JOURNAL OF COMPLEXITY **17**, 516–540 (2001)
doi:10.1006/jcom.2001.0584, available online at http://www.idealibrary.com on IDEAL®

A Probabilistic Analysis of Linear Operator Testing

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Received September 7, 1999; accepted February 2, 2000; published online June 21, 2001

We test for β-conformance of an implementation linear operator A to a specification linear operator S where the operator domain and range are separable Hilbert spaces and the domain space F is equipped with a Gaussian measure μ. Given an error bound $\varepsilon > 0$ and a tolerance parameter $\beta \in (0, 1)$, we want to determine either that there is an element f in a ball B_q of radius q in domain F such that $\|Sf - Af\| > \varepsilon$ or that A β-conforms to S on a set of measure at least $1-\beta$ in the ball B_q; i.e., $\mu_q(f\colon \|Sf - Af\| \leqslant \varepsilon) \geqslant 1-\beta$ where μ_q is the truncated Gaussian measure to B_q. We present a deterministic algorithm that solves this problem and uses almost a minimal number of tests where each test is an evaluation of operators S and A at an element of F. We prove that optimal tests are conducted on the eigenvectors of the covariance operator of μ. They are universal; they are independent of the operators under consideration and other problem parameters. We show that finite testing is conclusive in this probabilistic setting. In contrast, finite testing is inconclusive in the worst and average case settings; see [5, 7]. We discuss the upper and lower bounds on the minimal number of tests. For $q = \infty$ we derive the exact bounds on the minimal number of tests, which depend on β very weakly. On the other hand, for a finite q, the bounds on the minimal number of tests depend on β more significantly. We explain our approach by an example with the Wiener measure. © 2001 Academic Press

Key Words: β-conformance testing; universal testing sequence; linear operators; Gaussian measure.

[1] This work was done while the author was consulting at Bell Laboratories and was partially supported by a National Science Foundation grant.

0885-064X/01 $35.00

1. INTRODUCTION

Large computer systems are often built to fulfill complicated tasks. Such systems become less reliable as they get larger. Testing has become an indispensable part of system design and analysis.

Testing has a variety of forms in different areas of science and technology. We discuss one of them, the so-called "black-box testing," here. That is, we have a specification, of a system design and we want to test if a given implementation conforms to the specification based on the observed system input and output behavior.

As in [5, 7], we consider black-box testing of linear systems that are modeled by linear operators. The goal is to determine whether an implementation linear operator A conforms to a specification linear operator S. Conformance depends on the error settings. In an earlier paper [5], we studied a worst case error setting in which we test whether the error is no more than a given positive bound ε for all elements in an ellipsoid E of the domain Hilbert space; i.e., $\sup_{f \in E} \|Sf - Af\| \leqslant \varepsilon$. This setting may be too restrictive, and for certain applications a small error on the average can be acceptable. In a follow-up work [7], we studied instead an average error setting, in which we wanted to verify whether an average error $\sqrt{\int_{B_q} \|Sf - Af\|^2 \mu_q(df)} \leqslant \varepsilon$, where μ_q is the truncated Gaussian measure of the domain Hilbert space to the ball B_q of large radius q. For both settings, we considered algorithms that use a number of tests to solve the problems where each test is an evaluation of operators S and A at an element of the corresponding domain.

It turns out that for both worst and average settings any finite number of tests is in general inconclusive. However, the testing problem is still decidable in the limit; there is an algorithm to generate an infinite sequence of test-and-guess such that all but finitely many guesses are correct. We also obtain positive results for weak conformance testing; we allow a positive relaxation parameter α and test for weak conformance with an error bound $(1+\alpha)\,\varepsilon$. Specifically, we test whether there exists an element f in the ellipsoid E such that $\|Sf - Af\| > \varepsilon$, or if for all f in E we have $\|Sf - Af\| \leqslant (1+\alpha)\,\varepsilon$ in the worst case setting and $\sqrt{\int_{B_s} \|Sf - Af\|^2 \mu_q(df)}$ $\leqslant (1+\alpha)\,\varepsilon$ in the average case setting. Then for both worst and average case settings a finite number of tests is conclusive. Furthermore, we derived an optimal test sequence that minimizes the number of tests needed to solve the problem. An interesting result for both error settings is that there exists a *universal* optimal test sequence. This test sequence only depends on the input set and the measure (in the average case setting) and is independent of S, A, and the other parameters of the problem.

In this paper, we study a *probabilistic* setting with the input set B_q which is a ball of radius q in a separable Hilbert space F. The space F is equipped with a Gaussian measure μ and the ball B_q is equipped with the truncated Gaussian measure μ_q. If $q = \infty$ then B_q is the whole space. We also assume that the specification and implementation operators S and A are linear operators between Hilbert spaces F and G.

In this probabilistic setting, we want either to find an element $f \in B_q$ such that the error exceeds the bound $\|Sf - Af\| > \varepsilon$ or to show that A β-conforms to S; i.e., $\|Sf - Af\| \leqslant \varepsilon$ for a set of elements f in B_q with a measure at least $1 - \beta$ where $\beta \in (0, 1)$ and we are mostly interested in small β. We show that finite testing is conclusive even without a relaxation parameter α. This makes the testing problem in the probabilistic setting different than that in the worst and average settings. Of course, the tolerance parameter β in the probabilistic setting may be viewed similarly as the relaxation parameter α in the other settings.

We show that the eigenvectors of the covariance operator of the measure μ provide an almost optimal and universal test sequence. In fact, for the limiting case with $q = \infty$ the test sequence is optimal. This is similar to the results in the average case setting for the relaxed testing.

We now discuss the minimal number of tests $n^*(\varepsilon, \beta, q)$ needed in the probabilistic setting. It is fully determined by the eigenvalues of the covariance operator of the Gaussian measure μ as well as for other problem parameters. In particular, for $q = \infty$, the minimal number of tests depends on the relation between ε and β. If ε is fixed and β goes to zero then $n^*(\varepsilon, \beta) = n^*(\varepsilon, \beta, \infty)$ is the minimal n, for which the $(n+1)$st largest eigenvalue is at most $c\varepsilon^2/\ln \beta^{-1}$ with a positive constant c. On the other hand, if β is fixed and ε goes to zero then the minimal number of tests does *not* depend on β and is equal to the minimal n, for which the sum of the $(n+1)$st to $(n+d)$th largest eigenvalues is at most $c\varepsilon^2$ where d is the dimension of the range space of S and A and c is a positive constant. Hence, if S and A are linear functionals then $d = 1$, and both cases depend on how fast the eigenvalues approach zero. On the other hand, when $d = \infty$ there may be a big difference between the bounds since the sum of the tail of the eigenvalues may go to zero substantially slower than the $(n+1)$st eigenvalue. In either case, the error parameter ε is much more significant than β since the minimal number of tests depends on a positive power of ε^{-1} and yet on the same power of $1/\ln \beta^{-1}$.

For a finite q, the number of minimal tests may behave quite differently. Namely, if β goes to zero, the parameter β is almost as significant as ε since the number of minimal steps depends essentially on the product of ε and $\sqrt{\beta}$. This holds for $d = \infty$. The case of a finite d is still open. On the other hand, if β is fixed and q goes to infinity then the minimal number of tests is roughly the same as for $q = \infty$, and the parameters ε and β play different roles.

In Section 2, after introducing the testing problem in the probabilistic setting, we present an algorithm for the test generation. We then discuss the termination of this algorithm and the bounds on the number of tests. In Sections 3 and 4, we consider the case $q=\infty$, that is, when the input set is the whole domain space of the operators. We obtain an optimal test sequence and the minimal number of tests. We explain the results by an example of the Wiener measure. In Section 5, we study the case $q<\infty$ where the input set is a ball of finite radius q in the domain space. We derive a universal and almost optimal test sequence. We conclude the paper in Section 6 with a discussion on miscellaneous related issues.

2. TESTING ALGORITHM

Consider continuous linear operators from a separable Hilbert space F to a separable Hilbert space G of dimension d, $1 \leqslant d \leqslant +\infty$,

$$S, A : F \rightarrow G.$$

The space F is equipped with a zero mean Gaussian measure μ of the covariance operator $C_\mu : F \rightarrow F$; see, e.g., [10]. Let λ_i and e_i be the eigenpairs of C_μ,

$$C_\mu e_i = \lambda_i e_i, \qquad i = 1, 2, ..., \dim(F). \tag{1}$$

Here $\{e_i\}$ is an orthonormal sequence in F and λ_i are ordered, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_i$. For notational convenience, if $\dim(F)$ is finite then we formally set $e_i = 0$ and $\lambda_i = 0$ for $i > \dim(F)$. Without loss of generality, we assume that all λ_i's are positive for all $i \leqslant \dim(F)$.

We consider a ball of radius q in F,

$$B_q = \{ f \in F : \|f\| \leqslant q \}.$$

The induced measure of μ on B_q, denoted by μ_q, is the truncated Gaussian measure to B_q,

$$\mu_q(A) = \mu(A \cap B_q)/\mu(B_q),$$

for all measurable sets A.

Given an error bound $\varepsilon > 0$, we test the implementation operator A to determine whether it is *faulty*:

$$\text{There exists an element } f \in B_q \text{ such that } \|Sf - Af\| > \varepsilon \tag{2}$$

or for a given $\beta \in (0, 1)$ we test A to determine whether it β-*conforms* to the specification operator S,

$$\mu_q(\{f \in B_q : \|Sf - Af\| \leqslant \varepsilon\}) \geqslant 1 - \beta. \tag{3}$$

Note that the two concepts of being faulty and β-conformance are not mutually exclusive; an implementation operator can be faulty but also can β-conform to the specification operator.

As is often the case in practice, A is not likely to deviate from S drastically; otherwise, A can be easily detected to be faulty. Specifically, we assume that $\|S - A\| \leqslant K$ where $K > 0$ is a given known constant. Obviously, if $Kq \leqslant \varepsilon$ then for all $f \in B_q$ we have

$$\|Sf - Af\| \leqslant \|S - A\| \; \|f\| \leqslant Kq \leqslant \varepsilon.$$

This means that A β-conforms to S even with $\beta = 0$. To avoid this trivial case, from now on we assume that

$$\varepsilon < Kq. \tag{4}$$

We now present our testing algorithm which simply tests on the eigenvectors of the covariance operator in (1). The number of tests n^*, which depends on all the problem parameters ε, β, d, K, q and the eigenvalues λ_i's, will be specified later in Sections 3 and 5, depending on whether the radius of the ball q is infinite. We assume that for any element $f \in B_q$ we have a subroutine to compute the value of the specification operator Sf and the value of the implementation operator Af. Furthermore, we assume that we can compute inner products in the range space G.

TESTING ALGORITHM **T**.

Input. Subroutine to compute Sf and Af for any $f \in B_q$, error bound $\varepsilon > 0$, and maximal number of tests n^*.

Output. NO (A is faulty) or YES (A β-conforms to S).

begin
  for $i = 1, 2, ..., n^*$ **do**
    compute Se_i and Ae_i, and

$$\delta_i = \sup_{f \in B_q \cap \operatorname{span}\{e_1, ..., e_i\}} \|Sf - Af\|$$

    if $\delta_i > \varepsilon$
      return NO;
  end
  return YES;
  end

It is easy to see that δ_i's are fully determined by the values of $(S - A)\, e_j$ for $j = 1, 2, ..., i$. In fact,

$$\delta_i = q\rho(M_i) \qquad (\text{with the convention } \infty \cdot 0 = 0),$$

where M_i is the $i \times i$ symmetric and nonnegative definite matrix with coefficients

$$((S - A)\, e_j, (S - A)\, e_k), \qquad j, k = 1, 2, ..., i,$$

and $\rho(M_i)$ is the largest eigenvalue of M_i. The value $\rho(M_i)$ can be approximated efficiently by, for instance, the Lanczos algorithm [8].

There are two cases for which δ_i can be computed easily. The first case is for $d = 1$. Then we have $(S - A)\, f = (f, h)$ for some $h \in F$ with $\|h\| \leqslant K$ and matrix $M_i = a_i a_i^T$ with $a_i = [(h, e_1), (h, e_2), ..., (h, e_i)]^T$. Then we obtain $\rho(M_i) = \|a_i\|^2 = \sum_{j=1}^{i} (h, e_j)^2$. Hence, $\delta_i = q \sum_{j=1}^{i} |(S - A)\, e_j|^2$. The second case is for $q = \infty$. Then δ_i equals 0 or ∞. It is 0 if and only if $Se_j = Ae_j$, $j = 1, 2, ..., i$. It is ∞ otherwise. Indeed, if $\delta_i > 0$ for an $i \leqslant n^*$, then there exists $f \in F \cap \operatorname{span}\{e_1, ..., e_i\}$ for which $\|Sf - Af\| > 0$. Take an arbitrary positive α. Then $\alpha f \in F \cap \operatorname{span}\{e_1, ..., e_i\}$ and $\|S(\alpha f) - A(\alpha f)\| = \alpha\, \|Sf - Af\|$ goes to infinity with α. This implies that δ_i is infinity.

Note that the algorithm **T** and the negative result are deterministic and that the tests are from the eigenvectors of C_μ. However, as we shall see, the positive conclusion is probabilistic: the specification A conforms to S on an input set of a probability measure at least $1 - \beta$, provided n^* is properly defined.

We now discuss the number of tests n^*. There are two cases depending on the radius q of the ball B_q: (1) $q = \infty$ and (2) $0 < q < \infty$. Case (1) is studied in Sections 3 and 4 and Case (2) in Section 5.

3. INFINITE RADIUS

We first consider the case $q=\infty$. That is, $B_q=F$ and $\mu_q=\mu$. Recall that $d=\dim(G)$. We now show how to estimate n^* to guarantee the correctness of the testing algorithm $\mathbf{T}$.

THEOREM 1. *The testing algorithm* $\mathbf{T}$ *checks the* β*-conformance of implementation operator* A, *provided that*

$$n^* \geqslant n^*(\varepsilon, \beta) = \min \left\{ n : \mu\left(\left\{ f : \sum_{i=n+1}^{n+d} (f, e_i)^2 \leqslant (\varepsilon/K)^2 \right\}\right) \geqslant 1-\beta \right\}. \quad (5)$$

Proof. If the algorithm $\mathbf{T}$ terminates with $\delta_i > \varepsilon$ for $i \leqslant n^*$ then the implementation A is faulty. On the other hand, if $\delta_i \leqslant \varepsilon$ for all $i \leqslant n^*$ then the algorithm terminates after n^* steps and the output is YES. We need to show that A β-conforms to S. As already remarked, for $q=\infty$, we have

$$Ae_i = Se_i, \qquad i = 1, 2, ..., n^*. \quad (6)$$

We need the following two lemmas to show β-conformance.

LEMMA 1. *Let* μ *be an arbitrary zero mean Gaussian measure. Let* $P_n : F \to F$ *be an orthogonal projection such that* $\dim(P_n(F)) = n \leqslant \infty$ *and let* $Q_n f = \sum_{i=1}^{n} (f, e_i)\, e_i$ *be the orthogonal projection onto the eigenvectors of the covariance operator* C_μ *which correspond to the* n *largest eigenvalues. Then for an arbitrary* $r \geqslant 0$,

$$\mu(\{ f \in F : \|P_n f\| \leqslant r \}) \geqslant \mu(\{ f \in F : \|Q_n f\| \leqslant r \}). \quad (7)$$

If, in addition, there exists a finite integer k, $k \geqslant n$, *such that*

$$P_n(F) \subset \operatorname{span}\{ e_1, e_2, ..., e_k \}, \quad (8)$$

then

$$\mu(\{ f \in F : \|P_n f\| \leqslant r \}) \leqslant \mu(\{ f \in F : \|Q_{n,k} f\| \leqslant r \}), \quad (9)$$

where

$$Q_{n,k} f = \sum_{j=1}^{n} (f, e_{k-j+1})\, e_{k-j+1}.$$

If $k \to \infty$ *then* $\mu(\{ f \in F : \|Q_{n,k} f\| \leqslant r \}) \to 1$.

Proof. Assume first that $n=\infty$. Then $Q_n f = f$ and $\|P_n f\| \leqslant \|Q_n f\|$. Hence,

$$\{f\colon \|Q_n f\| \leqslant r\} \subset \{f\colon \|P_n f\| \leqslant r\}$$

and (7) trivially holds.

Assume now that $n<\infty$. Let $g = P_n f$ and $g^* = Q_n f$. Then

$$\mu(\{f \in F : \|P_n f\| \leqslant r\}) = \mu_{P_n}(\{g \in P_n(F) : \|g\| \leqslant r\}),$$
$$\mu(\{f \in F : \|Q_n f\| \leqslant r\}) = \mu_{Q_n}(\{g^* \in Q_n(F) : \|g^*\| \leqslant r\}),$$

where μ_{P_n} and μ_{Q_n} are zero mean Gaussian measures with covariance operators

$$C_{P_n} = P_n C_\mu P_n \qquad \text{and} \quad C_{Q_n} = Q_n C_\mu Q_n.$$

By Cauchy's interlacing theorem, see for instance [8, p. 186], the eigenvalues of C_{P_n} are $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant 0$, with $\beta_i \leqslant \lambda_i$ for $i = 1, 2, ..., n$ and $\beta_i = 0$ for $i = n+1, ...$. The eigenvalues of C_{Q_n} are clearly $\lambda_1, \lambda_2, ..., \lambda_n, 0, ..., 0$. Hence, $\beta_i \leqslant \lambda_i$, $\forall i$. It is known, see for instance [9, p. 470], that for Gaussian measures $\beta_i \leqslant \lambda_i$, $\forall i$, implies that

$$\mu_{P_n}(\{g \in P_n(F) : \|g\| \leqslant r\}) \geqslant \mu_{Q_n}(\{g^* \in Q_n(F) : \|g^*\| \leqslant r\}).$$

This concludes the proof of the first part.

We now show the second part of the lemma. Let C_k be defined as the truncation of C_μ to the first k eigenvectors, i.e., $C_k : \operatorname{span}\{e_1, ..., e_k\} \to \operatorname{span}\{e_1, ..., e_k\}$ and

$$C_k e_i = \lambda_i e_i, \qquad i = 1, 2, ..., k.$$

Due to (8), we have $C_\mu P_n = C_k P_n$ and hence $P_n C_\mu P_n = P_n C_k P_n$. We can now apply Cauchy's interlacing theorem for C_k and $P_n C_\mu P_n$. The eigenvalues of C_k are

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k.$$

Denoting the eigenvalues of $P_n C_\mu P_n$ by

$$\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_n,$$

we have

$$\lambda_{k-n+j} \leqslant \beta_j.$$

Similar to the first part of the proof, since the Gaussian measure of the ball increases as the eigenvalues decrease, we conclude (9).

Note that the trace of the measure $\mu Q_{n,k}^{-1}$ equals $\sum_{j=1}^{n} \lambda_{k-j+1}$. This trace goes to zero as k goes to infinity. It is known then that the measure of the ball goes to 1 as the trace goes to zero. This concludes the second part of the proof. ∎

We now estimate the measure of the set of elements for which the norms of Sf and Af differ by at most ε.

LEMMA 2. *Suppose that*

$$Ae_i = Se_i, \qquad i = 1, 2, ..., n, \tag{10}$$

Then

$$\mu(\{f \colon \|Sf - Af\| \leqslant \varepsilon\}) \geqslant \mu\left(\left\{f \colon K^2 \sum_{i=n+1}^{n+d} (f, e_i)^2 \leqslant \varepsilon^2\right\}\right). \tag{11}$$

Proof. Let $W = S - A$, $P_n f = \sum_{i=1}^{n} (f, e_i)\, e_i$, and $P_n^{\perp} f = f - P_n f$. Clearly, (10) implies that $Wf = WP_n^{\perp} f$. As in the proof of Lemma 1 in [7], let $M = W^* W \colon F \to F$. Clearly $M = M^* \geqslant 0$, $k = \dim(MF) \leqslant d$, and $\|M\| = \|W\|^2 \leqslant K^2$. There exist orthonormal $\eta_1, \eta_2, ..., \eta_k \in F$ such that $M\eta_i = \gamma_i \eta_i$, $0 < \gamma_i \leqslant K^2$ for $i \leqslant k$, and $Mf = 0$ for $f \perp \eta_i$, $i \leqslant k$. Clearly, $\eta_i \perp e_j$ for $i = 1, 2, ..., k$ and $j = 1, 2, ..., n$. If $k < d$ we formally set $\eta_i = 0$ and $\gamma_i = 0$ for $i = k+1, ..., d$. Hence $\|Wf\|^2 = (Mf, f) = \sum_{i=1}^{d} \gamma_i (f, \eta_i)^2$. Therefore we have

$$\|Sf - Af\|^2 = \sum_{i=1}^{d} \gamma_i (f, \eta_i)^2. \tag{12}$$

Then $K^2 \sum_{i=1}^{d} (P_n^{\perp} f, \eta_i)^2 \leqslant \varepsilon^2$ implies due to (12) that $\|Wf\| = \|WP_n^{\perp} f\| \leqslant \varepsilon$. We get

$$\mu(\{f \colon \|WF\| \leqslant \varepsilon\}) \geqslant \mu\left(\left\{f \colon \sum_{i=1}^{d} (P_n^{\perp} f, \eta_i)^2 \leqslant (\varepsilon/K)^2\right\}\right). \tag{13}$$

Changing variables, $h = P_n^{\perp} f$, we get

$$\mu\left(\left\{f\colon \sum_{i=1}^{d} (P_n^{\perp} f, \eta_i)^2 \leqslant (\varepsilon/K)^2\right\}\right) = \mu_n^{\perp}\left(\left\{h : \sum_{i=1}^{d} (h, \eta_i)^2 \leqslant (\varepsilon/K)^2\right\}\right) \tag{14}$$

with a zero mean Gaussian measure $\mu_n^{\perp}$ of covariance operator $C_n^{\perp} f = C_\mu(f - \sum_{i=1}^{n} (f, e_i)\, e_i)$ whose eigenvalues are $\lambda_{n+1}, \lambda_{n+2}, \ldots$.

Since $\sum_{i=1}^{d} (f, \eta_i)\, \eta_i$ is an orthogonal projection, then we can apply the first part of Lemma 1 for the measure $\mu_n^{\perp}$ to conclude

$$\begin{aligned}\mu_n^{\perp}\left(\left\{f\colon \sum_{i=1}^{d} (f, \eta_i)^2 \leqslant (\varepsilon/K)^2\right\}\right) &\geqslant \mu_n^{\perp}\left(\left\{f\colon \sum_{i=n+1}^{n+d} (f, e_i)^2 \leqslant (\varepsilon/K)^2\right\}\right) \\ &= \mu\left(\left\{f\colon \sum_{i=n+1}^{n+d} (f, e_i)^2 \leqslant (\varepsilon/K)^2\right\}\right).\end{aligned} \tag{15}$$

Hence (13), (14), and (15) conclude the proof of Lemma 2. ∎

From Lemma 2 and the conditions on n^*, we conclude that $\mu(\{f\colon \|Sf - Af\| \leqslant \varepsilon\}) \geqslant 1 - \beta$ and, therefore, A β-conforms to S. The proof of Theorem 1 is complete. ∎

Theorem 1 establishes an upper bound $n^*(\varepsilon, \beta)$ on the number of tests. We now show that $n^*(\varepsilon, \beta)$ is also a lower bound on the number of tests no matter what testing algorithms are used.

THEOREM 2. *Suppose that $q = \infty$. Then $n^*(\varepsilon, \beta)$ given by (5) is the minimal number of tests needed for checking β-conformance. Hence the eigenvectors e_i, $i = 1, 2, \ldots, n^*(\varepsilon, \beta)$, which correspond to the first $n^*(\varepsilon, \beta)$ largest eigenvalues of the covariance operator of the measure μ, provide an optimal and universal test sequence. Furthermore, the testing algorithm* **T** *minimizes the number of tests.*

Proof. We only have to prove that $n^*(\varepsilon, \beta)$ is a lower bound. Assume that we perform $n < n^*(\varepsilon, \beta)$ tests at $f_1, f_2, \ldots, f_n$. Due to (5) we have

$$\mu\left(\left\{f\colon \sum_{i=n+1}^{n+d} (f, e_i)^2 \leqslant (\varepsilon/K)^2\right\}\right) < 1 - \beta. \tag{16}$$

This implies that $\dim(F) \geqslant n + 1$.

Suppose that $Wf_i = 0$, for $W = S - A$, $i = 1, 2, \ldots, n$. Of course, it may happen that $W \equiv 0$, which implies that the corresponding A conforms. We now show that there exists an implementation A, which does not conform but provides the same test results.

Let

$$m = \min\{d, \dim(F) - n\}.$$

Observe that $m \geqslant 1$. If $\dim(F) = d = \infty$ then we have $m = \infty$.

For $i = 1, 2, ..., m$ choose $\eta_i \in \text{span}\{e_1, e_2, ..., e_{n+i}\}$ such that η_i is orthogonal to $f_1, f_2,, f_n, \eta_1, ..., \eta_{i-1}$. We also assume that $\|\eta_i\| = 1$. Define

$$Wf = K \sum_{i=1}^{m} (f, \eta_i)\, g_i, \tag{17}$$

where $g_1, g_2, ..., g_m$ are orthonormal elements of G. Then $Wf_i = 0$, $i = 1, 2, ..., n$ and $\|W\| = K$. We have

$$\mu(\{f\colon \|Wf\| \leqslant \varepsilon\}) = \mu\left(\left\{f\colon \sum_{i=1}^{m} (f, \eta_i)^2 \leqslant (\varepsilon/K)^2\right\}\right).$$

Let $P_m f = \sum_{i=1}^{m} (f, \eta_i)\, \eta_i$. The P_m is an orthogonal projection onto $\text{span}\{\eta_1, ..., \eta_m\} \subset \text{span}\{e_1, ..., e_{n+m}\}$.

Assume for a moment that m is finite. We may now apply the second part of Lemma 1 (with $n = m$ and $k = n + m$) to conclude that

$$\begin{aligned}\mu(\{f\colon \|Wf\| \leqslant \varepsilon\}) &= \mu\left(\left\{f\colon \sum_{i=1}^{m} (f, \eta_i)^2 \leqslant (\varepsilon/K)^2\right\}\right)\\ &\leqslant \mu\left(\left\{f\colon \sum_{i=n+1}^{n+m} (f, e_i)^2 \leqslant (\varepsilon/K)^2\right\}\right).\end{aligned} \tag{18}$$

If $m = d$, (16) and (18) imply that $\mu(\{f\colon \|Wf\| \leqslant \varepsilon\}) < 1 - \beta$.

On the other hand, if $m < d$ then $m + n = \dim(F)$ and $e_i = 0$ for $i = m + n + 1, ..., n + d$. Hence,

$$\mu\left(\left\{f\colon \sum_{i=n+1}^{n+m} (f, e_i)^2 \leqslant (\varepsilon/K)^2\right\}\right) = \mu\left(\left\{f\colon \sum_{i=n+1}^{n+d} (f, e_i)^2 \leqslant (\varepsilon/K)^2\right\}\right)$$

and (16) and (18) imply that

$$\mu(\{f\colon \|Wf\| \leqslant \varepsilon\}) < 1 - \beta.$$

Assume now that m is infinite. Let $W_k = K \sum_{i=1}^{k} (f, e_i)\, g_i$ for $k = 1, 2, \ldots$. Then, from (17), $Wf = \lim_{k\to\infty} W_k f$ and therefore

$$\begin{aligned}
\mu(\{f\colon \|Wf\| \leqslant \varepsilon\}) &= \lim_{k\to\infty} \mu(\{f\colon \|W_k f\| \leqslant \varepsilon\})\\
&\leqslant \lim_{k\to\infty} \mu\left(\left\{f\colon \sum_{i=n+1}^{n+k} (f, e_i)^2 \leqslant (K/\varepsilon)^2\right\}\right)\\
&= \mu\left(\left\{f\colon \sum_{i=n+1}^{\infty} (f, e_i)^2 \leqslant (K/\varepsilon)^2\right\}\right)\\
&\leqslant \mu\left(\left\{f\colon \sum_{i=n+1}^{n+d} (f, e_i)^2 \leqslant (K/\varepsilon)^2\right\}\right) < 1-\beta.
\end{aligned}$$

The last inequality is due to (16).

Hence, in both cases, the implementation $A = S - W$ does not conform. This concludes the proof. ∎

4. BOUNDS AND EXAMPLES

Bounds on the minimal number of tests $n^*(\varepsilon, \beta)$ in (5) may be found in [9, Chap. 8]. In particular, we have

$$n^*(\varepsilon, \beta) \geqslant n_L = \min\left\{n\colon \sqrt{\lambda_{n+1}} \leqslant \frac{\varepsilon}{K\psi^{-1}(1-\beta)}\right\}, \tag{19}$$

$$n^*(\varepsilon, \beta) \leqslant n_U = \min\left\{n\colon \sqrt{\sum_{i=n+1}^{n+d} \lambda_i} \leqslant \frac{\varepsilon}{K\sqrt{2\ln(5/\beta)}}\right\}, \tag{20}$$

where ψ is the probability integral, $\psi(x) = \sqrt{2/\pi} \int_0^x \exp(-t^2/2)\, dt$. For small β, we have $\psi^{-1}(1-\beta) \approx \sqrt{2\ln(1/\beta)}$.

For $d = 1$ we have equality in (19). For an arbitrary d, we now show that

$$n^*(\varepsilon, \beta) \leqslant \min\left\{n\colon \sqrt{\lambda_{n+1}} \leqslant \frac{\varepsilon}{2K\sqrt{\ln(1/\beta)}} \left(1 - \frac{4K^2 \ln 2}{\varepsilon^2} \sum_{i=n+1}^{n+d} \lambda_i\right)_+^{1/2}\right\}. \tag{21}$$

Note that the last estimate is meaningful only if ε is fixed and β goes to zero. In this case,

$$\left(1 - \frac{4K^2 \ln 2}{\varepsilon^2} \sum_{i=n+1}^{n+d} \lambda_i\right)_+ = 1 + o(1).$$

To prove (21) we need to estimate a Gaussian measure of a ball. Let μ_n be a Gaussian measure with mean zero and eigenvalues λ_{n+i} of its covariance operator. Let $B_{\varepsilon/K}$ be a ball with center zero and radius ε/K. Then from [9, p. 258] we have

$$1-\mu_n(B_{\varepsilon/K}) \leqslant e^{-\varepsilon^2 a/K^2} \prod_{i=n+1}^{n+d} (1-2a\lambda_i)^{-1/2} \qquad \forall a < \frac{1}{2\lambda_{n+1}}.$$

For simplicity we take $a=1/(4\lambda_{n+1})$ and note that $2a\lambda_i \in [0, 1/2]$. For $x \in [0, 1/2]$ we have $\ln(1-x) \geqslant -(2\ln 2)\, x$.

We need to show that $1-\mu_n(B_{\varepsilon/K}) \leqslant \beta$. This holds if

$$\frac{\varepsilon^2}{4K^2\lambda_{n+1}} + \frac{1}{2} \sum_{i=n+1}^{n+d} \ln\left(1-\frac{\lambda_i}{2\lambda_{n+1}}\right) \geqslant \ln 1/\beta$$

or if

$$\frac{\varepsilon^2}{4K^2\lambda_{n+1}} - \frac{\ln 2}{\lambda_{n+1}} \sum_{i=n+1}^{n+d} \lambda_i \geqslant \ln 1/\beta,$$

which proves (21). We stress a weak dependence on β in the upper bound of (21).

As we shall see, the parameter ε plays a much more important role than the parameter β. If $\beta \in (0, 1/2)$ is fixed, $\lambda_i = \Theta(i^{-p_1}(\ln(i+1))^{p_2})$ for $p_1 > 1$ and p_2 arbitrary, and ε goes to zero, then we have, see [9, p. 339],

$$n^*(\varepsilon, \beta) = \min\left\{n\colon \sum_{i=n+1}^{n+d} \lambda_i \leqslant \varepsilon^2(1+o(1))/K^2\right\}.$$

The last equation implies that the minimal number of tests depends asymptotically on ε and is independent of β.

Assume now that β is small. We approximate $\psi^{-1}(1-\beta)$ by $\sqrt{2\ln(1/\beta)}$. Assume also that $\lambda_i = \Theta(i^{-p})$ for $p > 1$.

Consider first the case $d=\infty$. Then $\sum_{i=n+1}^{\infty} \lambda_i = \Theta(n^{-p+1})$ and

$$n_L = \Theta\left(\left(\frac{K^2 \ln \frac{1}{\beta}}{\varepsilon^2}\right)^{1/p}\right), \qquad n_U = \Theta\left(\left(\frac{K^2 \ln \frac{1}{\beta}}{\varepsilon^2}\right)^{1/(p-1)}\right). \tag{22}$$

For the Wiener measure we have $p=2$ and

$$\lambda_i = \frac{4}{\pi^2(2i-1)^2}.$$

In this case

$$n_L = \frac{K}{\pi\varepsilon}\left(\sqrt{2\ln\frac{1}{\beta}}\,(1+o(1))\right), \qquad n_U = \frac{2K^2}{\pi^2\varepsilon^2}\ln\frac{1}{\beta}\,(1+o(1)).$$

Consider now the case $d<\infty$. Then $\sum_{i=n+1}^{n+d}\lambda_i = \Theta(d/n^p)$ and

$$n_L = \Theta\left(\left(\frac{K^2\ln\frac{1}{\beta}}{\varepsilon^2}\right)^{1/p}\right), \qquad n_U = \Theta\left(\left(\frac{K^2 d\ln\frac{1}{\beta}}{\varepsilon^2}\right)^{1/p}\right).$$

For the Wiener measure,

$$n_L = \frac{K}{\pi\varepsilon}\sqrt{2\ln\frac{1}{\beta}}\,(1+o(1)), \qquad n_U = \frac{K\sqrt{d}}{\pi\varepsilon}\sqrt{2\ln\frac{1}{\beta}}\,(1+o(1)).$$

5. FINITE RADIUS

In this section we assume that q is finite and we test for β-conformance on a ball B_q.

We first recall some estimates of the Gaussian measure of ball B_q. It is known, see [2, 3] as well as [9, p. 258], that

$$\mu(B_q) = 1 - e^{-q^2/2\lambda_1(1+\rho(q))}, \tag{23}$$

where λ_1 is, as before, the largest eigenvalue of C_μ and $\rho=\rho_\mu$ is a function such that $\lim_{q\to\infty}\rho(q)=0$. It is also known, see [4], that $\mu(B_q)$ is continuously differentiable for all positive q. This implies continuous differentiability of ρ for positive q, and since $\mu(B_q)$ is increasing in q we have

$$0 < \mu'(B_q) = \frac{q}{\lambda_1}\,e^{-q^2/2\lambda_1(1+\rho(q))}\left(1+\rho(q)+\frac{q\rho'(q)}{2}\right).$$

In particular, this means that

$$1+\rho(q)+q\rho'(q)/2 > 0, \qquad \forall q>0. \tag{24}$$

Let $\psi_q : [0, 1] \to [0, 1]$ be defined by

$$\psi_q(\eta) = \frac{\mu(B_{q(1-\eta)})}{\mu(B_q)}.$$

Then (23) yields

$$\psi_q(\eta) = 1 - \eta \, \frac{q^2(1+\rho(q)+q\rho'(q)/2+o(1))}{\lambda_1} \, \frac{e^{-q^2/2\lambda_1(1+\rho(q))}}{1-e^{-q^2/2\lambda_1(1+\rho(q))}} \quad \text{as} \quad \eta \to 0. \tag{25}$$

Obviously we also have

$$\lim_{q \to \infty} \psi_q(\eta) = 1, \qquad \forall \eta \in (0, 1). \tag{26}$$

In what follows, we need to define $\eta = \eta(\beta, q) \in [0, 1)$ as the unique solution of

$$\mu(B_{q(1-\eta)}) = (1-(1-\eta)\,\beta)\,\mu(B_q) \qquad \text{or, equivalently,} \quad \psi_q(\eta) = 1 - \beta + \beta\eta. \tag{27}$$

Observe that such an η exists. Indeed, the left-hand side of (27) is a monotonically decreasing continuous function such that for $\eta = 0$ we have $\mu(B_q)$ and for $\eta = 1$ we have 0. The right-hand side of (27) is a linear increasing function from $(1-\beta)\,\mu(B_q)$ to $\mu(B_q)$. Hence there exists a unique η at which the graphs of the two functions intersect. Furthermore, (25) yields

$$\begin{aligned}\eta &= \eta(\beta, q)\\ &= \beta\lambda_1 q^{-2} \frac{e^{q^2/2\lambda_1(1+\rho(q))}}{1+\rho(q)+q\rho'(q)/2} (1-e^{-q^2(1+\rho(q))/2\lambda_1})(1+o(1)) = \Theta(\beta)\\ &\qquad \text{as} \quad \beta \to 0.\end{aligned} \tag{28}$$

Note that η is well defined due to (24).

On the other hand, for a fixed β and q tending to infinity, it is easy to check that $\eta(\beta, q)$ goes to 1.

We are ready to define n^* for a finite q such that the testing algorithm **T** is correct.

THEOREM 3. *The testing algorithm* **T** *checks* β*-conformance of implementation* A *provided that*

$$n^* \geqslant \min\{n_1^*(\varepsilon, \beta, q), n_2^*(\varepsilon, \beta, q)\}, \tag{29}$$

where

$$n_1^*(\varepsilon, \beta, q) = \min \left\{ n \colon \mu \left(\left\{ f \colon \sum_{i=n+1}^{n+d} (f, e_i)^2 \right.\right.\right.$$
$$\left.\left.\left. \leqslant \frac{\varepsilon^2 \eta^2(\beta, q)}{K^2} \right\} \right) \geqslant 1 - \eta(\beta, q)\, \beta \mu(B_q) \right\},$$

$$n_2^*(\varepsilon, \beta, q) = \min \left\{ n \colon \mu \left(\left\{ f \colon \sum_{i=n+1}^{n+d} (f, e_i)^2 \right.\right.\right.$$
$$\left.\left.\left. \leqslant \frac{\varepsilon^2 \eta(\beta, q/\sqrt{2})}{2K^2} \right\} \right) \geqslant 1 - \eta(\beta, q/\sqrt{2})\, \beta \mu(B_q) \right\},$$

and $\eta(\beta, q)$ *and* $\eta(\beta, q/\sqrt{2})$ *are the corresponding solutions of* (27).

Observe that for $q = \infty$ we can take $\eta(\beta, \infty) = 1$, and the minimum in (29) is attained for $n_1^*(\varepsilon, \beta, \infty) = n^*(\varepsilon, \beta)$, the latter given by (5). For a fixed large q and small β, we have $\eta(\beta, q) = \Theta(\beta)$ due to (28), and the minimum in (29) is attained for $n_2(\varepsilon, q, \beta)$.

Proof. As before we only need to consider $\delta_i \leqslant \varepsilon$ in the testing algorithm **T**, for $i = 1, 2, ..., n^*$; otherwise, A is faulty. Hence the testing algorithm **T** terminates with YES and we need to show that A β-conforms to S. We first estimate

$$a := \mu(\{f \colon \|f\| \leqslant q \text{ and } \|Wf\| \leqslant \varepsilon\}) \tag{30}$$

where $W = S - A$. Obviously,

$$Wf = WP_n f + WP_n^{\perp} f,$$

where $n = n^*$ and

$$P_n f = \sum_{i=1}^{n} (f, e_i)\, e_i.$$

Since $\delta_n \leqslant \varepsilon$, we have

$$\| W|_{\mathrm{span}(e_1, \ldots, e_n)} \| = \delta_n / q \leqslant \varepsilon / q.$$

Then

$$\| W P_n f \| \leqslant \frac{\varepsilon}{q} \sqrt{\sum_{i=1}^{n} (f, e_i)^2}.$$

As in the proof of Lemma 2, we conclude

$$\| W f \| = \sqrt{\sum_{i=1}^{d} \gamma_i (f, \eta_i)^2},$$

where η_i's are orthonormal. Then

$$\| W P_n^{\perp} f \| \leqslant K \sqrt{\sum_{i=1}^{d} (P_n^{\perp} f, \eta_i)^2}.$$

Hence we have

$$\begin{aligned} \| W f \| &\leqslant \frac{\varepsilon}{q} \sqrt{\sum_{i=1}^{n} (f, e_i)^2} + K \sqrt{\sum_{i=1}^{d} (p_n^{\perp} f, \eta_i)^2} \\ &\leqslant \sqrt{2} \left(\frac{\varepsilon^2}{q^2} \sum_{i=1}^{n} (f, e_i)^2 + K^2 \sum_{i=1}^{d} (P_n^{\perp} f, \eta_i)^2 \right)^{1/2}. \end{aligned}$$

Thus

$$a \geqslant \mu(D_1) \geqslant \mu(D_2),$$

where

$$D_1 = \left\{ f \colon \| f \| \leqslant q, \frac{\varepsilon}{q} \sqrt{\sum_{i=1}^{n} (f, e_i)^2} + K \sqrt{\sum_{i=1}^{d} (P_n^{\perp} f, \eta_i)^2} \leqslant \varepsilon \right\},$$

$$D_2 = \left\{ f \colon \| f \| \leqslant q, \sqrt{2} \left(\frac{\varepsilon^2}{q^2} \sum_{i=1}^{n} (f, e_i)^2 + K^2 \sum_{i=1}^{d} (P_n^{\perp} f, \eta_i)^2 \right)^{1/2} \leqslant \varepsilon \right\}.$$

Define

$$C_1 = B_{q(1-\eta(\beta,q))} = \left\{ f\colon \sqrt{\sum_{i=1}^{\infty} (f, e_i)^2} \leqslant q(1-\eta(\beta, q)) \right\},$$

$$C_2 = B_{q\sqrt{(1-\eta(\beta, q/\sqrt{2}))/2}} = \left\{ f\colon \sqrt{\sum_{i=1}^{\infty} (f, e_i)^2} \leqslant q\sqrt{(1-\eta(\beta, q/\sqrt{2}))/2} \right\},$$

$$A_1 = \left\{ f\colon \sqrt{\sum_{i=1}^{d} (P_n^{\perp} f, \eta_i)^2} \leqslant \frac{\varepsilon}{K} \eta(\beta, q) \right\},$$

$$A_2 = \left\{ f\colon \sqrt{\sum_{i=1}^{d} (P_n^{\perp} f, \eta_i)^2} \leqslant \frac{\varepsilon}{\sqrt{2}\,K} \sqrt{\eta(\beta, q/\sqrt{2})} \right\}.$$

It is easy to check that for $j=1, 2$ we have $A_j \cap C_j \subset D_j$, and therefore

$$\mu(D_j) \geqslant \mu(A_j \cap C_j) = \mu(A_j) + \mu(C_j) - \mu(A_j \cup C_j) \geqslant \mu(A_j) + \mu(c_j) - 1.$$

We now estimate $\mu(A_j)$. Let $r_1 = \varepsilon\eta(\beta, q)/K$ and $r_2 = \varepsilon\sqrt{\eta(\beta, q/\sqrt{2})}/(\sqrt{2}\,K)$. Then

$$\mu\left(\left\{ f\colon \sum_{i=1}^{d} (P_n^{\perp} f, \eta_i)^2 \leqslant r_j^2 \right\}\right) = \mu_n^{\perp}\left(\left\{ h\colon \sum_{i=1}^{d} (h, \eta_i)^2 \leqslant r_j^2 \right\}\right),$$

where $\mu_n^{\perp}$ is as in the proof of Lemma 2. Applying Lemma 1 with $n=d$, we have

$$\begin{aligned}\mu_n^{\perp}\left(\left\{ h\colon \sum_{i=1}^{d} (h, \eta_i)^2 \leqslant r_j^2 \right\}\right) &\geqslant \mu_n^{\perp}\left(\left\{ f\colon \sum_{i=1}^{d} (f, e_{i+n})^2 \leqslant r_j^2 \right\}\right)\\ &= \mu\left(\left\{ f\colon \sum_{i=n+1}^{n+d} (f, e_i)^2 \leqslant r_j^2 \right\}\right).\end{aligned}$$

Due to (29), we have

$$\mu(A_1) \geqslant 1 - \eta(\beta, q)\,\beta\mu(B_q) \qquad \text{and} \qquad \mu(A_2) \geqslant 1 - \eta(\beta, q/\sqrt{2})\,\beta\mu(B_q).$$

Due to (27), we have

$$\mu(C_1) = \mu(B_{q(1-\eta(\beta,q))}) = \mu(B_q) - (1-\eta(\beta, q))\,\beta\mu(B_q),$$

$$\mu(C_2) = \mu(B_{q(1-\eta(\beta, q/\sqrt{2}))}) = \mu(B_q) - (1-\eta(\beta, q/\sqrt{2}))\,\beta\mu(B_q)$$

and therefore

$$a \geqslant \mu(D_2) \geqslant (1-\beta)\, \mu(B_q).$$

This means that A β-conforms to S and completes the proof. ∎

Let $n^*(\varepsilon, \beta, q)$ be the minimal number of tests needed for checking β-conformance. Recall that $n^*(\varepsilon, \beta) = n^*(\varepsilon, \beta, \infty)$ is given by (5). We now present bounds on $n^*(\varepsilon, \beta, q)$ in terms of $n^*(\varepsilon, \beta)$.

LEMMA 3. *Let* $x = 1 - \mu(B_q) = e^{-q^2(i+\rho(q))/(2\lambda_1)}$, *and let* $y_1 = \eta(\beta, q)$ *and* $y_2 = \eta(0, q/\sqrt{2})$ *be the solutions of* (27). *Then*

$$n^*(\varepsilon, \beta(1-x)+x) \leqslant n^*(\varepsilon, \beta, q)$$
$$\leqslant \min(n^*(\varepsilon y_1, \beta y_1 (1-x)), n^*(\varepsilon \sqrt{y_2/2}, \beta y_2(1-x)).$$

Proof. The upper bound on $n^*(\varepsilon, \beta, q)$ follows from Theorem 3. To prove the lower bound let $n = n^*(\varepsilon, \beta, q)$. Then n tests at, say, $f_1, f_2, ..., f_n$ are enough for β-conformance for a finite q. Let $W = S - A$. If A β-conforms to S then $\| Wf_i \| \leqslant \varepsilon$ for all $i \leqslant n$ and

$$\mu(\{ f \in B_q : \| Wf \| \leqslant \varepsilon \}) \geqslant (1-\beta)\, \mu(B_q) = 1 - \beta(1-x) - x.$$

It is easy to see that the same tests can be used for $q = \infty$. Indeed, if $Wf_i \neq 0$ for some $i \leqslant n$ then A is faulty, and if $Wf_i = 0$ for all $i \leqslant n$ then

$$\mu(\{ f \colon \| Wf \| \leqslant \varepsilon \}) \geqslant \mu(\{ f \in B_q : \| Wf \| \leqslant \varepsilon \}) \geqslant 1 - \beta(1-x) - x.$$

This means that n tests are enough for $(\beta(1-x)+x)$-conformance with $q = \infty$. Therefore $n \geqslant n^*(\varepsilon, \beta(1-x)+x)$, as claimed. ∎

Observe that for a fixed β and q tending to infinity we have $\beta(1-x)+x \approx \beta$ and $\eta(\beta, q) \approx 1$. In this case the bounds of Lemma 3 are sharp and we have

$$n^*(\varepsilon, \beta, q) \approx n^*(\varepsilon, \beta).$$

This means that in this case the minimal number of tests is more or less the same as for $q = \infty$.

On the other hand, if q is fixed and β goes to zero then we only have

$$n^*(\varepsilon, x + o(1)) \leqslant n^*(\varepsilon, \beta, q) \leqslant n^*(\Theta(\varepsilon \beta^{1/2}), \Theta(\beta^2)).$$

Observe that the upper bound is also sufficient for the number of steps of the testing algorithm **T** to check β-conformance. The lower bound on $n^*(\varepsilon, \beta, q)$ is now poor since it does not even go to infinity as β goes to zero. As we shall see, in this case the upper bound is essentially sharp for $d=\infty$, i.e., when the dimension of the range space is infinite. This means that the behavior of $n^*(\varepsilon, \beta, q)$ for small β and a fixed q is quite different than the behavior of $n^*(\varepsilon, \beta)$: The parameter β for a finite q plays a much more significant role than for $q=\infty$. The case of a finite d is left open. We suspect that for a finite d, the dependence of $n^*(\varepsilon, \beta, q)$ on β is not so crucial.

THEOREM 4. *For $d=\infty$ and a finite large q there exist positive numbers c_i, with $c_2<1$, such that*

$$n^*(c_1 \varepsilon \beta^{1/2}, c_2) \leqslant n^*(\varepsilon, \beta, q) \leqslant n^*(c_3 \varepsilon \beta^{1/2}, c_4 \beta^2) \qquad \text{for} \quad \varepsilon+\beta \leqslant c_5.$$

Since the dependence of $n^*(\varepsilon, \beta)$ is much more crucial on the first argument, this means that the testing algorithm **T** *almost minimizes the number of tests.*

Proof. We only need to prove a lower bound on $n^*(\varepsilon, \beta, q)$. We may assume that $\dim(F)=\infty$ since otherwise for any positive c_i with $i \leqslant 4$, $c_2<1$, we have

$$n^*(c_1 \varepsilon \beta^{1/2}, c_2)=n^*(\varepsilon, \beta, q)=n^*(c_3 \varepsilon \beta^{1/2}, c_4 \beta^2)=\dim(F),$$

for small $\varepsilon+\beta$.

Suppose then that we test at orthonormal $f_1, f_2, ..., f_n$ and $n=n(\varepsilon, \beta)$ is chosen such that we can verify β-conformance. For $i>n$, let f_i be as in the proof of Theorem 2. That is, f_i belongs to $\operatorname{span}\{e_1, ..., e_{n+i}\}$ and $\{f_i\}$ is an orthonormal sequence. Let

$$P_n f=\sum_{i=1}^{n} (f, f_i)\, g_i \qquad \text{and} \qquad P_n^{\perp} f=\sum_{i=n+1}^{\infty} (f, f_i)\, g_i$$

for an orthonormal sequence of g_i from the space G. Since $\dim(G)=\infty$ such a sequence exists. Define $W=S-A$ by

$$Wf=\varepsilon q^{-1} P_n f+aKP_n^{\perp} f,$$

where $a=0$ or $a=1$. In either case of a we obtain the same tests Wf_i for all $i \leqslant n$. For $a=0$, we have $\mu(\{f\colon \|Wf\| \leqslant \varepsilon\})=1$ and that A conforms to S. Hence, also for $a=1$ we must have β-conformance. This means that

$$\gamma:=\mu(\{f\colon \|f\| \leqslant q \text{ and } \|Wf\| \leqslant \varepsilon)\} \geqslant (1-\beta)\, \mu(B_q).$$

Since

$$\|Wf\|^2 = \varepsilon^2 q^{-2} \|P_n f\|^2 + K^2 \|P_n^{\perp} f\|^2,$$

with $\|P_n f\|^2 = \sum_{i=1}^{n} (f, f_i)^2$ and $\|P_n^{\perp} f\|^2 = \sum_{i=n+1}^{\infty} (f, f_i)^2$ we have

$$\gamma = \int_{\|P_n f\| \leqslant q} \mu_{P_n^{\perp}}(\{f\colon \|P_n^{\perp} f\| \leqslant \varepsilon K^{-1} (1 - \|P_n f\|^2/q^2)^{1/2}\})\, \mu_{P_n}(df),$$

where the zero Gaussian measures μ_{P_n} and $\mu_{P_n^{\perp}}$ have the covariance operators $C_{P_n} = P_n C_\mu P_n^*$ and $C_{P_n^{\perp}} = P_n^{\perp} C_\mu (P_n^{\perp})^*$. Here P_n^* and $(P_n^{\perp})^*$ act from G to F and are given by

$$P_n^* g = \sum_{i=1}^{n} (g_i, g)\, f_i \qquad \text{and} \qquad (P_n^{\perp})^* g = \sum_{i=n+1}^{\infty} (g_i, g)\, f_i.$$

Take now $\alpha = \alpha_1 \beta$ for some positive α_1. Let $\|P_n f\| \leqslant q\sqrt{1-\alpha}$. Since $\varepsilon < Kq$, see (4),

$$\|P_n^{\perp} f\| \leqslant \varepsilon K^{-1} (1 - \|P_n f\|^2/q^2)^{1/2}$$

implies that

$$\|P_n^{\perp} f\| \leqslant (q^2(1-\eta) - \|P_n f\|^2)^{1/2}, \qquad \eta = \alpha(1 - (\varepsilon/(Kq))^2).$$

Hence

$$\begin{aligned}
&\int_{\|P_n f\| \leqslant q\sqrt{1-\alpha}} \mu_{P_n^{\perp}}(\{f\colon \|P_n^{\perp} f\| \leqslant \varepsilon K^{-1} (1 - \|P_n f\|^2/q^2)^{1/2}\})\, \mu_{P_n}(df) \\
&\qquad \leqslant \int_{\|P_n f\| \leqslant q\sqrt{1-\alpha}} \mu_{P_n^{\perp}}(\{f\colon \|P_n^{\perp} f\| \leqslant (q^2(1-\eta) - \|P_n f\|^2)^{1/2}\})\, \mu_{P_n}(df) \\
&\qquad \leqslant \mu(B_{q\sqrt{1-\eta}}) = \mu(B_{q\sqrt{1-\alpha}})(1 + o(1)) \qquad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
\gamma &= \int_{\|P_n f\| \leqslant q\sqrt{1-\alpha}} \cdots + \int_{q\sqrt{1-\alpha} < \|P_n f\| \leqslant q} \cdots \\
&\leqslant \mu(B_{q\sqrt{1-\alpha}})(1 + o(1)) \\
&\qquad + \mu_{P_n^{\perp}}(\{\|P_n^{\perp} f\| \leqslant \varepsilon K^{-1} \sqrt{\alpha_1 \beta}\})(\mu_{P_n}(B_q) - \mu_{P_n}(B_{q\sqrt{1-\alpha}})).
\end{aligned}$$

Let

$$\alpha_1 = 2\lambda_1 q^{-2}(1+\rho(q)+q\rho'(q)/2)^{-1}\exp(q^2/(2\lambda_1))(1+o(1)) \qquad \text{as} \quad q \to \infty.$$

Note that α_1 is well defined due to (24). Then (25) implies that for small ε and β the first term $\mu(B_{q\sqrt{1-\alpha}})(1+o(1))$ is at most, say, $(1-2\beta)\,\mu(B_q)$. Therefore the second term must be at least $\beta\mu(B_q)$.

We now show that there exists a positive c_6 such that

$$\mu_{P_n}(B_q) - \mu_{P_n}(B_{q\sqrt{1-\alpha}}) \leqslant c_6 \alpha_1 \beta. \tag{31}$$

Let β_j, $j=1, 2, ..., n$, denote the eigenvalues of C_{P_n}. Then

$$\mu_{P_n}(B_q) - \mu_{P_n}(B_{q\sqrt{1-\alpha}}) = \mu_n^* \left(\frac{q^2(1-\alpha)}{\sum_{j=1}^n \beta_j} \leqslant \sum_{i=1}^n \frac{\beta_i}{\sum_{j=1}^n} \psi_i^2 \leqslant \frac{q^2}{\sum_{j=1}^n \beta_j} \right),$$

with independent standard Gaussian variables ψ_i and μ_n^* being the standard Gaussian mean sure on R^n, i.e., with mean zero and the identity covariance matrix.

We now show that $\beta_j \leqslant \lambda_j$. Indeed, let $Jf = \sum_{i=1}^\infty (f, f_i)\, g_i$ denote the embedding operator between the spaces F and G, and let $\tilde{P}_n f = \sum_{i=1}^\infty (f, f_i)\, f_i$ be an orthogonal projection in F. Clearly, we have $J^* = J^{-1}$ and $P_n = J\tilde{P}_n$. Therefore

$$C_{P_n} = J\tilde{P}_n C_\mu \tilde{P}_n J^{-1}$$

and the eigenvalues of C_{P_n} and $\tilde{P}_n C_\mu \tilde{P}_n$ are the same. Hence β_j are also the eigenvalues of $\tilde{P}_n C_\mu \tilde{P}_n$. For the operator $\tilde{P}_n C_\mu \tilde{P}_n$ we conclude as in the proof of Lemma 1 that $\beta_j \leqslant \lambda_j$, as claimed.

Assume that β is so small that $\alpha = \alpha_1 \beta \leqslant 1/2$. Then

$$\frac{q^2(1-\alpha)}{\sum_{j=1}^n \beta_j} \geqslant \frac{q^2}{2\sum_{j=1}^\infty \lambda_j}.$$

We assume that q is so large that $q^2 \geqslant 2(2+\sqrt{2}) \sum_{j=1}^\infty \lambda_j$. For such q we may apply Theorem 3 from [1] which states that

$$\mu_{P_n}(B_q) - \mu_{P_n}(B_{q\sqrt{1-\alpha}}) \leqslant \sqrt{\frac{2}{\pi}} \int_{y\sqrt{1-\alpha}}^{y} \exp(-x^2/2)\, dx$$

with $y=q/\sqrt{\sum_{j=1}^{n}\beta_j}$. We estimate the last integral by

$$\sqrt{\frac{2}{\pi}}\exp(-y^2(1-\alpha)/2)\ y(1-\sqrt{1-\alpha})\leqslant\sqrt{\frac{2}{\pi}}\exp(-y^2/4)\ y\frac{\alpha}{1+\sqrt{1-\alpha}}$$
$$\leqslant c_6\alpha_1\beta,$$

with

$$c_6=\max_{y\geqslant 0}\sqrt{\frac{2}{\pi}}\exp(-y^2/4)\ y\frac{1}{1+\sqrt{2^{-1}}}<\infty.$$

This proves (31).

We already noted that

$$\mu_{P_n^\perp}(\{\|P_n^\perp f\|\leqslant \varepsilon K^{-1}\sqrt{\alpha_1\beta}\})(\mu_{P_n}(B_q)-\mu_{P_n}(B_{q\sqrt{1-\alpha}}))\geqslant \beta\mu(B_q).$$

By (31) there exists $c_2\in(0,1)$ such that

$$\mu_{P_n^\perp}(\{\|P_n^\perp f\|\leqslant \varepsilon K^{-1}\sqrt{\alpha_1\beta}\})\geqslant c_2.$$

From Section 3 we know that

$$\mu\left(\left\{f\colon \sum_{i=n+1}^{\infty}(f,e_i)^2\leqslant \varepsilon K^{-1}\sqrt{\alpha_1\beta}\right\}\right)\geqslant \mu_{P_n^\perp}(\{\|P_n^\perp f\|\leqslant \varepsilon K^{-1}\sqrt{\alpha_1\beta}\}).$$

Hence

$$\mu\left(\left\{f\colon \sum_{i=n+1}^{\infty}(f,e_i)^2\leqslant \varepsilon K^{-1}\sqrt{\alpha_1\beta}\right\}\right)\geqslant c_2,$$

which means that $n\geqslant n^*(c_1\varepsilon\beta^{1/2},c_2)$ for some positive c_1. This completes the proof. ∎

6. CONCLUSION

We have studied a probabilistic setting of β-conformance testing of linear operators and described a simple test generation algorithm that tests on the eigenvectors of the covariance operator of the Gaussian measure on the input set. When the input set is the whole domain space we showed that the test sequence is optimal; the number of tests matches the lower bound.

On the other hand, when the input set is a ball of finite radius in the domain space, we obtained the lower and upper bounds on the number of tests. Partly due to the lack of precise estimation of the measure on the ball, we are not able to find the exact bound on the number of tests. However, we know that the bounds are essentially sharp in two cases. The first case is for a fixed confidence parameter β and large q. The second case is for a fixed q and $d = \infty$ when the range space is infinite dimensional.

Similar to the worst and average case settings [5, 7], our test sequences are universal: they are the eigenvectors of the covariance operator and hence are independent of the particular specification and implementation operators.

We have discussed testing of linear operators. Nonlinear operators are often encountered in practice and their testing is also of vital importance. In an earlier paper [6] we studied relaxed testing of nonlinear operators. We showed that finite testing is in general inconclusive. However, testing is decidable in the limit and finite tests are conclusive for weak conformance testing. On the other hand, the average and probabilistic settings for nonlinear operator testing remain to be explored.

The concept of β-conformance has been brought up in [6]. However, due to the lack of structure of nonlinear operators, deterministic test generation algorithms seem to be hard to design. Probabilistic algorithms were proposed based on random samplings in the input set according to its distribution. It is essentially a Bernoulli trial for faulty elements in the input set. Consequently, any positive conclusion of β-conformance is associated with a probability of an erroneous answer.

ACKNOWLEDGMENTS

We are grateful for valuable comments from Stefan Heinrich who carefully checked the paper and convinced us that the use of the o and O notations may sometimes be confusing. We also thank Stanisław Kwapień for pointing out to us the paper of N. K. Bakirov.

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