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Weighted local conditioning index of a positioning and orienting parallel manipulator

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Abstract In the presence of positioning and orienting tasks, the singular values of Jacobian matrices have different units, thereby making it impossible to order them and calculate the associated condition numbers. Here, this dimensional in-homogeneity is resolved by introducing a weighting factor. In this method, both the Jacobian and twist vector are made homogeneous, simultaneously. Moreover, relations between the weighting factors used here to the homogeneous Jacobian matrices derived by others are given. This factor should be constant throughout the workspace, while it is pose dependent in the latter methods. As a case study, both methods are applied to a Tricept parallel manipulator with complex degrees of freedom. A local conditioning index, as a dexterity index, is plotted in the workspace. Although both methods lead to homogeneous Jacobian matrices, obvious differences between the plotted local conditioning indices are revealed here. Therefore, those homogeneous Jacobian matrices derived by others, and the associated dexterity indices are unreliable.

1. Introduction

Parallel manipulators, in general, have some significant advantages over their serial counterparts, such as more rigidity and accuracy, higher force and torque capacity, and higher speed. Many of these manipulators have both rotational and translational Degrees Of Freedom (DOF) [1]. This leads to dimensionally in-homogeneous Jacobian matrices. Making the Jacobian matrices dimensionally homogeneous is very important whenever one wants to order their singular values in calculating their condition numbers [2–11].

Tandirci et al. [6] normalized the Jacobian by dividing a *characteristic length* out of all translational elements. This length which produces the best performance measure is dubbed the *natural length* by Ma and Angeles [12], and is used for design optimization. Chablat et al. [8] used *characteristic length* to determine that the design parameter of a planar parallel mechanism with PRR chains has an isotropic condition. Some authors proposed to choose a factor of L as that which makes the robot as far as possible from any singular configuration [13,14]. Gosselin [3] introduced a method for formulating a dimensionally homogeneous Jacobian matrix for a planar mechanism with one rotational and two translational DOF. This Jacobian matrix relates the actuator velocities to the velocities of the x- and y-coordinates of two points on the end-effector platform. Kim and Ryu [9] furthered this work by using the velocities of three points on the end-effector platform to develop a dimensionally homogeneous Jacobian matrix. Arsenault and Boudreau [15] used the same method to develop dimensional Jacobians for 3-RRR planar parallel manipulators, and compared the results with that of the conventional method during dexterity computation. Pond and Carretero [11] furthered this method again by using three independent coordinates of three points on an end-effector platform. Moreover, Angeles [10] introduced *engineering characteristic length* for a rigid body transformation matrix, to make it homogeneous.

Here, the inconsistency will be resolved by defining a *weighting factor* by which the Jacobian entries that have units of length are divided, thereby producing a new Jacobian that is dimensionally homogeneous. This may be considered as the *characteristic length*, but in contrast to the others, both the Jacobian and twist array are made homogeneous,

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simultaneously. Moreover, it will be shown that this weighting factor is position dependent for those methods that make the Jacobian's homogenous differently [9,11]. Furthermore, one might choose different weighting factors for different coordinates of twist array and the associated columns of the Jacobian matrix, even those with the same units.

This paper is organized as follows. In the next section, two methods to deal with inhomogeneous Jacobians are described. Then, some measures for dexterity are explained. Finally, as the case study, the Jacobian matrices of a Tricept parallel mechanism are derived by two different methods and the results are compared.

2. Characteristic length and weighting factor

The differential kinematic relations pertaining to parallel manipulators take on the form:

$$\mathbf{A}\dot{\boldsymbol{\theta}} = \mathbf{B}\mathbf{t}, \tag{1}$$

where **A** and **B** are two Jacobian matrices, $\dot{\boldsymbol{\theta}}$ is the vector of joint rates and **t** is the twist array, which is defined as:

$$\mathbf{t} \equiv \begin{bmatrix} \dot{\mathbf{c}} \\ \boldsymbol{\omega}_p \end{bmatrix}, \tag{2}$$

in which $\dot{\mathbf{c}}$ and $\boldsymbol{\omega}_p$ are linear and angular velocities of the end-effector, respectively.

The Jacobian matrix, **J**, can be written as follows, which relates the joint velocity vector and end-effector twist array:

$$\mathbf{J} = \mathbf{B}^{-1}\mathbf{A}. \tag{3}$$

The *condition number* of a given matrix is well known to provide a measure of invertibility of the matrix [16]. It is, thus, natural that this concept found its way in this context. Indeed, the condition number of the Jacobian matrix was proposed [17] as a figure of merit to minimize when designing manipulators for maximum accuracy. Condition numbers of the Jacobian matrices are known as a kinetostatic performance index of parallel manipulators [18]. Indeed, in order to determine the condition number of the Jacobian matrices, their singular values must be ordered from largest to smallest. However, in the presence of positioning and orienting tasks, three of these singular values, namely, those associated with positioning, are dimensionless, while those associated with orientation have units of length, thereby making impossible such an ordering. This dimensional inhomogeneity can be resolved by introducing a normalizing *characteristic length* [6]. Upon dividing the three *orientation columns*, i.e. the second three columns of the Jacobian by this length, a non-dimensional Jacobian is obtained whose singular values are non-dimensional as well.

Here, based on a new interpretation, we will call it *weighting factor*. Upon dividing the associated columns of the Jacobian by this length (factor), a non-dimensional Jacobian is obtained, whose singular values are non-dimensional as well. But one is not allowed to divide some terms of any equation by a length (factor); the resulting equation is different from the original one. In fact, it is only permissible to multiply to or divide by the same term by any nonzero value; i.e. one should make both the Jacobian and twist vector homogeneous, simultaneously. Therefore, Eq. (1) can be written as:

$$\hat{\mathbf{t}} = \hat{\mathbf{J}}\dot{\boldsymbol{\theta}}, \tag{4}$$

where the ith column of $\hat{\mathbf{J}}$ and $\hat{\mathbf{t}}$ are defined as:

$$\hat{\mathbf{j}}_i = \mathbf{j}_i/L, \quad \text{for } i = 1, 2, 3, \tag{5}$$

$$\hat{\mathbf{t}} \equiv \begin{bmatrix} \dot{\mathbf{c}} \\ L\boldsymbol{\omega}_p \end{bmatrix}. \tag{6}$$

Actually, here, our primary aim is to bring the Jacobian matrix as close as possible to the isotropic condition, while comparing one unit of the linear velocity vector with L units of the angular velocity vector. Moreover, one might assign different weights to the different components of linear and angular velocities in this optimization problem, as well.

One has the same problem in the manipulators, with different types of actuation, namely, prismatic and revolute ones. This can be resolved similarly.

3. Condition number and dexterity index

The Singular Value Decomposition (SVD) is an important technique used for factorization of a matrix. A SVD of a $m \times n$ positive semi-definite matrix, **A**, is any factorization of the form:

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T, \tag{7}$$

where **U** is a $m \times m$ orthogonal matrix, **V** is $n \times n$ orthogonal and $\boldsymbol{\Sigma}$ is a $m \times n$ diagonal matrix with elements $\sigma_{ij} = 0$, if $i \neq j$ and $\sigma_{ij} \geq 0$ in descending order along the diagonal. The columns of **U** and **V** are left hand and right hand singular vectors of **A**, respectively, and σ_{ij} are singular values of **A**.

Without losing generality, by assuming matrix $A_{2\times2}$, the **V** rotates a circular 2D space by θ_1 around the Z axis. Therefore, X and Y axes are mapped into U_2 and V_2. Then, $\boldsymbol{\Sigma}$ magnifies the rotated circular space in the direction of U_2 and V_2 by the associate singular values. Finally, **U** rotates the resulted ellipse around Z axis by θ_2 (Figure 1).

Moreover, one can define the condition number as the ratio of the maximum to the minimum singular values. The condition number can attain values from 1 to infinity. Clearly, it attains its minimum value of unity for matrices with identical singular values; such matrices are called isotropic. Based on the reciprocal of the condition number of Jacobian matrices, the kinematic conditioning index of robotic manipulators can be defined.

Recently, by comparing condition numbers and the absolute values of the positioning errors along different axes at several poses, Merlet questioned the credibility of the condition number as a measure of dexterity for parallel manipulators [19]. Although we share the same concern that this has to be carefully considered when talking about an optimal design for a robot, we should consider the rotations involved in the SVD of Jacobian matrices, i.e. the singular values give us the bounds for maximum and minimum relative errors. In other words, a larger condition number does not necessarily imply that the relative errors along the x-axis, or the other axes, are bigger! Moreover, defining different weighting factors, as proposed here, or the characteristic lengths by others drastically changes the condition number as well. We believe that condition numbers, being defined based on different norms, are credible measures for dexterity, by taking into account the weighting factor and the rotations involved in SVD decompositions.

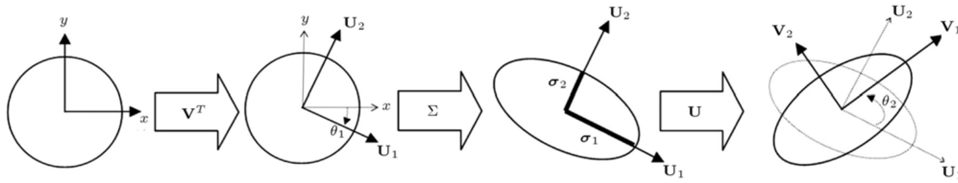

Figure 1: SVD of a 2 × 2 matrix **A**.

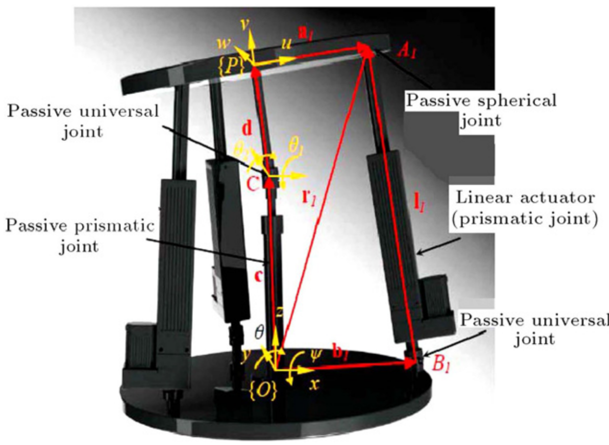

Figure 2: Tricept structure and geometric model.

4. Case study: the Tricept mechanism

The Tricept robot, as depicted in Figure 2, with two rotational and one translational DOF, was introduced by Neumann [20]. Siciliano [21] developed the kinematics and studied the manipulability of the Tricept. Pond and Carretero [11] formulated its square, dimensionally homogeneous, Jacobian matrices, by furthering the method proposed by Kim and Ryu [9]. Architectural optimization of the Tricept and similar mechanisms was undertaken by Zhang and Gosselin [7].

The manipulator consists of a base platform, a moving platform, three active legs and one passive leg. Active legs are linear (prismatic) actuators that connect the base to the moving platform by universal (or spherical) and spherical joints. The passive leg consists of two parts; the upper part is a link with a constant length that is connected to the moving platform by a spherical joint, while its lower part is a prismatic joint that is connected to the base and upper part by a passive universal joint. Moving and global frames, $\{P(uvw)\}$ and $\{O(xyz)\}$, are attached to the moving and base platforms, respectively.

4.1. Kinematic analysis

The geometric model of the ith leg of the Tricept is depicted in Figure 2. The closure equation for this leg can be written as:

$$\mathbf{c} + \mathbf{R}(\mathbf{a}_i + \mathbf{d}) = b_i \mathbf{n}_{bi} + l_i \mathbf{n}_{li}, \tag{8}$$

where $\mathbf{c}$ and $\mathbf{d}$ are the vectors from O to C and C to P, respectively. While $\mathbf{R}$ is the rotation matrix carrying frame $\{P\}$ into an orientation coincident with that of frame $\{O\}$; $\mathbf{a}_i$ is the position vector from P to A_i in frame $\{P\}$; $\mathbf{b}_i$ is the position vector of point B_i in the global frame. Moreover, $\mathbf{n}_{bi}$ and $\mathbf{n}_{li}$ are the unit vectors showing the directions of vectors $\mathbf{b}_i$ and $\mathbf{l}_i$, respectively.

Dot-multiplying both sides of Eq. (8) by $\mathbf{n}_{li}$, upon simplifications lead to:

$$\mathbf{c}^T \mathbf{n}_{li} + (\mathbf{R}(\mathbf{a}_i + \mathbf{d}))^T \mathbf{n}_{li} - \mathbf{b}_i \mathbf{n}_{bi}^T \mathbf{n}_{li} = l_i. \tag{9}$$

Rewriting Eq. (9), for $i = 1 \dots 3$, leads to three quadratic equations, which can be solved either numerically or theoretically.

4.2. Jacobian matrix

Taking the first time derivative of Eq. (8) yields:

$$\dot{\mathbf{c}} + \boldsymbol{\omega}_p \times (\mathbf{R}(\mathbf{a}_i + \mathbf{d})) = \dot{l}_i \mathbf{n}_{li} + \boldsymbol{\omega}_l \times l_i \mathbf{n}_{li}, \tag{10}$$

where $\boldsymbol{\omega}_p$ and $\boldsymbol{\omega}_l$ are the end-effector and limb angular velocity vectors. Dot-multiplying both sides of Eq. (10) by $\mathbf{n}_{li}$, upon simplifications lead to:

$$\mathbf{n}_{li}^T \dot{\mathbf{c}} + \mathbf{n}_{li}^T \boldsymbol{\omega}_p \times (\mathbf{R}(\mathbf{a}_i + \mathbf{d})) = \dot{l}_i. \tag{11}$$

Written Eq. (11), for $i = 1 \dots 3$, upon simplification leads to Eq. (1), where $\mathbf{t}$ is the three dimensional twist vector; $\dot{\boldsymbol{\theta}}$ is the three dimensional actuator velocity vector; $\mathbf{A}$ and $\mathbf{B}$ are the Jacobian matrices, namely:

$$\mathbf{t} = \begin{bmatrix} \dot{c} & \dot{\psi} & \dot{\theta} \end{bmatrix}^{\mathbf{T}}, \tag{12}$$

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{l}_1 & \dot{l}_2 & \dot{l}_3 \end{bmatrix}^{\mathbf{T}}, \tag{13}$$

$$\mathbf{B} = \begin{bmatrix} n_{l1z} & (\mathbf{R}(\mathbf{a}_1 + \mathbf{d}) \times \mathbf{n}_{l1})_x & (\mathbf{R}(\mathbf{a}_1 + \mathbf{d}) \times \mathbf{n}_{l1})_y \\ n_{l2z} & (\mathbf{R}(\mathbf{a}_2 + \mathbf{d}) \times \mathbf{n}_{l2})_x & (\mathbf{R}(\mathbf{a}_2 + \mathbf{d}) \times \mathbf{n}_{l2})_y \\ n_{l3z} & (\mathbf{R}(\mathbf{a}_3 + \mathbf{d}) \times \mathbf{n}_{l3})_x & (\mathbf{R}(\mathbf{a}_3 + \mathbf{d}) \times \mathbf{n}_{l3})_y \end{bmatrix}, \tag{14}$$

$$\mathbf{A} = \mathbf{I}_{3\times 3}, \tag{15}$$

in which $\dot{c}$ is translational velocity along z axis, and $\dot{\psi}$ and $\dot{\theta}$ are rotational velocities around x and y axes, respectively.

Moreover, the Jacobian matrix, $\mathbf{J}$, is defined as:

$$\mathbf{J} \equiv \mathbf{B}^{-1}\mathbf{A} = \mathbf{B}^{-1}. \tag{16}$$

4.2.1. Homogeneous Jacobian and twist vector

As expected, the Jacobian matrix contains entries of different units. This is true with the twist vector, as well. We resolve these dimensional in-homogeneities by introducing a normalizing weighting factor, as explained earlier.

Dividing the second and the third columns of the Jacobian matrix by a length, and multiplying the second and the third coordinates of the twist vector to the same length leads to the following relation:

$$\begin{bmatrix} n_{l1z} & \dfrac{(\mathbf{R}(\mathbf{a}_1 + \mathbf{d}) \times \mathbf{n}_{l1})_x}{L} & \dfrac{(\mathbf{R}(\mathbf{a}_1 + \mathbf{d}) \times \mathbf{n}_{l1})_y}{L} \\ n_{l2z} & \dfrac{(\mathbf{R}(\mathbf{a}_2 + \mathbf{d}) \times \mathbf{n}_{l2})_x}{L} & \dfrac{(\mathbf{R}(\mathbf{a}_2 + \mathbf{d}) \times \mathbf{n}_{l2})_y}{L} \\ n_{l3z} & \dfrac{(\mathbf{R}(\mathbf{a}_3 + \mathbf{d}) \times \mathbf{n}_{l3})_x}{L} & \dfrac{(\mathbf{R}(\mathbf{a}_3 + \mathbf{d}) \times \mathbf{n}_{l3})_y}{L} \end{bmatrix} \begin{bmatrix} \dot{c} \\ L\dot{\psi} \\ L\dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{l}_3 \end{bmatrix}. \tag{17}$$

Moreover, one might assign different weighting factors to the different angular velocity coordinates, as well. If this is the

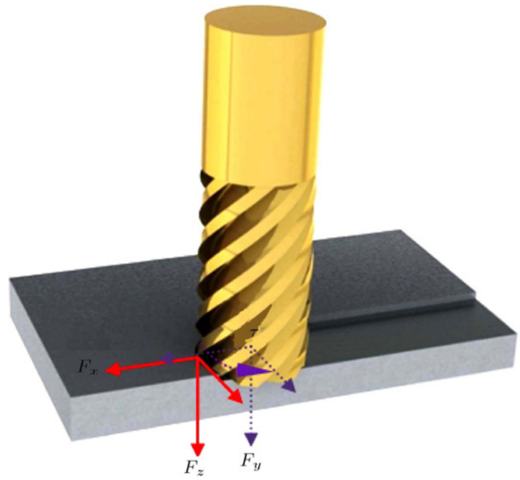

Figure 3: End-milling operation and choosing a sufficient *L*.

case, one should introduce two other scalars, namely m and n. Therefore, Eq. (17) takes the form:

$$\begin{bmatrix} n_{l1z} & \dfrac{(\mathbf{R}(\mathbf{a}_1+\mathbf{d})\times\mathbf{n}_{l1})_x}{mL} & \dfrac{(\mathbf{R}(\mathbf{a}_1+\mathbf{d})\times\mathbf{n}_{l1})_y}{nL} \\ n_{l2z} & \dfrac{(\mathbf{R}(\mathbf{a}_2+\mathbf{d})\times\mathbf{n}_{l2})_x}{mL} & \dfrac{(\mathbf{R}(\mathbf{a}_2+\mathbf{d})\times\mathbf{n}_{l2})_y}{nL} \\ n_{l3z} & \dfrac{(\mathbf{R}(\mathbf{a}_3+\mathbf{d})\times\mathbf{n}_{l3})_x}{mL} & \dfrac{(\mathbf{R}(\mathbf{a}_3+\mathbf{d})\times\mathbf{n}_{l3})_y}{nL} \end{bmatrix} \begin{bmatrix} \dot{c} \\ mL\dot{\psi} \\ nL\dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{l}_3 \end{bmatrix}. \tag{18}$$

As a result, one unit of the linear velocity is compared with $m \times L$ units of angular velocity around x axis and $n \times L$ units of angular velocity around y axis. So, different weighting factors can be assigned to the different coordinates of the twist vector.

The designer should choose the weighting factor based on the application of the mechanism. For example, in a milling operation, as depicted in Figure 3, the radius of the end-milling cutter is the length that relates the tangential cutting force to the torque, and should be chosen as the weighting factor by the designer. It is noteworthy that the same factor relates angular velocity to linear velocity.

4.2.2. Point based Jacobian matrix

Here, the method for deriving the homogeneous Jacobian matrices proposed by Pond and Carretero [11] is adopted. The position vector of any points of the end-effector platform ($\mathbf{v}_i$) in the fixed frame can be expressed by the following equation:

$$\mathbf{v}_i = \mathbf{r}_3 + k_{i,1}(\mathbf{r}_1 - \mathbf{r}_2) + k_{i,2}(\mathbf{r}_2 - \mathbf{r}_3), \tag{19}$$

where $\mathbf{r}_i$ is the position vector of A_i in the fixed frame and $k_{i,j}$ $(i, j = 1, \dots, 3)$ are defined as:

$$\begin{Bmatrix} k_{i,j} = 1; & i = j \\ k_{i,j} = 0; & i \neq j \end{Bmatrix}. \tag{20}$$

If $\mathbf{v}_i$ is the A_i position vector, the above equation can be simplified as follows:

$$\mathbf{r}_i = k_{i,1}\mathbf{r}_1 + k_{i,2}\mathbf{r}_2 + k_{i,3}\mathbf{r}_3. \tag{21}$$

On the other hand, we have the following closure equation:

$$\mathbf{r}_i = \mathbf{b}_i + \mathbf{l}_i = b_i\mathbf{n}_{bi} + l_i\mathbf{n}_{li}. \tag{22}$$

Substituting the values of $\mathbf{r}_i$ from Eq. (21) into Eq. (22), and taking the first time derivative, yields:

$$k_{i,1}\dot{\mathbf{r}}_1 + k_{i,2}\dot{\mathbf{r}}_2 + k_{i,3}\dot{\mathbf{r}}_3 = \boldsymbol{\omega}_{li} \times l_i\mathbf{n}_{li} + \dot{l}_i\mathbf{n}_{li}. \tag{23}$$

Dot-multiplying Eq. (23) by $\mathbf{n}_{li}$, upon simplifications yields:

$$k_{i,1}\mathbf{n}_{li}^T\dot{\mathbf{r}}_1 + k_{i,2}\mathbf{n}_{li}^T\dot{\mathbf{r}}_2 + k_{i,3}\mathbf{n}_{li}^T\dot{\mathbf{r}}_3 = \dot{l}_i. \tag{24}$$

Writing this equation, for $i = 1, 2, 3$, leads to:

$$\hat{\mathbf{A}}\dot{\hat{\boldsymbol{\theta}}} = \hat{\mathbf{B}}\hat{\mathbf{t}}, \tag{25}$$

where:

$$\widehat{\mathbf{B}} = \begin{bmatrix} k_{1,1}\mathbf{n}_{l1}^T & k_{1,2}\mathbf{n}_{l1}^T & k_{1,3}\mathbf{n}_{l1}^T \\ k_{2,1}\mathbf{n}_{l2}^T & k_{2,2}\mathbf{n}_{l2}^T & k_{2,3}\mathbf{n}_{l2}^T \\ k_{3,1}\mathbf{n}_{l3}^T & k_{3,2}\mathbf{n}_{l3}^T & k_{3,3}\mathbf{n}_{l3}^T \end{bmatrix}_{3\times 9}, \tag{26}$$

$$\dot{\hat{\boldsymbol{\theta}}} = \begin{bmatrix} \dot{\mathbf{r}}_1 & \dot{\mathbf{r}}_2 & \dot{\mathbf{r}}_3 \end{bmatrix}_{9\times 1}^T, \tag{27}$$

$$\widehat{\mathbf{A}} = \mathbf{I}_{3\times 3}, \tag{28}$$

in which the components of $\dot{\hat{\boldsymbol{\theta}}}$ and $\mathbf{t}$ have units of *length/time*, and the components of $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are dimensionless. Eq. (27) rewrites as the following equation:

$$\widehat{\mathbf{B}}^{-1}\widehat{\mathbf{A}}\dot{\hat{\boldsymbol{\theta}}} = \hat{\mathbf{t}}, \qquad (\widehat{\mathbf{B}}^{-1}\widehat{\mathbf{A}}) = \widehat{\mathbf{J}}. \tag{29}$$

There are only three independent variables in $\dot{\hat{\boldsymbol{\theta}}}$ that are z coordinates of A_i coordinates. These variables can be defined as a function of structure parameters, namely:

$$A_{1z} = c + d\cos\psi\cos\theta - r_a\sin\theta, \tag{30}$$

$$A_{2z} = c + d\cos\psi\cos\theta + r_a(-\sin\theta\cos\alpha + \sin\psi\cos\theta\sin\alpha), \tag{31}$$

$$A_{3z} = c + d\cos\psi\cos\theta + r_a(-\sin\theta\cos\beta + \sin\psi\cos\theta\sin\beta), \tag{32}$$

where $\mathbf{r_a}$ is the radius of the moving platform, α is the angle between $\mathbf{a}_1$ and $\mathbf{a}_2$ and β is the angle between $\mathbf{a}_1$ and $\mathbf{a}_3$. By considering the symmetric structure, they should be 120° and 240°.

Eq. (29) can be written as:

$$\hat{\mathbf{J}} = \frac{\partial \mathbf{l}}{\partial \hat{\boldsymbol{\theta}}} = \begin{bmatrix} \dfrac{\partial l_1}{\partial A_{1x}} & \dfrac{\partial l_1}{\partial A_{1y}} \cdots & \dfrac{\partial l_1}{\partial A_{3z}} \\ \dfrac{\partial l_2}{\partial A_{1x}} & \dfrac{\partial l_2}{\partial A_{1y}} \cdots & \dfrac{\partial l_2}{\partial A_{3z}} \\ \dfrac{\partial l_3}{\partial A_{1x}} & \dfrac{\partial l_3}{\partial A_{1y}} \cdots & \dfrac{\partial l_3}{\partial A_{3z}} \end{bmatrix}, \tag{33}$$

in which $\mathbf{l}$ is the vector of leg length. Moreover, $\dot{\hat{\boldsymbol{\theta}}}$ from Eq. (29) can be written as:

$$\dot{\hat{\boldsymbol{\theta}}} = \mathbf{P}\dot{\hat{\theta}}', \tag{34}$$

where:

$$\dot{\hat{\boldsymbol{\theta}}}' = \begin{bmatrix} \dot{A}_{1z} & \dot{A}_{2z} & \dot{A}_{3z} \end{bmatrix}_{3\times 1}^T, \tag{35}$$

$$\mathbf{P} = \begin{bmatrix} \dfrac{\partial A_{1x}}{\partial A_{1z}} & \dfrac{\partial A_{1y}}{\partial A_{1z}} & 1 & \dfrac{\partial A_{2x}}{\partial A_{1z}} & \dfrac{\partial A_{2y}}{\partial A_{1z}} & 0 & \dfrac{\partial A_{3x}}{\partial A_{1z}} & \dfrac{\partial A_{3y}}{\partial A_{1z}} & 0 \\ \dfrac{\partial A_{1x}}{\partial A_{2z}} & \dfrac{\partial A_{1y}}{\partial A_{2z}} & 0 & \dfrac{\partial A_{2x}}{\partial A_{2z}} & \dfrac{\partial A_{2y}}{\partial A_{2z}} & 1 & \dfrac{\partial A_{3x}}{\partial A_{2z}} & \dfrac{\partial A_{3y}}{\partial A_{2z}} & 0 \\ \dfrac{\partial A_{1x}}{\partial A_{3z}} & \dfrac{\partial A_{1y}}{\partial A_{3z}} & 0 & \dfrac{\partial A_{2x}}{\partial A_{3z}} & \dfrac{\partial A_{2y}}{\partial A_{3z}} & 0 & \dfrac{\partial A_{3x}}{\partial A_{3z}} & \dfrac{\partial A_{3y}}{\partial A_{3z}} & 1 \end{bmatrix}_{9\times 3}^T, \tag{36}$$

Table 1: Parameters of structure.

Actuator Stroke (mm)	Actuator minimum length (mm)	C stroke (mm)	d (mm)	$\mathbf{r}_b$ (mm)	$\mathbf{r}_a$ (mm)
350	400	200–400	200	500	200

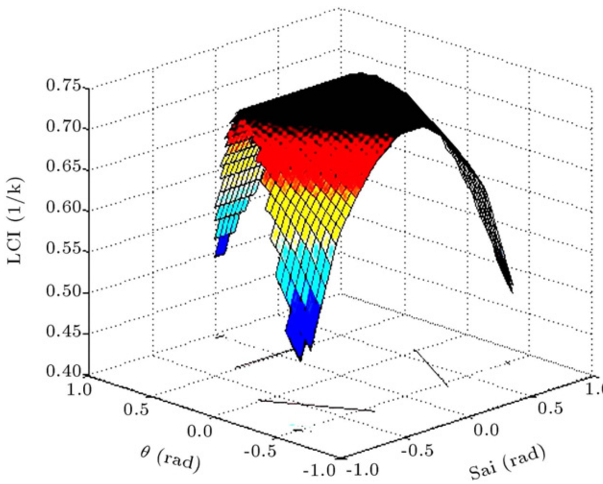

Figure 4: Loci of LCI in prescribed altitude ($z = 500$ mm) by weighting factor 200.

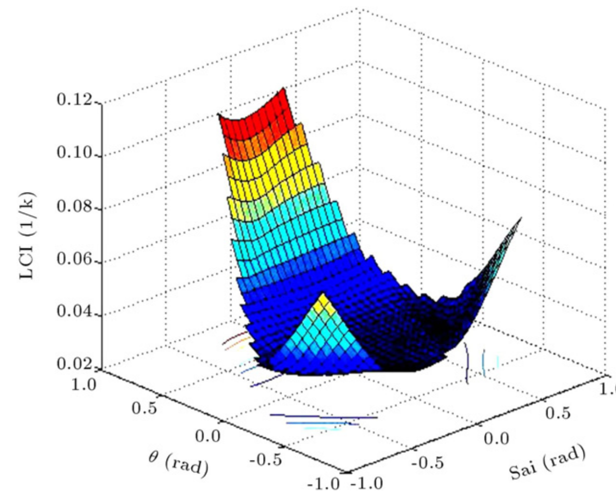

Figure 5: Loci of LCI in prescribed altitude ($z = 500$ mm) by point coordinates based formulation.

$$\mathbf{J}_d = \hat{\mathbf{J}}\mathbf{P} = \begin{bmatrix} \frac{\partial l_1}{\partial A_{1z}} & \frac{\partial l_1}{\partial A_{2z}} & \frac{\partial l_1}{\partial A_{3z}} \\ \frac{\partial l_2}{\partial A_{1z}} & \frac{\partial l_2}{\partial A_{2z}} & \frac{\partial l_2}{\partial A_{3z}} \\ \frac{\partial l_3}{\partial A_{1z}} & \frac{\partial l_3}{\partial A_{2z}} & \frac{\partial l_3}{\partial A_{3z}} \end{bmatrix}_{3\times 3}. \tag{37}$$

The foregoing is a dimensionless Jacobian matrix of the manipulator, in which:

$$l_i = \sqrt{(A_{ix} - B_{ix})^2 + (A_{iy} - B_{iy})^2 + (A_{iz} - B_{iz})^2}, \tag{38}$$

in which B_i, for $i = 1, 2, 3$, is the coordinate of ith passive universal joints connected to the fixed platform.

4.3. Local conditioning index

Here, the local conditioning index (LCI) will be derived based on the above mentioned methods.

4.3.1. Homogeneous Jacobian and twist vector

For the manipulator with the data of Table 1, the condition number is calculated with a weighting factor, namely $L = 200$ mm, throughout the workspace. Moreover, LCI for $z = 500$ mm is plotted in Figure 4.

4.3.2. Point based method

By applying the point based method [11] to formulate the homogeneous Jacobian matrix, we compute the condition number of the Jacobian matrix for the same structure of Table 1. The LCI for $z = 500$ mm is plotted in Figure 5.

4.3.3. Comparison study

Although both methods lead to homogeneous Jacobian matrices, one can see the obvious differences between the plotted LCIs. Those points with higher LCI, based on the first method, have a poor LCI, according to the second method, and vice versa.

Herein, the weighting factor of the first method, based on the parameters of the second one, is computed. Eq. (35) can be written as:

$$\mathbf{J}_d \dot{\hat{\boldsymbol{\theta}}}' = \mathbf{t}, \tag{39}$$

in which the components of $\dot{\hat{\boldsymbol{\theta}}}'$ are given as:

$$\begin{aligned} \dot{A}_{1z} &= \dot{c} + \dot{\psi}(-d\sin\psi\cos\theta) \\ &\quad + \dot{\theta}(-d\sin\theta\cos\psi - r_a\cos\theta), \end{aligned} \tag{40}$$

$$\begin{aligned} \dot{A}_{2z} &= \dot{c} + \dot{\psi}(-d\sin\psi\cos\theta + r_a\cos\psi\cos\theta\sin\alpha) \\ &\quad + \dot{\theta}(-d\sin\theta\cos\psi - r_a\cos\theta\cos\alpha \\ &\quad - r_a\sin\theta\sin\psi\sin\alpha), \end{aligned} \tag{41}$$

$$\begin{aligned} \dot{A}_{3z} &= \dot{c} + \dot{\psi}(-d\sin\psi\cos\theta + r_a\cos\psi\cos\theta\sin\beta) \\ &\quad + \dot{\theta}(-d\sin\theta\cos\psi - r_a\cos\theta\cos\beta \\ &\quad - r_a\sin\theta\sin\psi\sin\beta). \end{aligned} \tag{42}$$

According to the above equations, the independent variable velocity vector ($\dot{\hat{\boldsymbol{\theta}}}'$) can be expressed as:

$$\dot{\hat{\boldsymbol{\theta}}}' = \mathbf{D}\dot{\boldsymbol{\theta}}, \tag{43}$$

where:

$$\mathbf{D}_{3\times 3} = \begin{bmatrix} 1 & p_1 & t_1 \\ 1 & p_2 & t_2 \\ 1 & p_3 & t_3 \end{bmatrix}, \tag{44}$$

in which:

$$p_1 = -d\sin\psi\cos\theta, \tag{45}$$

$$p_2 = -d\sin\psi\cos\theta + r_a\cos\psi\cos\theta\sin\alpha, \tag{46}$$

$$p_3 = -d\sin\psi\cos\theta + r_a\cos\psi\cos\theta\sin\beta, \tag{47}$$

$$t_1 = -d\sin\theta\cos\psi - r_a\cos\theta, \tag{48}$$

$$t_2 = -d \sin\theta \cos\psi - r_a \cos\theta \cos\alpha - r_a \sin\theta \sin\psi \sin\alpha, \quad (49)$$

$$t_3 = -d \sin\theta \cos\psi - r_a \cos\theta \cos\beta - r_a \sin\theta \sin\psi \sin\beta. \quad (50)$$

As shown, the dimensions of the entries of matrix **D** are *length*. In fact, there is a matrix factor instead of a weighting factor. In addition, these entries are a function of the end-effector position. This means that the second method is using a position dependent weighting factor for making the Jacobian matrix dimensionally homogeneous.

5. Conclusions

The Jacobian entries are divided by units of length, thereby producing a new Jacobian that is dimensionally homogeneous. By multiplying the associated entries of the twist vector to the same factor, this vector is made homogeneous, as well. Moreover, it has been shown that the method used by others to define homogeneous Jacobian matrices leads to some pose dependent weighting factors. Therefore, it is better to rely on the characteristic length with the new interpretation (weighting factor) in order to produce homogeneous Jacobian matrices.

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