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Supergravity one-loop corrections on AdS_7 and AdS_3, higher spins and AdS/CFT

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Abstract

As was shown earlier, the one-loop correction in 10d supergravity on $AdS_5 \times S^5$ corresponds to the contributions to the vacuum energy and 4d boundary conformal anomaly which are minus the values for one $\mathcal{N} = 4$ Maxwell supermultiplet, thus reproducing the subleading term in the $N^2 - 1$ coefficient in the dual $SU(N)$ SYM theory. We perform similar one-loop computations in 11d supergravity on $AdS_7 \times S^4$ and 10d supergravity on $AdS_3 \times S^3 \times T^4$. In the AdS_7 case we find that the corrections to the 6d conformal anomaly a-coefficient and the vacuum energy are again minus the ones for one $(2, 0)$ tensor multiplet, suggesting that the total a-anomaly coefficient for the dual $(2, 0)$ theory is $4N^3 - 9/4N - 7/4$ and thus vanishes for $N = 1$. In the AdS_3 case the one-loop correction to the vacuum energy or 2d central charge turns out to be equal to that of one free $(4, 4)$ scalar multiplet, i.e. is $c = +6$. This reproduces the subleading term in the central charge $c = 6(Q_1 Q_5 + 1)$ of the dual 2d CFT describing decoupling limit of D5–D1 system. We also present the expressions for the 6d a-anomaly coefficient and vacuum energy contributions of general-symmetry higher spin field in AdS_7 and consider their application to tests of vectorial AdS/CFT with the boundary conformal 6d theory represented by free scalars, spinors or rank-2 antisymmetric tensors.

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1. Introduction

One of the key probes of the AdS/CFT correspondence [1–3] is the boundary theory conformal anomaly which is closely related to the simplest correlators of the stress tensor [4–6]. In the case of the duality between $\mathcal{N}=4$ $SU(N)$ SYM theory and string theory in $AdS_5\times S^5$ the gauge-theory result for the Weyl anomaly is $\mathcal{A}_4=-\mathrm{a}\mathcal{E}_4+\mathrm{c}\mathcal{W}_4$, $\mathrm{a}=\mathrm{c}=(N^2-1)k_1$ ($k_1=\frac{1}{4}$ is the contribution of a single $\mathcal{N}=4$ vector multiplet). It is determined by the 2- and 3-point correlators of stress tensor and should thus be exact. The N^2 term is indeed reproduced at strong coupling by the classical supergravity action [5].

It was suggested in [7,8][2] that the -1 term in N^2-1 coefficient should come from the one-loop 10d supergravity correction (the contribution of all massive string mode multiplets should vanish). This was recently confirmed in [10] where it was found that the contributions of the massless 5d supergravity modes and the massive S^5 KK modes to the boundary conformal anomaly can be universally described by a simple formula: $\mathrm{a}_p=\mathrm{c}_p=pk_1$, where $p=1$ for a vector multiplet (or boundary doubleton to be omitted), $p=2$ for the massless 5d supergravity modes, and $p=3,4,\dots$ for the massive KK levels. Summing over p using a special regularization prescription $\sum_{p=1}^{\infty}p=0$ (which is, in fact, required for consistency with the standard ζ-function regularization for the Casimir energy in 10d) gives indeed $(\mathrm{a}=\mathrm{c})_{\text{1-loop sugra}}=-1$.

Below will perform a similar one-loop computation of the boundary a-anomaly in the case of 11d supergravity on $AdS_7\times S^4$ (correcting an earlier attempt in [11]). This will determine the subleading N^0 term in the a-coefficient of conformal anomaly of the 6d (2, 0) theory describing N coincident M5-branes which should be dual to M-theory on $AdS_7\times S^4$.

In addition to the duality examples based on $AdS_5\times S^5$ and $AdS_7\times S^4$ supergravity backgrounds there is also the duality [1,12] between string theory in $AdS_3\times S^3\times T^4$ space supported by RR 3-form flux and 2d CFT corresponding to gauge theory describing low-energy limit D5–D1 system. The central charge of this CFT is $c=6(Q_1Q_5+1)$ [13,12] (Q_i are the number of branes). The leading $6Q_1Q_5$ can be reproduced from the classical action of 10d supergravity on $S^3\times T^4$ [5,14]. Here we shall demonstrate that the subleading $+6$ term is reproduced by the one-loop 10d supergravity contribution. This provides a non-trivial test of this AdS$_3$/CFT$_2$ duality.

1.1. AdS_7/CFT_6

The conformal anomaly of a classical Weyl invariant theory in 6d has the following general form [15–17]

$$\mathcal{A}_6=\mathrm{a}\mathcal{E}_6+W_6+D_6,\qquad W_6=\mathrm{c}_1I_1+\mathrm{c}_2I_2+\mathrm{c}_3I_3, \tag{1.1}$$

where $\mathcal{E}_6$ is the Euler density in six dimensions, W_6 is a combination of three independent Weyl invariants and D_6 is a total derivative term (which can be changed by adding a local counterterm and thus depends on a scheme). Omitting the derivative D_6 term, the conformal anomaly corresponding to a single 6d tensor multiplet [17] and the 6d conformal anomaly contribution coming from the classical 11d supergravity action on S^7 [5] (that should be representing the large N limit of the $(2,0)$ theory result) may be written as

[2] This follows the analogy with what happens in the case of the R-symmetry anomaly [9].

$$\mathcal{A}_6 = \mathrm{a}\mathcal{E}_6 + \mathrm{c}\mathcal{W}_6, \qquad \mathcal{W}_6 \equiv 96 I_1 + 24 I_2 - 8 I_3, \tag{1.2}$$

$$\mathrm{a}_{\mathrm{tens.}} = \frac{7}{4}, \qquad \mathrm{c}_{\mathrm{tens.}} = 1, \qquad \mathrm{a}_{(2,0)} = 4N^3 + \cdots, \qquad \mathrm{c}_{(2,0)} = 4N^3 + \cdots. \tag{1.3}$$

The fact that the anomaly in these two cases contains the same Weyl-invariant combination $\mathcal{W}_6$ (so that its Weyl-tensor or B-anomaly part is effectively parametrized by just one overall coefficient c) is related to non-renormalization of the ratio of the 2- and 3-point correlation functions of the corresponding stress tensor [18].[3]

By analogy with a subleading order-N term in the R-symmetry anomaly of $(2,0)$ theory [19] it was suggested in [20] that there should be also order N contributions to $\mathrm{a}_{(2,0)}$ and $\mathrm{c}_{(2,0)}$ coming from the R^4 term in the M-theory 11d effective action,

$$\mathrm{a}_{(2,0)} = 4N^3 - \frac{9}{4}N + a_1, \qquad \mathrm{c}_{(2,0)} = 4N^3 - 3N + c_1. \tag{1.4}$$

In [20] the further N^0 corrections a_1, c_1 were ignored, while the coefficients of order N terms were fixed so that the resulting $N^3 + N$ terms interpolated to $N = 1$ match the single tensor-multiplet anomalies in (1.3). As in the case of 10d supergravity on S^5, one may expect that a_1 and c_1 should be determined by the one-loop 11d supergravity correction [11].

Following the example of the D3-brane-based $AdS_5 \times S^5$ duality where the full anomaly coefficient $N^2 - 1$ vanishes for $N = 1$ it is natural to expect that here too the boundary singleton (single M5-brane tensor multiplet) should decouple and thus the full 6d anomaly of the $(2,0)$ theory should vanish for $N = 1$. This suggests that a_1 and c_1 should be non-zero and given by minus the values for a single tensor multiplet in (1.3)

$$a_1 = -\mathrm{a}_{\mathrm{tens.}} = -\frac{7}{4}, \qquad c_1 = -\mathrm{c}_{\mathrm{tens.}} = -1. \tag{1.5}$$

It was noted in [21] that the expression $\mathrm{c}_{(2,0)} = 4N^3 - 3N - 1 = (N-1)(2N+1)^2$ is exactly the same as the central charge of the A_{N-1} Toda theory at the "symmetric" coupling point (cf. also [22,23]).[4]

Here we shall provide support for (1.5) by showing that the one-loop 11d supergravity correction indeed produces the value $a_1 = -\mathrm{a}_{\mathrm{tens.}}$. Then the expected exact value of $\mathrm{a}_{(2,0)}$ is[5]

$$\mathrm{a}_{(2,0)} = 4N^3 - \frac{9}{4}N - \frac{7}{4} = (N-1)\left(4N^2 + 4N + \frac{7}{4}\right). \tag{1.6}$$

Below we shall consider the one-loop 11d supergravity on S^7 supergravity contributions in the case when the 6d boundary of AdS_7 is either S^6 (determining the a-anomaly part of $\mathcal{A}_6$) or

[3] The a-coefficient in 6d is related to 4-point stress tensor correlator and may thus receive a more non-trivial renormalization.

[4] 6d CFT with $(2,0)$ supersymmetry possess a protected sector of operators and observables related to a 2d chiral algebra [21] which is $\mathcal{W}$-algebra labeled by a simply-laced Lie algebra $\mathfrak{g}$ for a specific value of the central charge. In the $\mathfrak{g} = A_{N-1}$ case this leads to $c_1 = -1$.

[5] The non-vanishing 1-loop supergravity correction to the conformal anomaly implies that there should be also a similar correction also to the corresponding R-symmetry anomaly (i.e. $N \to N - 1$ in the I_8 term in the anomaly [24]) implying its vanishing for $N = 1$. The chiral anomaly of the boundary theory is accounted for by the Chern–Simons terms in the supergravity action. In the case of $AdS_5 \times S^5$ the 1-loop supergravity correction shifts the Chern–Simons coefficient $N^2 \to N^2 - 1$ [9]. A similar shift is then expected in the $AdS_7 \times S^7$ case where the CS term reproduces the leading N^3 anomaly and also the $\mathcal{O}(N)$ correction [19].

$R \times S^5$ (finding the vacuum or Casimir energy E_c). We will find that in both cases the result is minus that of a single tensor multiplet

$$\mathrm{a}_{\text{1-loop sugra}} = -\mathrm{a}_{\text{tens.}}, \qquad E_{c\,\text{1-loop sugra}} = -E_{c\,\text{tens.}}. \tag{1.7}$$

We shall use similar methods as in the $AdS_5 \times S^5$ case in [10], i.e. first determine the contributions to a and E_c coming from a generic AdS_7 higher spin filed in representation $(\Delta; h_1, h_2, h_3)$ of $SO(2,6)$ and then sum up the contributions of the relevant fields appearing in the supergravity spectrum.

We shall also apply our general expressions for $\mathrm{a}(\Delta; h_1, h_2, h_3)$ and $E_c(\Delta; h_1, h_2, h_3)$ to provide tests of the vectorial AdS/CFT duality [25–27] in the case when the boundary theory is represented by a free scalar, spinor or tensor singleton.

1.2. AdS3/CFT2

The 2d CFT dual to superstring in $AdS_3 \times S^3 \times T^4$ with RR charges Q_5, Q_1 is described by a coupled system of three $(4,4)$ supersymmetric multiplets (see [13,12] and [28] for a recent review): $U(Q_1)$ adjoint vector multiplet, $U(Q_1)$ adjoint hypermultiplet, and $U(Q_1) \times U(Q_5)$ bi-fundamental hypermultiplet. The contribution to 2d conformal anomaly of a single free $(4,4)$ hypermultiplet (with 4 real scalars and 4 real fermions) is $c = 4 + 4 \times \frac{1}{2} = 6$.[6] The 2d vector multiplet has an irrelevant kinetic term and thus contributes to anomaly only through measure (or ghost) factor, with single $U(1)$ vector giving negative contribution $c = -1$.[7] The $U(1)$ part of the vector multiplet is decoupled (representing the c.o.m. of the bound D5–D1 system) and thus the total central charge count is[8]

$$c = 6Q_1Q_5 + Q_1^2 - 6\big(Q_1^2 - 1\big) = 6Q_1Q_5 + 6, \tag{1.8}$$

where the first term is the contribution of bi-fundamental hypers, the second of adjoint hypers and the third one of the vectors (with the $U(1)$ part subtracted).[9]

A peculiarity of the 2d case is that here the subleading (for large Q_5) term in the central charge which is responsible for subtraction of the decoupled c.o.m. modes enters with plus rather than minus sign (as was in 4d and 6d examples). Still, we shall demonstrate below that as in the AdS_5 and AdS_7 cases this extra $+6$ term (which should be protected and thus receive contributions only from the BPS modes) is also reproduced on the dual AdS theory side by the corresponding one-loop correction in 10d supergravity on $AdS_3 \times S^3 \times M^4$ with $M^4 = T^4$ or $K3$.

More precisely, instead of computing directly the correction to the central c we shall determine the one-loop correction to the AdS_3 vacuum energy or S^1 Casimir energy in 2d; the latter should be directly related to the central charge [31]

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6 In 2d the conformal anomaly is $\mathcal{A}_2 = 4\pi b_2 = \mathrm{a}R$, $\mathrm{a} = \frac{1}{6}c$, so that $c = 1$ for one real scalar.

7 The contribution of "non-dynamical" 2d vector gauge field to the central charge is negative (-1) [29] just like that of non-dynamical 2d gravity (-26) [30]. The reason for this -1 contribution can be understood also by giving vector a mass by coupling it to a complex scalar so that it will not contribute to c; then the central charge of the scalar part is reduced by 1 as one scalar component is absorbed by the vector.

8 The same result is found by counting the $SU(2)$ chiral anomaly of the $(4,4)$ superconformal algebra [13,12].

9 The $(4,4)$ vector multiplet contains one 2d vector A_m, 4 scalars ϕ_i, 4 real spinors ψ_k and 3 auxiliary fields D_r, all having canonical dimensions (i.e. 1 for A_m and ϕ_i, $\frac{1}{2}$ for ψ_k and 2 for D_r). With these dimension assignments the corresponding 2d conformal anomaly can be found from the following dimensionless action (same as the standard one but with each kinetic term containing an extra ∂^{-2} factor) $\int d^2x[(\mathcal{A}_m^{\perp})^2 + \phi_i^2 + D_r\partial^{-2}D_r + \psi_k\partial^{-1}\psi_k]$. As a result, the total central charge contribution is $c = -1 + 0 + 3 \times (-1) + 4 \times (-\frac{1}{2}) = -6$.

$$E_c = -\frac{1}{12}c, \quad \text{i.e. } c = 6 \leftrightarrow E_c = -\frac{1}{2}. \tag{1.9}$$

We shall find that the one-loop supergravity contribution gives indeed $E_c = -\frac{1}{2}$ after summing over the contributions of the KK modes of 10d supergravity on $S^3 \times M^4$.

The rest of this paper is organized as follows. In Section 2 we shall present the expressions for the a-anomaly coefficient and the vacuum energy of a higher-spin field in AdS_7 corresponding to an arbitrary (massive or massless) representation of $SO(2,6)$, generalizing earlier results for symmetric tensors to mixed symmetry case.

In Section 3 we shall apply these results to compute the one-loop corrections to the 6d boundary a-anomaly and vacuum energy in 11d supergravity compactified on S^7 obtaining Eq. (1.7). As another application, in Section 4 we shall perform checks of vectorial AdS$_7$/CFT$_4$ duality in the cases when the boundary 6d theory is represented by free scalars, spinors or (self-dual) rank-2 tensors. We shall find that matching of both a-anomaly and Casimir energy requires particular shifts of the inverse coupling of the AdS_7 higher spin theory.

In Section 5 we shall turn to the case of 10d supergravity in $AdS_3 \times S^3 \times M^4$ and compute the corresponding one-loop correction to the vacuum energy, demonstrating that it is equal to $-\frac{1}{2}$ as in (1.8), thus deriving the subleading term in the central charge (1.9) on the dual string theory side.

There are several technical appendices. In Appendix A we present the expressions for the Casimir energy, a-anomaly and partition function for the fields of the free (2, 0) multiplet in 6d. In Appendix B we derive the 6d boundary a-anomaly coefficient corresponding to a generic higher spin field on AdS_7 using spectral ζ-function method. Appendix C collects decompositions of tensor products of two $SO(2,6)$ singleton representations with spin 0, $\frac{1}{2}$, 1 into infinite sums of other representations and the corresponding relations for the characters. These Flato–Fronsdal like relations are used in the discussion of applications to vectorial AdS/CFT duality in Section 4. Appendix D contains discussion of some properties of the Casimir energy of spin 0, $\frac{1}{2}$, 1 singletons in AdS_{d+1} for general d. They are useful in comparing the 6d results to the previously studied 4d case. In Appendix E we list the explicit field content of the $SU(2,2|1) \times SU(2,2|1)$ building blocks appearing in the Kaluza–Klein towers of 6d supergravity compactified on S^3. Appendix F contains the discussion of the relation between the expression for the 2d Casimir energy in Section 5 and the 2d central charge derived [32] using AdS_3 method for short $SU(2,2|1) \times SU(2,2|1)$ multiplets.

2. Casimir energy and a-anomaly for generic higher spin fields in AdS_7

Given a generic conformal field in 6d we may associate to it a field in AdS_7 corresponding to the same representation of $SO(2,6)$. That allows to interpret the one-loop contributions for a field in AdS_7 in terms of Casimir energy and conformal anomaly of the boundary field (see [10] and refs. there).

The $SO(2,6)$ conformal group representations will be denoted as $(\Delta; \mathbf{h})$ where $\mathbf{h} = (h_1, h_2, h_3)$ are the $SO(6)$ highest weights or Young tableu labels (h_i are all integers or all half-integers with $h_1 \geq h_2 \geq |h_3|$).[10]

[10] An alternative is to use $SO(6)$ Dynkin labels $[r_1, r_2, r_3] = (h_2 - h_3, h_1 - h_2, h_2 + h_3)$.

The unitary irreducible representations of $SO(2,6)$ have (see, e.g., [33])

$$\begin{array}{llll} \text{(i)} & \Delta \geq \Delta = h_1 + 4, & \text{for } h_1 > h_2 \geq |h_3|, \\ \text{(ii)} & \Delta \geq \Delta = h_1 + 3, & \text{for } h_1 = h_2 > |h_3|, \\ \text{(iii)} & \Delta \geq \Delta = h_1 + 2, & \text{for } h_1 = h_2 = \pm h_3, \\ \text{(iv)} & \Delta \geq 2 \text{ or } \Delta = 0 & \text{for } h_1 = h_2 = h_3 = 0. \end{array} \tag{2.1}$$

If Δ does not saturate the above inequalities then the character of the corresponding *massive* representation is[11]

$$\widehat{\mathcal{Z}}^+(\Delta; \mathbf{h}) = \mathrm{d}(\mathbf{h}) \frac{q^\Delta}{(1-q)^6}, \tag{2.2}$$

where $\mathrm{d}(\mathbf{h})$ is the multiplicity of the representation

$$\begin{aligned} \mathrm{d}(\mathbf{h}) = \frac{1}{12}&(1 + h_1 - h_2)(1 + h_2 - h_3)(1 + h_2 + h_3)(2 + h_1 - h_3)(2 + h_1 + h_3) \\ &\times (3 + h_1 + h_2). \end{aligned} \tag{2.3}$$

If Δ is at one of the unitarity bounds the corresponding representation is short or *massless* (i.e. corresponds to a massless field in AdS_7 space)[12] and its character requires a proper subtraction of null states and their descendants. For the $\Delta = h_1 + 4$ case in (i) in (2.1) we have the following massless representation character

$$\mathcal{Z}^+(h_1 + 4; h_1, h_2, h_3) = \widehat{\mathcal{Z}}^+(h_1 + 4; h_1, h_2, h_3) - \widehat{\mathcal{Z}}^+(h_1 + 5; h_1 - 1, h_2, h_3), \tag{2.4}$$

where $\widehat{\mathcal{Z}}^+$ is given in (2.2). For the massless $\Delta = h_1 + 3$ case with $h_1 = h_2 = h > |h_3|$ in (ii) we get

$$\begin{aligned} \mathcal{Z}^+(h + 3; h, h, h_3) = \widehat{\mathcal{Z}}^+(h_1 + 3; h, h, h_3) &- \widehat{\mathcal{Z}}^+(h + 4; h, h - 1, h_3) \\ &+ \widehat{\mathcal{Z}}^+(h + 5; h - 1, h - 1, h_3). \end{aligned} \tag{2.5}$$

In the massless case of (iii) with $\Delta = h+2$ and $\mathbf{h} = (h, h, \pm h)$ which corresponds to the *singleton* representation the character is

$$\begin{aligned} \mathcal{Z}^+(h + 2; h, h, \pm h) = \widehat{\mathcal{Z}}^+(h + 2; h, h, \pm h) &- \widehat{\mathcal{Z}}^+\big(h + 3; h, h, \pm(h - 1)\big) \\ &+ \widehat{\mathcal{Z}}^+\big(h + 4; h, h - 1, \pm(h - 1)\big) \\ &- \widehat{\mathcal{Z}}^+\big(h + 5; h - 1, h - 1, \pm(h - 1)\big). \end{aligned} \tag{2.6}$$

In particular, it is possible to view the $(2, 0)$ tensor multiplet as supersingleton [35] which is a combination of 6d singletons with $h = 0, \frac{1}{2}, 1$: the one-particle partition functions for a scalar ϕ, Majorana–Weyl fermion ψ and self-dual tensor T are the characters of the corresponding singleton representations (see also Appendices A and C)

[11] The label $^+$ indicates that this will represent the partition function of the corresponding AdS_7 field with standard (Dirichlet) boundary conditions. Same quantity without $^+$ corresponds to associated conformal field in boundary theory (see [10] for details). $\widehat{\ }$ indicates massive representation character.

[12] In general [34], given a field in AdS_{d+1} (with even d) corresponding to $SO(2,d)$ representation $(\Delta; h_1, h_2, \dots, h_{\frac{d}{2}})$ where first $k = 0, 1, 2, \dots$ raws of the $SO(d)$ Young tableu may be equal, i.e. $h_1 = \dots = h_k > h_{k+1} \geq h_{k+2} \geq \dots \geq h_{\frac{d}{2}}$, this field is massless if $\Delta = h_k - k + d - 2$. In the case of (2.1) where $d = 6$ the lower bounds in (i), (ii) and (iii) correspond to $k = 0, 1, 2$.

$$\mathcal{Z}_\phi = \mathcal{Z}_{\{0\}} = \mathcal{Z}^+(2;0,0,0), \qquad \mathcal{Z}_\psi = \mathcal{Z}_{\{\frac{1}{2}\}} = \mathcal{Z}^+\Big(\frac{5}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2}\Big),$$
$$\mathcal{Z}_T = \mathcal{Z}_{\{1\}} = \mathcal{Z}^+(3;1,1,1). \tag{2.7}$$

From one-particle partition function $\mathcal{Z}(q)$ given by the corresponding $SO(2,6)$ character one can extract the expression for the Casimir energy E_c as [36]

$$E_c = \frac{1}{2}(-1)^F \sum_n \mathrm{d}_n \omega_n = \frac{1}{2}(-1)^F \zeta_E(-1), \tag{2.8}$$

$$\zeta_E(z) = \sum_n \frac{\mathrm{d}_n}{\omega_n^z} = \frac{1}{\Gamma(z)} \int_0^\infty d\beta\, \beta^{z-1} \mathcal{Z}\big(e^{-\beta}\big). \tag{2.9}$$

For a generic massive representation $(\Delta;\mathbf{h})$ with the character (2.2) the corresponding Casimir energy is found to be ($\bar{h} \equiv h_1 + h_2 + h_3$)

$$\widehat{E}_c^+(\Delta;\mathbf{h}) = \frac{(-1)^{2\bar{h}}\mathrm{d}(\mathbf{h})}{120\,960}(\Delta-3) \\ \times \big[12(\Delta-3)^6 - 126(\Delta-3)^4 + 336(\Delta-3)^2 - 191\big]. \tag{2.10}$$

The expression for the a-anomaly can be found from the one-loop partition function on Euclidean AdS_7 as explained in Appendix B

$$\begin{aligned}\widehat{\mathrm{a}}^+(\Delta;\mathbf{h}) = {} & \frac{(-1)^{2\bar{h}}\mathrm{d}(\mathbf{h})}{2\times 96\times 37\,800}(\Delta-3)\big[15(\Delta-3)^6 \\ & - 21(\Delta-3)^4\big[h_3^2 + h_1(h_1+4) + h_2(h_2+2) + 5\big] \\ & + 35(\Delta-3)^2\big[(h_1+2)^2(h_2+1)^2 + \big(h_1(h_1+4) + h_2(h_2+2) + 5\big)h_3^2\big] \\ & - 105(h_1+2)^2(h_2+1)^2h_3^2\big].\end{aligned} \tag{2.11}$$

In the case of short representations saturating a unitarity bound one needs to combine the massive representation expression as in (2.4), (2.5), (2.6).

In the special case of the totally symmetric massive spin s tensor representation with $\mathbf{h} = (s,0,0)$, the expression (2.11) can be written in the following alternative form

$$\begin{aligned}\widehat{\mathrm{a}}^+(\Delta;s,0,0) = {} & \frac{5(s+2)(s+3)!}{8(6!)^2\pi s!} \\ & \times \int_3^\Delta dx\,(x-3)(x+s-1)(x-s-5)\Gamma(x-1)\Gamma(5-x)\sin(\pi x),\end{aligned} \tag{2.12}$$

which is in agreement with the earlier result in [32,26].

3. One-loop correction to vacuum energy and a-anomaly in 11d supergravity on $AdS_7 \times S^4$

Let us now apply the above results (2.10) and (2.11) to compute the corresponding total contribution of the fields in the spectrum of 11d supergravity compactified on S^4. The corresponding

Table 1
$SO(2,6)\times USp(4)$ representations of the fields of 11d supergravity on $AdS_7\times S^4$.

	$(\Delta;h_1,h_2,h_3)$	$USp(4)$
$p\geq 2$	$(2p;0,0,0)$	$[0,p]$
	$(2p+\frac{1}{2};\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$[1,p-1]$
	$(2p+1;1,1,-1)$	$[0,p-1]$
	$(2p+1;1,0,0)$	$[2,p-2]$
	$(2p+\frac{3}{2};\frac{3}{2},\frac{1}{2},-\frac{1}{2})$	$[1,p-2]$
	$(2p+2;2,0,0)$	$[0,p-2]$
$p\geq 3$	$(2p+\frac{3}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$[3,p-3]$
	$(2p+2;1,1,0)$	$[2,p-3]$
	$(2p+\frac{5}{2}\frac{3}{2},\frac{1}{2},\frac{1}{2})$	$[1,p-3]$
	$(2p+3;1,1,1)$	$[0,p-3]$
$p\geq 4$	$(2p+2;0,0,0)$	$[4,p-4]$
	$(2p+\frac{5}{2};\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$[3,p-4]$
	$(2p+3;1,0,0)$	$[2,p-4]$
	$(2p+\frac{7}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$[1,p-4]$
	$(2p+4;0,0,0)$	$[0,p-4]$

KK spectrum [37,35,38] is given in Table 1 (see also [39]). The massless level $p=2$ correspond to the fields of maximal gauged 7d supergravity with AdS_7 vacuum.

Contributions of the AdS_7 fields should be summed with multiplicities corresponding to their $USp(4)=SO(5)$ representations.[13]

Using (2.10) to sum of the vacuum energy contributions at each level p we find

$$E^+_{c,p=2}=-\frac{325}{384},\qquad E^+_{c,p=3}=-\frac{925}{384},\qquad E^+_{c,p\geq 4}=-\frac{25}{384}\big(6p^2-6p+1\big). \tag{3.1}$$

The value for the massless multiplet $p=2$ is in agreement with [36]. The expressions for the a-anomaly are similar

$$\mathrm{a}^+_{p=2}=-\frac{91}{1152},\qquad \mathrm{a}^+_{p=3}=-\frac{259}{1152},\qquad \mathrm{a}^+_{p\geq 4}=-\frac{7}{1152}\big(6p^2-6p+1\big). \tag{3.2}$$

Recalling that for one (2, 0) tensor multiplet (see Appendix A)

$$E_{c,\text{tens.}}=E^+_{c,1}=-\frac{25}{384},\qquad \mathrm{a}_{\text{tens.}}=\mathrm{a}^+_1=-\frac{7}{1152}. \tag{3.3}$$

we observe that, remarkably, both the vacuum energy and a-anomaly has the following universal expressions for any value of $p=1,2,3,\dots$

$$E^+_{c,p}=\big(6p^2-6p+1\big)E_{c,\text{tens.}},\qquad \mathrm{a}^+_p=\big(6p^2-6p+1\big)\mathrm{a}_{\text{tens.}}. \tag{3.4}$$

This is the direct analog to what was found in the case of 10d supergravity on $AdS_5\times S^5$ in [10] where the role of tensor multiplet was played by $\mathcal{N}=4$ vector one (or superdoubleton) and instead of the coefficient $6p^2-6p+1$ we had simply p.[14]

To sum over p we shall use the same prescription as in [10], i.e. introducing a sharp cutoff and dropping all divergent terms[15]

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13 The dimension of the $USp(4)$ representation $[a,b]$ (a,b are Dynkin labels) is $\dim(a,b)=\frac{1}{6}(a+1)(b+1)(a+b+2)(a+2b+3)$.

14 For comparison, in the case of 11d supergravity on $AdS_4\times S^7$ one finds [40,41] that the contributions to the AdS_4 vacuum energy sum up to zero at each level p separately, i.e. $E^+_{c,p}=0$. The boundary conformal anomaly also vanishes as the boundary is 3-dimensional.

15 Explicitly, $\sum_{p=1}^{P}(6p^2-6p+1)=2P^3-P\to 0$.

$$\sum_{p=1}^{\infty}\big(6p^2-6p+1\big)=0. \tag{3.5}$$

This prescription can be justified by using the spectral ζ-function regularization directly in 11d, i.e. before explicitly expanding in modes of S^4 (see below); it is such a regularization that should be consistent with diffeomorphism symmetry of 11d theory.

Assuming (3.5) we conclude that if the boundary (2, 0) singleton were included in the spectrum of 11d supergravity, the total vacuum energy and a-anomaly would vanish. However, it should be left out representing gauge degrees of freedom. Thus we conclude that the total one-loop supergravity contributions are exactly minus the tensor multiplet ones

$$\sum_{p=2}^{\infty}E^{+}_{c,p}=-E^{+}_{c,1}=-E_{c,\text{tens.}},\qquad \sum_{p=2}^{\infty}\mathrm{a}^{+}_{p}=-\mathrm{a}^{+}_{1}=-\mathrm{a}_{\text{tens.}}, \tag{3.6}$$

as claimed in (1.7).

Let us now demonstrate that the prescription (3.5) is indeed equivalent to the use of spectral ζ-function directly in 11d theory. We shall consider the case of the Casimir energy (for a similar discussion on 10d case see [10]). For a massive 7d field in representation $(\Delta;\mathbf{h})$ the vacuum energy can be extracted from the partition function (2.2) that we may write in the form

$$\mathcal{Z}^{+}(\Delta;\mathbf{h})=\mathrm{d}(\mathbf{h})\sum_{n=0}^{\infty}\binom{n+5}{5}q^{\Delta+n}. \tag{3.7}$$

Then from (2.8), (2.9) we obtain a formal (divergent) expression for E_c

$$\widehat{E}^{+}_{c}(\Delta;\mathbf{h})=\sum_{n=0}^{\infty}e_n(\Delta;\mathbf{h}),\qquad e_n(\Delta;\mathbf{h})=\frac{1}{2}(-1)^{2\bar{h}}\mathrm{d}(\mathbf{h})\binom{n+5}{5}(\Delta+n). \tag{3.8}$$

This sum can be computed using the ζ-function regularization applied to the full effective energy eigenvalue $\Delta+n$, or, equivalently, by introducing an exponential cutoff via $e_n\to e_n e^{-\epsilon(\Delta+n)}$, doing the sum, expanding in $\epsilon\to 0$, and finally dropping all singular terms. Keeping ϵ finite we may find the contribution to the sum (3.8) from all KK states (taking into account that $p=2$ states are massless, cf. (2.4), (2.5)). Denoting the total summand from level p as $e_n(p;\epsilon)$ and, summing over both n and $p=1,2,\dots$, we obtain

$$\begin{aligned}\sum_{p=1}^{\infty}\sum_{n=0}^{\infty}e_n(p;\epsilon)&=\frac{e^{2\epsilon}}{(e^{\epsilon/2}-1)^3(e^{\epsilon/2}+1)^{11}(e^{\epsilon}+1)^5}\big(20e^{\epsilon/2}+50e^{\epsilon}+100e^{3\epsilon/2}+178e^{2\epsilon}\\&\quad+260e^{5\epsilon/2}+343e^{3\epsilon}+400e^{7\epsilon/2}+428e^{4\epsilon}+400e^{9\epsilon/2}+343e^{5\epsilon}\\&\quad+260e^{11\epsilon/2}+178e^{6\epsilon}+100e^{13\epsilon/2}+50e^{7\epsilon}+20e^{15\epsilon/2}+5e^{8\epsilon}+5\big)\\&=\frac{785}{2048\epsilon^3}+\mathcal{O}(\epsilon).\end{aligned} \tag{3.9}$$

Thus the finite part of the sum over $p\geq 1$ vanishes in agreement with (3.5). Equivalently,

$$\sum_{p=2}^{\infty}\sum_{n=0}^{\infty}e_n(p;\epsilon)=-\sum_{n=0}^{\infty}e_n(1;\epsilon)=-\frac{5}{16\epsilon^2}+\frac{25}{384}+\cdots, \tag{3.10}$$

in agreement with (3.3), (3.6).

4. Vectorial AdS$_7$/CFT$_6$ duality

As in lower dimensions, we may start with a free CFT in 6d described, e.g., by N (complex or real) scalars, spinors or rank-2 antisymmetric tensors and consider the duality between its singlet sector represented by the corresponding bilinear conserved currents and higher spin theory in AdS_7 (see, e.g., [26]). The representation content of the 7d theory is determined from the Flato–Fronsdal type decomposition of the product of 2 singleton representations into sum of higher-spin $SO(2,6)$ representations described in Appendix C (see also [10]). Then using the general expressions for the Casimir energy (2.10) and a-anomaly coefficient (2.11) given in Section 2 we may study the matching of these quantities on the two sides of the duality. In what follows we shall denote by K^+ the two quantities a^+ and E_c^+ corresponding to AdS_7 field in a generic massless $SO(2,6)$ representation and also use $K = -2K^+$ for the associated boundary conformal field values.

Starting with the case of a free conformal scalar boundary 6d theory, the corresponding fields of the dual AdS_7 theory ("type A" theory) are massless totally symmetric tensors with spin s, for which we find from (2.10), (2.11)

$$E_c^+(s+4;s,0,0) = -\frac{1}{483\,840}\nu^2\big(12\nu^3 - 58\nu^2 - 6\nu + 117\big), \qquad \nu \equiv (s+1)(s+2) \tag{4.1}$$

$$\mathrm{a}^+(s+4;s,0,0) = -\frac{1}{29\,030\,400}\nu^2\big(22\nu^3 - 55\nu^2 - 4\nu + 2\big). \tag{4.2}$$

The Casimir energy (4.1) is a simple extension of the results in [27]. The a-anomaly expression (4.2) is the same as found in [26]. To sum over spins we shall follow the spectral ζ-function prescription of [26] which is equivalent to introducing the cutoff $e^{-\epsilon(s+\frac{d-3}{2})} = e^{-\epsilon(s+\frac{3}{2})}$ and dropping all singular terms in the limit $\epsilon \to 0$, i.e.

$$\sum_{s=1}^{\infty} K(s) \equiv \sum_{s=1}^{\infty} e^{-\epsilon(s+\frac{3}{2})} K(s)\Big|_{\text{finite part, } \epsilon\to 0}. \tag{4.3}$$

Below we shall use the same prescription also for mixed representations with $s \equiv \Delta - 4$.

One can then verify the following relations

$$K^+(4;0,0,0) + \sum_{s=1}^{\infty} K^+(4+s;s,0,0) = 0, \tag{4.4}$$

$$K^+(4;0,0,0) + \sum_{s=2,4,\dots}^{\infty} K^+(4+s;s,0,0) = K_\phi, \tag{4.5}$$

where $K_\phi = (\mathrm{a}_\phi, E_{c\,\phi})$ are the real scalar values from (A.1) and (A.5). As discussed in Appendix C, the l.h.s. of (4.4) corresponds to the representation content of the tensor product of two scalar singletons and the associated sum of characters is equal to the partition function of the singlet sector of the 6d $U(N)$ invariant theory of N free complex scalars, see (C.5). The vanishing to the r.h.s. of (4.5) is consistent with the expectation that the a-anomaly and Casimir energy of the $U(N)$ 6d CFT which are proportional to N should be exactly reproduced by the classical action of "non-minimal" type A higher spin theory in AdS_7 with the inverse coupling $G^{-1}_{\text{non-min}} \sim N$, so that the one-loop HS correction should vanish [25,26].

The l.h.s. of (4.5) corresponds the field content of the "minimal" type A theory in AdS_7 which should be dual to singlet sector of $O(N)$ invariant free real scalar 6d theory, with the partition function relation given by (C.8) (for similar relations in the case of 3d and 4d cases see [27,10]). Here the non-vanishing r.h.s. may be canceled against part of the classical contribution of non-minimal type A theory if one assumed that in this case $G^{-1}_{\rm min} \sim N-1$ [25,27].

Similarly, in the case when the boundary 6d theory is the $U(N)$ invariant free complex (Weyl) fermion theory or $O(N)$ invariant free Majorana–Weyl fermion theory (with the dual theory being non-minimal or minimal type B theory in AdS_7) we get

$$\sum_{s=1}^{\infty}\big[K^+(4+s;s,1,1)+K^+(4+s;s,0,0)\big]=0, \tag{4.6}$$

$$\sum_{s=2,4,\dots}^{\infty} K^+(4+s;s,1,1)+\sum_{s=1,3,\dots}^{\infty} K^+(4+s;s,0,0)=K_\psi, \tag{4.7}$$

where the field content corresponds to the one in the r.h.s. of (C.3), (C.6) and (C.9) and K_ψ is given in (A.1), (A.5). Here we have also other representations than totally symmetric tensors and thus require general expressions in (2.10), (2.11). As in the scalar case, the non-vanishing r.h.s. of (4.7) may be compensated by assuming that the coupling constant of minimal type B theory is $G^{-1}_{\rm min} \sim N-1$.

When the 6d boundary theory is described by N real or complex self-dual 2-tensors with dual theory being non-minimal or minimal "type C" theory in AdS_7 we find (see (C.4), (C.7), (C.10) and (A.1), (A.5))

$$\sum_{s=2}^{\infty}\big[K^+(4+s;s,2,2)+K^+(4+s;s,1,1)+K^+(4+s;s,0,0)\big]=-K_T, \tag{4.8}$$

$$\begin{aligned}\sum_{s=2,4,\dots}^{\infty}\big[K^+(4+s;s,2,2)+K^+(4+s;s,0,0)\big]+\sum_{s=3,5,\dots}^{\infty} K^+(4+s;s,1,1)\\ =\frac{1}{2}K_T.\end{aligned} \tag{4.9}$$

Here the non-vanishing result is found in both non-minimal and minimal cases. This is similar to what was found in the case of the AdS$_5$/CFT$_4$ duality with the boundary theory represented by N complex or real Maxwell vectors [10,42]. The (real) vector corresponds to the parity invariant singleton combination $\{1\}_c=(2;1,0)+(2;0,1)$ in the $SO(2,4)$ notation.[16] There the r.h.s. of the analogs of Eqs. (4.8) and (4.9) for the non-minimal and minimal type C theories was the same $2K_V$, i.e. twice a single 4d real vector contribution, implying the same -2 shift of couplings, i.e. $G^{-1}_{\rm non\text{-}min} \sim 2N-2$ and $G^{-1}_{\rm min} \sim N-2$.

In the present case of the 6d self-dual tensor multiplet theory corresponding to chiral $\{1\}$ singleton Eqs. (4.8) and (4.9) imply instead $G^{-1}_{\rm non\text{-}min} \sim 2N+1$ and $G^{-1}_{\rm min} \sim N-\frac{1}{2}$. Considering instead the full (self-dual + anti-self-dual) tensor represented by $\{1\}_c=(3;1,1,1)+(3;1,1,-1)$ (see (C.1), (C.12)) one finds that the r.h.s. of the analogs of (4.8) and (4.9) become $-2K_T$ and 0 respectively (for the values of E_c see (D.6), (D.10)). This implies that in the AdS_7 theory dual to the 6d theory of N complex 6d tensors $G^{-1}_{\rm non\text{-}min} \sim 2N-1$ and $G^{-1}_{\rm min} \sim 2N$.

The l.h.s. of the above relations (4.4), (4.6) and (4.8) correspond to K of the products of singletons $\{0\}\times\{0\}$, $\{\frac{1}{2}\}\times\{\frac{1}{2}\}$, and $\{1\}\times\{1\}$ (see (C.2), (C.3), (C.4)). One can also consider

[16] Here we follow [10] and use the $SU(2)\times SU(2)$ weight notation for $SO(2,4)$ representation: $(\Delta; j_1, j_2)$, where $h_1=j_1+j_2$, $h_2=j_1-j_2$.

a generalization when each factor in the product is a linear combination of the singletons, i.e. $n_\phi\{0\}+n_\psi\{\frac{1}{2}\}+n_T\{1\}$. Then (4.4), (4.6), (4.8) are generalized to

$$K^+\left[\left(n_\phi\{0\}+n_\psi\left\{\frac{1}{2}\right\}+n_T\{1\}\right)\times\left(n_\phi\{0\}+n_\psi\left\{\frac{1}{2}\right\}+n_T\{1\}\right)\right] \\ =-n_T(n_\phi K_\phi+n_\psi K_\psi+n_T K_T), \tag{4.10}$$

where the l.h.s. is computed for the representation content appearing in the character relation in (C.11). For example, in the case when the boundary theory is described by N complex $(2,0)$ tensor multiplets we have $n_\phi=5$, $n_\psi=4$, $n_T=1$ we get

$$K^+\big(\{\text{tens.}\}\times\{\text{tens.}\}\big)=-K_{\text{tens.}},\qquad \{\text{tens.}\}=\{1\}+4\left\{\frac{1}{2}\right\}+5\{0\}, \tag{4.11}$$

where the tensor multiplet values of $K_{\text{tens.}}$ are given in (3.3). This may be compared with the relation found in the case of $\mathcal{N}=4$ vector multiplet in 4d [10,42]: $K^+(\{\text{vect.}\}\times\{\text{vect.}\})=2K_{\text{vect.}}$.

5. One-loop vacuum energy in 10d supergravity on $AdS_3\times S^3\times M^4$

As discussed in the Introduction, one may also perform a similar one-loop computations in the supergravity sector of type IIB superstring on $AdS_3\times S^3\times T^4$ to determine the subleading term in the central charge (1.8) or the vacuum energy (1.9).[17]

The one-loop AdS_3 vacuum energy can be computed by starting with the spectrum of 6d supergravity on $AdS_3\times S^3$ as massive KK multiplets on $M^4=T^4$ should not contribute due to supersymmetric cancellation. More generally, we may consider in parallel the cases of IIA or IIB supergravities on $M^4=T^4$ or $K3$. The results for the one-loop vacuum energy are expected to be the same.[18]

The list of relevant 6d supergravities with $\mathcal{N}=(n_L,n_R)$ supersymmetry was given in [43], where an algorithm for construction of the corresponding KK spectrum on S^3 was presented. Below we shall consider the following cases:

10d	M^4	(n_L,n_R)
IIB	$K3$	$(2,0)$
IIA	$K3$	$(1,1)$
IIA or IIB	T^4	$(2,2)$

(5.1)

5.1. KK towers of states on S^3

The 6d supergravity fields transform in representations (j_1,j_2) of the 6d little group $SO'(4)\simeq SU(2)\times SU(2)$ (of $SO(1,5)$ in the tangent space). This gives a set Φ of representations of the diagonal subgroup $SO(3)\simeq SU(2)$ of $SO(4)$. Considering compactification on S^3, the above $SO(3)$ can be identified with the factor in $S^3=SO(4)/SO(3)$. Each representation $R\in\Phi$ is

[17] String modes corresponding to massive unprotected multiplets are expected not to contribute to c.

[18] For example, type IIB theory on $AdS_3\times S^3\times T^4$ with RR 3-form flux is S-dual to type IIB theory with NSNS flux and as the supergravity theory is S-duality invariant the same should be true for the value of E_c. Since NS–NS sector is common to IIB and IIA theories, the same result should be found also in the corresponding IIA theory.

associated with a tower of KK states with $SO(4)$ representations containing R under restriction to their diagonal $SO(3)$.

These KK fields carry also representation of the AdS_3 isometry group $SO(2,2)$ (or global part of 2d conformal group) which are are labeled by scaling dimension and spin (Δ, s), with $\Delta \geq |s|$. The values of (Δ, s) can be determined by re-organizing the KK towers in short supermultiplets of $SU(2,2|1) \times SU(2,2|1)$ since its generators include the dilatation (Virasoro L_0) and spin operators. The relevant short representations $(J)_\mathrm{s}$ of $SU(2,2|1)$ have the following content

$$(J)_\mathrm{s}: \quad \begin{array}{c|cc} \text{States} & j & L_0 \\ \hline |0\rangle & J & J \\ Q_\pm|0\rangle & J-\frac{1}{2} & J+\frac{1}{2} \\ Q_+Q_-|0\rangle & J-1 & J+1 \end{array} \tag{5.2}$$

where $|0\rangle$ is the lowest weight of the representation in the usual oscillator construction [44], $Q_\pm$ are the supercharges, and j is $SU(2)$ spin. Thus, in general, each short $(J)_\mathrm{s}$ representation contains four $SO(2,2)$ representations. Using (5.2) and that $\Delta = L_0 + \overline{L}_0$, $s = L_0 - \overline{L}_0$ one obtains the quantum numbers of representations in the tensor products $(\overline{J}, J)_\mathrm{s}$.

Let us now list the KK towers that appear in the theories in (5.1). For (2, 0) 6d supergravity, or IIB theory dimensionally reduced on K3 the field content is a graviton, five self-dual two-forms, four gravitinos, and $n_T = 21$ tensor multiplet of one anti-self-dual two-form, four fermions and five scalars (see also [45,46])[19]

$$\Phi^{(2,0)} = \underset{g_{\mu\nu}}{(1,1)} + 4\underset{\psi_\mu}{\left(\frac{1}{2},1\right)} + 5\underset{B}{(0,1)} + n_T\Big[\underset{\widetilde{B}}{(1,0)} + 4\underset{\psi}{\left(\frac{1}{2},0\right)} + 5\underset{\varphi}{(0,0)}\Big]. \tag{5.3}$$

Reorganizing KK towers in short multiplets of $SU(2,2|1) \times SU(2,2|1)$, we find

$$\Phi^{(2,0)}_{\mathrm{KK}} = \sum_{\ell=0}^{\infty} \Phi_2(\ell) + (n_T+1)\sum_{\ell=0}^{\infty} \Phi_1(\ell) + n_T\left(\frac{1}{2},\frac{1}{2}\right)_\mathrm{s}, \tag{5.4}$$

where

$$\Phi_2(\ell) = \left(\frac{\ell+1}{2},\frac{\ell+3}{2}\right)_\mathrm{s} + \left(\frac{\ell+3}{2},\frac{\ell+1}{2}\right)_\mathrm{s}, \qquad \Phi_1(\ell) = \left(\frac{\ell+2}{2},\frac{\ell+2}{2}\right)_\mathrm{s}. \tag{5.5}$$

The towers in the first and second sums are called spin-2 and spin-1 towers because of the maximum spin of their bottom floor $\ell = 0$. The explicit field content is collected in Appendix E and their 6d origin is discussed in [45].

For (1, 1) 6d supergravity, or 10d IIA supergravity reduced on K3, the field content is the sum of 6d graviton multiplet and $n_V = 20$ vector multiplets [47]. The $SO(4)$ little group representations are[20]

$$\begin{aligned}\Phi^{(1,1)} = {} & (1,1) + 4\left(\frac{1}{2},1\right) + 2(0,1) + 4\left(\frac{1}{2},\frac{1}{2}\right) + 4\left(\frac{1}{2},0\right) + (0,0) \\ & + n_V\Big[\left(\frac{1}{2},\frac{1}{2}\right) + 4\left(\frac{1}{2},0\right) + 4(0,0)\Big],\end{aligned} \tag{5.6}$$

---

[19] We shall keep n_T generic because this will be useful in comparing with IIA case.

[20] Here we combine representations related by conjugation $(j_1, j_2) \to (j_2, j_1)$ since they give same contribution to KK spectrum.

and the KK towers are

$$\Phi^{(1,1)}_{\rm KK} = \sum_{\ell=0}^{\infty} \Phi_2(\ell) + (n_V + 2) \sum_{\ell=0}^{\infty} \Phi_1(\ell) + (n_V + 1)\left(\frac{1}{2}, \frac{1}{2}\right)_{\rm s}. \tag{5.7}$$

Comparing (5.4) and (5.7), we see that they are equal under the identification $n_V + 1 = n_T$ that is indeed true for the physical values. Thus we should find that E_c(IIB on K3) $= E_c$(IIA on K3) (as was already mentioned above, this is implied by S-duality of IIB theory and NS–NS sector being common for IIA and IIB theories).

Finally, for (2, 2) 6d supergravity, or IIA or IIB theory dimensionally reduced on T^4 the field content is a graviton, five self-dual and five anti-self-dual two-forms, eight gravitinos, 16 gauge fields, 40 fermions and 25 scalars:

$$\Phi^{(2,2)} = \underset{g_{\mu\nu}}{(1,1)} + 8\underset{\psi_\mu}{\left(\frac{1}{2}, 1\right)} + 5\underset{B}{(0,1)} + 5\underset{\widetilde{B}}{(1,0)} + 16\underset{V_\mu}{\left(\frac{1}{2}, \frac{1}{2}\right)} + 40\underset{\psi}{\left(\frac{1}{2}, 0\right)} + 25\underset{\varphi}{(0,0)}. \tag{5.8}$$

The KK towers here are

$$\Phi^{(2,2)}_{\rm KK} = \sum_{\ell=0}^{\infty} \Phi_2(\ell) + 4 \sum_{\ell=0}^{\infty} \Phi_{\frac{3}{2}}(\ell) + 6 \sum_{\ell=0}^{\infty} \Phi_1(\ell) + 5\left(\frac{1}{2}, \frac{1}{2}\right)_{\rm s}, \tag{5.9}$$

where

$$\Phi_{\frac{3}{2}}(\ell) = \left(\frac{\ell+1}{2}, \frac{\ell+2}{2}\right)_{\rm s} + \left(\frac{\ell+2}{2}, \frac{\ell+1}{2}\right)_{\rm s} \tag{5.10}$$

is a fermionic spin-$\frac{3}{2}$ tower (see Appendix E).

5.2. Vacuum energy

The AdS_3 vacuum energy contributions of the above KK towers can be computed using the expressions for the characters or one-particle partition functions of the corresponding $SO(2,2)$ representations which we shall first recall.

$SO(2,2)$ viewed as global conformal group in 2d is generated by the $L_0, L_{\pm 1}$ and $\overline{L}_0, \overline{L}_{\pm 1}$ Virasoro generators. Unitary irreducible representations of $SO(2,2)$ are massive for $\Delta > |s|$ and massless for $\Delta = |s|$. A massive representation is built on a ground state $|h, \bar{h}\rangle$ with $h\bar{h} > 0$. Thus, both L_{-1} and $\overline{L}_{-1}$ give a non=zero result and the resulting character is (see, e.g., [44])[21]

$$\Delta > |s|: \quad \widehat{\mathcal{Z}}^+(\Delta; s) = \frac{q^\Delta}{(1-q)^2}, \tag{5.11}$$

A massless representation with $\Delta = |s| > 0$ has conformal weights $(h, 0)$ or $(0, \bar{h})$. Acting with the lowering operators L_{-1} and $\overline{L}_{-1}$ on $|h, \bar{h}\rangle$ only one of them gives a non-zero result. As a consequence, here

$$\Delta = |s|: \quad \mathcal{Z}^+\big(|s|; s\big) = \frac{q^\Delta}{1-q} = \frac{q^\Delta - q^{\Delta+1}}{(1-q)^2} = \widehat{\mathcal{Z}}^+(\Delta; s) - \widehat{\mathcal{Z}}^+(\Delta+1; s). \tag{5.12}$$

[21] The double factor of $1/(1-q)$ takes into account multiple applications of both L_{-1} and $\overline{L}_{-1}$.

Finally, for $\Delta = s = 0$, we have only the ground state $|0, 0\rangle$ and $\mathcal{Z}^+(0; 0) = 1$. The expressions (5.11) and (5.12) can be used to prove that $SU(1, 1 \mid 2)$ short multiplets obey the important relation $E_c = -\frac{1}{12} c$, see (1.9). We discuss this in details in Appendix F.

The contribution from a particular $SO(2, 2)$ representation to the AdS_3 vacuum or S^1 2d Casimir energy E_c can then be computed using (2.8),(2.9). Explicitly, for a massive field in AdS_3, we may write the partition function (5.11) as

$$\widehat{\mathcal{Z}}^+(\Delta; s) = \sum_{n=0}^{\infty} (n+1) q^{\Delta+n}. \tag{5.13}$$

We then obtain a formal (divergent) expression for the corresponding E_c as (cf. (3.7), (3.8))

$$\widehat{E}_c^+(\Delta; s) = \sum_{n=0}^{\infty} e_n(\Delta; s), \quad e_n(\Delta; s) = \frac{1}{2}(-1)^{2s}(n+1)(\Delta+n). \tag{5.14}$$

In addition, we then need to sum over the KK states.

There will be divergences coming from the sum over n, but also from the sum over the KK level ℓ. Like in $AdS_5 \times S^5$ case [10] and $AdS_7 \times S^4$ case in Section 3 the total sum may be again computed using the ζ-function regularization applied to the full effective 6d energy eigenvalue $\Delta + n$, or, equivalently, by introducing the cutoff $e_n \to e_n e^{-\epsilon(\Delta+n)}$, doing the sum, expanding in $\epsilon \to 0$, and dropping all singular terms. Applying this procedure to the KK towers appearing in (5.4), (5.7) and (5.9), we obtain

$$E_{c,2} = E_c\left[\sum_{\ell=0}^{\infty} \Phi_2(\ell)\right] = -\frac{89}{192}, \qquad E_{c,\frac{3}{2}} = E_c\left[\sum_{\ell=0}^{\infty} \Phi_{\frac{3}{2}}(\ell)\right] = \frac{19}{96}, \tag{5.15}$$

$$E_{c,1} = E_c\left[\sum_{\ell=0}^{\infty} \Phi_1(\ell)\right] = -\frac{101}{384}, \qquad E_{c,\text{extra}} = E_c\left[\left(\frac{1}{2}, \frac{1}{2}\right)_{\text{s}}\right] = \frac{1}{4}, \tag{5.16}$$

where $E_{c,\text{extra}}$ is the contribution from the $(\frac{1}{2}, \frac{1}{2})_{\text{s}}$ representation appearing in (5.4), (5.7), and (5.9) in the bottom part of the KK towers.

The above are the contributions from the massive $SO(2, 2)$ representations. As discussed in [45,43], the resolution of the missing states puzzle raised in [48] amounts to the re-introduction of the massless representations ($\ell = -1$ states in the spin 2 and $\frac{3}{2}$ towers). These are massless multiplets in AdS_3 that do not carry propagating degrees of freedom. Their structure is presented in Appendix E. For these multiplets we find

$$\begin{aligned} E_{c,2}^{\text{massless}} &= E_c\big[(0, 1)_{\text{s}} + (1, 0)_{\text{s}}\big] = \frac{1}{2}, \\ E_{c,\frac{3}{2}}^{\text{massless}} &= E_c\left[\left(0, \frac{1}{2}\right)_{\text{s}} + \left(\frac{1}{2}, 0\right)_{\text{s}}\right] = -\frac{1}{4}. \end{aligned} \tag{5.17}$$

Collecting all contributions of states in (5.4), we find in the case of for IIB theory on K3

$$\begin{aligned} E_c^{(2,0)} &= E_{c,2}^{\text{massless}} + E_{c,2} + (n_T + 1) E_{c,1} + n_T E_{c,\text{extra}} \\ &= \frac{1}{2} - \frac{89}{192} - (n_T + 1)\frac{101}{384} + n_T \frac{1}{4} = -\frac{29}{128} - \frac{5}{384} n_T \xrightarrow{n_T = 21} -\frac{1}{2}. \end{aligned} \tag{5.18}$$

This is also the result for IIA theory on K3, as follows from (5.7). From (5.9) we also get exactly the same result for IIA or IIB theory on T^4,

$$E_c^{(2,2)} = E_{c,2}^{\text{massless}} + E_{c,2} + 4\big(E_{c,\frac{3}{2}}^{\text{massless}} + E_{c,\frac{3}{2}}\big) + 6E_{c,1} + 5E_{c,\text{extra}}$$
$$= \frac{1}{2} - \frac{89}{192} + 4\left(-\frac{1}{4} + \frac{19}{96}\right) - 6\frac{101}{384} + 5\frac{1}{4} = -\frac{1}{2}, \tag{5.19}$$

in agreement with the claim in (1.9).

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Appendix A. Free (2, 0) multiplet in 6d

The field content of the $(2,0)$ tensor multiplet is composed of five scalars fields ϕ^a, two complex Weyl fermions ψ_L^I or 4 Majorana–Weyl fermions (each with 4 real components), and an antisymmetric tensor T_{ij} with (anti-)self-dual strength. It represents a free 6d CFT invariant under superconformal $\mathcal{N} = (2,0)$ group [49,50] containing the conformal group $SO(2,6)$ and the R-symmetry group $SO(5) \simeq USp(4)$.

The Weyl anomaly of the $(2,0)$ multiplet was discussed in [17]. The values of the a-anomaly coefficients for the individual fields are (here ψ stands for one 6d Majorana–Weyl fermion)

$$\mathrm{a}_\phi = -\frac{1}{72\,576}, \qquad \mathrm{a}_\psi = -\frac{191}{1\,451\,520}, \qquad \mathrm{a}_T = -\frac{221}{40\,320}. \tag{A.1}$$

The total a-anomaly of one free $(2,0)$ tensor multiplet is thus

$$\mathrm{a}_{\text{tens.}} = 5\mathrm{a}_\phi + 4\mathrm{a}_\psi + \mathrm{a}_T = -\frac{7}{1152}. \tag{A.2}$$

Considering $(2,0)$ multiplet on $S^1 \times S^5$ one may compute the corresponding thermal partition function. The canonical (or one-particle) partition function of a free CFT in $S^1 \times S^{d-1}$ can be computed by direct evaluation of the free QFT path-integral in terms of the eigenmodes of the quadratic kinetic operator. An alternative approach is the operator counting method [51–53]. From the spectrum of eigenvalues of the Hamiltonian or dilatation operator $\omega_n = \Delta_n$ and their degeneracies d_n one gets

$$\mathcal{Z}(q) = \mathrm{Tr}\, e^{-\beta H} = \sum_n \mathrm{d}_n\, e^{-\beta\omega_n} = \sum_n \mathrm{d}_n\, q^{\Delta_n}, \qquad q \equiv e^{-\beta}. \tag{A.3}$$

In the approach based on counting of states one needs to consider the contribution of off-shell components (and their derivative descendants) of a suitable gauge invariant field strength modulo non-trivial gauge identities and then subtract the components of the equations of motion for the field strength (and their derivatives). The single particle partition functions for the 5 scalars, 4 Majorana–Weyl fermions, and self-dual tensor in $S^1 \times S^5$ are [52]

$$\mathcal{Z}_\phi(q) = \frac{1}{12}\sum_{n=0}^{\infty}(n+1)(n+2)^2(n+3)q^{n+2} = \frac{q^2 - q^4}{(1-q)^6},$$

$$\mathcal{Z}_\psi(q) = \frac{1}{6}\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)(n+4)q^{n+\frac{5}{2}} = \frac{4q^{\frac{5}{2}} - 4q^{\frac{7}{2}}}{(1-q)^6},$$

$$\mathcal{Z}_T(q) = \frac{1}{4}\sum_{n=0}^{\infty}(n+1)(n+2)(n+4)(n+5)q^{n+3} = \frac{10q^3 - 15q^4 + 6q^5 - q^6}{(1-q)^6}. \tag{A.4}$$

These expressions are in agreement with (2.7), (2.6), (2.2).

The related Casimir energy on S^5 can be computed from the one-particle partition function $\mathcal{Z}(q)$ using (2.8), (2.9):

$$E_{c,\phi} = -\frac{31}{60\,480}, \qquad E_{c,\psi} = -\frac{367}{96\,768}, \qquad E_{c,T} = -\frac{191}{4032}. \tag{A.5}$$

Then the total Casimir energy for the free (2, 0) tensor multiplet is

$$E_{c,\text{tens.}} = 5E_{c,\phi} + 4E_{c,\psi} + E_{c,T} = -\frac{25}{384}. \tag{A.6}$$

This agrees with the value found in [36].[22]

Let us note that the expressions in (A.4) admit also AdS_7 interpretation. In general, given a conformal 6d field, the corresponding one-particle partition function $\mathcal{Z}(q)$ may be expressed as [53]

$$\mathcal{Z}(q) = \mathcal{Z}^-(q) - \mathcal{Z}^+(q), \tag{A.7}$$

where $\mathcal{Z}^\pm(q)$ are the one-particle partition functions for the one-loop partition function $Z^\pm$ of the associated higher spin field in (thermal quotient of) AdS_7 computed with the standard ("Dirichlet") or alternative ("Neumann") boundary conditions. The canonical dimension of the conformal 6d field is equal to $\Delta_- = 6 - \Delta$, $\Delta = \Delta_+$. For generic representation (A.7) may be written as

$$\mathcal{Z}(q) = \mathcal{Z}^+(\Delta;\mathbf{h})\big(q^{-1}\big) - \mathcal{Z}^+(\Delta;\mathbf{h})(q) + \sigma(q), \tag{A.8}$$

where $\sigma(q)$ can be interpreted as a Killing tensor character associated with missing gauge invariances [53]. This term is a polynomial in q and $1/q$ which is symmetric under $q \to 1/q$. For a 6d conformal scalar with canonical dimension 2 we find that (see (2.2), (2.6), (2.7), (A.4))

$$\mathcal{Z}_\phi(q) = \mathcal{Z}^+(4;0,0,0)\big(q^{-1}\big) - \mathcal{Z}^+(4;0,0,0)(q). \tag{A.9}$$

For the Majorana–Weyl 6d fermion with canonical dimension $\frac{5}{2}$ we get

$$\mathcal{Z}_\psi(q) = \mathcal{Z}^+\left(\frac{7}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\big(q^{-1}\big) - \mathcal{Z}^+\left(\frac{7}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)(q). \tag{A.10}$$

In the case of rank-2 tensor of dimension 2 let us define (see (2.2), (2.5))

$$\mathcal{Z}_T^+ \equiv \mathcal{Z}^+(4;1,1,0) = \widehat{\mathcal{Z}}^+(4;1,1,0) - \widehat{\mathcal{Z}}^+(5;1,0,0) + \widehat{\mathcal{Z}}^+(6;0,0,0). \tag{A.11}$$

[22] Note that the ratio of the vacuum energy (A.6) and the a-anomaly (A.2), i.e. $E_{c,\text{tens.}}/\mathrm{a}_{\text{tens.}} = \frac{75}{7}$, differs from the expression in [54]. The reason is that the Casimir energy is computed in the standard ζ-function regularization scheme in which derivative terms D_6 in the conformal anomaly (1.1) do not vanish [17] while Ref. [54] assumed an abstract scheme where there are no derivative terms in the anomaly (see also a related discussion in [10]).

This is a $SO(2,6)$ character corresponding to the massless case (unitarity bound) in (ii) in (2.1).[23] One observes then that

$$2\mathcal{Z}_T(q) = \mathcal{Z}_T^+\big(q^{-1}\big) - \mathcal{Z}_T^+(q) - 1, \tag{A.12}$$

where $\mathcal{Z}_T$ is the self-dual tensor partition function in (2.7), (A.4). The -1 term should be interpreted as a subtraction of a non-normalizable gauge transformation.

Note that in the case of the (2, 0) tensor multiplet corresponding formally to the $p = 1$ singleton level of KK tower in Table 1 we find

$$\mathcal{Z}_{\text{tens.}}(q) = 5\mathcal{Z}_\phi + 4\mathcal{Z}_\psi + \mathcal{Z}_T = \frac{5q^2 + 16q^{\frac{5}{2}} + 15q^3 - 5q^4 + q^5}{(1-q)^5}, \tag{A.13}$$

which satisfies the relation

$$\mathcal{Z}_{\text{tens.}}(q) + \mathcal{Z}_{\text{tens.}}\big(q^{-1}\big) + 1 = 0. \tag{A.14}$$

The general relations for the boundary conformal anomaly and Casimir energy are [10]

$$\mathrm{a} = -2\mathrm{a}^+, \qquad E_c = -2E_c^+. \tag{A.15}$$

Denoting $K = (E_c, \mathrm{a})$ and $K^+ = (E_c^+, \mathrm{a}^+)$, we have[24]

$$\begin{aligned} &K^+(4; 0, 0, 0) = -\frac{1}{2}K_\phi, \qquad K^+\left(\frac{7}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{2}K_\psi, \\ &K^+(4; 1, 1, 0) = -K_T, \end{aligned} \tag{A.16}$$

where K_ϕ, K_ψ, K_T are given by (A.1) and (A.5).

Appendix B. a-anomaly from spectral ζ-function in AdS_7

The a-coefficient of the boundary conformal anomaly can be determined from the logarithmic IR singular part of the one-loop partition function in Euclidean AdS_7 with boundary S^6, i.e. hyperboloid $\mathbb{H}^7$ (see, e.g., [55,32])

$$\log Z^+ = -\frac{1}{2}\log\det{}_+\mathcal{O} = \frac{1}{2}\zeta'(0) = -96\mathrm{a}^+\log \mathrm{R} + \cdots. \tag{B.1}$$

Here $\zeta(z)$ is the spectral zeta function found by evaluating the trace of the $\mathbb{H}^7$ heat kernel [56] associated with the 7d operator $\mathcal{O}$ and R is an IR cutoff regularizing the volume of $\mathbb{H}^7$.

Below we shall consider the operator $\mathcal{O}$ corresponding to a generic massive (or massless) higher spin field in representation $(\Delta; \mathbf{h})$ generalizing the expression in [32] found in the totally symmetric tensor case $h_1 = s$, $h_2 = h_3 = 0$[25]

$$\mathcal{O} = -D^2 + X, \quad X = \Delta(\Delta - 6) - h_1 - h_2 - |h_3|. \tag{B.2}$$

Here D^2 is the standard Laplacian in AdS_7 defined on transverse field. The discussion will be parallel to the one in AdS_5 case in [10].

[23] Here the two additional terms are related to gauge freedom in the rank-2 tensor potential.

[24] In the tensor case, the factor $\frac{1}{2}$ is absent due to the self-duality condition.

[25] For the general form of X see [57,34,58].

The spectral ζ-function of the operator $\mathcal{O}$ can be expressed in terms of the heat kernel

$$\zeta(z)=\frac{1}{\Gamma(z)}\int_0^\infty dt\, t^{z-1}\,\mathrm{Tr}\,K,\qquad K(x,y;t)=\langle x|e^{-t\mathcal{O}}|y\rangle. \tag{B.3}$$

Since $\mathbb{H}^7$ is homogeneous, the trace over the position x gives a factor of (regularized) volume, i.e.

$$\zeta(z)=\mathrm{Vol}\big(\mathbb{H}^7\big)\zeta(z;x),\qquad \zeta(z;x)\equiv\frac{1}{\Gamma(z)}\int_0^\infty dt\, t^{z-1}\,\mathrm{tr}\,K(x,x;t), \tag{B.4}$$

where tr is the trace over the representation indices of the operator and $\zeta(z;x)$ does not actually depend on x.

One can use the results for the heat kernel of the Laplacian in AdS_{2n+1} with even n derived in [56,59] applying them to the case of $n=3$. It is convenient to start with heat-kernel for the sphere S^7 and then analytically continue to AdS_7. Let us consider a field on S^7 transforming under the tangent space rotations in a representation G of $SO(7)$. Since $\mathrm{S}^7=SO(8)/SO(7)$, the heat kernel receives contributions from each representation R of $SO(8)$ that contains G when restricted to $SO(7)$. Let us denote R and G by the corresponding weights as

$$\begin{aligned}&\mathsf{R}=(\ell_1,\ell_2,\ell_3,\ell_4),\quad \ell_1\ge\ell_2\ge\ell_3\ge|\ell_4|,\\&\mathsf{G}=(g_1,g_2,g_3),\quad g_1\ge g_2\ge g_3\ge 0,\end{aligned} \tag{B.5}$$

were all labels are integer or half integer. The branching condition on the representation R is

$$\ell_1\ge g_1\ge\ell_2\ge g_2\ge\ell_3\ge g_3\ge|\ell_4|, \tag{B.6}$$

with the additional requirement that $\ell_i-g_i\in\mathbb{Z}$. The heat kernel at the coincident points, traced over representation indices, can be written as

$$\mathrm{tr}\,K(x,x;t)=\frac{3}{\pi^4}\sum_{\ell_i}\mathrm{d}_{\mathsf{R}}\,e^{-tE^{(\mathsf{G})}_{\mathsf{R}}}, \tag{B.7}$$

where $E^{(\mathsf{H})}_{\mathsf{R}}$ are the eigenvalues of the Laplacian $-D^2$ on S^7 expressed in terms of the second Casimir values for the two representations and d_{R} is the dimension of R

$$-D^2\big|_{\mathrm{S}^7}\to E^{(\mathsf{G})}_{\mathsf{R}}=C_2(\mathsf{R})-C_2(\mathsf{G}), \tag{B.8}$$

$$C_2(\mathsf{R})=\ell_4^2+\ell_1(\ell_1+6)+\ell_2(\ell_2+4)+\ell_3(\ell_3+2), \tag{B.9}$$

$$C_2(\mathsf{G})=g_3^2+g_3+g_1(g_1+5)+g_2(g_2+3), \tag{B.10}$$

$$\begin{aligned}\mathrm{d}_{\mathsf{R}}=\;&\frac{1}{4320}\big[(\ell_1+3)^2-(\ell_2+2)^2\big]\big[(\ell_1+3)^2-(\ell_3+1)^2\big]\\&\times\big[(\ell_2+2)^2-(\ell_3+1)^2\big]\big[(\ell_1+3)^2-\ell_4^2\big]\big[(\ell_2+2)^2-\ell_4^2\big]\big[(\ell_3+1)^2-\ell_4^2\big].\end{aligned} \tag{B.11}$$

The analytic continuation from S^7 to AdS_7 amounts to [56,59]

$$\ell_1\to i\lambda-3, \tag{B.12}$$

with the sum over ℓ_1 becoming an integral over $\lambda\ge 0$. Finally, considering states saturating the inequalities (B.6) and identifying $(\ell_2,\ell_3,\ell_4)=\mathbf{h}=(h_1,h_2,h_3)$, we find that the eigenvalues of the operator (B.2) for the representation $(\Delta;\mathbf{h})$ are

$$\left(-D^2 + X\right)\big|_{\mathrm{AdS}_7} \to \lambda^2 + (\Delta - 3)^2. \tag{B.13}$$

The regularized volume may be written as $\mathrm{Vol}(\mathbb{H}^7) = \frac{1}{3}\pi^3 \log \mathrm{R} + \cdots$ where the IR cutoff R is the radius of S^6 measured in 7d metric $d\rho^2 + \sinh^2 \rho \, d\Omega_6^2$ at large ρ. Doing the analytic continuation (B.12) in the dimension $\mathrm{d_R}$ in (B.11) we finally obtain

$$\begin{aligned}\zeta(z) &= \mathrm{Vol}(\mathbb{H}^7)\zeta(z;x)\\ &\to -\frac{\log \mathrm{R}}{4320\pi}(h_1 - h_2 + 1)(h_1 + h_2 + 3)\big[(h_1+2)^2 - h_3^2)((h_2+1)^2 - h_3^2\big]\\ &\quad\times \int_0^\infty d\lambda\, \frac{[h_1(h_1+4) + \lambda^2 + 4][h_2(h_2+2) + \lambda^2 + 1](h_3^2 + \lambda^2)}{[\lambda^2 + (\Delta-3)^2]^z}.\end{aligned} \tag{B.14}$$

Integrating over λ and taking the z-derivative at $z = 0$ we may then use (B.1) to find the expression for $\hat{\mathrm{a}}^+$ in (2.11).

Appendix C. Tensor products of $SO(2, 6)$ singleton representations and associated character relations

Let us introduce the following notation for the spin $j = 0, \frac{1}{2}, 1, \ldots$ singleton representations of $SO(2, 6)$

$$\{j\} = (2 + j; j, j, j). \tag{C.1}$$

Here $\{0\}$ corresponds to a real scalar ϕ, $\{\frac{1}{2}\}$ to MW fermion ψ, and $\{1\}$ to self-dual tensor T. We shall also use the notation $(\Delta; h_1, h_2, h_3)_c \equiv (\Delta; h_1, h_2, h_3) + (\Delta; h_1, h_2, -h_3)$, so that $\{j\}_c = (2 + j; j, j, j) + (2 + j; j, j, -j)$.

From the general Flato–Fronsdal relations in [60,33] we get in the present 6d case

$$\{0\} \times \{0\} = (4; 0, 0, 0) + \bigoplus_{s=1}^{\infty}(4 + s; s, 0, 0), \tag{C.2}$$

$$\left\{\frac{1}{2}\right\} \times \left\{\frac{1}{2}\right\} = \bigoplus_{s=1}^{\infty}\big[(4 + s; s, 1, 1) + (4 + s; s, 0, 0)\big], \tag{C.3}$$

$$\{1\} \times \{1\} = \bigoplus_{s=2}^{\infty}\big[(4 + s; s, 2, 2) + (4 + s; s, 1, 1) + (4 + s; s, 0, 0)\big]. \tag{C.4}$$

The above relations imply analogous relations for the characters or one-particle partition functions

$$\big[\mathcal{Z}_\phi(q)\big]^2 = \mathcal{Z}^+(4; 0, 0, 0) + \sum_{s=1}^{\infty}\mathcal{Z}^+(4 + s; s, 0, 0), \tag{C.5}$$

$$\big[\mathcal{Z}_\psi(q)\big]^2 = \sum_{s=1}^{\infty}\big[\mathcal{Z}^+(4 + s; s, 1, 1) + \mathcal{Z}^+(4 + s; s, 0, 0)\big], \tag{C.6}$$

$$\big[\mathcal{Z}_T(q)\big]^2 = \sum_{s=2}^{\infty}\big[\mathcal{Z}^+(4 + s; s, 2, 2) + \mathcal{Z}^+(4 + s; s, 1, 1) + \mathcal{Z}^+(4 + s; s, 0, 0)\big]. \tag{C.7}$$

Here the l.h.s. may be interpreted as the one-particle partition functions corresponding to the single sector of the $U(N)$ boundary theory, with $\mathcal{Z}_\phi = \mathcal{Z}_{\{0\}}$, $\mathcal{Z}_\psi = \mathcal{Z}_{\{\frac{1}{2}\}}$, $\mathcal{Z}_T = \mathcal{Z}_{\{1\}}$ given in (2.7), (A.4).

The case of the real $O(N)$ invariant theory is found by an appropriate $\mathbb{Z}_2$ projection. The corresponding sums then represent the partition functions of the singlet sector of $O(N)$ invariant free real scalar, Majorana fermion, and real self-dual tensor theories in 6d:

$$\frac{1}{2}\big[\mathcal{Z}_\phi(q)\big]^2 + \frac{1}{2}\mathcal{Z}_\phi\big(q^2\big) = \mathcal{Z}^+(4;\mathbf{0}) + \sum_{s=2,4,\dots}^{\infty} \mathcal{Z}^+(4+s;s,0,0), \tag{C.8}$$

$$\frac{1}{2}\big[\mathcal{Z}_\psi(q)\big]^2 - \frac{1}{2}\mathcal{Z}_\psi\big(q^2\big) = \sum_{s=2,4,\dots}^{\infty} \mathcal{Z}^+(4+s;s,1,1) + \sum_{s=1,3,\dots}^{\infty} \mathcal{Z}^+(4+s;s,0,0), \tag{C.9}$$

$$\begin{aligned}\frac{1}{2}\big[\mathcal{Z}_T(q)\big]^2 + \frac{1}{2}\mathcal{Z}_T\big(q^2\big) &= \sum_{s=2,4,\dots}^{\infty} \big[\mathcal{Z}^+(4+s;s,2,2) + \mathcal{Z}^+(4+s;s,0,0)\big] \\ &\quad + \sum_{s=3,5,\dots}^{\infty} \mathcal{Z}^+(4+s;s,1,1).\end{aligned} \tag{C.10}$$

These relations (C.5)–(C.7) can be generalized by considering a tensor product of the linear combination of singletons: $[n_\phi\{0\} + n_\psi\{\frac{1}{2}\} + n_T\{1\}] \times [n_\phi\{0\} + n_\psi\{\frac{1}{2}\} + n_T\{1\}]$. This gives for the corresponding characters

$$\begin{aligned}&\big[n_\phi \mathcal{Z}_\phi(q) + n_\psi \mathcal{Z}_\psi(q) + n_T \mathcal{Z}_T(q)\big]^2 \\ &\quad = n_\phi^2 \sum_{s=0}^{\infty} \mathcal{Z}^+(s+4;s,0,0) + n_\psi^2 \sum_{s=1}^{\infty} \big[\mathcal{Z}^+(s+4;s,0,0) + \mathcal{Z}^+(s+4;s,1,1)\big] \\ &\qquad + n_T^2 \sum_{s=2}^{\infty} \big[\mathcal{Z}^+(s+4;s,0,0) + \mathcal{Z}^+(s+4;s,1,1) + \mathcal{Z}^+(s+4;s,2,2)\big] \\ &\qquad + 2n_\phi n_\psi \sum_{s=0}^{\infty} \mathcal{Z}^+\left(\frac{9}{2}+s, \frac{1}{2}+s, \frac{1}{2}, \frac{1}{2}\right) + 2n_\phi n_T \sum_{s=1}^{\infty} \mathcal{Z}^+(s+4;s,1,1) \\ &\qquad + 2n_\psi n_T \sum_{s=1}^{\infty} \left[\mathcal{Z}^+\left(\frac{9}{2}+s; \frac{1}{2}+s, \frac{1}{2}, \frac{1}{2}\right) + \mathcal{Z}^+\left(\frac{9}{2}+s, \frac{1}{2}+s, \frac{3}{2}, \frac{3}{2}\right)\right].\end{aligned} \tag{C.11}$$

It is of interest to consider also the reducible case when the boundary theory is represented by an unrestricted 2-tensor, i.e. the parity-invariant combination of self-dual and anti-self-dual tensors, i.e. $\{1\}_c = (3;1,1,1) + (3;1,1,-1)$. Then the corresponding Flato–Fronsdal type relation becomes (cf. (C.4))

$$\begin{aligned}\{1\}_c \times \{1\}_c &= 2\big[(6;2,2,0) + (6;1,1,0) + (6;0,0,0)\big] + 2\bigoplus_{s=3}^{\infty}(4+s;s,2,0) \\ &\quad + \bigoplus_{s=2}^{\infty}\big[(4+s;s,2,2)_c + (4+s;s,1,1)_c + 2(4+s;s,0,0)\big].\end{aligned} \tag{C.12}$$

Then $\mathcal{Z}_{\{1\}_c} = 2\mathcal{Z}_T$ and one finds that (C.7) is replaced by

$$\begin{aligned}\big[2\mathcal{Z}_T(q)\big]^2 &= 2\big[\mathcal{Z}^+(6;2,2,0)+\mathcal{Z}^+(6;1,1,0)+\mathcal{Z}^+(6;0,0,0)\big]\\ &\quad +2\bigoplus_{s=3}^{\infty}\mathcal{Z}^+(4+s;s,2,0)\\ &\quad +\bigoplus_{s=2}^{\infty}\big[\mathcal{Z}^+(4+s;s,2,2)_c+\mathcal{Z}^+(4+s;s,1,1)_c+2\mathcal{Z}^+(4+s;s,0,0)\big].\end{aligned} \tag{C.13}$$

Also, the analog of (C.10) is

$$\begin{aligned}\frac{1}{2}\big[2\,\mathcal{Z}_T(q)\big]^2+\frac{1}{2}\big[2\,\mathcal{Z}_T\big(q^2\big)\big] &= \mathcal{Z}^+(6;2,2,0)+\mathcal{Z}^+(6;1,1,0)+\mathcal{Z}^+(6;0,0,0)\\ &\quad +\sum_{s=2,4,\ldots}^{\infty}\big[\mathcal{Z}^+(4+s;s,2,2)_c+2\mathcal{Z}^+(4+s;s,0,0)\big]\\ &\quad +\sum_{s=3,5,\ldots}\mathcal{Z}^+(4+s;s,1,1)_c+\sum_{s=3}^{\infty}\mathcal{Z}^+(4+s;s,2,0).\end{aligned} \tag{C.14}$$

Appendix D. Casimir energy for spin 0, $\frac{1}{2}$, 1 singletons in AdS_{d+1}

It is useful to derive the general expressions for the Casimir energy for spin $j=0,\frac{1}{2},1$ $SO(2,d)$ singletons in the general case of even dimension $d=2,4,6,\ldots$ of the boundary.

For $j=0,\frac{1}{2}$ the corresponding character or one-particle partition functions are readily found, e.g., by counting states of free scalar or fermion in d dimensions[26]

$$\mathcal{Z}_0(q)=\frac{q^{\frac{d-2}{2}}(1-q^2)}{(1-q)^d},\qquad \mathcal{Z}_{\frac{1}{2}}(q)=2^{\frac{d}{2}}\frac{q^{\frac{d-1}{2}}(1-q)}{(1-q)^d}. \tag{D.1}$$

These satisfy

$$\mathcal{Z}_0(q)+\mathcal{Z}_0\big(q^{-1}\big)=0,\qquad \mathcal{Z}_{\frac{1}{2}}(q)+\mathcal{Z}_{\frac{1}{2}}\big(q^{-1}\big)=0. \tag{D.2}$$

As a consequence, in any even d the Casimir energy associated with the $U(N)$ singlet partition functions $[\mathcal{Z}_0]^2$ and $[\mathcal{Z}_{\frac{1}{2}}]^2$ vanishes because these functions are invariant under $q\to q^{-1}$.

The character of the $j=1$ singleton representation is [33][27]

$$\begin{aligned}\mathcal{Z}_1(q) &= \frac{1}{[(\frac{d}{2}-1)!]^2}\sum_{n=0}^{\infty}\frac{(n+d-1)!}{n!(n+\frac{d}{2})}q^{n+\frac{d}{2}}\\ &= \frac{d!}{2(\frac{d}{2})!^2}\frac{q^{\frac{d}{2}}}{(1-q)^{d-1}}\,{}_2F_1\bigg(1,1-\frac{d}{2};1+\frac{d}{2};q\bigg).\end{aligned} \tag{D.3}$$

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26 The singleton with spin j occurs with two possible chiralities. Here we consider one of them.

27 Explicitly, we find $\mathcal{Z}_1(q)=P_d(q)/(1-q)^d$ where

$$\begin{aligned}&P_4(q)=3q^2-4q^3+q^4, && P_8(q)=35q^4-56q^5+28q^6-8q^7+q^8,\\ &P_6(q)=10q^3-15q^4+6q^5-q^6, && P_{10}(q)=126q^5-210q^6+120q^7-45q^8+10q^9-q^{10}.\end{aligned}$$

One can check that

$$\mathcal{Z}_1(q)+\mathcal{Z}_1\big(q^{-1}\big)=(-1)^{d/2}, \tag{D.4}$$

so that

$$\big[\mathcal{Z}_1(q)\big]^2=\frac{1}{2}\Big(\big[\mathcal{Z}_1(q)\big]^2+\big[\mathcal{Z}_1\big(q^{-1}\big)\big]^2-1\Big)+(-1)^{\frac{d}{2}}\mathcal{Z}_1(q). \tag{D.5}$$

The first term in the r.h.s. is symmetric under $q\to q^{-1}$ and thus it does not contribute the Casimir energy. As a result, we find for the Casimir energy of the product of two spin 1 singletons (see (2.9), (A.5))

$$E_c\big(\{1\}\times\{1\}\big)=(-1)^{\frac{d}{2}}E_c\big(\{1\}\big). \tag{D.6}$$

For example, for the boundary $U(N)$ theory described by the $d=4$ vector corresponding to $\{1\}_{\rm c}$ (self-dual and anti-self-dual strength) we get [10]

$$d=4:\quad E_c\big(\{1\}_{\rm c}\times\{1\}_{\rm c}\big)=4E_c\big(\{1\}\times\{1\}\big)=4E_c\big(\{1\}\big)=2E_c\big(\{1\}_{\rm c}\big), \tag{D.7}$$

while for chiral singleton in 6d, i.e. self-dual (or anti-self-dual) 6d tensor

$$d=6:\quad E_c\big(\{1\}\times\{1\}\big)=-E_c\big(\{1\}\big), \tag{D.8}$$

in agreement with (4.8), (4.10).

Similar results can be obtained in the $O(N)$ case of real boundary singleton theory, i.e. for the singlet partition functions [27,42]

$$\mathcal{Z}_{j,\rm real}(q)=\frac{1}{2}\big[\mathcal{Z}_j(q)\big]^2+\frac{1}{2}(-1)^{2s}\mathcal{Z}_j\big(q^2\big). \tag{D.9}$$

In this case[28]

$$\begin{aligned}
&E_c\big(\{0\}\times\{0\}\big)_{\rm real}=E_c\big(\{0\}\big),\qquad E_c\left(n\cdot\left\{\frac{1}{2}\right\}\times n\cdot\left\{\frac{1}{2}\right\}\right)_{\rm real}=nE_c\left(\left\{\frac{1}{2}\right\}\right),\\
&E_c\big(n\cdot\{1\}\times n\cdot\{1\}\big)_{\rm real}=\frac{2+(-1)^{d/2}n}{2}E_c\big(\{1\}\big).
\end{aligned} \tag{D.10}$$

Then in 4d for the scalar, Dirac fermion and the vector we recover the results from [27,42]

$$\begin{aligned}
d=4:\quad &E_c\big(\{0\}\times\{0\}\big)_{\rm real}=E_c\big(\{0\}\big),\\
&E_c\left(\left\{\frac{1}{2}\right\}_{\rm c}\times\left\{\frac{1}{2}\right\}_{\rm c}\right)_{\rm real}=2E_c\left(\left\{\frac{1}{2}\right\}\right)=E_c\left(\left\{\frac{1}{2}\right\}_{\rm c}\right),\\
&E_c\big(\{1\}_{\rm c}\times\{1\}_{\rm c}\big)_{\rm real}=4E_c\big(\{1\}\big)=2E_c\big(\{1\}_{\rm c}\big).
\end{aligned} \tag{D.11}$$

In 6d we get instead

$$\begin{aligned}
d=6:\quad &E_c\big(\{0\}\times\{0\}\big)_{\rm real}=E_c\big(\{0\}\big),\qquad E_c\left(\left\{\frac{1}{2}\right\}\times\left\{\frac{1}{2}\right\}\right)_{\rm real}=E_c\left(\left\{\frac{1}{2}\right\}\right),\\
&E_c\big(\{1\}\times\{1\}\big)_{\rm real}=\frac{1}{2}E_c\big(\{1\}\big),
\end{aligned} \tag{D.12}$$

in agreement with (4.5), (4.7), (4.9).

---

28 Here $n\cdot\{j\}$ denotes n copies of the singleton, with partition function $n\mathcal{Z}_j$. Note also that $E_c(\{j\}_c)=2E_c(\{j\})$.

Appendix E. Field content of KK towers in 6d supergravity on S^3

Here we collect the field content of the KK towers discussed in Section 5. Let us list the representations of fields transforming in the $(\Delta; s) \times (j_1, j_2)$ representations of $SO(2,2) \times SO(4)$ as a formal sum of the form

$$\sum n_{\Delta,s;j_1,j_2} q^{\Delta} x^s R_{j_1,j_2}. \tag{E.1}$$

For states of the spin-2 tower in (5.5) with $\ell \geq 0$ we get

$$\begin{aligned}
\Phi_2(\ell) &= \left(\frac{\ell+1}{2}, \frac{\ell+3}{2}\right)_{\mathrm{s}} + \left(\frac{\ell+3}{2}, \frac{\ell+1}{2}\right)_{\mathrm{s}} \\
&= q^{\ell}\big[q^2\big(xR_{\frac{\ell+1}{2},\frac{\ell+3}{2}} + x^{-1}R_{\frac{\ell+3}{2},\frac{\ell+1}{2}}\big) \\
&\quad + q^{5/2}\big(2x^{3/2}R_{\frac{\ell+1}{2},\frac{\ell+2}{2}} + 2x^{-3/2}R_{\frac{\ell+2}{2},\frac{\ell+1}{2}} + 2x^{1/2}R_{\frac{\ell}{2},\frac{\ell+3}{2}} + 2x^{-1/2}R_{\frac{\ell+3}{2},\frac{\ell}{2}}\big) \\
&\quad + q^3\big(x^2R_{\frac{\ell+1}{2},\frac{\ell+1}{2}} + x^{-2}R_{\frac{\ell+1}{2},\frac{\ell+1}{2}} + 4xR_{\frac{\ell}{2},\frac{\ell}{2}+1} + 4x^{-1}R_{\frac{\ell}{2}+1,\frac{\ell}{2}} \\
&\quad + R_{\frac{\ell-1}{2},\frac{\ell+3}{2}} + R_{\frac{\ell+3}{2},\frac{\ell-1}{2}}\big) \\
&\quad + q^{7/2}\big(2x^{3/2}R_{\frac{\ell}{2},\frac{\ell+1}{2}} + 2x^{-3/2}R_{\frac{\ell+1}{2},\frac{\ell}{2}} + 2x^{1/2}R_{\frac{\ell-1}{2},\frac{\ell+2}{2}} + 2x^{-1/2}R_{\frac{\ell+2}{2},\frac{\ell-1}{2}}\big) \\
&\quad + q^4\big(xR_{\frac{\ell-1}{2},\frac{\ell+1}{2}} + x^{-1}R_{\frac{\ell+1}{2},\frac{\ell-1}{2}}\big)\big].
\end{aligned} \tag{E.2}$$

The massless states at $\ell = -1$ are

$$\begin{aligned}
(0,1)_{\mathrm{s}} + (1,0)_{\mathrm{s}} &= q\big(xR_{0,1} + x^{-1}R_{1,0}\big) + q^{3/2}\big(2x^{3/2}R_{0,\frac{1}{2}} + 2x^{-3/2}R_{\frac{1}{2},0}\big) \\
&\quad + q^2\big(x^2R_{0,0} + x^{-2}R_{0,0}\big).
\end{aligned} \tag{E.3}$$

For the spin $\frac{3}{2}$ tower (5.10) we get

$$\begin{aligned}
\Phi_{\frac{3}{2}}(\ell) &= \left(\frac{\ell+1}{2}, \frac{\ell+2}{2}\right)_{\mathrm{s}} + \left(\frac{\ell+2}{2}, \frac{\ell+1}{2}\right)_{\mathrm{s}} \\
&= q^{\ell}\big[q^{3/2}\big(\sqrt{x}R_{\frac{\ell+1}{2},\frac{\ell+2}{2}} + x^{-1/2}R_{\frac{\ell+2}{2},\frac{\ell+1}{2}}\big) \\
&\quad + q^2\big(2xR_{\frac{\ell+1}{2},\frac{\ell+1}{2}} + 2x^{-1}R_{\frac{\ell+1}{2},\frac{\ell+1}{2}} + 2(R_{\frac{\ell}{2}+1,\frac{\ell}{2}} + R_{\frac{\ell}{2},\frac{\ell}{2}+1})\big) \\
&\quad + q^{5/2}\big(x^{3/2}R_{\frac{\ell+1}{2},\frac{\ell}{2}} + x^{-3/2}R_{\frac{\ell}{2},\frac{\ell+1}{2}} + \sqrt{x}(4R_{\frac{\ell}{2},\frac{\ell+1}{2}} + R_{\frac{\ell+2}{2},\frac{\ell-1}{2}}) \\
&\quad + x^{-1/2}(R_{\frac{\ell-1}{2},\frac{\ell+2}{2}} + 4R_{\frac{\ell+1}{2},\frac{\ell}{2}})\big) \\
&\quad + q^3\big(2xR_{\frac{\ell}{2},\frac{\ell}{2}} + 2x^{-1}R_{\frac{\ell}{2},\frac{\ell}{2}} + 2(R_{\frac{\ell-1}{2},\frac{\ell+1}{2}} + R_{\frac{\ell+1}{2},\frac{\ell-1}{2}})\big) \\
&\quad + q^{7/2}\big(\sqrt{x}R_{\frac{\ell-1}{2},\frac{\ell}{2}} + x^{-1/2}R_{\frac{\ell}{2},\frac{\ell-1}{2}}\big)\big],
\end{aligned} \tag{E.4}$$

and its massless part at $\ell = -1$ is

$$\left(0, \frac{1}{2}\right)_{\mathrm{s}} + \left(\frac{1}{2}, 0\right)_{\mathrm{s}} = \sqrt{q}\big(x^{1/2}R_{0,\frac{1}{2}} + x^{-1/2}R_{\frac{1}{2},0}\big) + q\big(2xR_{0,0} + 2x^{-1}R_{0,0}\big). \tag{E.5}$$

For the spin-1 tower in (5.5) we have for $\ell \geq 0$

$$
\begin{aligned}
\Phi_1(\ell) &= \left(\frac{\ell+2}{2}, \frac{\ell+2}{2}\right)_{\mathrm{s}} \\
&= q^{\ell}\big[q^2 R_{\frac{\ell+2}{2},\frac{\ell+2}{2}} + q^{5/2}\big(2x^{-1/2} R_{\frac{\ell+1}{2},\frac{\ell+2}{2}} + 2x^{1/2} R_{\frac{\ell+2}{2},\frac{\ell+1}{2}}\big) \\
&\quad + q^3\big(x R_{\frac{\ell}{2}+1,\frac{\ell}{2}} + x^{-1} R_{\frac{\ell}{2},\frac{\ell}{2}+1} + 4R_{\frac{\ell+1}{2},\frac{\ell+1}{2}}\big) \\
&\quad + q^{7/2}\big(2x^{-1/2} R_{\frac{\ell}{2},\frac{\ell+1}{2}} + 2x^{1/2} R_{\frac{\ell+1}{2},\frac{\ell}{2}}\big) + q^4 R_{\frac{\ell}{2},\frac{\ell}{2}}\big].
\end{aligned} \tag{E.6}
$$

Finally, the extra term in (4.5), (5.7), and (5.9) is

$$
\left(\frac{1}{2}, \frac{1}{2}\right)_{\mathrm{s}} = q R_{\frac{1}{2},\frac{1}{2}} + q^{3/2}\big(2x^{-1/2} R_{0,\frac{1}{2}} + 2x^{1/2} R_{\frac{1}{2},0}\big) + 4q^2 R_{0,0}. \tag{E.7}
$$

Appendix F. Relation between Casimir energy and 2d central charge computed from AdS_3 for short $SU(2,2\,|\,1) \times SU(2,2\,|\,1)$ multiplets

The short multiplet $(J_1, J_2)_{\mathrm{s}}$ of $SU(2,2\,|\,1) \times SU(2,2\,|\,1)$ contains, for generic j_1, j_2, the following representations $(\Delta; s)_{(j_1,j_2)}$ of $SO(2,2) \times SO(4)$ (see (5.2)):

$$
\begin{aligned}
(J_1, J_2)_{\mathrm{s}} &= (J_1 + J_2; J_1 - J_2)_{(J_1,J_2)} + 2\left(J_1 + J_2 + \frac{1}{2}; J_1 - J_2 - \frac{1}{2}\right)_{(J_1,J_2-\frac{1}{2})} \\
&\quad + 2\left(J_1 + J_2 + \frac{1}{2}; J_1 - J_2 + \frac{1}{2}\right)_{(J_1-\frac{1}{2},J_2)} \\
&\quad + (J_1 + J_2 + 1; J_1 - J_2 - 1)_{(J_1,J_2-1)} \\
&\quad + 4(J_1 + J_2 + 1; J_1 - J_2)_{(J_1-\frac{1}{2},J_2-\frac{1}{2})} + (J_1 + J_2 + 1; J_1 - J_2 + 1)_{(J_1-1,J_2)} \\
&\quad + 2\left(J_1 + J_2 + \frac{3}{2}; J_1 - J_2 - \frac{1}{2}\right)_{(J_1-\frac{1}{2},J_2-1)} \\
&\quad + 2\left(J_1 + J_2 + \frac{3}{2}; J_1 - J_2 + \frac{1}{2}\right)_{(J_1-1,J_2-\frac{1}{2})} \\
&\quad + (J_1 + J_2 + 2; J_1 - J_2)_{(J_1-1,J_2-1)}.
\end{aligned} \tag{F.1}
$$

The S^1 Casimir energy for a 2d conformal field in the $SO(2,2)$ representation $(\Delta; s)$ can be found from the partition functions $\mathcal{Z}^+$ in (5.11) and (5.12) and using $E_c = -2E_c^+$:

$$
E_c(\Delta; s) = -\frac{1}{12}(-1)^{2s}(\Delta - 1)\big[2(\Delta - 1)^2 - 1\big]. \tag{F.2}
$$

At the same time, the 2d central charge can be computed via "dual" route as the coefficient of the logarithmic IR divergence of 1-loop partition function of the corresponding higher spin field in AdS_3 [32,25,26]; for a single chiral spin s component it reads

$$
c_{\mathrm{AdS}_3}(\Delta; s) = (-1)^{2s}(\Delta - 1)\big[(\Delta - 1)^2 - 3s^2\big]. \tag{F.3}
$$

Comparing (F.2) and (F.3) we observe that the 2d relation $E_c = -\frac{1}{12}c$ in (1.9) does not hold for a single massive field. Nevertheless, this relation holds for a massless field with $\Delta = s$, because

$$
\begin{aligned}
E_c(s; s) - E_c(s+1; s-1) &= -\frac{1}{12}\big[c_{\mathrm{AdS}_3}(s; s) - c_{\mathrm{AdS}_3}(s+1, s-1)\big] \\
&= \frac{1}{12}(-1)^{2s}\big[1 - 6s(1-s)\big].
\end{aligned} \tag{F.4}
$$

It also holds if we evaluate the total E_c and $c = c_{\mathrm{AdS}_3}$ for a short multiplet $(J_1, J_2)_\mathrm{s}$, i.e.

$$E_c\big[(J_1, J_2)_\mathrm{s}\big] = -\frac{1}{12}c\big[(J_1, J_2)_\mathrm{s}\big] = -\frac{1}{2}(-1)^{2(J_1+J_2)}(J_1 + J_2). \tag{F.5}$$

Different expressions are found when additional shortening occur due to particular low values of J_1 or J_2, but we checked that the relation $E_c = -\frac{1}{12}c$ always holds.

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