Close-To-Convexity, Starlikeness, and Convexity of Certain Analytic Functions

S. OWA
Department of Mathematics, Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
owaQmath.kindai.ac.jp

M. NUNOKAWA
Department of Mathematics, Gunma University
Aramaki (Maebashi), Gunma 371-8510, Japan
nunokawaQedu.gunma-u.ac.jp

H. SAITOH
Department of Mathematics, Gunma National College of Technology
Toriba (Maebashi), Gunma 371-8530, Japan
saitohQnat.gunma-ct.ac.jp

H. M. SRIVASTAVA
Department of Mathematics and Statistics, University of Victoria
Victoria, British Columbia V8W 3P4, Canada
harimsriQmath.uvic.ca

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Abstract—The main object of the present paper is to derive several sufficient conditions for close-to-convexity, starlikeness, and convexity of certain (normalized) analytic functions. Relevant connections of some of the results obtained in this paper with those in earlier works are also provided.

Keywords—Analytic functions, Starlike functions, Close-to-convex functions, Convex functions, Subordination principle, Univalent functions.

1. INTRODUCTION AND DEFINITIONS

Let \( \mathcal{A} \) denote the class of functions \( f \) normalized by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disk

\[ \mathcal{U} := \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}. \]

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Also let $S^*(\alpha)$, $K(\alpha)$, and $C(\alpha)$ denote the subclasses of $A$ consisting of functions which are, respectively, starlike, convex, and close-to-convex of order $\alpha$ in $U$ ($0 \leq \alpha < 1$). Thus, we have (see, for details, [1, 2]; see also [3])

$$S^*(\alpha) := \left\{ f : f \in A \text{ and } \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ (z \in U; \ 0 \leq \alpha < 1) \right\},$$

$$K(\alpha) := \left\{ f : f \in A \text{ and } \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ (z \in U; \ 0 \leq \alpha < 1) \right\},$$

and

$$C(\alpha) := \left\{ f : f \in A \text{ and } \Re \left( \frac{f'(z)}{g'(z)} \right) > \alpha, \ (z \in U; \ 0 \leq \alpha < 1; \ g \in K) \right\};$$

where, for convenience,

$$S^* := S^*(0), \quad K := K(0), \quad \text{and} \quad C := C(0).$$

Next, with a view to recalling the principle of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $U$. Then we say that the function $f$ is subordinate to $g$ if there exists a function $h$, analytic in $U$, with

$$h(0) = 0 \quad \text{and} \quad |h(z)| < 1, \quad (z \in U),$$

such that

$$f(z) = g(h(z)), \quad (z \in U).$$

We denote this subordination by

$$f(z) \prec g(z).$$

In particular, if the function $g$ is univalent in $U$, the subordination (1.8) is equivalent to (cf. [1, p. 190])

$$f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Recently, Singh and Singh [4] proved several interesting results involving univalence and starlikeness of functions $f \in A$. In our attempt here to generalize these results of Singh and Singh [4], we are led naturally to several sufficient conditions for close-to-convexity, starlikeness, and convexity of functions $f \in A$.

The following lemma (popularly known as Jack's lemma) will be required in our present investigation.

**Lemma 1.** (See [5, 6].) Let the (nonconstant) function $w(z)$ be analytic in $U$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then

$$z_0 w'(z_0) = c \cdot w(z_0),$$

where $c$ is a real number and $c \geq 1$.

2. **SUFFICIENT CONDITIONS FOR CLOSE-TO-CONVEXITY**

Our first result (Theorem 1 below) provides a sufficient condition for close-to-convexity of functions $f \in A$.

**Theorem 1.** Let the function $f \in A$ satisfy the inequality

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \frac{1 + 3\alpha}{2(1 + \alpha)}, \quad (z \in U; \ 0 \leq \alpha < 1).$$

(2.1)
Then
\[ \Re \{ f'(z) \} > \frac{1+\alpha}{2}, \quad (z \in \mathcal{U}; \ 0 \leq \alpha < 1), \quad (2.2) \]
or equivalently,
\[ f \in \mathcal{C} \left( \frac{1+\alpha}{2} \right), \quad (0 \leq \alpha < 1). \quad (2.3) \]

**PROOF.** We begin by defining a function \( w \) by
\[ f'(z) = \frac{1 + \alpha w(z)}{1 + w(z)}, \quad (w(z) \neq -1; \ z \in \mathcal{U}; \ 0 \leq \alpha < 1). \quad (2.4) \]

Then, clearly, \( w \) is analytic in \( \mathcal{U} \) with \( w(0) = 0 \). We also find from (2.4) that
\[ 1 + z f''(z) f'(z) = 1 + \frac{\alpha z w'(z)}{1 + \alpha w(z)} - \frac{z w'(z)}{1 + w(z)}, \quad (z \in \mathcal{U}). \quad (2.5) \]

Suppose now that there exists a point \( z_0 \in \mathcal{U} \) such that
\[ |w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1, \quad \text{when } |z| < |z_0|. \quad (2.6) \]

Then, by applying Lemma 1, we have
\[ z_0 w'(z_0) = c w(z_0), \quad (c \geq 1; \ w(z_0) = e^{i\theta}; \ \theta \in \mathbb{R}). \quad (2.7) \]

Thus, we find from (2.5) and (2.7) that
\[
\Re \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) = 1 + \Re \left( \frac{c e^{i\theta}}{1 + \alpha e^{i\theta}} \right) - \Re \left( \frac{c e^{i\theta}}{1 + e^{i\theta}} \right)
\]
\[ = 1 + \frac{\alpha (\alpha + \cos \theta)}{1 + \alpha^2 + 2\alpha \cos \theta} - \frac{c}{2}
\]
\[ \leq \frac{1 + 3\alpha}{2(1 + \alpha)}, \quad (z_0 \in \mathcal{U}; \ 0 \leq \alpha < 1), \]

which obviously contradicts our hypothesis (2.1). It follows that
\[ |w(z)| < 1, \quad (z \in \mathcal{U}), \]
that is, that
\[ \left| \frac{1 - f'(z)}{f'(z) - \alpha} \right| < 1, \quad (z \in \mathcal{U}; \ 0 \leq \alpha < 1). \quad (2.8) \]

This evidently completes the proof of Theorem 1.

**THEOREM 2.** If the function \( f \in \mathcal{A} \) satisfies the inequality
\[ \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3 + 2\alpha}{2 + \alpha}, \quad (z \in \mathcal{U}; \ 0 \leq \alpha < 1), \quad (2.9) \]
then
\[ |f'(z) - 1| < 1 + \alpha, \quad (z \in \mathcal{U}; \ 0 \leq \alpha < 1). \quad (2.10) \]

**PROOF.** Our proof of Theorem 2, also based upon Lemma 1, is similar to that of Theorem 1. Indeed, in place of definition (2.4), here we let the function \( w \) be given by
\[ f'(z) = (1 + \alpha)w(z) + 1, \quad (z \in \mathcal{U}; \ 0 \leq \alpha < 1). \quad (2.11) \]

The details may be omitted.
REMARK 1. Since the inequality (2.10) implies that
\[ \Re \{ f'(z) \} > -\alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1), \] (2.12)
by setting \( \alpha = 0 \) in Theorem 2, we readily obtain the following.

COROLLARY 1. (See [4, p. 311, Corollary 2].) If the function \( f \in \mathcal{A} \) satisfies the inequality
\[ \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2}, \quad (z \in \mathcal{U}), \] (2.13)
then
\[ |f''(z) - 1| < 1, \quad (z \in \mathcal{U}), \] (2.14)
that is, \( f \in \mathcal{C} \).

Next we prove the following.

THEOREM 3. If the function \( f \in \mathcal{A} \) satisfies the inequality
\[ |f'(z) - 1|^\beta |zf''(z)|^\gamma < \frac{(1 - \alpha)^{\beta + \gamma}}{2^{\beta + 2\gamma}}, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1; \beta, \gamma \geq 0), \] (2.15)
then
\[ \Re \{ f'(z) \} > \frac{1 + \alpha}{2}, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1). \] (2.16)

PROOF. We define the function \( w \) by
\[ f'(z) = \frac{1 + \alpha w(z)}{1 + w(z)}, \quad (w(z) \neq -1; z \in \mathcal{U}; 0 \leq \alpha < 1). \] (2.17)
Then, clearly, \( w \) is analytic in \( \mathcal{U} \) with \( w(0) = 0 \). We also find from (2.17) that
\[ |f'(z) - 1|^\beta |zf''(z)|^\gamma = \frac{(1 - \alpha)^{\beta + \gamma} |w(z)|^\beta |zw'(z)|^\gamma}{|1 + w(z)|^{\beta + 2\gamma}}, \quad (z \in \mathcal{U}). \] (2.18)
Supposing now that there exists a point \( z_0 \in \mathcal{U} \) such that
\[ |w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1, \quad \text{when} \quad |z| < |z_0|, \]
if we apply Lemma 1 just as we did in the proof of Theorem 1, we shall obtain
\[ |f'(z_0) - 1|^\beta |z_0f''(z_0)|^\gamma = \frac{(1 - \alpha)^{\beta + \gamma} c^\gamma}{|1 + c^\iota|^{\beta + 2\gamma}} \]
\[ \geq \frac{(1 - \alpha)^{\beta + \gamma}}{2^{\beta + 2\gamma}}, \quad (z_0 \in \mathcal{U}; 0 \leq \alpha < 1), \]
which obviously contradicts our hypothesis (2.15). Thus, we have
\[ |w(z)| < 1, \quad (z \in \mathcal{U}), \]
which implies that
\[ \left| \frac{f'(z) - 1}{f'(z) - \alpha} \right| < 1, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1), \] (2.19)
that is, that (2.16) holds true.
By letting
\[ \beta = \gamma - 1 = 0 \]
in Theorem 2, we arrive at the following.

**COROLLARY 2.** If the function \( f \in A \) satisfies the inequality
\[
|zf''(z)| < \frac{1 - \alpha}{4}, \quad (z \in U; \ 0 \leq \alpha < 1),
\] (2.20)
then
\[
\Re \{f'(z)\} > \frac{1 + \alpha}{2}, \quad (z \in U; \ 0 \leq \alpha < 1). \quad (2.21)
\]

**REMARK 2.** An analogous result (which apparently is not contained in Corollary 2) was proven earlier by Singh and Singh [4, p. 310, Corollary 1], which asserted that, if the function \( f \in A \) satisfies the inequality
\[
|zf''(z)| < 1, \quad (z \in U),
\]
then \( f \in C \).

### 3. STARLIKENESS AND CONVEXITY

In this section, we first prove the following result (Theorem 4 below), which involves the already introduced principle of subordination between analytic functions (see Section 1).

**THEOREM 4.** If the function \( f \in A \) satisfies the inequality
\[
\Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > \begin{cases} 
\frac{5\lambda - 1}{2(\lambda + 1)}, & (z \in U; \ 1 < \lambda \leq 2), \\
\frac{\lambda + 1}{2(\lambda - 1)}, & (z \in U; \ 2 < \lambda < 3),
\end{cases} \quad (3.1)
\]
for some \( \lambda(1 < \lambda < 3) \), then
\[
\frac{zf'(z)}{f(z)} < \frac{\lambda(1 - z)}{\lambda - z}. \quad (3.2)
\]
The result is sharp for the function \( f \) given by
\[
f(z) = z \left(1 - \frac{z}{\lambda}\right)^{\lambda - 1}. \quad (3.3)
\]

**PROOF.** Let us define the function \( w \) by
\[
\frac{zf'(z)}{f(z)} = \frac{\lambda[1 - w(z)]}{\lambda - w(z)}, \quad (w(z) \neq \lambda; \ z \in U; \ 1 < \lambda < 3). \quad (3.4)
\]
Then, clearly, \( w \) is analytic in \( U \) with \( w(0) = 0 \). By logarithmic differentiation of both sides of (3.4), we also find that
\[
1 + \frac{zf''(z)}{f'(z)} = \frac{\lambda[1 - w(z)]}{\lambda - w(z)} - \frac{zw'(z)}{1 - w(z)} + \frac{zw'(z)}{\lambda - w(z)}, \quad (z \in U). \quad (3.5)
\]
Assuming now that there exists a point \( z_0 \in U \) such that
\[
|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1, \quad \text{when} \ |z| < |z_0|,
\]
if we apply Lemma 1 just as we did in the proof of Theorem 1, we shall obtain
\[
\Re \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) = \Re \left( \frac{\lambda(\lambda - e^{i\theta})}{\lambda - e^{i\theta}} \right) - \Re \left( \frac{ce^{i\theta}}{1 - e^{i\theta}} \right) + \Re \left( \frac{ce^{i\theta}}{1 - e^{i\theta}} \right)
\]
\[
= \lambda(\lambda + 1)(1 - \cos \theta) + \frac{c}{2} + \frac{c(\lambda \cos \theta - 1)}{1 + \lambda^2 - 2\lambda \cos \theta}
\]
\[
= \frac{\lambda + 1}{2} + \frac{(\lambda^2 - 1)(c + 1 - \lambda)}{2(1 + \lambda^2 - 2\lambda \cos \theta)}
\]
\[
\geq \frac{\lambda + 1}{2} + \frac{(\lambda^2 - 1)(2 - \lambda)}{2(1 + \lambda^2 - 2\lambda \cos \theta)}, \quad (z_0 \in \mathcal{U}; 1 < \lambda < 3),
\]
which yields the inequality
\[
\Re \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) \geq \begin{cases} 
\frac{5\lambda - 1}{2(\lambda + 1)}, & (z_0 \in \mathcal{U}; 1 < \lambda \leq 2), \\
\frac{\lambda + 1}{2(\lambda - 1)}, & (z_0 \in \mathcal{U}; 2 < \lambda < 3).
\end{cases} \tag{3.6}
\]
Since (3.6) obviously contradicts our hypothesis (3.1), we conclude that
\[
|w(z)| < 1, \quad (z \in \mathcal{U}),
\]
that is, that
\[
\left| \frac{zf'(z)}{f(z)} - \frac{\lambda}{\lambda + 1} \right| < \frac{\lambda}{\lambda + 1}, \quad (z \in \mathcal{U}; 1 < \lambda < 3), \tag{3.7}
\]
which implies the subordination (3.2) asserted by Theorem 4.

Finally, for the function \( f \) given by (3.3), we have
\[
\frac{zf'(z)}{f(z)} = \frac{\lambda(1 - z)}{\lambda - z}, \tag{3.8}
\]
which evidently completes our proof of Theorem 4.

REMARK 3. A special case of Theorem 4 when \( \lambda = 2 \) was given earlier by Singh and Singh [4, p. 313, Theorem 6].

Finally, since
\[
f(z) \in \mathcal{K}(\alpha) \iff zf'(z) \in S^*(\alpha), \quad (0 \leq \alpha < 1), \tag{3.9}
\]
whose special case, when \( \alpha = 0 \), is the familiar Alexander theorem (cf., e.g., [1, p. 43, Theorem 2.12]), Theorem 4 can be applied in order to deduce the following.

COROLLARY 3. If the function \( f \in \mathcal{A} \) satisfies the inequality
\[
\Re \left( 2zf''(z) + z^2f'''(z) \right) < \begin{cases} 
\frac{3(\lambda - 1)}{2(\lambda + 1)}, & (z \in \mathcal{U}; 1 < \lambda \leq 2), \\
\frac{3 - \lambda}{2(\lambda - 1)}, & (z \in \mathcal{U}; 2 < \lambda < 3),
\end{cases} \tag{3.10}
\]
for some \( 1 < \lambda < 3 \), then
\[
1 + \frac{zf''(z)}{f'(z)} < \frac{\lambda(1 - z)}{\lambda - z}. \tag{3.11}
\]
The result is sharp for the function \( f \) given by
\[
f'(z) = \left( 1 - \frac{z}{\lambda} \right)^{\lambda-1}. \tag{3.12}
\]
REFERENCES