A Pattern for the Asymptotic Number of Rooted Maps on Surfaces

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Let \( T_g(n) \) \((P_g(n))\) be the number of \( n \)-edged rooted maps (in a certain class) on an orientable (non-orientable) surface of type \( g \), and let \( t_g \) and \( p_g \) be the positive constants defined in [3]. In [10], Bender and Wormald observed the following pattern:

\[
T_g(n) \sim t_g(n)^{2(g-1)/2} n^n,
\]

\[
P_g(n) \sim p_g(n)^{2(g-1)/2} n^n,
\]

for all maps, 2-connected maps and smooth maps. In this paper, we show that many classes of maps fit the following modified pattern:

\[
T_g(n) \sim a_t(n)^{2(g-1)/2} n^n,
\]

\[
P_g(n) \sim a_p(n)^{2(g-1)/2} n^n.
\]


1. INTRODUCTION

A map on a surface \( S \) is a graph \( G \) embedded on \( S \) such that all components of \( S - G \) are simply connected regions. These components are called faces of the map. A map is called orientable (non-orientable) if it is embedded in a orientable (non-orientable) surface. Maps and orientations can also be defined purely combinatorially without using the concept of surface (cf. [18, Chap. X]). If a map has \( v \) vertices, \( e \) edges and \( f \) faces, then its Euler characteristic is \( \chi = v - e + f \), and its type is \( g = 1 - \chi/2 \) (cf. [3] for further details about type). A map is called rooted if a vertex of the map, together with an edge adjacent to it and a side of the edge are specified. The concept of rooting a map was first introduced by W. T. Tutte [19, 20, 21] in the 1960's. Since then, much work has been done by several mathematicians on counting various classes of rooted maps on the sphere (See
[9] for a survey). But it seems very difficult to obtain any nice explicit formula for the exact number of non-planar maps (cf. [1, 4, 11, 12, 13, 16, 17, 22, 23]). Instead, people started working on asymptotic formulas for such numbers. In 1986–1988, E. A. Bender, E. R. Canfield, and N. C. Wormald [3, 10] studied various classes of rooted maps on general surfaces and they obtained asymptotic formulas for the number of rooted maps, rooted smooth maps and rooted 2-c maps (Throughout this paper, k-connected is abbreviated to k-c.). Letting $T_g(n)$ ($P_g(n)$) be the number of orientable (non-orientable) rooted maps (in a certain class) with $n$ edges of type $g$, they observed that these asymptotic formulas fit the following pattern:

$$T_g(n) \sim t_g(\beta n)^{5(g-1)/2} \gamma^n,$$

$$P_g(n) \sim p_g(\beta n)^{5(g-1)/2} \gamma^n,$$

where $t_g$ and $p_g$ are positive constants defined in [3], $\beta$ and $\gamma$ are independent of $n$ and $g$ (but they depend on the class of maps), and

$$a_n \sim b_n \quad \text{means} \quad \frac{a_n}{b_n} \to 1 \quad \text{as} \quad n \to \infty.$$

But this pattern is not satisfied by the triangular maps and a large class of degree restricted maps (cf. [13–15]). Instead, they satisfy the following modified pattern:

$$T_g(n) \sim \alpha t_g(\beta n)^{5(g-1)/2} \gamma^n, \quad (1)$$

$$P_g(n) \sim \alpha p_g(\beta n)^{5(g-1)/2} \gamma^n, \quad (2)$$

where $\alpha$ is the gcd of the face valencies of the class of maps.

We shall show that pattern (1) and (2) are also satisfied by the following classes of maps:

1. loopless maps;
2. simple maps, i.e., maps without loops or multiple edges;
3. 3-c triangular maps.

Asymptotic formulas like (1) and (2) played an important role in proving some asymptotic properties of maps such as 0–1 laws for submaps of maps and chromatic properties of maps (cf. [5–7]). It is believed that there should be some purely combinatorial explanation for pattern (1) and (2), but no such one has been found yet. (1) and (2) are derived through some delicate asymptotic analysis about “typical recursions” which are satisfied by many classes of maps and were first described in [3]. We
shall show that the three classes of maps listed above also satisfy the typical recursions with some extra "negligible terms." These extra terms correspond to non-contractible cycles of length 1 or 2 (A non-contractible cycle, denoted by nc-cycle, is a cycle of a map which is not homotopic to a point). This situation also occurred when we dealt with the 2-c triangular maps [14]. A rigorous argument was given there to show that these extra terms were indeed negligible. Since the argument is quite lengthy and need to be modified case by case, including such a detailed argument would make this paper much too long, we shall not repeat it here. Interested readers may refer to [14] for the details. The following theorem [5, Theorem 1] should be helpful for understanding this fact.

**THEOREM 1.** Let $\mathcal{M}_n(S)$ be a class of $n$-edged ($n$-vertex) rooted maps on a surface $S$ of type $g$, $M_n(S)$ be the number of such maps. If

$$M_n(S) \sim C(S) n^{5(g-1)/2} R^n,$$

then for any given constant $l$, almost all maps in $\mathcal{M}_n(S)$ do not have any nc-cycle with length less than $l$.

The main purpose of this paper is to extend the list of maps which satisfy (1) and (2) and to provide a table of the parameters $\alpha, \beta$ and $\gamma$ which may be useful for further investigations. To avoid considerable repetition, we rely heavily on [3, 13]. For those notations and terminologies not defined here, we also refer to them.

2. **OUTLINE OF THE PROOF OF THE PATTERN**

In what follows, $I = \{i_1 < i_2 < \cdots\}$ is a finite set of positive integers, and the empty set is denoted by $\emptyset$. We use $\alpha$ to denote a vector of non-negative integers such that $\alpha_i = 0$ for $i \notin I$, $\mathbf{0}$ to denote the vector of all zeros, and $\mathbf{e}_w$ to denote the vector of all zeros except a 1 in the $w$th coordinate. For $J \subseteq I$, $\alpha|_J$ is $\alpha$ restricted to $J$. We use $\sum_{i=k/2}^{g}$ to mean the summation over half integers $k/2, (k+1)/2, ..., g$.

Consider rooted maps in a certain class with some distinguished faces indexed by $I$. Let $\hat{M}_g(x, y, z_I)$ ($\tilde{M}_g(x, y, z_I)$) be the generating function of orientable (non-orientable) rooted maps (in a certain class) of type $g$, where $x$ marks the number of edges (or vertices), $y$ marks the root face valency and $z_I$ marks the valencies of the distinguished faces. Let

$$M_g(x, y, z_I) = \hat{M}_g(x, y, z_I) + \tilde{M}_g(x, y, z_I).$$

For convenience, we shall use $M_g(x, y, I)$ to denote $M_g(x, y, z_I)$ etc. in the rest of the paper. We shall show, for the three classes of maps listed above,
that for \((g, I) \neq (0, \varnothing)\), \(M_g(x, y, I)\) and \(\hat{M}_g(x, y, I)\) satisfy the following recursions:

\[
A(x, y, M_0(x, y, \varnothing)) M_g(x, y, I) \\
= - p_1(x, y) \sum_{l=0/2}^g \sum_{J \subseteq I} M_l(x, y, J) M_{g-l}(x, y, I-J) \\
- 2y^2 p_2(x, y) \frac{\partial}{\partial z_w} M_{g-1}(x, y, I+\{w\}) \bigg|_{z_w=y} \\
- y p_2(x, y) \frac{\partial}{\partial y} (y M_{g-1/2}(x, y, I)) \\
- p_2(x, y) \sum_{i \in I} \frac{z_i}{z_i-y} \left[ z_i M_g(x, z_i, I-\{i\}) - y M_g(x, y, I-\{i\}) \right] \\
+ q(x, y) m_g(x, I) + E_g(x, y, I),
\]

\[(3)\]

\[
A(x, y, M_0(x, y, \varnothing)) \hat{M}_g(x, y, I) \\
= - p_1(x, y) \sum_{l=0/2}^g \sum_{J \subseteq I} \hat{M}_l(x, y, J) \hat{M}_{g-l}(x, y, I-J) \\
- y^2 p_2(x, y) \frac{\partial}{\partial z_w} \hat{M}_{g-1}(x, y, I+\{w\}) \bigg|_{z_w=y} \\
- p_2(x, y) \sum_{i \in I} \frac{z_i}{z_i-y} \left[ z_i \hat{M}_g(x, z_i, I-\{i\}) - y \hat{M}_g(x, y, I-\{i\}) \right] \\
+ q(x, y) \hat{m}_g(x, I) + \hat{E}_g(x, y, I),
\]

\[(4)\]

where \(E_g(x, y, I)\) and \(\hat{E}_g(x, y, I)\) are negligible terms corresponding to maps with \(n_c\)-cycles of length 1 or 2; \(m_g(x, \varnothing)\) and \(\hat{m}_g(x, \varnothing)\) are generating functions for maps under consideration; \(p_1(x, y), p_2(x, y)\) and \(q(x, y)\) are known functions.

The case \((g, I) = (0, \varnothing)\) is covered by

\[
[A(x, y, M_0(x,y,\varnothing))]^2 = B(x, y, m_0(x, \varnothing)),
\]

\[(5)\]

where \(B\) is a polynomial. Equations like (5) are usually solved by the "quadratic method" (cf. [3, 12]). Let \(y=f(x)\) be a power series satisfying \(B = \frac{\partial B}{\partial y} = 0\) and use \(B^{(k)}\) to denote

\[
\frac{\partial^k B}{\partial y^k} \bigg|_{y=f(x)}.
\]
It will be shown that

\[ f(x) = y_0 + \text{higher power terms in } (1 - \gamma x)^{1/2}, \]

\[ m_0(x, \emptyset) = b_0 + b_1 (1 - \gamma x)^{3/2} + \text{higher power terms in } (1 - \gamma x)^{1/2}, \]

\[ B^{(2)}(x) = c_2 (1 - \gamma x)^{1/2} + \text{higher power terms in } (1 - \gamma x)^{1/2}, \]

\[ B^{(3)} = -c_3 + \text{higher power terms in } (1 - \gamma x)^{1/2}, \]

where \( y_0, b_0, c_2 \) and \( c_3 \) are positive constants. Let

\[ M^{(n)}_g(x, I, \alpha) = \frac{\partial^n + |\alpha|}{\partial y^n \prod_{i \in I} \partial z_i^{x_i}} M_g(x, y, I) \bigg|_{y = z_1 = f(x)} \]

\[ \hat{M}^{(n)}_g(x, I, \alpha) = \frac{\partial^n + |\alpha|}{\partial y^n \prod_{i \in I} \partial z_i^{x_i}} \hat{M}_g(x, y, I) \bigg|_{y = z_1 = f(x)} \]

Applying \( -2p_1(x_0, y_0) \binom{n + 1}{n} M^{(1)}_0(\emptyset, 0) M^{(n)}_g(I, \alpha) \)

\[ = 2p_1(x_0, y_0) \sum_{k=0}^{n-1} \binom{n + 1}{k} M^{(n + 1 - k)}_0(\emptyset, 0) M^{(k)}_g(I, \alpha) \]

\[ + p_1(x_0, y_0) \sum_{i=0}^{g} \sum_{J \neq \emptyset} \sum_{k=0}^{n+1} \binom{n + 1}{k} \times M^{(k)}(J, \alpha | J) M^{(n+1-k)}_g(I - J, \alpha | I - J) \]

\[ + 2y_0^2 p_2(x_0, y_0) \sum_{k=0}^{n+1} \binom{n + 1}{k} M^{(n+1-k)}(I + \{w\}, \alpha + (k+1) e_w) \]

\[ + y_0^2 p_2(x_0, y_0) M^{(n+2)}_{g-1/2}(I, \alpha) \]

\[ + y_0^2 p_2(x_0, y_0) \sum_{i \in I} \binom{n + 1}{n+\alpha_i + 2} M^{(n+\alpha_i+2)}_{g}(I - \{i\}, \alpha | I - \{i\}) \]
and

\[-2p_1(x_0, y_0)\binom{n+1}{n} M_0^{(1)}(\emptyset, 0) M_g^{(n)}(I, \alpha)\]

\[= 2p_1(x_0, y_0) \sum_{k=0}^{n-1} \binom{n+1}{k} M_0^{(n+1-k)}(\emptyset, 0) M_g^{(k)}(I, \alpha)\]

\[+ p_1(x_0, y_0) \sum_{(i, J) \neq (0, \emptyset), (g, l)} \sum_{k=0}^{n+1} \binom{n+1}{k} \times M_l^{(k)}(J, \alpha | J) M_g^{(n+1-k)}(I - J, \alpha | I - J)\]

\[+ y_0^2 p_2(x_0, y_0) \sum_{k=0}^{n+1} \binom{n+1}{k} M_g^{(n+1-k)}(I + \{w\}, \alpha + (k+1)e_w)\]

\[+ y_0^2 p_2(x_0, y_0) \sum_{i \in I} \binom{n+1}{n + x_i + 2} M_g^{(n + x_i + 2)}(I - \{i\}, \alpha | I - \{i\})\]  

(7)

with the initial conditions (cf. [3, Lemma 2])

\[M_0^{(n)}(\emptyset, 0) = M_0^{(n)}(\emptyset, 0) = \frac{3c_2}{2c_3 p_1(x_0, y_0)} \sqrt{\frac{c_2}{2}} \left( -\frac{c_3}{3c_2} \right)^{n/2} (n-1)! \]  

(8)

In [3], \(H_g^{(n)}(x, I, \alpha), \tilde{H}_g^{(n)}(x, I, \alpha), \phi_g^{(n)}(I, \alpha), \) and \(\tilde{\phi}_g^{(n)}(I, \alpha)\) played the same role as \(M_g^{(n)}(x, I, \alpha), \tilde{M}_g^{(n)}(x, I, \alpha), \tilde{M}_g^{(n)}(I, \alpha), \) and \(\tilde{M}_g^{(n)}(I, \alpha)\) do here, respectively. There,

\[p_1(x, y) = xy^2(1 - y), \quad p_2(x, y) = xy(y - 1), \quad q(x, y) = xy, \]

\[x_0 = \frac{1}{12}, \quad \text{and} \quad y_0 = \frac{6}{5}.\]

Comparing [3, (3.4), (4.2)] with (7) and (8), we have

**Theorem 2.** There exist positive constants \(\beta_0, \beta_1, \) and \(\beta_2,\) such that

\[M_g^{(n)}(I, \alpha) = \beta_0 \beta_1^{n+|\alpha|} \beta_2^{n+|\alpha|+2} \phi_g^{(n)}(I, \alpha),\]  

(9)

\[\tilde{M}_g^{(n)}(I, \alpha) = \beta_0 \beta_1^{n+|\alpha|} \beta_2^{n+|\alpha|+2} \tilde{\phi}_g^{(n)}(I, \alpha).\]  

(10)

**Proof.** We just prove (9) since the proof of (10) is essentially the same. Set

\[\frac{p_1(x_0, y_0) \beta_0 \beta_1 \beta_2}{y_0^2 p_2(x_0, y_0)} = \frac{3/125}{18/625}.\]  

(11)
Using \([3, (3.4)]\), we see that two sequences \(\{f_{0}\beta^{n} + \sum_{k=0}^{n} \beta_{k} \Phi_{g}^{(k)}(I, \alpha)\}\) and \(\{M_{g}^{(n)}(I, \alpha)\}\) satisfy the same recursion. They also have the same initial values by setting

\[M_{0}^{(n)}(\emptyset, 0) = \beta_{0} \beta_{1}^{n} \Phi_{0}^{(n)}(\emptyset, 0).\]  

Therefore the two sequences are the same. Combining \([3, \text{Lemma 2}, (8),\) and \((12)\) we obtain

\[5 \sqrt{6} \beta_{0} = \frac{3c_{2}}{2c_{3} p_{1}(x_{0}, y_{0})} \sqrt{\frac{c_{2}}{2}}; \quad \frac{25}{18} \beta_{1} = \frac{c_{3}}{3c_{2}}.\]  

Therefore \(\beta_{0}, \beta_{1},\) and \(\beta_{2}\) are uniquely determined by \((11)\) and \((13)\).}

Setting \(y = f(x)\) and \(I = \emptyset\) in \((3)\) and \((4)\), we have

\[m_{g}(x, \emptyset) \approx \left(\frac{p_{1}(x_{0}, y_{0})}{q(x_{0}, y_{0})} \sum_{t=1}^{g} M_{t}^{(0)}(\emptyset, 0) M_{g-t}^{(0)}(\emptyset, 0) + \frac{2y_{0}^{2} p_{2}(x_{0}, y_{0})}{q(x_{0}, y_{0})} M_{g-1}^{(0)}(\emptyset, 0) q(x_{0}, y_{0}) + \frac{y_{0}^{2} p_{2}(x_{0}, y_{0})}{q(x_{0}, y_{0})} M_{g-1}^{(1)}(\emptyset, 0) (1 - \gamma x)^{(5g-3)/2},\right.\]

\[\left.\hat{m}_{g}(x, \emptyset) \approx \left(\frac{p_{1}(x_{0}, y_{0})}{q(x_{0}, y_{0})} \sum_{t=1}^{g} \hat{M}_{t}^{(0)}(\emptyset, 0) \hat{M}_{g-t}^{(0)}(\emptyset, 0) + \frac{y_{0}^{2} p_{2}(x_{0}, y_{0})}{q(x_{0}, y_{0})} \hat{M}_{g-1}^{(0)}(\emptyset, 0) (1 - \gamma x)^{(5g-3)/2}.\right.\]

Let \(T_{g}(n) = \lfloor x^{n} \rfloor \hat{m}_{g}(x, \emptyset)\) and \(P_{g}(n) = \lfloor x^{n} \rfloor m_{g}(x, \emptyset) - T_{g}(n)\). Applying \([3, \text{Theorem 4}]\) to \((14)\) and \((15)\) and using \([3, \text{Theorem 1}, (3.3), (4.1)]\) and Theorem 2, we obtain \((1)\) and \((2)\) with

\[\alpha = \frac{25}{6} \frac{p_{1}(x_{0}, y_{0})}{q(x_{0}, y_{0})} \beta_{0}^{2} \beta_{2}^{2} \quad \text{and} \quad \beta = \beta_{2}^{4/5}.\]  

### 3. Loopless Maps

In this section, let \(\hat{M}_{g}(x, y, I) (M_{g}(x, y, I))\) be the generating function, by edges, of orientable (non-orientable) rooted loopless maps of type \(g\). We shall prove
THEOREM 3. Let \( T_g(n) = [x^n] \hat{M}_g(x, 1, \emptyset) \) (\( P_g(n) = [x^n] \tilde{M}_g(x, 1, \emptyset) \)) be the number of orientable (non-orientable) rooted loopless maps with \( n \) edges of type \( g \), then \( T_g(n) \) and \( P_g(n) \) satisfy (1) and (2) with \( \alpha = 1 \), \( \beta = \frac{3}{2} \), and \( \gamma = \frac{256}{27} \).

We first give functional equations for \( M_g(x, y, I) \) and \( \hat{M}_g(x, y, I) \).

THEOREM 4.

\[
M_0(x, y, \emptyset) = xy^2 (M_0(x, y, \emptyset))^2 + \frac{xy}{y-1} (yM_0(x, y, \emptyset) - M_0(x, 1, \emptyset))
- xyM_0(x, 1, \emptyset) M_0(x, y, \emptyset) + 1,
\]

and for \( (g, I) \neq (0, \emptyset) \) and \( \emptyset \notin I \), we have

\[
M_g(x, y, I) = xy^2 \sum_{t=0/2}^{g} \sum_{J \subseteq I} M_t(x, y, J) M_{g-t}(x, y, I-J)
+ 2xy^3 \frac{\partial}{\partial z_w} M_{g-1}(x, y, I + \{w\}) \bigg|_{z_w = y}
+ xy^2 \frac{\partial}{\partial y} (yM_{g-1/2}(x, y, I))
+x \sum_{i \in I} \frac{yz_i}{z_i - y} \left[ z_i M_g(x, z_i, I - \{i\}) - y M_g(x, y, I - \{i\}) \right]
+ \frac{xy}{y-1} (yM_g(x, y, I) - M_g(x, 1, I))
- xyM_0(x, 1, \emptyset) M_g(x, y, I) - xyM_g(x, 1, I) M_0(x, y, \emptyset)
+ N_g(x, y, I),
\]

\[
\hat{M}_g(x, y, I) = xy^2 \sum_{t=0}^{g} \sum_{J \subseteq I} \hat{M}_t(x, y, J) \hat{M}_{g-t}(x, y, I-J)
+ xy^3 \frac{\partial}{\partial z_w} \hat{M}_{g-1}(x, y, I + \{w\}) \bigg|_{z_w = y}
+ x \sum_{i \in I} \frac{yz_i}{z_i - y} \left[ z_i \hat{M}_g(x, z_i, I - \{i\}) - y \hat{M}_g(x, y, I - \{i\}) \right]
+ \frac{xy}{y-1} (y\hat{M}_g(x, y, I) - \hat{M}_g(x, 1, I))
- xyM_0(x, 1, \emptyset) \hat{M}_g(x, y, I) - xy\hat{M}_g(x, 1, I) M_0(x, y, \emptyset)
+ \hat{N}_g(x, y, I),
\]

where \( N_g(x, y, I) \) and \( \hat{N}_g(x, y, I) \) correspond to maps having \( nc \)-loops.
Proof. We prove only (17) and (18) since (19) follows similarly. Let $M$ be a map of type $g$ with root edge $e$. As in [3], we distinguish the following four cases.

(A) The root face borders both sides of $e$ and
   (A1) removal of $e$ disconnects $M$, or
   (A2) removal of $e$ does not disconnect $M$.
(B) The root face borders only one side of $e$ and
   (B1) none of the distinguished faces borders $e$, or
   (B2) a distinguished face borders $e$.

The contribution from each case is similar to that described in [3, Theorem 2], the only difference here is to exclude the possible occurrence of loops. Since all contributions from $nc$-loops are collected in $N_g(x, y, I)$, we only need to take care of contractible loops. No loop occurs in case (A1). If a loop occurs in (A2), since $e$ crosses a handle or a crosscap, it must be an $nc$-loop. If a loop occurs in case (B1) and it is contractible, then cutting along the loop $e$ and filling the holes with disks yields a planar map and another map of type $g$. We only need to consider the case that the planar map has no distinguished faces because otherwise, when all distinguished faces are joined by handles, the loop will eventually become an $nc$-loop. Therefore the contractible loop occurred in this case contributes

$$xyM_0(x, 1, \emptyset) M_0(x, y, \emptyset)$$

for $(g, I) = (0, \emptyset)$ and

$$xyM_0(x, 1, \emptyset) M_g(x, y, I) + xyM_g(x, 1, I) M_0(x, y, \emptyset)$$

for $(g, I) \neq (0, \emptyset)$ (See Fig. 1). The loop occurring in (B2) will eventually become an $nc$-loop because it is on the boundary of a distinguished face. Combining all the cases gives Theorem 4.
Proof of Theorem 3. Let \( m_g(x, I) = M_g(x, 1, I), \) \( \tilde{m}_g(x, I) = \tilde{M}_g(x, 1, I). \) Equations (17), (18), and (19) can then be rewritten as (5), (3), and (4), with

\[
A(x, y, M_0(x, y, \emptyset)) = 2x(y - 1) y^2 M_0(x, y, \emptyset) + xy^2 - x(y - 1) y m_0(x, \emptyset) + 1 - y,
\]
\[
B(x, y, M_0(x, \emptyset)) = (xy^2 - x(y - 1) y m_0(x, \emptyset) + 1 - y)^2 + 4x(y - 1) y^2 (1 - y + x y m_0(x, \emptyset)),
\]
\[
p_1(x, y) = x(y - 1) y^2, \quad p_2(x, y) = x(y - 1) y,
\]
and
\[
q(x, y) = xy + xy(y - 1) M_0(x, y, \emptyset).
\]

We solve \( B = \partial B / \partial y = 0 \) to obtain

\[
x = t(1 - t)^3, \tag{20}
\]
\[
y = f(x) = \frac{1}{1 - t + t^2}, \tag{21}
\]
\[
x m_0(x, \emptyset) = t(1 - 2t). \tag{22}
\]

From (20)–(22) we obtain the singularity \( x_0 = \frac{27}{256}, \) and

\[
f(x) = \frac{16}{13} + \text{higher power terms in } \left( 1 - \frac{256}{27} x \right)^{1/2},
\]
\[
m_0(x, \emptyset) = \frac{32}{27} + \frac{32}{81} \sqrt{6} \left( 1 - \frac{256}{27} x \right)^{3/2} + \text{higher power terms in } \left( 1 - \frac{256}{27} x \right)^{1/2},
\]
\[
B^{(2)} = \frac{9}{16} \sqrt{6} \left( 1 - \frac{256}{27} x \right)^{1/2} + \text{higher power terms in } \left( 1 - \frac{256}{27} x \right)^{1/2},
\]
\[
B^{(3)} = - \left( \frac{39}{16} \right)^2 + \text{higher power terms in } \left( 1 - \frac{256}{27} x \right)^{1/2}.
\]

Using (11), (13) and (16), we obtain

\[
\beta_0 = \frac{26}{45} \left( \frac{3}{2} \right)^{1/4}, \quad \beta_1 = \left( \frac{13}{20} \right)^2 \sqrt{6}, \quad \beta_2 = \left( \frac{3}{2} \right)^{5/4},
\]
and

\[ \alpha = 1, \quad \beta = \frac{3}{2}. \]

This gives Theorem 3.

4. Simple Maps

A map is called simple if it has no loops or multiple edges. In this section, let \( \tilde{M}_g(x, y, I) \) (\( \tilde{M}_g(x, y, I) \)) be the generating function, by edges, of orientable (non-orientable) rooted simple maps of type \( g \). We prove

**Theorem 5.** Let \( T_g(n) = \lceil x^n \rceil \tilde{M}_g(x, 1, \emptyset) \) (\( P_g(n) = \lceil x^n \rceil \tilde{M}_g(x, 1 \emptyset) \)) be the number of orientable (non-orientable) rooted simple maps with \( n \) edges of type \( g \), then \( T_g(n) \) and \( P_g(n) \) satisfy (1) and (2) with \( \alpha = 1, \beta = (3/2)^2, \) and \( \gamma = 8 \).

The following theorem gives functional equations for \( M_g(x, y, I) \) and \( \tilde{M}_g(x, y, I) \).

**Theorem 6.**

\[
M_0(x, y, \emptyset) = xy^2(M_0(x, y, \emptyset))^2 + \frac{xy}{y-1} (yM_0(x, y, \emptyset) - M_0(x, 1, \emptyset))
\]

\[- xyM_0(x, 1, \emptyset) M_0(x, y, \emptyset)
\]

\[- (M_0(x, 1, \emptyset) - 1)(M_0(x, y, \emptyset) - 1) + 1, \tag{23}\]

and for \( (g, I) \neq (0, \emptyset) \) and \( w \notin I \), we have

\[
M_g(x, y, I) = xy^2 \sum_{i=0/2}^g \sum_{J \subseteq I} M_i(x, y, J) M_{g-1}(x, y, I-J)
\]

\[
+ 2xy^3 \frac{\partial}{\partial z_w} M_{g-1}(x, y, I + \{w\}) \bigg|_{z_w = y}
\]

\[
+ xy^2 \frac{\partial}{\partial y} (yM_{g-1/2}(x, y, I))
\]

\[
+ x \sum_{i \in I} \frac{yz_i}{z_i - y} \left[ z_i M_g(x, z_i, I - \{i\}) - yM_g(x, y, I - \{i\}) \right]
\]

\[
+ \frac{xy}{y-1} (yM_g(x, y, I) - M_g(x, 1, I))
\]

\[- xyM_0(x, 1, \emptyset) M_g(x, y, I) - xyM_g(x, 1, I) M_0(x, y, \emptyset)
\]

\[- (M_0(x, 1, \emptyset) - 1) M_g(x, y, I)
\]

\[- M_g(x, 1, I)(M_0(x, y, \emptyset) - 1) + N_g(x, y, I), \tag{24}\]

\[
(23) \quad \frac{xy}{y-1} (yM_{g-1/2}(x, y, I))
\]

\[
+ x \sum_{i \in I} \frac{yz_i}{z_i - y} \left[ z_i M_g(x, z_i, I - \{i\}) - yM_g(x, y, I - \{i\}) \right]
\]

\[
+ xy^2 \frac{\partial}{\partial y} (yM_{g-1/2}(x, y, I))
\]

\[- xyM_0(x, 1, \emptyset) M_g(x, y, I) - xyM_g(x, 1, I) M_0(x, y, \emptyset)
\]

\[- (M_0(x, 1, \emptyset) - 1) M_g(x, y, I)
\]

\[- M_g(x, 1, I)(M_0(x, y, \emptyset) - 1) + N_g(x, y, I), \tag{24}\]
\[
\tilde{M}_g(x, y, I) = xy^2 \sum_{t=0}^{g} \sum_{J \subseteq I} \tilde{M}_t(x, y, J) \tilde{M}_{g-t}(x, y, I-J)
\]
\[
+ xy^3 \frac{\partial}{\partial z_w} \tilde{M}_{g-1}(x, y, I + \{w\}) \bigg|_{z_w = y}
\]
\[
x \sum_{i \in I} \frac{y z_i}{z_i - y} \left[ z_i \tilde{M}_g(x, z_i, I - \{i\}) - y \tilde{M}_g(x, y, I - \{i\}) \right]
\]
\[
+ \frac{xy}{y-1} (y \tilde{M}_g(x, y, I) - \tilde{M}_g(x, 1, I))
\]
\[
- xyM_0(x, 1, \emptyset) \tilde{M}_g(x, y, I) - xy\tilde{M}_g(x, 1, I) M_0(x, y, \emptyset)
\]
\[
- (M_0(x, 1, \emptyset) - 1) \tilde{M}_g(x, y, I)
\]
\[
- \tilde{M}_g(x, 1, I)(M_0(x, y, \emptyset) - 1)
\]
\[
+ \tilde{N}_g(x, y, I),
\]
(25)

where \(N_g(x, y, I)\) and \(\tilde{N}_g(x, y, I)\) correspond to maps having nc-cycles of length 1 or 2.

**Proof.** The proof is similar to that of Theorem 4, we only need to take care of the contractible multiple edge occurring in the case (B1) as shown in Fig. 2. This contributes the row next to the last row in (24) and (25) for \((g, I) \neq (0, \emptyset)\) and the term next to the last term in (23) for \((g, I) = (0, \emptyset)\).

**Proof of Theorem 5.** Let \(m_g(x, I) = M_g(x, 1, I)\) and \(\hat{m}_g(x, I) = \tilde{M}_g(x, 1, I)\). Equations (23), (24), and (25) can then be rewritten as (5), (3), and (4), with

\[
A(x, y, M_0(x, y, \emptyset)) = 2x(y-1) y^2 M_0(x, y, \emptyset)
\]
\[
+ xy^2 - (y-1)(1+xy) m_0(x, \emptyset),
\]

\[
B(x, y, m_0(x, \emptyset)) = (xy^2 - (y-1)(1+xy) m_0(x, \emptyset))^2
\]
\[
+ 4x(y-1) y^2 (1-y+xy) m_0(x, \emptyset),
\]

\[
p_1(x, y) = x(y-1) y^2, \quad p_2(x, y) = x(y-1) y,
\]

**Fig. 2.** Adding a contractible multiple edge.
and

\[ q(x, y) = xy + xy(y - 1) M_0(x, y, \emptyset) + (y - 1)(M_0(x, y, \emptyset) - 1). \]

We solve \( B = \partial B/\partial y = 0 \) to obtain

\[ x = t(1 - 2t), \quad (26) \]

\[ y = f(x) = \frac{1}{1 - t + t^2}, \quad (27) \]

\[ m_0(x, \emptyset) = \frac{1 - 2t}{(1 - t)^3}. \quad (28) \]

From (26)-(28) we obtain the singularity \( x_0 = \frac{1}{8} \), and

\[ f(x) = \frac{16}{13} + \text{higher power terms in } (1 - 8x)^{1/2}, \]

\[ m_0(x, \emptyset) = \frac{32}{27} + \left( \frac{16}{27} \right)^2 (1 - 8x)^{3/2} + \text{higher power terms in } (1 - 8x)^{1/2}, \]

\[ B^{(2)} = \frac{128}{81} (1 - 8x)^{1/2} + \text{higher power terms in } (1 - 8x)^{1/2}, \]

\[ B^{(3)} = -\left( \frac{26}{9} \right)^2 + \text{higher power terms in } (1 - 8x)^{3/2}. \]

Using (11), (13), and (16), we obtain

\[ \beta_0 = \frac{26 \sqrt{6}}{135}, \quad \beta_1 = 3 \left( \frac{13}{20} \right)^2, \quad \beta_2 = \left( \frac{3}{2} \right)^{5/2}, \]

and

\[ \alpha = 1, \quad \beta = \left( \frac{3}{2} \right)^2. \]

This gives Theorem 5. \[ \square \]

5. 3-CONNECTED TRIANGULAR MAPS

A map is called \textit{triangular} if all its faces have valency 3; a map is called \textit{near-triangular} if all its faces except possibly the root face and some other distinguished faces have valency 3. Let \( \bar{M}_g(x, y, I) \) (\( \bar{M}_g(x, y, I) \)) be the
generating function, by vertices, of orientable (non-orientable) rooted near-triangular maps without loops or multiple edges of type $g$. Let

$$\hat{M}_g(x, y, I) = [y^r] \hat{M}_g(x, y, I), \quad \hat{M}_g(x, y, I) = [y^r] \hat{M}_g(x, y, I).$$

Using the fact that a triangular map is 3-connected if and only if it has no loops or multiple edges, $\hat{M}_g^3(x, \emptyset)(\hat{M}_g^3(x, \emptyset))$ is therefore the generating function of orientable (non-orientable) rooted 3-c triangular maps of type $g$. Let

$$M_g(x, y, I) = \hat{M}_g(x, y, I) + \hat{M}_g(x, y, I).$$

The main purpose of this section is to prove the following theorem.

**Theorem 7.** Let $T_g(n) = [x^n] M_g^3(x, \emptyset)$ ($P_g(n) = [x^n] \hat{M}_g^3(x, \emptyset)$) be the number of orientable (non-orientable) rooted 3-c triangular maps with $n$ vertices of type $g$, then $T_g(n)$ and $P_g(n)$ satisfy (1) and (2) with $\alpha = 3$, $\beta = \frac{256}{9} (\frac{2}{3})^{1/5}$, and $\gamma = \frac{256}{27}$.

We first give the functional equations for $M_g(x, y, I)$ and $\hat{M}_g(x, y, I)$.

**Theorem 8.**

$$M_0(x, y, \emptyset) = y^2 (M_0(x, y, \emptyset) - x)$$

$$-xyM_0(x, y, \emptyset)$$

$$-x^{-2}M_0^3(x, y, \emptyset)(M_0(x, y, \emptyset) - x) + x,$$

(29)

and for $(g, I) \neq (0, \emptyset)$ and $w \notin I$, we have

$$M_g(x, y, I) = y^2 \sum_{i=0/2}^g \sum_{J \subseteq I} M_i(x, y, J) M_{g-i}(x, y, I-J)$$

$$+ 2y^3 \frac{\partial}{\partial z_w} M_{g-1}(x, y, I + \{w\}) \bigg|_{z_w = y}$$

$$+ y^2 \frac{\partial}{\partial y} (y M_{g-1/2}(x, y, I))$$

$$+ \sum_{i \in I} \frac{y z_i}{z_i - y} [z_i M_g(x, z_i, I - \{i\}) - y M_g(x, y, I - \{i\})]$$

$$+ y^{-1}M_g(x, y, I)$$

$$- xyM_g(x, y, I) - x^{-2}M_g^3(x, y, \emptyset)(M_g(x, y, \emptyset) - x)$$

$$- x^{-2}M_0^3(x, \emptyset) M_g(x, y, I)$$

$$+ N_g(x, y, I),$$

(30)
\[ \tilde{M}_g(x, y, I) = y^2 \sum_{t=0}^g \sum_{J \subseteq I} \tilde{M}_t(x, y, J) \tilde{M}_{g-t}(x, y, I - J) + y^3 \frac{\partial}{\partial z_w} \tilde{M}_{g-1}(x, y, I + \{w\}) \bigg|_{z_w = y} \\
+ \sum_{i \in I} \frac{yz_i}{z_i - y} \left[ z_i \tilde{M}_g(x, z_i, I - \{i\}) - y \tilde{M}_g(x, y, I - \{i\}) \right] + y^{-1} \tilde{M}_g(x, y, I) - xy \tilde{M}_g(x, y, I) - x^{-2} \tilde{M}_g^3(x, I) (\tilde{M}_0(x, y, I) - x) - x^{-2} \tilde{M}_0^3(x, \emptyset) \tilde{M}_g(x, y, I) + \tilde{N}_g(x, y, I), \]  

where \( N_g(x, y, I) \) and \( \tilde{N}_g(x, y, I) \) correspond to nc-cycles of length 1 or 2.

**Proof.** Let \( T \) be any rooted near-triangular map without loops or multiple edges. Let \( e \) be its root edge. For \((g, I) = (0, \emptyset)\), the case (A) contributes the first term of the right hand side of (29). In the case (B), deleting \( e \) yields another such map with root face valency increased by 1. Reversing this process gives the second term which includes the case that \( e \) is a loop or multiple edge as shown in Fig. 3. Maps as shown in Fig. 3 contribute the third and the forth terms. The last term corresponds to the single vertex map. For \((g, I) \neq (0, \emptyset)\), the proof is similar to that of [13, Theorem 2]. The case (A) contributes the first three terms of the right hand side of (30) and (31), the case (B2) contributes the forth term. In the case (B1), the contractible cycle of length 1 or 2 as shown in Fig. 3 contributes the row next to the last row in (30) and (31).

**Proof of Theorem 7.** Let \( m_g(x, I) = M_g^2(x, I), \tilde{m}_g(x, I) = \tilde{M}_g^3(x, I) \). Equations (29), (30), and (31) can then be rewritten as (5), (3), and (4), with

\[
A(x, y, M_0(x, y, \emptyset)) = 2x^2y^3M_0(x, y, \emptyset) + x^2 - x^2y - x^3y^2 - ym_0(x, \emptyset),
\]

\[
B(x, y, m_0(x, \emptyset)) = (x^2 - x^2y - x^3y^2 - ym_0(x, \emptyset))^2 - 4x^2y^3(x^3y - x^3 + ym_0(x, \emptyset)),
\]

**Fig. 3.** Adding a contractible cycle of length 1 or 2.
and
\[ p_1(x, y) = x^2y^3, \quad p_2(x, y) = x^2y^2, \quad q(x, y) = y(M_0(x, y, \emptyset) - x). \]

We solve \( B = \frac{\partial B}{\partial y} = 0 \) to obtain
\[ x = t(1 - t)^3, \quad (32) \]
\[ y = f(x) = \frac{1}{1 - t^2}, \quad (33) \]
\[ \frac{m_0(x, \emptyset)}{x^2} = t(1 - 2t). \quad (34) \]

From (32)–(34) we obtain the singularity \( x_0 = \frac{27}{256} \), and
\[ f(x) = \frac{16}{15} + \text{higher power terms in} \left(1 - \frac{256}{27} x\right)^{1/2}, \]
\[ m_0(x, \emptyset) = \frac{1}{8} \left( \frac{27}{256} \right)^2 + \frac{\sqrt{6}}{24} \left( \frac{27}{256} \right)^2 \left(1 - \frac{256}{27} x\right)^{3/2} \]
\[ + \text{higher power terms in} \left(1 - \frac{256}{27} x\right)^{1/2}, \]
\[ B^{(2)} = \frac{9}{16} \left( \frac{27}{256} \right)^4 \left(1 - \frac{256}{27} x\right)^{1/2} \]
\[ + \text{higher power terms in} \left(1 - \frac{256}{27} x\right)^{1/2}, \]
\[ B^{(3)} = -\left( \frac{45}{16} \right)^2 \left( \frac{27}{256} \right)^4 \]
\[ + \text{higher power terms in} \left(1 - \frac{256}{27} x\right)^{1/2}. \]

Using (11), (13), and (16), we obtain
\[ \beta_0 = \frac{27}{2048} \left( \frac{3}{2} \right)^{1/4}, \quad \beta_1 = \frac{9\sqrt{6}}{16}, \quad \beta_2 = \frac{2048}{27} \left( \frac{3}{2} \right)^{1/4}, \]

and
\[ \alpha = 3, \quad \beta = \frac{256}{9} \left( \frac{8}{3} \right)^{1/5}. \]

This gives Theorem 7.
Let $D$ be a set of positive integers and let $\mathcal{M}(D)$ be a class of maps whose face valencies all lie in $D$. Let $M(S, D; x)$ be the generating function, by edges, of such maps on the surface $S$ and let $M_n(S, D) = \left[ x^n \right] M(S, D; x)$ be the number of such map with $n$ edges. We rewrite patterns (1) and (2) in the form

$$M_n(S, D) \sim \alpha \mu(S)(\beta n)^{-\delta} \gamma^n,$$  

(35)

where $\mu(S)$ equals $t_g(p_g)$ if $S$ is an orientable (a non-orientable) surface of type $g$, and $\alpha = \gcd(D)$. The parameters $\alpha$, $\beta$, and $\gamma$ are listed in Table I for some classes of maps, where we have used Euler's formula to convert the asymptotic formulas for triangular maps with respect to vertices to that with respect to edges and have used the following theorem to obtain asymptotic formulas for loopless $2d$-regular maps.

**THEOREM 9.** If $\gcd(D)$ is even, almost all maps in $\mathcal{M}(D)$ are loopless.

**Proof:** First, no maps in $\mathcal{M}(D)$ can have contractible loops, otherwise, we shall obtain a map in which all faces have even valencies except a loop face which is a contradiction. Now our claim follows from Theorem 1.

The following theorem indicates that pattern (35) is satisfied by many more classes of maps.

**TABLE I**

A Table for Parameters $\alpha$, $\beta$, and $\gamma$ w.r.t. Edges

<table>
<thead>
<tr>
<th>Classes of maps</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>All maps</td>
<td>1</td>
<td>1</td>
<td>12</td>
<td>[3]</td>
</tr>
<tr>
<td>2-c maps</td>
<td>1</td>
<td>3</td>
<td>27/4</td>
<td>[10]</td>
</tr>
<tr>
<td>3-c maps</td>
<td>1</td>
<td>9</td>
<td>4</td>
<td>[8]</td>
</tr>
<tr>
<td>Smooth maps</td>
<td>1</td>
<td>(3/2)$^{1/2}$</td>
<td>5 + 2$\sqrt{6}$</td>
<td>[3]</td>
</tr>
<tr>
<td>Loopless maps</td>
<td>1</td>
<td>3/2</td>
<td>256/27</td>
<td></td>
</tr>
<tr>
<td>Simple maps</td>
<td>1</td>
<td>(3/2)$^2$</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Triangular maps</td>
<td>3</td>
<td>(1/3)$^{6^{1/5}}$</td>
<td>2$^{2/3}$ $\times$ 3$^{1/2}$</td>
<td>[13]</td>
</tr>
<tr>
<td>2-c triangular maps</td>
<td>3</td>
<td>(2/3)$^{6^{1/5}}$</td>
<td>3 $\times$ 2$^{-1/3}$</td>
<td>[14]</td>
</tr>
<tr>
<td>3-c triangular maps</td>
<td>3</td>
<td>(4/3)$^{6^{1/5}}$</td>
<td>(8/3)$^{2-1/3}$</td>
<td></td>
</tr>
<tr>
<td>$2d$-regular maps</td>
<td>2d</td>
<td>(8d)$^{1/5}$ $(d-1)/d$</td>
<td>$d$ $\frac{d-1}{2}$ $\left( \frac{2d}{d} \right)^{1/d}$</td>
<td>[15]</td>
</tr>
<tr>
<td>Loopless $2d$-regular maps</td>
<td>2d</td>
<td>(8d)$^{1/5}$ $(d-1)/d$</td>
<td>$d$ $\frac{d-1}{2}$ $\left( \frac{2d}{d} \right)^{1/d}$</td>
<td></td>
</tr>
</tbody>
</table>
Theorem 10. Suppose $2 \notin D$, $\overline{D} = D \cup \{2\}$, and $\mathcal{M}(D)$ satisfies (35).

1. If $\gcd(D) = d$ is odd and

$$M(S, D; x) \approx C(S)(1 - (\gamma x)^d)^{5x/4 - 1}$$

where

$$C(S) = \Gamma(1 - 5\chi/4) d\mu(S)(\beta d)^{-5\chi/4},$$

then $\mathcal{M}(\overline{D})$ also satisfies (35) with parameters $1, \beta\gamma/(1 + \gamma)$, and $1 + \gamma$.

2. If $\gcd(D) = 2d$ and

$$M(S, D; x) \approx C(S)(1 - (\gamma x)^d)^{5x/4 - 1}$$

where

$$C(S) = \Gamma(1 - 5\chi/4)(2d) \mu(S)(\beta d)^{-5\chi/4},$$

then $\mathcal{M}(\overline{D})$ also satisfies (35) with parameters $2, \beta\gamma/(1 + \gamma)$, and $1 + \gamma$.

Proof. Closing all digons of a map in $\mathcal{M}(\overline{D})$, we obtain a map in $\mathcal{M}(\overline{D})$; conversely, replacing each non-root edge of a map in $\mathcal{M}(D)$ with a sequence of digons and replacing the root edge with two sequences of digons (one for each side), we obtain a map in $\mathcal{M}(\overline{D})$. Thus, we have

$$M(S, \overline{D}; x) = \frac{1}{1 - x} M \left( S, D; \frac{x}{1 - x} \right) + \frac{1}{1 - x}$$

if $S$ is the sphere or the projective plane, and otherwise

$$M(S, \overline{D}; x) = \frac{1}{1 - x} M \left( S, D; \frac{x}{1 - x} \right).$$

It follows from (36) and (37) that $M(S, \overline{D}; x)$ has a unique singularity $x = 1/(1 + \gamma)$ on its circle of convergence. Using

$$1 - \left( \frac{\gamma x}{1 - x} \right)^d \approx \frac{d(1 + \gamma)}{\gamma} (1 - (1 + \gamma)x)$$

as $x \to 1/(1 + \gamma)$, and [2, Theorem 4], we complete the proof.

References


5. E. A. Bender, Z. C. Gao, and L. B. Richmond, Almost all rooted maps have large representativity, *J. Graph Theory*, to appear.


