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Flat and Coherent Functors*

ULRICH OBERST

Department of Mathematics, University of Chicago, Chicago, Ill.

AND

HELMUT ROHRL

Department of Mathematics, University of California, San Diego, Calif.

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The main objective of this paper is to characterize flat group valued functors. We obtain the following theorem, announced in [*7*]: Let X be a small additive category with dual X^0 and S an object in $[X^0, AB]$, the category of all additive functors from X to the category Ab of Abelian groups. Then S is flat, i.e., the functor $S \otimes_X : [X, AB] \to AB$ is exact if and only if the fiber X/S of the Yoneda embedding $X \to [X^0, AB]$ over S is filtered from above, or if and only if S is a filtered direct limit of representable functors. There are several other equivalent statements, and it is, ***mutatis mutandis***, enough to assume X preadditive.

A similar theorem has been obtained by B. Stenstrom in [*11*]. He proves that a functor is flat if and only if it is a filtered direct limit of projective (instead of representable) functors. For Abelian X the result was obtained by J. Fisher [*5*]; in this particular case "flat" means "left exact," and a short proof is possible. Our result is a generalization of the well-known characterization of flat modules by means of generators and relations, and has applications in the study of the exactness of the direct limit functor [*7*], and in the singular homology theory of sheaves [*8*].

Using the above characterization of flat functors we show in analogy to the results of S. U. Chase [*4*] on coherent rings that the category $[X^0, AB]$ is locally coherent, i.e., has a family of coherent generators, if and only if

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a product of flat functors in $[X, AB]$ is again flat. Moreover, if $[X^0, AB]$ is locally coherent, then the weak global dimension of X and the global dimension of Coh$[X^0, AB]$, the (Abelian) category of all coherent functors from X^0 to AB, coincide, and X is a full subcategory of almost generating projectives of Coh$[X^0, AB]$. This last result is due to P. Freyd [6], however he does not use the coherence notion. If X is Abelian this weak global dimension of X is at most two; this case has been investigated by M. Auslander [1] and J. Fisher [5]. Without Abelianess hypotheses we show that $[X^0, AB]$ is locally coherent and the above dimension is at most two if and only if flat functors are closed under inverse limits in $[X, AB]$.

The first two sections of this paper contain preliminary material, the main results on flatness resp. coherence are contained in the third resp. fourth section. The proofs of the preliminary lemmas and of the corollaries of the main results have mostly been omitted.

1. PRELIMINARIES

Let K be a commutative ring with unit, and Mod K the category of unital K-modules. A K-preadditive category $\mathfrak{A}$ is a preadditive category $\mathfrak{A}$ together with a unital ring homomorphism from K into the center of $\mathfrak{A}$, i.e., the endomorphism ring of the identity functor of $\mathfrak{A}$. If X and $\mathfrak{A}$ are K-preadditive categories a functor $F : X \to \mathfrak{A}$ is called K-additive if for all $x, y \in X$ the induced map

$$F(x, y) : X(x, y) \to \mathfrak{A}(Fx, Fy)$$

is K-linear. The category of all K-additive functors from X to $\mathfrak{A}$ is denoted by $[X, \mathfrak{A}]$. A category X is called K-additive if it is K-preadditive and admits finite direct sums. If X is K-preadditive let $\bar{X}$ be the universal K-additive category generated by X. The objects of $\bar{X}$ are m-tuples $(x_1, ..., x_m)$, $m \geqslant 0$, of objects in X. If $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_n)$ are two such objects then a morphism from x to y is an $n \times m$ matrix (β_{ji}) of morphisms $\beta_{ji} : x_i \to y_j$ in X. The composition of such morphisms is the matrix multiplication, the coefficients being composed as in X. Obviously $\bar{X}$ contains X as a full subcategory and is K-additive. For any K-additive category $\mathfrak{A}$ the restriction

$$[\bar{X}, \mathfrak{A}] \to [X, \mathfrak{A}]$$

is an equivalence. We will identify $[\bar{X}, \mathfrak{A}]$ and $[X, \mathfrak{A}]$ in the sequel. The use of $\bar{X}$ instead of X has notational advantages; for if Y is K-additive then a finite direct sum of representable functors in $[Y, \text{Mod } K]$ is again representable. Also remark that if X is a K-preadditive category then

$$[X, \text{Mod } K] = [X, AB],$$

where the K-structure of X has been forgotten on the right. For most of the sequel the consideration of the case $K = \mathbf{Z}$ = ring of rational integers would be enough; nevertheless we deal with a general K throughout since this does not require any more work.

If $S \in [X^0, \text{Mod } K]$, $x \in X$, $\xi \in Sx$, and $\alpha : y \to x$ in X we write $\xi\alpha := (S\alpha)(\xi)$. Obviously, $(\xi\alpha)\beta = \xi(\alpha\beta)$ if at least one side is defined. In the same manner, if $F \in [X, \text{Mod } K]$, $x \in X$, $\xi \in Fy$, and $\alpha : y \to x$ we write $\alpha\xi := (F\alpha)(\xi)$. Also, in the same situation, we write

$$\tilde{\xi} : X(-, x) \to S,$$

for the unique morphism with $\tilde{\xi}(x)(id_x) = \xi$, given by the Yoneda isomorphism $j : [X^0, \text{Mod } K](X(-, x), S) \to Sx$. Often we will simply write ξ for $\tilde{\xi}$. If $F \in [X, \text{Mod } K]$, $x \in X$, and if I is an ideal (i.e., subobject) of $X(-, x)$ let IF be the submodule of Fx, generated by all $\alpha\xi$, where $\alpha \in I(y)$, $y \in X$, $\xi \in Fy$. More generally, IF can be defined for functors $F \in [X, \mathfrak{A}]$ where $\mathfrak{A}$ is any K-additive Abelian category with arbitrary direct sums. Indeed, one defines

$$IF := \text{Image}(\textstyle\coprod(Fy;\, y \xrightarrow{\alpha} x \text{ in } I(y),\, y \in X) \xrightarrow{(F\alpha)} Fx).$$

If X is a small K-preadditive category and if $\mathfrak{A}$ is a K-additive Abelian category with arbitrary direct sums then the tensor product

$$\otimes_X : [X^0, \text{Mod } K] \times [X, \mathfrak{A}] \to \mathfrak{A}$$
$$(S, F) \rightsquigarrow S \otimes_X F$$

exists; it is defined by the isomorphism

$$\text{(1.1)} \qquad \varphi : \mathfrak{A}(S \otimes_X F, A) \cong [X^0, \text{Mod } K](S, \mathfrak{A}(F, A)),$$

functorial in $S \in [X^0, \text{Mod } K]$, $F \in [X, \mathfrak{A}]$, and $A \in \mathfrak{A}$. In the same way the tensor product $\otimes_X : [X^0, \mathfrak{A}] \times [X, \text{Mod } K] \to \mathfrak{A}$ exists. If $S \in [X^0, \text{Mod } K]$ and $F \in [X, \mathfrak{A}]$ one has the functorial isomorphism $S \otimes_X F \cong F \otimes_{X^0} S$. The tensor product is the unique right continuous functor satisfying the normalization conditions $X(-, x) \otimes_X F \cong Fx$, and then also

$$G \otimes_X X(x, -) \cong Gx, \qquad \text{for} \quad G \in [X^0, \text{Mod } K].$$

The latter isomorphisms follow from (1.1) and the Yoneda isomorphism. If $S \in [X^0, \text{Mod } K]$ and $F \in [X, \text{Mod } K]$ the tensor product can be constructed as

$$S \otimes_X F = \textstyle\coprod(Sx \otimes_K Fx;\, x \in X)/B,$$

where B is the K-submodule of $\coprod(Sx \otimes_K Fx; x \in X)$ generated by all $\eta\alpha \otimes \xi - \eta \otimes \alpha\xi$, where $\alpha \in X(x, y)$, $\eta \in Sy$, $\xi \in Fx$ (see [*12*]). The image of $\eta \otimes \xi \in Sx \otimes_K Fx$ in $S \otimes_X F$ is again denoted by $\eta \otimes \xi$. With this notation the canonical isomorphism $X(-, x) \otimes_X F \to Fx$, is given by $\alpha \otimes \xi \rightsquigarrow \alpha\xi$, $\alpha \in X(y, x)$, $\xi \in Fy$.

(1.2) Lemma. *Let* $F \in [X, \operatorname{Mod} K]$ *and* I *an ideal of* $X(-, x)$, $x \in X$. *Then the correspondence* $\operatorname{cl}(\alpha) \otimes \xi \rightsquigarrow \operatorname{cl}(\alpha\xi)$ *defines an isomorphism*

$$X(-, x)/I \otimes_X F \to Fx/IF.$$

(Here $\alpha \in X(y, x)$, $\xi \in Fy$ and $\operatorname{cl}(\alpha)$ resp. $\operatorname{cl}(\alpha\xi)$ denote the image of α resp. $\alpha\xi$ in $X(y, x)/I(y)$ resp. Fx/IF).

The proof is easy. The preceding lemma also holds (with the obvious changes) if $F \in [X, \mathfrak{A}]$.

2. Coherent Functors

Let K be a commutative ring and X a small K-preadditive category. We consider finiteness properties for the functors in $[X^0, \operatorname{Mod} K]$. The following definitions are essentially contained in [*2*], Ch. 1, [*4*], and [*10*].

We say that a set has cardinality φ (for finite) if it is finite; φ is called "the" finite cardinal. In the sequel "cardinal" means the finite cardinal φ or any infinite cardinal in the usual sense. We define $\varphi < \aleph_0$. If α is a cardinal in this sense, an ordered set I is called *α-directed* (or α-filtered from above) if every subset of I of cardinality at most α has an upper bound in I. An object $S \in [X^0, \operatorname{Mod} K]$ is called of *type* α if every α-directed set of subobjects of S whose supremum is S contains S (same definition for any category). In particular the objects of type φ are the *finitely generated* objects or objects of *finite type*.

(2.1) Lemma. *Let* $S \in [X^0, \operatorname{Mod} K]$ *and let* α *be a cardinal. The following statements are equivalent*:

(1) *S is of type α.*

(2) *There is a family* $(x_\lambda; \lambda \in \Lambda)$ *of objects in* X *with* $|\Lambda| \leqslant \alpha$ *and an exact sequence*

$$\coprod(X(-, x_\lambda); \lambda \in \Lambda) \to S \to 0.$$

(3) *There are a family* $(x_\lambda; \lambda \in \Lambda)$ *of objects in* X *and a family* $(\xi_\lambda; \lambda \in \Lambda)$ *of elements* $\xi_\lambda \in Sx_\lambda$ *such that* $|\Lambda| \leqslant \alpha$ *and such that for all* $x \in X$

$$Sx: = \left\{\sum_{\lambda \in \Lambda} \xi_\lambda \alpha_\lambda \mid \alpha_\lambda : x \to x_\lambda\,,\ \alpha_\lambda = 0 \quad \text{for almost all } \lambda\right\}.$$

The proof of the lemma is easy (see also [10], Prop. 1).

A functor $S \in [X^0, \text{Mod } K]$ is called *finitely presented* if there are finite families $(x_i; i \in I)$ and $(y_j; j \in J)$ of objects in X and an exact sequence

$$\coprod(X(-, x_i); i \in I) \to \coprod(X(-, y_j); j \in J) \to S \to 0.$$

A functor $S \in [X^0, \text{Mod } K]$ is called *coherent* if S is of finite type and if for every morphism $f : S' \to S$ with S' of finite type also the kernel of f is of finite type.

(2.2) LEMMA. *Let S_1, S_2 be two finitely presented subfunctors of a functor $S \in [X^0, \text{Mod } K]$. Then $S_1 + S_2$ is finitely presented if and only if $S_1 \cap S_2$ is of finite type.*

The proof of this lemma is analogous to that of [2], I. 1, Ex. 6, f.

(2.3) LEMMA. (i) *Let $S \in [X^0, \text{Mod } K]$ be of finite type. Then the following assertions are equivalent.*

(1) *S is coherent.*

(2) *Every subobject of S of finite type is finitely presented.*

(3) *For every finite family $(x_i; i \in I)$ of objects of X and every morphism $f : \coprod(X(-, x_i); i \in I] \to S$ the kernel of f is of finite type.*

(ii) *The full subcategory* $\text{Coh}[X^0, \text{Mod } K]$ *of* $[X^0, \text{Mod } K]$ *of all coherent functors is closed under finite limits and colimits in* $[X^0, \text{Mod } K]$. *Moreover* $\text{Coh}[X^0, \text{Mod } K]$ *is equivalent to a small category.*

This lemma originated in the theory of sheaves. For modules over a ring it is contained in [2], I. 2, Ex. 11, or originally in [4].

(2.4) LEMMA. *Any $S \in [X^0, \text{Mod } K]$ is the direct limit of a filtered direct system of finitely presented functors.*

The proof of this lemma is along the lines suggested in [2], I. 2, Ex. 10. This lemma is an improvement of Theorem 1.5 in [5] where it is assumed that X is additive and admits cokernels.

3. FLAT FUNCTORS

If $j : X \to Y$ is a functor and $y \in Y$ then X/y, the fiber of j over y, is the category whose objects are pairs (x, β) of an $x \in X$ and a morphism $\beta : j(x) \to y$. In particular, if X is K-preadditive and $S \in [X^0, \text{Mod } K]$ then the fiber X/S of the Yoneda embedding

$$X \to [X^0, \text{Mod } K] : x \rightsquigarrow X(-, x),$$

over S has objects (x, ξ), where $x \in X$ and $\xi \in Sx$. Here we identify $\xi \in Sx$ with $\xi : X(-, x) \to S$.

We say that a category X is *filtered from above* if it satisfies the conditions

(F1). If $x, y \in X$ there is a $s \in X$ with

$$X(x, z) \neq \varnothing \neq X(y, z)$$

(F2). Any diagram $x \rightrightarrows y$ can be extended to a commutative diagram

$$x \rightrightarrows y \to z.$$

(3.1) Lemma. *Let K be a commutative ring and X a small K-preadditive category. If $n \geqslant 0$ and $F \in [X, \text{Mod}\, K]$, and if $\text{Tor}_n{}^X(S, F) = 0$, for all finitely presented S, then $\text{Tor}_n{}^X = 0$.*

Here $\text{Tor}_n{}^X$ is the nth left-derived functor of the biadditive, right-exact, and balanced functor $\otimes_X$.

This lemma follows from Lemma (2.4) since Tor commutes with filtered direct limits.

(3.2) Theorem. *Let K be a commutative ring with unit X a small K-preadditive category. Let $S \in [X^0, \text{Mod}\, K]$. Then the following statements are equivalent:*

(1) *S is flat*, i.e., *the functor*

$$S \otimes_X : [X, \text{Mod}\, K] \to \text{Mod}\, K$$

is exact.

(2) *For every K-additive Abelian category $\mathfrak{A}$, with exact filtered direct limits, the functor*

$$S \otimes_X : [X, \mathfrak{A}] \to \mathfrak{A}$$

is exact.

(3) *For every $z \in X$ and every ideal I of $X(z, -)$: $\text{Tor}_1{}^X(S, X(z, -)/I) = 0$.*

(4) *If $z, \beta_j : z \to y_j$, $j = 1,\ldots, n$, are in X and $\eta_j \in Sy_j$, all j, with $\sum_j \eta_j\beta_j = 0$ then there are objects x_i, $i = 1,\ldots, m$, elements $\xi_i \in Sx_i$ and morphisms $\alpha_{ij} : y_j \to x_i$, all i, j, such that*

$$\sum_i \xi_i\alpha_{ij} = \eta_j, \qquad \textit{all } j$$

$$\sum_j \alpha_{ij}\beta_j = 0, \qquad \textit{all } i.$$

(5) *The functor S is a filtered limit of finite direct sums of representable functors, i.e., there is a small category J, filtered from above and a functor*

$$J \to \bar{X} : j \rightsquigarrow \bar{x}_j$$

such that $S \cong \operatorname{inj\,lim}_J \bar{X}(-, \bar{x}_j)$.

(5′) *The statement of* (5) *is true with a filtered ordered set J.*

(6) *The fiber $\bar{X}/S$ of the Yoneda embedding*

$$\bar{X} \to [X^0, \operatorname{Mod} K] : \bar{x} = (x_1, ..., x_m) \to \bar{X}(-, \bar{x}) = \coprod_i X(-, x_i),$$

over S is filtered from above.

Proof. We prove (1) ⇒ (4) ⇒ (3) ⇒ (1) ⇒ (6) ⇒ (5) ⇒ (5′) ⇒ (2) ⇒ (1).

(1) ⇒ (4). Given the data of (4) let I be the ideal of $X(z, -)$, generated by the β_j, $j = 1, ..., n$, i.e., for $x \in X$

$$I(x) = \left\{\sum \gamma_j \beta_j \mid \gamma_j : y_j \to x\right\}.$$

Since the functors $X(-, x)$, $x \in X$, form a system of generators of $[X^0, \operatorname{Mod} K]$, there is an exact sequence

$$0 \to K \subset F = \coprod (X(-, x_\lambda); \lambda \in \Lambda) \xrightarrow{p} S \to 0,$$

where $(x_\lambda; \lambda \in \Lambda)$ is some family of objects of X. Let $\xi_\lambda : X(-, x_\lambda) \to S$, $\lambda \in \Lambda$, be the λth component of p, where $\xi_\lambda \in Sx_\lambda$, $\lambda \in \Lambda$.

Since S is flat the sequence

$$0 \to K \otimes_X X(z, -)/I \to F \otimes_X X(z, -)/I$$

is exact. By Lemma (1.2) we obtain that $KI = Kz \cap FI$.

Since p is an epimorphism there are elements $\alpha_j' \in Fy_j$, $j = 1, ..., n$, such that $p(y_j)(\alpha_j') = \eta_j$, all j. The relation $\sum_j \eta_j \beta_j = 0$ implies that $\sum_j \alpha_j' \beta_j \in Kz$, and by definition of I we have $\sum_j \alpha_j' \beta_j \in FI$. Hence $\sum_j \alpha_j' \beta_j \in Kz \cap FI = KI$. By definition of KI and since I is generated by the β_j, there are $\alpha_j'' \in Ky_j$, $j = 1, ..., n$, with $\sum_j \alpha_j' \beta_j = \sum_j \alpha_j'' \beta_j$. Defining $\alpha_j = \alpha_j' - \alpha_j''$, $j = 1, ..., n$, we obtain $\sum_j \alpha_j \beta_j = 0$, and $p(y_j)(\alpha_j) = p(y_j)(\alpha_j') = \eta_j$, all j.

But $\alpha_j = \sum_\lambda \alpha_{\lambda j} \in \coprod_\lambda X(y_j, x_\lambda) = Fy_j$. Hence $\sum_j \alpha_{\lambda j} \beta_j = 0$, all $\lambda \in \Lambda$, and

$$\eta_j = \sum_\lambda \xi_\lambda \alpha_{\lambda j}, \qquad \text{all } j.$$

This is the desired result since there are only finitely many j; and hence one needs only finitely many λ.

(4) ⇒ (3). Let $z \in X$ and I be an ideal of $X(z, -)$. In order that $\mathrm{Tor}_1{}^X(S, X(z, -)/I) = 0$, it is necessary and sufficient that the map

$$S \otimes_X I \to Sz : \sum_j \eta_j \otimes \beta_j \rightsquigarrow \sum_j \eta_j\beta_j ,$$

is injective. But assume that $\beta_j \in I(y_j) \subset X(z, y_j)$, $j = 1,..., n$, and $\eta_j \in Sy_j$ such that $\sum_j \eta_j\beta_j = 0$. By (4) there are finitely many objects x_i, $i = 1,..., m$, in X, elements $\xi_i \in Sx_i$, and morphisms $\alpha_{ij} : y_j \to x_i$ such that $\eta_j = \sum_i \xi_i\alpha_{ij}$, all j, and $\sum_j \alpha_{ij}\beta_j = 0$, all i. This implies that

$$\sum_j \eta_j \otimes \beta_j = \sum_{i,j} \xi_i\alpha_{ij} \otimes \beta_j = \sum_i \xi_i \otimes \Big(\sum_j \alpha_{ij}\beta_j\Big) = 0,$$

(3) ⇒ (1)

Lemma. *Let $T : \mathfrak{A} \to \mathfrak{B}$ be a half-exact functor between Abelian categories $\mathfrak{A}$ and $\mathfrak{B}$. If A_1, $A_2 \in \mathfrak{A}$, and if T vanishes on all factor objects of A_1 and A_2, then T vanishes on all factor objects of $A_1 \oplus A_2$.*

The proof is easy, and known.

We apply the lemma to the half-exact functor $\mathrm{Tor}_1{}^X(S, -)$, and obtain from (3) that $\mathrm{Tor}_1{}^X(S, -)$ vanishes on all functors of finite type. By Lemma 3.1 this implies that $\mathrm{Tor}_1{}^X(S, -) = 0$, i.e., S is flat.

(1) ⇒ (6). Without loss of generality we assume that $X = \bar{X}$, i.e., that X is additive. We show that X/S is filtered from above. The condition (F1) is trivially satisfied since X and S are additive.

Let then $\beta_j : (z, \xi) \to (y, \eta)$, $j = 1, 2$, be two morphisms in X/S where $\xi = \eta\beta_1 = \eta\beta_2$, so $\eta(\beta_1 - \beta_2) = 0$. By (4) which is equivalent to (1) by the above proof there are objects x_1, $i = 1,..., m$, in X, elements $\xi_i \in Sx_i$ and morphisms $\alpha_i : y \to x_i$ such that $\sum_i \xi_i\alpha_i = \eta$, and $\alpha_i(\beta_1 - \beta_2) = 0$, all i. Let $x = \coprod_i x_i$, $\xi = (\xi_i)_i \in Sx = \coprod_i Sx_i$, and $\alpha : y \to x$ be the morphism with components α_i. Then $\alpha(\beta_1 - \beta_2) = 0$, and $\eta = \xi\alpha$. This implies that $\alpha : (y, \eta) \to (x, \xi)$ is a morphism in X/S and $\alpha\beta_1 = \alpha\beta_2$. But this is the condition (F2).

(6) ⇒ (5) follows directly from the fact that $\bar{X}$ is additive and thus $S \cong \mathrm{inj\,lim}_{\bar{X}/S}\, \bar{X}(-, \bar{x})$.

(5) ⇒ (5′) follows from the unpublished result of *R*. Swan, that for every small category J, which is filtered from above, there is an ordered set J' which is filtered from above and a cofinal functor $J' \to J$.

(5′) ⇒ (2). If $S = \mathrm{inj\,lim}_J\, \bar{X}(-, \bar{x}_j)$ then for every $F \in [X, \mathfrak{A}]$ one has

$$S \otimes_X F = \mathrm{inj\,lim}_J\, \bar{X}(-, \bar{x}_j) \otimes_X F = \mathrm{inj\,lim}_J F\bar{x}_j .$$

Since the filtered limits in $\mathfrak{A}$ are assumed to be exact the functor

$$S \otimes_X : [X, \mathfrak{A}] \to \mathfrak{A}$$

is exact.

(2) ⇒ (1) obvious.

Remark. The isomorphism $S \otimes_X F \cong F \otimes_{X^0} S$ and the Theorem (3.2) show that a functor $F \in [X, \text{Mod } K]$ is flat if and only if, e.g., X^0/F is filtered from above.

Remark. The preceding theorem is essentially a generalization of [3], Ch. VI, Ex. 6, to functor categories.

(3.3) COROLLARY. *Assumptions as in theorem. If X is K-additive, then S is flat if and only if X/S is filtered from above.*

(3.4) COROLLARY ([5], 2.4, 2.9). *Assumptions as in Theorem* 3.2. *If in addition X is K-additive and admits cokernels then S is flat if and only if S is left exact.*

Theorem 3.2 can also be applied to non-additive functors. Let X be any small category, and let KX denote the universal K-preadditive category generated by X. The objects of KX are those of X. If $x, y \in X$ then $(KX)(x, y)$ is the free K-module generated by $X(x, y)$. The category X is a subcategory of KX, and KX is characterized by the property that for every K-preadditive category $\mathfrak{A}$ the restriction

$$[KX, \mathfrak{A}] \to \mathfrak{A}^X$$

is an equivalence. Here $\mathfrak{A}^X$ denotes the category of all functors from X to $\mathfrak{A}$. We identify $[KX, \mathfrak{A}] = \mathfrak{A}^X$. With this identification, if $S \in (\text{Mod } K)^{X^0}$ and $F \in \mathfrak{A}^X$ one has

$$S \otimes_X F = S \otimes_{KX} F.$$

(3.5) COROLLARY. *Let X be any small category and $S \in (\text{Mod } K)^{X^0}$. Then S is flat if and only if $\overline{KX}/S$ is filtered from above.*

Under special assumptions one can replace $\overline{KX}/S$ by KX/S in the preceding corollary.

(3.6) COROLLARY. *Let X be a small category with a zero object e and finite direct sums or finite direct products. Let $S \in (\text{Mod } K)^{X^0}$ such that $Se = 0$. Then S is flat if and only if KX/S is filtered from above. If this is the case then KX/S is cofinal in $\overline{KX}/S$, and in particular*

$$S \cong \text{inj lim}_{KX/S}\, KX(-, x).$$

Under the assumptions of the preceding corollary any functor $S \in (\text{Mod } K)^{X^0}$ can be written as $S = S_1 \oplus S_2$, where $S_1 x = \text{Ker}(Sx \xrightarrow{so} Se)$ and $S_2 x \cong Se$. This decomposition is functorial in S. Then S is flat if and only if S_1 and S_2 are, and the preceding corollary can be applied to S_1.

The next corollary is similar to a result on functors into the category of sets, due to A. Grothendieck (see, e.g., [8], p. 326).

(3.7) Corollary. *Let X be a small K-additive category. Let E be a multiplicatively closed set of epimorphisms of X such that*

(1) *for all $x \in X$, the ordered set $I = \{\text{cl}(\epsilon); \text{domain } \epsilon = x, \epsilon \in E\}$ is Artinian.*

(Here cl(ϵ) means the quotient object i.e., the equivalence class of isomorphic epimorphisms, which is represented by ϵ. The set I is ordered by: $\text{cl}(\epsilon') \subseteq \text{cl}(\epsilon)$, if there is a morphism α in X with $\epsilon' = \alpha\epsilon$. An ordered set is called Artinian if it satisfies the descending chain condition).

(2) *every morphism of X is a product $\mu\epsilon$ of a monomorphism μ and a morphism $\epsilon \in E$.*

Then a functor $S \in [X^0, \text{Mod } K]$ is flat if and only if it is a filtered direct limit of representable subfunctors.

Proof. (i) The condition is sufficient by Theorem 3.2.

(ii) Assume now that S is flat, i.e., that X/S is filtered from above. We show first that any morphism $X(-, r) \xrightarrow{\tilde{\rho}} S$ can be factorized in the form $X(-, r) \xrightarrow{X(-,\epsilon)} X(-, y) \xrightarrow{\tilde{\eta}} S$ where $\tilde{\eta}$ is a monomorphism ($\rho \in Sr$, $\eta \in Sy$). Indeed let $\epsilon : r \to y$ be minimal among the epimorphisms in E with domain r such that there is a factorization $\tilde{\rho} = \tilde{\eta}X(-, \epsilon)$. We show that any such $\tilde{\eta}$ is a monomorphism. Let $z \in X$ and $\beta_1, \beta_2 \in X(z, y)$ such that $\tilde{\eta}(z)(\beta_1) = \tilde{\eta}(z)(\beta_2)$, i.e., $\eta\beta_1 = \eta\beta_2$. Then β_1 and β_2 are morphisms in X/S from $(z, \eta\beta_1)$ to (y, η). Since X/S is filtered from above there is a morphism $\alpha : (y, \eta) \to (x, \xi)$ in X/S with $\alpha\beta_1 = \alpha\beta_2$. By (2), $\alpha = \mu\delta$, where $\delta \in E$ and μ is a monomorphism. We first conclude $\delta\beta_1 = \delta\beta_2$. Moreover $\eta = \xi\alpha$ implies that

$$\tilde{\eta} = \tilde{\xi}X(-, \alpha), \qquad \text{hence}$$
$$\tilde{\rho} = \tilde{\eta}X(-, \epsilon) = (\tilde{\xi}X(-, \mu))\, X(-, \delta\epsilon).$$

By the minimality of ϵ we obtain that δ is an isomorphism, so $\delta\beta_1 = \delta\beta_2$ implies $\beta_1 = \beta_2$. But this means that η is a monomorphism.

(iii) Since $S = \text{inj lim}_{X/S}\, X(-, x)$ and by (ii), we get that S is the sum of its representable subfunctors. Since the direct sum of two representable functors is again representable and by (ii) we obtain that the set of representable subfunctors of S is filtered from above. Hence S is the filtered union of its representable subfunctors.

EXAMPLE. Let R be right Noetherian and right hereditary ring. Let $\mathrm{Mod}_r R$ resp. $\mathrm{Mod}_l R$ be the categories of all right- resp. left R-modules, and let X be the full subcategory of $\mathrm{Mod}_r R$ of all finitely generated projective R-right modules. The category X is equivalent to a small category, and the class E of all surjections, i.e., of all split epimorphisms, satisfies the conditions of the preceding theorem ([3], I. 6, Prop. 6.2). Moreover the functors

$$[X, AB] \to \mathrm{Mod}_l R : F \rightsquigarrow FR,$$

and

$$[X^0, AB] \to \mathrm{Mod}_r R : S \rightsquigarrow SR,$$

are equivalences such that

$$S \otimes_X F \cong SR \otimes_R FR.$$

Hence an R-right module N is flat if and only if it is the filtered union of its finitely generated, projective submodules. Of course, this result can also be shown directly.

4. Flatness and Coherence

This section represents a generalization of some of the results of S. U. Chase [4] to functor categories. The proof of the next result is modelled after a suggestion in [2], I. 2, Ex. 12. Let K be a commutative ring.

(4.1) THEOREM. *Let X be a small K-preadditive category. Let $\aleph$ be a cardinal such that for all $x \in X$ any ideal I of $X(-, x)$ is of type $\aleph$* (see Section 2). *Then the following statements are equivalent.*

(1) $[X^0, \mathrm{Mod}\, K]$ *is locally coherent*, i.e., *admits a family of coherent generators.*

(2) *For each $x \in X$ the functor $X(-, x)$ is coherent.*

(3) *All finitely presented objects in* $[X^0, \mathrm{Mod}\, K]$ *are coherent.*

(4) *For any $x \in X$ the intersection of two ideals of finite type in $X(-, x)$ is again of finite type, and for any morphism $\alpha : x \to y$ in X the kernel of $X(-, \alpha)$ is of finite type.*

(5) *The product of flat functors in* $[X, \mathrm{Mod}\, K]$ *is flat.*

(6) *If $(x_\lambda; \lambda \in \Lambda)$ is a family of objects in X with $|\Lambda| \leqslant \aleph$, then $\prod(X(x_\lambda, -), \lambda \in \Lambda)$ is flat.*

Remark that some such cardinal $\aleph$ always exists. Hence the statements (1) to (5) are equivalent for any small K-preadditive category.

Proof. We show (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (2). (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is easy by using Lemma (2.3).

(4) $\Rightarrow$ (2). Let $(x_i; i \in I)$ be a finite family of objects in X and

$$f : \coprod(X(-, x_i); i \in I) \to X(-, x)$$

a morphism. By induction on the number n of elements in I, we show that $\operatorname{Im} f$ is finitely presented. This is trivial for $n = 0$, and true for $n = 1$ by the hypothesis. Assume then that $n \geqslant 1$, let $j \in I$, and

$$S_1 = \sum_{i \neq j} f(X(-, x_i)), \qquad \text{and}$$

$$S_2 = f(X(-, x_j)).$$

Then $\operatorname{Im} f = S_1 + S_2$. By induction hypothesis S_1 and S_2 are finitely presented and $S_1 \cap S_2$ is of finite type by assumption (4), hence Im f is finitely presented by Lemma (2.2). Hence any ideal of finite type in $X(-, x)$ is finitely presented. Thus $X(-, x)$ is coherent.

(2) $\Rightarrow$ (5) (a) If $S \in [X^0, \operatorname{Mod} K]$ is finitely presented then for any Abelian category $\mathfrak{A}$ with exact direct products the functor

$$S \otimes_X : [X, \mathfrak{A}] \to \mathfrak{A}$$

commutes with direct products. The proof is analogous to [3], Ch. II, Ex. 1.

(b) Under the hypothesis (2) (and hence (3)) any finitely-presented functor S has a projective resolution of finitely-presented functors. Using (a) we obtain that the functors

$$\operatorname{Tor}_n{}^X(S, -) : [X, \operatorname{Mod} K] \to \operatorname{Mod} K$$

commute with products.

(c) Lemma (3.1) and (b) imply that the product of flat functors in $[X, \operatorname{Mod} K]$ is flat.

(5) $\Rightarrow$ (6). Obvious.

(6) $\Rightarrow$ (2). For simplicity we assume that X is additive. Let $z \in Z$. We show that $X(-, z)$ is coherent by showing that for any morphism $\beta : y \to z$ the kernel S of $X(-, \beta)$ is of finite type. By assumption there is a family $(\gamma_\lambda; \lambda \in \Lambda)$ of generators of S with $| \Lambda | \leqslant \aleph$, i.e., a family of morphisms $\gamma_\lambda : u_\lambda \to y$, $\lambda \in \Lambda$, such that the morphism

$$\coprod(X(-, u_\lambda); \lambda \in \Lambda) \to X(-, y),$$

whose components are the $X(-, y_\lambda)$ has image S. By (6) the functor $F = \prod(X(u_\lambda, -); \lambda \in \Lambda)$ is flat, and $(y_\lambda; \lambda \in \Lambda) \in Fy$ with $\beta(y_\lambda; \lambda \in \Lambda) = (\beta y_\lambda; \lambda \in \Lambda) = 0$. Hence, by Theorem (3.2) and since X is additive, there are an object $x \in X$, a morphism $\alpha : x \to y$, and $(\delta_\lambda; \lambda \in \Lambda) \in Fx$ such that

$$(y_\lambda; \lambda \in \Lambda) = \alpha(\delta_\lambda; \lambda \in \Lambda) \quad \text{and} \quad \beta\alpha = 0.$$

The second of these relations shows that $\alpha \in Sx$; the first implies that S is generated by α, and in particular is of finite type.

(4.2) COROLLARY. *Assume that the equivalent conditions of the preceding theorem hold. Then X is a full subcategory of projectives of* Coh[X^0, Mod K] *(up to equivalence) such that every object of* Coh[X^0, Mod K] *is an epimorphic image of a finite direct sum of objects in X. Every additive functor from X into an Abelian category can be extended to a right-exact functor on* Coh[X^0, Mod K], *uniquely up to isomorphism.*

This corollary essentially coincides with Corollary 1.6 in [6]; the conditions in that corollary just mean that all functors $X(-, x)$, $x \in X$, are coherent. Remark that without any condition on X the category Coh[X^0, Mod K] is abelian, the only question is whether X is contained in it.

Special instances where the equivalent conditions of the preceding theorem are satisfied are

(1) X is K-additive and admits weak kernels ([6], Section 1).

(2) X is K-additive and admits kernels ([1], Section 2).

(3) There is a cardinal $\aleph$ such that X admits direct sums of families of cardinal $\aleph$, and such that every ideal of an $X(-, x)$; $x \in X$, is of type $\aleph$. For then

$$\prod(X(x_\lambda, -); \lambda \in \Lambda) = X(\textstyle\coprod_\lambda x_\lambda, -).$$

If X is a small K-preadditive category the weak global dimension of X is the infimum of all nonnegative integers m, such that for all $S \in [X^0, \text{Mod } K]$ and $F \in [X, \text{Mod } K]$, one has

$$\text{Tor}^X_{m+1}(S, F) = 0.$$

It is clear that this dimension does not depend on K and that the weak global dimensions of X and X^0 coincide.

(4.3) THEOREM. *Assume that the equivalent statements of Theorem* 4.1 *hold. Then the global dimension of* Coh[X^0, Mod K] *equals the weak global dimension of X.*

Proof. Let m resp. n be the global resp. weak global dimension of $\mathrm{Coh}[X^0, \mathrm{Mod}\, K]$ resp. X. Here m and n are nonnegative integers.

We show first that $n \leqslant m$. Assume $m < \infty$. Let then S be a finitely presented, hence a coherent functor in $\mathrm{Coh}[X^0, \mathrm{mod}\, K]$. By assumption there is a projective resolution

$$0 \to P_m \to \cdots \to P_0 \to S \to 0,$$

in $\mathrm{Coh}[X^0, \mathrm{Mod}\, K]$. This is also a projective resolution in $[X^0, \mathrm{Mod}\, K]$, so $\mathrm{Tor}_{m+1}^X(S, -) = 0$. But then, by Lemma 3.1, $\mathrm{Tor}_{m+1}^X = 0$, and hence $n \leqslant m$.

Show now $m \leqslant n$. Assume $n < \infty$. Let $S \in \mathrm{Coh}[X^0, \mathrm{Mod}\, K]$. Then there is an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to S \to 0$$

in $\mathrm{Coh}[X^0, \mathrm{Mod}\, K]$, where P_i is projective for $0 \leqslant i \leqslant n-1$. Then

$$\mathrm{Tor}_1^X(P_n, -) = \mathrm{Tor}_{n+1}^X(S, -) = 0.$$

Hence P_n is flat and finitely presented, so projective by [2], I. 2, Ex. 15, or [5], Th. 4.4. This means that $m \leqslant n$.

(4.4) Corollary. (See [4], Th. 4.1): *Let X be a small K-preadditive category. The following statements are equivalent.*

(1) *X^0 is semi-hereditary, i.e., every ideal of finite type of an $X(-, x)$, $x \in X$, is projective.*

(2) *$[X^0, \mathrm{Mod}\, K]$ is locally coherent, and $\mathrm{Coh}[X^0, \mathrm{Mod}\, K]$ has global dimension at most 1.*

(3) *A product of flat functors and a subfunctor of a flat functor in $[X, \mathrm{Mod}\, K]$ are flat.*

(4.5) Corollary. *Let X be a small K-additive category. The following statements are equivalent.*

(1) *$[X^0, \mathrm{Mod}\, K]$ is locally coherent, and the global dimension of $\mathrm{Coh}[X^0, \mathrm{Mod}\, K]$ is at most 2.*

(2) *For every morphism $\alpha : x \to y$ in X, the kernel of $X(-, \alpha)$ is of finite type and projective.*

(3) *An inverse limit of flat functors in $[X, \mathrm{Mod}\, K]$ is flat.*

Example. ([1], Th. 2.2). If X is K-additive and admits kernels then the statements of the preceding corollary are true; for a functor is flat if and only if it is left-exact, and an inverse limit of left-exact functors is left-exact.

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