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Generalized Inverses of Hankel and Toeplitz Mosaic Matrices

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ABSTRACT

Hankel and Toeplitz mosaic matrices are block matrices with Hankel or Toeplitz blocks, respectively. It is shown that Hankel and Toeplitz mosaic matrices possess reflexive generalized inverses which are Bezoutians. Furthermore the Bezoutian structure of the Moore-Penrose and group inverses is investigated.

1. INTRODUCTION

Hankel matrices $[s_{i+j}]$ and Toeplitz matrices $[t_{i-j}]$ occur in signal processing, systems theory (partial realization), approximation theory (Padé approximation), moment problems, orthogonal polynomials, numerical solution of integral equations, and many other fields. A striking feature of this class is that their structure can be exploited in order to construct fast inversion algorithms. One basic fact which is behind these constructions is that the inverses of these matrices have the structure of a Bezoutian (for the definition see below). In particular, this leads to formulas of Gohberg-Semencul type. With their help Hankel and Toeplitz systems can be solved fast (see [16], [11], and references therein). Similar results are known for block Hankel and Toeplitz matrices [6], which also occur in many applications.

It is natural to ask whether generalized inverses of Hankel and Toeplitz matrices have also a Bezoutian or at least a Bezoutian-like structure. An affirmative answer was given in [11] for (1,2)-generalized inverses in [9] for the Moore-Penrose

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inverse. These results led to the question of the block matrix case. It turned out that, in order to give an answer to this question, it is natural to consider a slightly larger class than block Hankel and Toeplitz matrices, namely Hankel and Toeplitz mosaic matrices.

A matrix A is said to be a *Hankel* (*Toeplitz*) *mosaic matrix* if it can be partitioned into blocks $A = [A_{ij}]_1^p{}_1^q$ such that the blocks A_{ij} are Hankel (Toeplitz) and p and q are small compared with the size of the matrix. By rearranging columns and rows, block Hankel and Toeplitz matrices can be transformed into Hankel and Toeplitz mosaic matrices with equal block sizes. In this sense the classes of Hankel and Toeplitz mosaic matrices are more general than the classes of the corresponding block matrices.

Note that in many applications one encounters Hankel and Toeplitz mosaic matrices with unequal block sizes, i.e. matrices which cannot be transformed straightforwardly into corresponding block matrices. For example, Moore-Penrose inverses of scalar Hankel matrices are related to certain Hankel mosaic matrices (see [9]). The resultant matrix in classical algebra is a special Toeplitz mosaic matrix. More general, systems of polynomial (called Bezout or Diophantine) equations

$$\sum_{i=1}^{q} a_{ij}(\lambda)x_j(\lambda) = b_i(\lambda) \qquad (i = 1, \ldots, p),$$

where $a_{ij}(\lambda)$ and $b_i(\lambda)$ are given and $x_j(\lambda)$ are unknown polynomials with fixed degrees, are equivalent to certain Toeplitz mosaic systems. The coefficient matrix is obtained by comparing the coefficients of the polynomials. Furthermore, convenient discretizations of integral equations

$$\phi(t) - \int_\Omega k(t-s)\phi(s)\,ds = \psi(t) \quad (t \in \Omega)$$

over a convex domain Ω in the plane $\mathbf{R}^2$ lead to Toeplitz mosaic matrices, the blocks of which have equal size only if Ω is a rectangle. The application of the reduction method for singular integral equations on the unit circle includes the solution of so-called paired systems which are special Toeplitz mosaic systems (see [5]). Moreover, generalized Padé-Hermite approximation problems are related to Hankel mosaic matrices. In fact, let us consider the following problem.

Given integers μ_i $(i = 1, \ldots, p)$ and ν_j $(j = 1, \ldots, q)$ and formal power series

$$f_{ij}(\lambda) = \sum_{k=0}^{\infty} f_{ij}^{(k)}\lambda^k.$$

The problem is to find polynomials P_j with degree less than ν_j ($j = 1, \dots, q$) such that

$$\sum_{j=1}^{q} f_{ij}(\lambda)P_j(\lambda) = O(\lambda^{\mu_i}) \qquad (i = 1, \dots, p). \tag{1.1}$$

Comparing the coefficients in (1.1), one obtains a homogeneous system of equations with a Hankel mosaic coefficient matrix $\left[f_{ij}^{(k-l)}\right]$ ($k = 0, \dots, \mu_i - 1, l = 0, \dots, \nu_j - 1$) and a solution vector consisting of the coefficients of the polynomials P_j. For the special cases $p = 1$ and $q = 1$ this problem is called the Padé-Hermite approximation problem. It is widely discussed in the literature (see for example [2, 4, 13, 16, 18–20]). Very similar to the Padé approximation problems is the realization problem in systems theory (see [14] and references therein).

The aim of this paper is to investigate the structure of generalized inverses of Hankel and Toeplitz mosaic and block Hankel and Toeplitz matrices, generalizing the corresponding results for scalar Hankel and Toeplitz matrices presented in [11] and [9].

To begin with let us recall some definitions. Throughout the paper we consider matrices with complex entries. Let A be an $m \times n$ matrix. An $n \times m$ matrix B is said to be a *reflexive generalized inverse* (*g-inverse*) or $(1, 2)$-*generalized inverse* of A if

$$ABA = A \tag{1}$$

and

$$BAB = B. \tag{2}$$

If only (1) is satisfied, then B is called an *inner g-inverse* of A (or 1-*inverse* or *von Neumann g-inverse*).

If in addition to (1) and (2) the equalities

$$(AB)^* = AB \tag{3}$$

and

$$(BA)^* = BA \tag{4}$$

hold, then B is called the *Moore-Penrose inverse* or *pseudoinverse* of A. It is well known that the pseudoinverse always exists and is unique. We shall denote it by $A^\dagger$.

If $m = n$ and B is a reflexive g-inverse of A satisfying the additional condition

$$AB = BA, \tag{5}$$

then B is called a *group inverse* of A. The group inverse exists if and only if the kernel and range of A are direct complements, and the group inverse is always unique. We shall denote it by $A^{\#}$.

For $\lambda \in \mathbf{C}$, let $l_n(\lambda)(n = 1, 2, \ldots)$ denote the column vector $l_n(\lambda) := [1\ \lambda \cdots \lambda^{n-1}]^T$. If A is an $m \times n$ matrix, then the polynomial

$$\widehat{A}(\lambda, \mu) = l_m(\lambda)^T A l_n(\mu)$$

is called the *generating function* of A. For a column vector $x \in \mathbf{C}^n$ we write $\widehat{x}(\lambda)$ instead of $\widehat{x}(\lambda, \mu)$, since there is no dependence on μ.

A matrix B is said to be a *Hankel Bezoutian* if it has a generating function of the form

$$\widehat{B}(\lambda, \mu) = \frac{1}{\lambda - \mu} \sum_{i=1}^{r} a_i(\lambda) b_i(\mu) \tag{1.2}$$

for $r = 2$, where $a_i(\lambda)$ and $b_i(\mu)$ are polynomials; it is said to be a *Toeplitz Bezoutian* if

$$\widehat{B}(\lambda, \mu) = \frac{1}{1 - \lambda\mu} \sum_{i=1}^{r} a_i(\lambda) b_i(\mu) \tag{1.3}$$

for $r = 2$.

The $n \times n$ Hankel [Toeplitz] Bezoutian is called *classical* in the case that $b_1 = a_2$ and $b_2 = -a_1$ $[b_1(\lambda) = a_2(\lambda^{-1})\lambda^n,\ b_2(\lambda) = -a_1(\lambda^{-1})\lambda^n]$. Any nonsingular Bezoutian and any symmetric Bezoutian is classical (see [11]).

It can be easily checked that the Moore-Penrose inverse of a scalar Hankel matrix is, in general, no Bezoutian, but it is a certain generalized Bezoutian in sense of the following definition.

DEFINITION. A matrix is said to be a Hankel (Toeplitz) *r-Bezoutian*, for a nonnegative integer r, if its generating function has the form (1.2) or (1.3), respectively.

In this paper we shall formulate all results only for Hankel matrices. In every case there is a corresponding Toeplitz analogue, which can be formulated and proved in the same way.

Concerning scalar Hankel matrices the following is known.

THEOREM 1.1. *Let H be an $m \times n$ Hankel matrix. Then*

(1) [11] *There exists a Bezoutian which is a reflexive generalized inverse of H. In the case $m = n$ this g-inverse can be chosen as a classical Bezoutian.*

(2) [11] *There is a matrix consisting of the first n rows and first m columns of a nonsingular Bezoutian of order $m + n - \operatorname{rank} H$ which is an inner inverse of H.*

(3) [9] *The Moore-Penrose inverse of H is a 4-Bezoutian. Moreover, for $m = n$, it is the sum of two classical Bezoutians.*

(4) [unpublished] *The group inverse of H provided that it exists, is the sum of two classical Bezoutians.*

Actually in [11] parts (1) and (2) are proved only for the case when H does not have full rank, but the general case is covered by the results below. The proof of (4) is completely analogous to that of (3) in [9].

The aim of the present paper is to generalize Theorem 1.1, except for the assertions containing the word "classical," to Hankel and Toeplitz mosaic matrices.

Let us agree upon calling a partitioned matrix with $\leq p$ block rows and $\leq q$ block columns a *(p, q)-mosaic matrix.* In this sense a (p, q)-Hankel (Toeplitz) mosaic matrix is a partitioned matrix with $\leq p$ Hankel (Toeplitz) block rows and $\leq q$ Hankel (Toeplitz) block columns.

Let $A = [A_{ij}]_{1\,1}^{qp}$ be a (q, p)-mosaic matrix with blocks $A_{ij} \in \mathbf{C}^{m_i \times n_j}$. Then the *generating function* of A is, by definition, the $q \times p$ matrix polynomial

$$\widehat{A}(\lambda, \mu) = [\widehat{A}_{ij}(\lambda, \mu)]_{i=1}^{q}{}_{j=1}^{p}.$$

A (q, p)-mosaic matrix B is said to be a *(q, p)-Hankel (or Toeplitz) Bezoutian* if its generating function admits a representation

$$\widehat{B}(\lambda, \mu) = \frac{1}{\lambda - \mu}\widehat{U}(\lambda)\widehat{V}(\mu)^T$$

or

$$\widehat{B}(\lambda, \mu) = \frac{1}{1 - \lambda\mu}\widehat{U}(\lambda)\widehat{V}(\mu)^T,$$

respectively, where $\widehat{U}(\lambda)$ is a $q \times (p+q)$ and $\widehat{V}(\lambda)$ is a $p \times (p+q)$ matrix polynomial. This Bezoutian concept is a slight generalization of the one introduced by B. D. O. Anderson and E. I. Jury in [1] (see also [22]). The Anderson-Jury Bezoutian corresponds to the case that all integers m_i and n_j are equal.

The following result, generalizing Lander's theorem and some results on block Toeplitz matrices in [6], is proved in [12].

THEOREM 1.2. *The inverse of a nonsingular (p, q)-Hankel (Toeplitz) mosaic matrix is a (q, p)-Bezoutian; the inverse of a nonsingular (q, p)-Bezoutian is a (p, q)-Hankel matrix.*

The organization of the paper is easily described. In Section $k + 1$, part (k) of Theorem 1.1 will be generalized to the mosaic case. Similarly to the scalar case, for the first part a restriction approach will be applied, and for the second part an extension approach. In order to describe the Bezoutian structure of the pseudoinverse we also exploit the extension approach and some kernel structure properties for Hankel mosaic matrices generalizing the corresponding results for block Hankel matrices (see [10]).

Let us note that in the paper [22] of H. Wimmer a similar problem is treated. It is shown in principle that (p,p)-Bezoutians possess under some conditions generalized inverses which are (p,p)-Hankel matrices, and it is shown how these Hankel matrices are, related to the Bezoutians. It is now very natural to conjecture that every (q,p)-Bezoutian possesses a generalized inverse which is a (p,q)-Hankel matrix. However, this problem is still open.

2. RESTRICTION APPROACH

In this section we prove the following result.

THEOREM 2.1. *Let H be a (p,q)-Hankel mosaic matrix (Toeplitz mosaic matrix). Then there exists a reflexive generalized inverse B of H which is a (q,p)-Bezoutian.*

COROLLARY 2.1. *For any block Hankel matrix H there exists an Anderson-Jury Bezoutian which is a reflexive generalized inverse of H.*

For the proof we employ the familiar restriction idea which is formulated next.

LEMMA 2.1. *Let A be a given $m \times n$ matrix with* rank r, *and let $C_1 \in \mathbf{C}^{n\times r}$, $C_2 \in \mathbf{C}^{r\times m}$ be matrices such that $\widetilde{A} := C_2AC_1$ is nonsingular. Then $B := C_1\widetilde{A}^{-1}C_2$ is a reflexive generalized inverse of A.*

PROOF. We have

$$BAB = C_1\widetilde{A}^{-1}C_2AC_1\widetilde{A}^{-1}C_2 = C_1\widetilde{A}^{-1}C_2 = B$$

and

$$ABAC_1 = AC_1\widetilde{A}^{-1}C_2AC_1 = AC_1.$$

Hence ABA coincides with A on the range of C_1. Furthermore, $ABA = A$ trivially holds on the kernel of A. Since $\ker AC_1 = \{0\}$, we have $\ker A \cap \operatorname{im} C_1 = \{0\}$. It follows by a dimension argument that $\operatorname{im} C_1$ is a direct complement of $\ker A$. Thus we have $ABA = A$ on the whole space. ■

is again a (p,q)-Hankel mosaic matrix.

where

$$D_2 = \operatorname{diag}(D_{m_i - t_i}(v_i))_1^p, \qquad D_1 = \operatorname{diag}(D_{n_j - r_j}(u_j))_1^q, \tag{2.5}$$

$$\widetilde{H} = D_2 H D_1^T,$$

where

LEMMA 2.3. (1) *Let H be a (p,q)-Hankel mosaic matrix with $m_i \times n_j$ blocks, and let $u_j \in \mathbf{C}^{r_j+1}$, $v_i \in \mathbf{C}^{t_i+1}$, where $r_j < n_j$, $t_i < m_i$. Then*

$$= u(\lambda)B(\lambda,\mu)v(\mu). \qquad \blacksquare$$

Lemma 2.2 generalizes to (p,q)-Hankel mosaic matrices and (p,q)-Bezoutians.

$$\widehat{B}(\lambda,\mu) = l_n(\lambda)^T D_{n-r}(u)^T \widetilde{B} D_{m-t}(v) l_m(\mu)$$

$$= \widehat{u}(\lambda)\widehat{\widetilde{B}}(\lambda,\mu)\widehat{v}(\mu) \qquad \blacksquare$$

PROOF. The first assertion can be immediately checked. The second one follows from

$$B := D_{n-r}(u)^T B D_{m-t}(v)$$

is again a Bezoutian.

is Hankel again.

(2) *Let $\widetilde{B}$ be an $(n-r)\times(m-t)$ Hankel Bezoutian, and u, v as above. Then*

$$\widetilde{H} := D_{m-t}(v) H D_{n-r}(u)^T$$

is Hankel matrix

LEMMA 2.2. (1) *Let H be an $m \times n$ Hankel matrix, $u \in \mathbf{C}^{r+1}$, $v \in \mathbf{C}^{t+1}$, where $r < n$ and $t < m$. Then*

$$D_k(u) = \left.\begin{pmatrix} \vdots & \ddots & \ddots & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & u_0 & u_1 & \cdots & u_r \end{pmatrix}\right\} k \text{ rows}. \tag{2.4}$$

$$\begin{pmatrix} u_0 & u_1 & \cdots & u_r & 0 & \cdots & 0 \\ 0 & u_0 & u_1 & \cdots & u_r & \cdots & 0 \end{pmatrix}$$

For expository reasons let us consider first the case $p = q = 1$.

Let $u \in \mathbf{C}^{r+1}$ be a given vector, $u = (u_i)_0^r$, and k a natural number. Then we denote by $D_k(u)$ the $k \times (r+k)$ matrix

Our aim is to find convenient matrices C_1, C_2 such that, for a (p,q)-Hankel mosaic matrix H, the matrix $\widetilde{H} = C_2 H C_1$ is a nonsingular Hankel mosaic matrix and $C_1 \widetilde{H}^{-1} C_2$ is a Bezoutian.

(2) *Let $\widetilde{B}$ be a (q,p)-Hankel Bezoutian and u_j, v_i, D_1, D_2 as above. Then*

$$B := D_1^T \widetilde{B} D_2$$

is again a (q,p)-Bezoutian.

The matrices D_1, D_2 will be constructed successively. We shall utilize the fact that, for $u \in \mathbf{C}^{r+1}, D_k(u)^T$ is the matrix of the operator of multiplication by the polynomial $\widehat{u}(\lambda)$, i.e.

$$(D_k(u)^T x)^\wedge = \widehat{u}(\lambda)\widehat{x}(\lambda).$$

A consequence of this fact is that, for suitable choices of j, k, and l,

$$D_k(u)D_j(v) = D_l(w), \tag{2.6}$$

where $\widehat{w} = \widehat{u}\widehat{v}$. This multiplication rule generalizes to block diagonal matrices with blocks of the form $D_k(u)$ of feasible size.

The construction of D_1, D_2 is based on the following fact.

LEMMA 2.4. *Let A be an $m \times n$ matrix such that* $\ker A^T \neq \{0\}$, *and let A be represented in the form*

$$A = \begin{pmatrix} A_1 \\ * \end{pmatrix} = \begin{pmatrix} * \\ A_2 \end{pmatrix},$$

where $A_1, A_2 \in \mathbf{C}^{(m-1)\times n}$. Then for all $\xi \in \mathbf{C}$ with the possible exception of a finite number of points the equality

$$\ker A = \ker(A_2 - \xi A_1) \tag{2.7}$$

holds.

PROOF. Obviously, $\ker(A_2 - \xi A_1) \supseteq \ker A$ for all ξ. Let $w = (w_k)_1^m \neq 0$ belong to $\ker A^T$, and let

$$\widehat{w}(\xi) := \sum_{k=1}^{n} w_k \xi^{k-1} \neq 0. \tag{2.8}$$

We show that in this case (2.7) is fulfilled.

Let f_i denote the ith row of A ($i = 1, \ldots, m$). Then the equation $(A_2 - \xi A_1)x = 0$ is equivalent with

$$f_{i+1}x - \xi f_i x = 0 \qquad (i = 1, \ldots, m-1).$$

Hence

$$f_i x = c\xi^{i-1}$$

for a certain $c \in \mathbf{C}$. That means

$$Ax = cl(\xi), \tag{2.9}$$

where $l(\xi) := \operatorname{col}(\xi^{i-1})_1^m$. Since $w^T l(\xi) = w(\xi) \neq 0$ by assumption, the vector $l(\xi)$ does not belong to $\operatorname{im} A$. Hence (2.9) implies $Ax = 0$. That means we have $\ker(A_2 - \xi A_1) \subseteq \ker A$.

We have proved that (2.7) is true for all ξ with the possible exception of the roots of $\widehat{w}$, which is a finite set. ■

REMARK 2.1. It is important to observe that

$$A_2 - \xi A_1 = D_{m-1}(u)A,$$

where $u(\lambda) = \lambda - \xi$.

Applying Lemma 2.4 successively, we obtain the following conclusion.

COROLLARY 2.2. *Suppose that $A \in \mathbf{C}^{p\times q}$ and $r = \operatorname{rank} A$. Then there are polynomials $\widehat{u}$ and $\widehat{v}$ with $\deg \widehat{u} = p - r$, $\deg \widehat{v} = q - r$ such that*

$$A_0 := D_r(u)AD_r(v)^T$$

is nonsingular.

We shall need a slight generalization of Lemma 2.4.

LEMMA 2.5. *Suppose that $A \in \mathbf{C}^{m\times n}$ and $1 \leq m_1 < m_2 \leq m$, and let A_1 denote the $(m-1)\times n$ matrix obtained from A by removing the m_1th row, and A_2 the matrix obtained by removing the m_2th row. Assume that there exists a $w = (w_k)_1^m \in \ker A^T$ such that $(w_k)_{m_1}^{m_2} \neq 0$. Then for all $\xi \in \mathbf{C}$ with the possible exception of a finite number of points the equality*

$$\ker A = \ker(A_2 - \xi A_1)$$

holds.

The proof is analogous to that of Lemma 2.4.

We also need the following elementary fact.

LEMMA 2.6. *Suppose that A_1 is obtained from A after removing the m_1th row. Assume that there exists a $w = (w_k)_1^m \in \ker A^T$ such that $w_{m_1} \neq 0$. Then*

$$\ker A = \ker A_1. \tag{2.10}$$

PROOF. Obviously, $\ker A \subseteq \ker A_1$. Now let $A_1 x = 0$. Then $Ax = ce_{m_1}$ for a certain $c \in \mathbf{C}$. Here e_k denotes the kth unit vector. Since $w^T e_{m_1} = w_{m_1} \neq 0$, we have $e_{m_1} \notin \operatorname{im} A$. Hence $x \in \ker A$, which implies (2.10). ■

Now we are able to prove the main result of this section.

Proof of Theorem 2.1. Let H be a (p, q)-Hankel mosaic matrix with block sizes $m_i \times n_j$ $(i = 1, \dots, p; j = 1, \dots, q)$. Our goal is to find matrices $D_1 \in \mathbf{C}^{r \times n}$ and $D_2 \in \mathbf{C}^{r \times m}$, $r = \operatorname{rank} H$, $n = \sum n_j, m = \sum m_i$, such that $\widetilde{H} := D_2 H D_1^T$ is a nonsingular Hankel mosaic matrix. In view of Lemma 2.3 we have to look for matrices D_1 and D_2 of the form (2.5). These matrices will be constructed successively by applying Lemma 2.5 or Lemma 2.6 and taking Remark 2.1 and (2.6) into account.

Suppose that $\ker H^T \neq \{0\}$ and $w = (w_i)_1^m \in \ker H^T$ $(w_i \in \mathbf{C}^{m_i}, w \neq 0)$. We choose $i = i_0$ such that $w_{i_0} \neq 0$. In case $m_{i_0} > 1$ we form matrices H_1 and H_2 by removing the first or the last row, respectively, of the i_0th block row and apply Lemma 2.5 According to this lemma there exists a ξ such that $\ker H = \ker(H_2 - \xi H_1)$. Now we form the matrix $\widetilde{H} = D_2 H$, where $D_2 = \operatorname{diag}(D_{21}, \dots, D_{2p})$:

$$D_{2i} = \begin{cases} I_{m_i} & \text{if } i \neq i_0, \\ D_{m_{i_0}-1}(u) & \text{if } i = i_0 \end{cases}$$

for $\widehat{u}(\lambda) = \lambda - \xi$. By construction, the kernels of H and $\widetilde{H}$ coincide and the kernel dimension of $\widetilde{H}^T$ is one less than that of H^T.

In case $m_{i_0} = 1$, we form the matrix $\widetilde{H}$ by removing the i_0th block row of H. According to Lemma 2.6 we have again $\ker \widetilde{H} = \ker H$ and $\dim\ker \widetilde{H}^T = \dim\ker H^T - 1$.

If now $\dim\ker \widetilde{H}^T > 0$, we repeat the construction after replacing H by $\widetilde{H}$. After a finite number of steps we arrive at a matrix $\widetilde{H} = D_2 H$ with trivial kernel of $\widetilde{H}^T$ and D_2 of the form (2.5). In case the kernel of $\widetilde{H}$ is nontrivial, we repeat the construction for the transpose of H and get a matrix D_1 of the form (2.5) such that the matrix $\widetilde{H} := D_2 H D_1^T$ has the desired properties, i.e., it is a nonsingular (p, q)-Hankel mosaic matrix.

By Theorem 1.2, $\widetilde{B} := \widetilde{H}^{-1}$ is a (q, p)-Bezoutian, and by Lemma 2.3, $B := D_1^T \widetilde{B}\, D_2$ is also a (q, p)-Bezoutian. According to Lemma 2.1, B is a generalized inverse of H. This proves the theorem. ■

3. EXTENSION APPROACH

In this section we employ the familiar extension approach for generalized inversion, which is described in the following lemma.

LEMMA 3.1. *Let A be an $m \times n$ matrix, $\alpha = \dim\ker A$, and $\beta = \dim\ker A^T$. Furthermore, let A_1 be an $m \times \beta$ matrix the columns of which span a direct complement to the range of A, A_2^T an $n \times \alpha$ matrix the columns of which span a direct complement to the range of A^T, and A_3 an arbitrary $\alpha \times \beta$ matrix. Then*

$$\widetilde{A} = \begin{bmatrix} A & A_1 \\ A_2 & A_3 \end{bmatrix}$$

is nonsingular. Suppose that

$$\widetilde{A}^{-1} = \begin{bmatrix} B & * \\ * & * \end{bmatrix}$$

for B of appropriate size. Then B is an inner inverse of A. In case $A_3 = 0$, B is a reflexive generalized inverse of A.

With the help of this lemma we shall prove the following theorem.

THEOREM 3.1. *Let H be a (p,q)-Hankel mosaic matrix of size $m \times n$, $\alpha = \dim\ker H$, $\beta = \dim\ker H^T$. Then there exists a nonsingular (q,p)-Bezoutian B of order $m+\alpha = n+\beta$ such that an inner inverse of H can be obtained by removing β rows and α columns of B.*

In order to prove this theorem we start with a nonsingularity criterion. For this we introduce some notation.

For a Hankel matrix $H = [s(k+l)]_{k=0}^{m-1}\,{}_{l=0}^{n-1}$, let $g(H)$ denote the column vector $[s(n) \cdots s(m+n-2)\, 0]^T$. If H is a Hankel mosaic matrix $H = [H_{ij}]_{1}^{p}\,{}_{1}^{q}$, then we define $g_j(H) := [g(H_{1j})^T, \ldots, g(H_{pj})^T]^T$. Furthermore we denote by E_i the vector consisting of p block components which are equal to zero except for the ith component, which is equal to the last unit vector e_{m_i} in $\mathbf{C}^{m_i}$.

We consider the following $p+q$ systems of equations

$$Hx_i = E_i \qquad (i = 1, \ldots, p), \tag{3.11}$$

$$Hy_j = g_j(H) \qquad (j = 1, \ldots, q). \tag{3.12}$$

LEMMA 3.2. *If the equations* (3.11) *and* (3.12) *are solvable, then H has full row rank, i.e., the kernel of H^T is trivial.*

PROOF. Suppose that $w^T H = 0, w = (w_i)_1^p, w_i = (w_{ik})_{k=1}^{m_i}$. Then, due to the solvability of (3.11), we have

$$0 = w^T H x_i = w^T E_i = w_{im_i}. \tag{3.13}$$

Let Z_m denote the forward shift in $\mathbf{C}^m$. We set $w_i' := Z_{m_i} w_i$ and $w' = (w_i')_1^p$. It is easily seen that $(w')^T H$ vanishes with the possible exception of the last component of each block. The last components of the blocks are equal to $w^T g_j(H)$. But we have $w^T g_j(H) = w^T H y_j = 0$. Thus $w'^T H = 0$. Repeating the arguments above with w replaced by w', we conclude $w_{i,m_i-1} = 0$ and $(u')'^T H = 0$. Proceeding in this way, we shall arrive at $w = 0$. That means H has full row rank. ■

Clearly, an analogous lemma holds with H replaced by its transpose.

We shall apply the following consequence of Lemma 3.2.

COROLLARY 3.1. *Suppose that the (p,q)-Hankel mosaic matrix $H = [H_{ij}]_1^p{}_1^q$ has no full row rank. Then there exists a one column extension of H to a (p,q)-Hankel mosaic matrix $\widetilde{H} = [\widetilde{H}_{ij}]_1^p{}_1^q$, where, for a certain j_0,*

$$\widetilde{H}_{ij} = \begin{cases} H_{ij}, & j \neq j_0, \\ [H_{ij}\ g], & j = j_0 \end{cases},$$

and g is equal to $g_{j_0}(H)$ plus a multiple of one of the vectors E_i $(i = 1, \dots, p)$, such that

$$\operatorname{rank} \widetilde{H} = \operatorname{rank} H + 1. \tag{3.14}$$

PROOF. Let H have no full row rank. Then by Lemma 3.2 at least one of the equations (3.11) or (3.12) is not solvable. Hence there are $j = j_0, i = i_0$, and $t \in \mathbf{C}$ such that $g := g_{j_0}(H) + tE_{i_0}$ does not belong to the range of H. The corresponding extension $\widetilde{H}$ fulfills (3.14). ■

If we apply Corollary 3.1 successively to H and its transpose, we arrive at the following theorem. In order to simplify the formulation of the theorem, let us agree upon calling a Hankel matrix $\widehat{H}$ a (μ,ν)-*extension* of the Hankel matrix $H = [s(k+l)]_0^{m-1}{}_0^{n-1}$ if

$$\widehat{H} = [s(k+l)]_0^{m+\mu-1}{}_0^{n+\nu-1}$$

for certain $s(m+n-1), \dots, s(m+n+\mu+\nu-2) \in \mathbf{C}$. A (p,q)-Hankel mosaic matrix $\widetilde{H}$ will be called a (μ,ν)-extension of the (p,q)-Hankel mosaic matrix H if the (i,j) block of $\widetilde{H}$ is a (μ_i,ν_j)-extension of the (i,j) block of H and $\sum \mu_i = \mu$ and $\sum \nu_j = \nu$.

THEOREM 3.2. *Let H be a (p,q)-Hankel mosaic matrix, $\alpha :=$ dim ker H, and $\beta :=$ dim ker H^T. Then there exists an (α, β)-extension $\widetilde{H}$ of H which is a nonsingular (p,q)-Hankel mosaic matrix.*

Proof of Theorem 3.1. We extend H according to Theorem 3.2 to a nonsingular (p,q)-Hankel mosaic matrix. Due to Theorem 1.2, $\widetilde{H}^{-1}$ is a (q,p)-Bezoutian. In view of the construction of $\widetilde{H}$, the assumptions of Lemma 3.1 are fulfilled. It remains now to apply this lemma in order to get the assertion of the theorem. ■

4. MOORE-PENROSE INVERSES

It is easily checked that the Moore-Penrose inverse of a Hankel matrix is, in general, not a Bezoutian, but according to Theorem 1.1 it is an r-Bezoutian for $r = 4$. In order to generalize this result we have to define a Bezoutian concept generalizing both the concepts of r-Bezoutian and of (q,p)-Bezoutian.

A (q,p)-mosaic matrix B is said to be a (generalized, Hankel) (q,p,r)-*Bezoutian* if its generating function admits a representation

$$\widehat{B}(\lambda, \mu) = \frac{1}{\lambda - \mu} \widehat{U}(\lambda) \widehat{V}(\mu)^T,$$

where $\widehat{U}(\lambda)$ is a $p \times (p+q+r)$ and $\widehat{V}(\lambda)$ is a $p \times (p+q+r)$ matrix polynomial.

That means the (q,p)-Bezoutians in the usual sense are $(q,p,0)$-Bezoutians. The scalar r-Bezoutians are $(1,1,r-2)$-Bezoutians. Bezoutian concepts of this generality were introduced in [17] and [8].

In this section we prove the following.

THEOREM 4.1. *The Moore-Penrose inverse of a (p,q)-Hankel mosaic matrix is a $(q,p,p+q)$-Bezoutian.*

In order to prove this theorem we apply the following well-known lemma (see for example [3]).

LEMMA 4.1. *Let A be an $m \times n$ matrix with $\alpha =$ dim ker A and $\beta =$ dim ker A^*, where A^* denotes the adjoint (i.e. conjugate transpose) matrix. Furthermore, let U be a matrix the columns of which span the kernel of A, and V a matrix the columns of which span the kernel of A^*. Then the matrix*

$$\widetilde{A} = \begin{bmatrix} A & V \\ U^* & 0 \end{bmatrix} \tag{4.15}$$

is nonsingular, and $\widetilde{A}^{-1}$ *has the form*

$$\widetilde{A}^{-1} = \begin{bmatrix} A^{\dagger} & (U^{*})^{\dagger} \\ V^{\dagger} & 0 \end{bmatrix},$$

where the dagger indicates the Moore-Penrose inverse.

Lemma 4.1 tells us that in order to describe the Moore-Penrose inverse of Hankel mosaic matrices one has to study the kernel structure of such matrices. This will be done next. In that way we shall generalize some results of [10].

Our aim is to find bases of $\ker H$ and $\ker H^*$ consisting of shift chains. Recall that a sequence of vectors $x, Z_m x, \ldots, Z_m^{\nu-1} x \in \mathbf{C}^m$, where Z_m is the forward shift, is called a shift chain of length ν. A sequence of block vectors will be said to be a shift chain if all its block components form shift chains.

In order to motivate the subsequent considerations, let us briefly explain the idea of our proof of Theorem 4.1. Suppose that there is a basis of $\ker H$ consisting of a shift chains, and a basis of $\ker H^*$ consisting of b shift chains. Then the matrices V and U occurring in Lemma 4.1, which are formed from these bases (in reverse order), are (p, a)- and (q, b)-Hankel mosaic matrices, respectively. Hence $\widetilde{H}$ is $(p + a, q + b)$-Hankel mosaic matrix. Therefore $\widetilde{H}^{-1}$ is, by Theorem 1.2, a $(q + b, p + a)$-Bezoutian. This implies that the restriction $H^{\dagger}$ is a $(q, p, a + b)$-Bezoutian. That means it remains to show that $a + b \leq p + q$.

We introduce some notation. For an $m \times n$ Hankel matrix $H = [s(k+l)]_0^{m-1}{}_0^{n-1}$, let $H^{(k)}$ $(k = 0, \pm 1, \pm 2, \ldots)$ denote the $(m + k) \times (n - k)$ Hankel matrix

$$H^{(k)} := [s(k + l)]_0^{m+k-1}{}_0^{n+k+1}.$$

If $H = [H_{ij}]_1^p{}_1^q$ is a (p, q)-Hankel mosaic matrix, then we set $H^{(k)} = \left[H_{ij}^{(k)}\right]_1^p{}_1^q$.

Besides the kernel of H, we study the kernels of the matrices $H^{(k)}$. For their description it is convenient to use polynomial language. Define $\mathcal{C}_k := \{\widehat{u}(\lambda) : u \in \ker H^{(k)}\}$. The advantage of the polynomial notation is that we have natural imbeddings $\cdots \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_0 \subseteq \mathcal{C}_{-1} \subseteq \cdots$.

Note that the sequence $\{x_j\}$ forms a shift chain if and only if $\widehat{x}_j(\lambda) = \lambda^j \widehat{x}_0(\lambda)$. Furthermore, one has, for $k \geq 0$, $\widehat{x} \in \mathcal{C}_k$ if and only if the elements of the shift chain $\widehat{x}, \lambda\widehat{x}, \ldots, \lambda^{k-1}\widehat{x}$ belong to $\mathcal{C}_0$. It can also easily be checked that $\lambda\mathcal{C}_{k+1} \subseteq \mathcal{C}_k$ for all k. Hence

$$\mathcal{C}_{k+1} + \lambda\mathcal{C}_{k+1} \subseteq \mathcal{C}_k. \tag{4.16}$$

Another relation which is immediately verified is

$$\mathcal{C}_k \cap \lambda\mathcal{C}_k = \lambda\mathcal{C}_{k+1}. \tag{4.17}$$

We introduce now nonnegative integers $a_k := \dim \mathcal{C}_k - \dim \mathcal{C}_{k+1}$. Obviously, $a_k \le p + q$. From (4.16) and (4.17) we get

$$\begin{aligned} 0 &\le \dim\, \mathcal{C}_k - \dim(\mathcal{C}_{k+1} + \lambda\mathcal{C}_{k+1}) \\ &= \dim\, \mathcal{C}_k - 2\,\dim \mathcal{C}_{k+1} + \dim \mathcal{C}_{k+1} \cap \lambda\mathcal{C}_{k+1} \\ &= a_k - a_{k+1}. \end{aligned}$$

We choose now, for each $k \le 0$ with $\delta_k := a_k - a_{k+1} > 0$, linearly independent vector polynomials $\widehat{x}_{k1}, \dots, \widehat{x}_{k,\delta_k}$ spanning a direct complement of $\mathcal{C}_{k+1} + \lambda\mathcal{C}_{k+1}$ in $\mathcal{C}_k$. In this way we obtain a system of $a_0 = \sum_{k\ge 0} \delta_k \le p+q$ vector polynomials such that the vector polynomials $\lambda^i \widehat{x}_{kj}$ $(i = 0 \dots,\ k-1;\ j = 1,\ \dots,\ \delta_k)$ form a basis of $\mathcal{C}_0$. Translating this into vector language, we obtain the following.

LEMMA 4.2. *The kernel of a (p, q)-Hankel mosaic matrix H possesses a basis consisting of* $a_0 := \dim\ \ker H - \dim\ \ker H^{(1)} \le p + q$ *shift chains.*

Analogously, the kernel of the adjoint matrix H^* is spanned by $b_0 := \dim \ker H^* - \dim \ker (H^*)^{(1)}$ $(\le p + q)$ shift chains. It is remarkable that there is a relation between the numbers of shift chains spanning $\ker H$ and $\ker H^*$.

LEMMA 4.3. *Let a_0 and b_0 be defined as above. Then $a_0 + b_0 \le p + q$.*

PROOF. We introduce the abbreviations $\alpha_k := \dim \ker H^{(k)}$, $\beta_k := \dim \ker (H^{(k)})^* = \dim \ker (H^*)^{(-k)}$ and, for the index of the matrix $H^{(k)}$, $\kappa_k := \alpha_k - \beta_k$. Then we have

$$\kappa_{-1} = \kappa_0 + p + q \quad \text{and} \quad \kappa_1 = \kappa_0 - p - q. \tag{4.18}$$

With the help of (4.18) we obtain

$$\begin{aligned} a_0 + b_0 &= \alpha_0 - \alpha_1 + \beta_0 - \beta_1 \\ &= \alpha_0 - \alpha_1 + \alpha_0 - \kappa_0 - (\alpha_{-1} - \kappa_0 - p - q) \\ &= p + q - (\alpha_{-1} - \alpha_0) + (\alpha_0 - \alpha_1) \\ &= p + q - (a_{-1} - a_0). \end{aligned}$$

Since $a_0 \le a_{-1}$, we get from this the estimate $a_0 + b_0 \le p + q$. ■

Proof of Theorem 4.1. We choose a basis of $\ker H$ consisting of a_0 shift chains and form from them in reverse order a (q, a_0)-Hankel mosaic matrix U. Analogously we form from a basis of $\ker H^*$ consisting of b_0 shift chains a (p, b_0)-Hankel mosaic matrix V. Now we define $\widetilde{H}$ according to (4.15). $\widetilde{H}$ is a $(p + a_0, q +$

b_0)-Hankel mosaic matrix. By Theorem 1.2, $\widetilde{H}^{-1}$ is a $(q+b_0,\ p+a_0)$-Bezoutian. It remains to apply Lemma 4.1. ■

5. GROUP INVERSES

In order to describe the group inverses of Hankel mosaic matrices we apply the following lemma (see [21]).

LEMMA 5.1. *Let A be an $m \times n$ matrix, U a matrix the columns of which form a basis of* $\ker A$, *and V a matrix the columns of which form a basis of* $\ker A^T$. *Then A possesses a group inverse if and only if the matrix*

$$\widetilde{A} := \begin{bmatrix} A & U \\ V^T & 0 \end{bmatrix}$$

is nonsingular. Furthermore, if $\widetilde{A}$ is nonsingular, then $\widetilde{A}^{-1}$ admits a representation

$$\widetilde{A}^{-1} = \begin{bmatrix} A^{\#} & * \\ * & * \end{bmatrix}.$$

With the help of this lemma and the arguments of the proof of Theorem 4.1 one can prove the following.

THEOREM 5.1. *The group inverse of a (p,q)-Hankel mosaic matrix is, provided that it exists, a $(q,p,p+q)$-Bezoutian.*

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