'] \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n, b_n\}$.
2. L: a list of all sequences $(i_2, \ldots, i_{n-1}, N)$ such that, with σ_1 denoting $(\alpha_1, b_1]$, we have for $2 \leq j \leq n-1$, A_j has at least i_j sections over σ_{j-1} adjacent to $(\alpha_1, \alpha_2, \ldots, \alpha_j)$, with σ_j denoting the i_jth such section; and $N > 0$ is the number of sections of A_n over σ_{n-1} which are adjacent to $(\alpha_1, \ldots, \alpha_n)$.

Steps:

1. The goal of the first two steps is to refine the interval (a_1, b_1) to an interval (a_1', b_1') such that $A_2, \ldots, A_n$ have no common zeros in $[\alpha_1, b_1'] \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n, b_n\}$. First, carry out the following loop:

   set $i \leftarrow n-1$;
   set $success \leftarrow true$;
   set $R \leftarrow A_n(x_1, \ldots, x_{n-1}, a_n)$;
   while $i \geq 2$ and $success$ do
      set $R \leftarrow \mathrm{res}_{x_i}(A_i, R)$;
      if $R = 0$ then
         set $success \leftarrow false$;
      set $i \leftarrow i-1$;
   if $success = false$ then
      report "failure" and exit;

2. Refine (a_1, b_1) to an interval (a_1', b_1') such that $(\alpha_1, b_1']$ contains no root of $R(x_1)$. [The correctness proof for this algorithm will show that steps 1 and 2 guarantee that $A_2, \ldots, A_n$ have no common zeros in $[\alpha_1, b_1'] \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$.] Steps 1 and 2 must be repeated exactly once, with "a_n" replaced by "b_n". Before repeating step 2 set $b_1 \leftarrow b_1'$.

3. Set $L \leftarrow ()$. Set $\beta_1 \leftarrow b_1'$. Set $T \leftarrow$ a tree with one node β_1. Compute isolating intervals for the real roots $\gamma_1 < \gamma_2 < \cdots < \gamma_N$ of $A_2(\beta_1, x_2)$ which are contained in (a_2, b_2). Store in T as the children of the root every γ_k. If $N > 0$ then set $L \leftarrow ((N))$.
4. For $l = 2, \ldots, n-1$ do

   set $L' \leftarrow ()$;
   while $L \neq ()$ do

      set $\lambda \leftarrow$ the first element of L;

      [λ is a sequence of the form $(i_2, \ldots, i_{l-1}, N)$]

      remove the first element from L;

      for $j = 2, \ldots, l-1$ do

         set $\beta_j \leftarrow$ the i_jth real root of $A_j(\beta_1, \ldots, \beta_{j-1}, x_j)$ in (a_j, b_j), previously stored in T;

      for $i = 1, \ldots, N$ do

         set $\beta_l \leftarrow$ the ith real root of $A_l(\beta_1, \ldots, \beta_{l-1}, x_l)$ in (a_l, b_l), previously stored in T;

         compute isolating intervals for the real roots $\gamma_1 < \gamma_2 < \cdots < \gamma_{N'}$ of $A_{l+1}(\beta_1, \ldots, \beta_l, x_{l+1})$ in (a_{l+1}, b_{l+1});

         store in T as the children of the node containing β_l every γ_k;

         if $N' > 0$ then

            append $(i_2, \ldots, i_{l-1}, i, N')$ to L';

   set $L \leftarrow L'$;

5. Return b_1' and L. □

Theorem 5.2. *The above algorithm is correct.*

Proof. We shall show that, if the algorithm does not report failure, then $A_2, \ldots, A_n$ have no common zeros in $[\alpha_1, b_1'] \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$. A symmetric argument will show that, if failure is not reported, then $A_2, \ldots, A_n$ have no common zeros in $[\alpha_1, b_1'] \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{b_n\}$. Assume that failure is not reported. Then step 1 successively computes a sequence of non-zero polynomials $R_{n-1}(x_1, \ldots, x_{n-1}), \ldots, R_1(x_1)$, which are the successive values of the program variable R.

Now step 2 ensures that $R_1(x_1)$ has no root in $(\alpha_1, b_1']$. Assume that there is some common zero $(\beta_1, \ldots, \beta_n)$ of the polynomials $A_2, \ldots, A_n$ in $(\alpha_1, b_1'] \times \mathbf{R} \times \cdots \times \mathbf{R} \times \{a_n\}$. Then, for $2 \leq i \leq n-1$, $A_i(\beta_1, \ldots, \beta_i) = 0$, and $R_{n-1}(\beta_1, \ldots, \beta_{n-1}) = 0$. Therefore, for $i = n-2, n-3, \ldots, 1$, $R_i(\beta_1, \ldots, \beta_i) = 0$, by Theorem 5 of Collins (1971). In particular, $R_1(\beta_1) = 0$. But $\beta_1 \in (\alpha_1, b_1']$, contradicting the property that $(\alpha_1, b_1']$ contains no root of $R_1(x_1)$. We conclude that $A_2, \ldots, A_n$ have no common zeros in $(\alpha_1, b_1'] \times \mathbf{R} \times \cdots \times \mathbf{R} \times \{a_n\}$, hence in $[\alpha_1, b_1'] \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$.

It remains to show that $A_2, \ldots, A_n$ have no common zeros in $\{\alpha_1\} \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$. Assume that there is some common zero $(\beta_1, \ldots, \beta_n)$ of $A_2, \ldots, A_n$ in

$\{\alpha_1\} \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$. Then, for $2 \le i \le n-1$, $A_i(\alpha_1, \beta_2, \ldots, \beta_i) = 0$. Hence, for $i = 2, 3, \ldots, n-1$, $\beta_i = \alpha_i$, because (a_i, b_i) is a strong isolating interval for α_i as a root of $A_i(\alpha_1, \ldots, \alpha_{i-1}, x_i) \neq 0$. Therefore $A_n(\alpha_1, \ldots, \alpha_{n-1}, a_n) = 0$, contrary to the hypothesis that (a_n, b_n) is a strong isolating interval for α_n as a root of $A_n(\alpha_1, \ldots, \alpha_{n-1}, x_n) \neq 0$. We conclude that $\{\alpha_1\} \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$ contains no common zeros of $A_2, \ldots, A_n$. The proof that $A_2, \ldots, A_n$ have no common zeros in $[\alpha_1, b_1'] \times [a_2, b_2] \times \cdots \times [a_{n-1}, b_{n-1}] \times \{a_n\}$ is complete.

In order to prove that the algorithm correctly delivers a list L according to the specification stated as output assertion 2, we need to consider the following assertion, which we will denote by I. It will be shown that I is an invariant of the main loop of step 4.

> $l \ge 2$; T is a tree of algebraic numbers such that T has root $\beta_1 = b_1'$ and T has height at most l; L is a list of $(l-1)$-tuples such that $(i_2, \ldots, i_{l-1}, N)$ belongs to L if and only if, with σ_1 denoting $(\alpha_1, b_1]$, there exists a path $\beta_1, \ldots, \beta_{l-1}$ in T such that:
>
> for $2 \le j \le l-1$, A_j has at least i_j sections over σ_{j-1} adjacent to $(\alpha_1, \ldots, \alpha_j)$, with σ_j denoting the i_jth such section, β_j is the i_jth real root of $A_j(\beta_1, \ldots, \beta_{j-1}, x_j)$ in (a_j, b_j), and the point $(\beta_1, \ldots, \beta_j)$ belongs to σ_j; $N > 0$ is the number of sections of A_l over σ_{l-1} which are adjacent to $(\alpha_1, \ldots, \alpha_l)$, the children of β_{l-1} are the N real roots of $A_l(\beta_1, \ldots, \beta_{l-1}, x_l)$ in (a_l, b_l); for $1 \le i \le N$, if β_l denotes the ith child of β_{l-1} and σ_l denotes the ith section of A_l over σ_{l-1} adjacent to $(\alpha_1, \ldots, \alpha_l)$, then $(\beta_1, \ldots, \beta_l) \in \sigma_l$.

Assertion I is established by step 3, if the reader allows the first statement of step 4, namely the assignment of 2 to the variable l, to be considered as part of step 3. [That N is equal to the number of sections of A_2 over σ_1 which are adjacent to (α_1, α_2) is proved by exhibiting a bijection ϕ between the set of sections of A_2 over σ_1 adjacent to (α_1, α_2) and the set of real roots of $A_2(\beta_1, x_2)$ in (a_2, b_2). The mapping ϕ is defined and proved bijective by analogy with the definition and proof of bijectivity of the mapping ϕ in the proof of Theorem 2.3.]

Let us now show that our assertion I is indeed an invariant of the outermost loop ("for $l = 2, \ldots, n-1$ do ...") of step 4. Assume that I is true prior to an iteration of this outermost loop. The "while" loop ("while $L \neq ()$ do ...") processes every element of L, in turn. Let us consider what happens for a typical element $(i_2, \ldots, i_{l-1}, N)$ of L. By the assertion I, there is a path $\beta_1, \ldots, \beta_{l-1}$ in T such that, for $2 \le j \le l-1$, β_j is the i_jth real root of $A_j(\beta_1, \ldots, \beta_{j-1}, x_j)$ in (a_j, b_j). So the first "for" loop ("for $j = 2, \ldots, l-1$ do ...") works correctly.

Let $1 \le i \le N$, let β_l be the ith child of β_{l-1} and let σ_l be the ith section of A_l over σ_{l-1} adjacent to $(\alpha_1, \ldots, \alpha_l)$. [Note that β_l and σ_l are well defined, by the assertion I.] We compute isolating intervals for the real roots $\gamma_1 < \cdots < \gamma_{N'}$ of $A_{l+1}(\beta_1, \ldots, \beta_l, x_{l+1})$ in (a_{l+1}, b_{l+1}). We must prove that A_{l+1} has N' sections over σ_l which are adjacent to $(\alpha_1, \ldots, \alpha_{l+1})$. We shall do this by exhibiting a bijection ϕ from the set of sections of A_{l+1} over σ_l adjacent to $(\alpha_1, \ldots, \alpha_{l+1})$ to the set of real roots of $A_{l+1}(\beta_1, \ldots, \beta_l, x_{l+1})$ in (a_{l+1}, b_{l+1}).

Let σ_{l+1} be a section of A_{l+1} over σ_l which is adjacent to $(\alpha_1, \ldots, \alpha_{l+1})$. By Theorem 5.1, in which we take $n = l+1$ and $b_1 = \beta_1 (= b_1')$, σ_{l+1} and $\{\beta_1\} \times [a_2, b_2] \times \cdots \times [a_{l+1}, b_{l+1}]$ have non-empty intersection. That is, there is a point $(\beta_1, \beta_2', \ldots, \beta_{l+1}')$ of σ_{l+1} such that $\beta_j' \in [a_j, b_j]$, for all j in the range $2 \le j \le l+1$. It follows from the assertion I that $\beta_j' = \beta_j$, for all j, $2 \le j \le l+1$. Therefore $\beta_{l+1}' \in (a_{l+1}, b_{l+1})$. We define $\phi(\sigma_{l+1}) = \beta'_{l+1}$. Now ϕ is a one-to-one mapping since A_{l+1} is delineable over σ_l, by hypothesis.

Let β'_{l+1} be a real root of $A_{l+1}(\beta_1, \ldots, \beta_l, x_{l+1})$ in (a_{l+1}, b_{l+1}). Then the point $(\beta_1, \ldots, \beta_l)$ lies in σ_l, by assertion I, and hence the point $(\beta_1, \ldots, \beta_l, \beta'_{l+1})$ lies in some section σ_{l+1} of A_{l+1} over σ_l. By Theorem 5.1, in which we take $n = l+1$ and $b_1 = \beta_1 (= b'_1)$, σ_{l+1} is adjacent to $(\alpha_1, \ldots, \alpha_{l+1})$. Clearly $\phi(\sigma_{l+1}) = \beta'_{l+1}$. We have shown that ϕ is also an onto mapping, completing the proof of bijectivity of ϕ, and of the assertion that A_{l+1} has N' sections over σ_l which are adjacent to $(\alpha_1, \ldots, \alpha_{l+1})$.

Therefore, the "for" loop ("for $i = 1, \ldots, N$ do …") extends T and L' in precisely the manner required so that, when every element of L has been processed, the assertion I will be valid, with $l+1$ in place of l and L' in place of L. Since the last two actions of the outermost loop body set L equal to L' and (effectively) l equal to $l+1$, the truth of assertion I is restored at the end of the loop iteration being considered.

We have shown that assertion I is established by step 3, and is preserved by each iteration of the main loop of step 4. Hence I is true upon termination of step 4, at which point $l = n$. The required assertion about L can now be read off from part of the invariant assertion I (in which we put $l = n$). □

We can give an intuitive argument that this algorithm is unlikely to fail. It suffices to argue that none of the resultants computed in Step 1 of the algorithm is likely to be zero. Let us call these successive resultants $R_{n-2}, R_{n-3}, \ldots, R_1$. A_{n-1} has only a finite number of irreducible factors. There is no relation between A_n and A_{n-1} that would lead one to expect that one of these irreducible factors would also be a factor of $A(x_1, \ldots, x_{n-1}, a_n)$, and thus R_{n-2} is unlikely to be zero. (A_{n-1} might be a factor of the discriminant of A_n, or a factor of the resultant of A_n with some other polynomial, but there is no reason to expect that this would lead to a common factor after the rational number a_n has been substituted for x_n in A_n.) Continuing, there is similarly no reason to expect that A_{n-2} and R_{n-2} would have a common factor. The argument is the same for $n-3, \ldots, 1$.

In spite of the probabilities, the algorithm might fail. But then there is an easy remedy: simply change the isolating interval $[a_n, b_n]$, for example, by refining the interval.

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