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Multivariate aging properties of epoch times of nonhomogeneous processes

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Abstract

The purpose of this paper is to give conditions on the parameters of nonhomogeneous Poisson and nonhomogeneous pure birth processes, under which the corresponding random vector of the first n epoch times has some multivariate stochastic properties. These results provide an inside to understand the effect of the time over the occurrence of events in such processes. Some applications of these results are given.

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1. Introduction

Nonhomogeneous processes are of great interest in applied probability. Epoch times of nonhomogeneous Poisson processes correspond to the times of repair of a unit which is being minimally repaired (see [3]), where by minimal repair we mean that the unit is restored to a working condition just prior to the failure. Also epoch times of nonhomogeneous Poisson processes are the consecutive record values of a sequence of independent and identically distributed nonnegative random variables (see, for example, [15]).

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Nonhomogeneous Poisson processes can be extended to nonhomogeneous pure birth processes. Applications of this general model in insurance, reliability theory, epidemiology and load-sharing models can be seen in [6,17,26].

Therefore, it is of interest to study stochastic properties of epoch times of nonhomogeneous processes. In the literature there are several papers which study aging notions of epoch times under conditions on the parameters of the nonhomogeneous process. Gupta and Kirmani [15] give conditions under which the epoch times of a nonhomogeneous Poisson processes are IFR, IFRA and NBU and Pellerey et al. [26] give conditions for the logconcavity of the density function of epoch times. Other results about stochastic comparisons of epoch times have been given, recently, by Belzunce et al. [5], Belzunce et al. [6], Belzunce and Ruiz [8] and Belzunce and Shaked [10]. However when we consider a single nonhomogeneous process we are more interested about the occurrence of events when time is going on. In this paper we study stochastic properties of the random vector of the first n epoch times, which describe the effect of time on the occurrence of events, also some other consequences are described through the paper.

The organization of this paper is as follows. In Section 2 we will give the definitions and some properties of the multivariate stochastic notions that we will study through the paper. Also we will introduce nonhomogeneous Poisson and nonhomogeneous pure birth processes. Later in Section 3 we will give conditions on the parameters of nonhomogeneous Poisson and pure birth processes under which the random vector of the first n epoch times has some of the multivariate stochastic properties given in Section 2. Examples of processes which satisfy these conditions are given. To finish in Section 4 we consider some applications of these results.

In this paper "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing," respectively. The notation $=_{\rm st}$ stands for equality in law. Also given a random variable X with distribution function F, $\bar{F} \equiv 1 - F$ denotes the survival function.

2. Preliminaries on multivariate classification and nonhomogeneous processes

In reliability theory it has been found useful to classify univariate distributions according to the process of aging. These univariate classifications have been extended to a multivariate setting in several ways. Some of these extensions can be seen in [11,18,23,27–29]. Although these extensions are of mathematical interest, not all are based on physical considerations. Arjas [1] and Shaked and Shanthikumar [33] provide new extensions, not just technical but from a dynamic point of view. Next we introduce the notation needed to give the multivariate aging notions given by Arjas [1] and Shaked and Shanthikumar [33]. In what follows, the vector of ones will be denoted by $\mathbf{e}$, and the dimension of $\mathbf{e}$ is always possible to determine from the expression in which appears. Also given two vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$, we denote $\mathbf{x} \leqslant \mathbf{y}$, if $x_i \leqslant y_i$ for $i = 1, \dots, n$.

Let $\mathbf{T} = (T_1, \dots, T_n)$ be a nonnegative random vector with an absolutely continuous distribution function. The coordinates, T_i, can be considered as lifetimes of n devices or times of failure of n processes. In this paper these will be the times in which a series of n events occurs. Through the paper we will use indistinctly these two interpretations. For $t \geqslant 0$ let h_t denotes the list of devices (events) which have failed (occurred) and their failure (occurrence) times. More explicitly, a history h_t will denote

$$h_t = \{\mathbf{T}_I = \mathbf{t}_I, \mathbf{T}_{\bar{I}} > t\mathbf{e}\},$$

where $I = \{i_1, \dots, i_k\}$ is a subset of $\{1, \dots, n\}$, $\bar{I}$ is its complement with respect to $\{1, \dots, n\}$, $\mathbf{T}_I$ will denote the vector formed by the components of $\mathbf{T}$ with index in I and $0 < \mathbf{t}_{i_j} < t$ for all $j = 1, \dots, k$. For every vector $\mathbf{x} = (x_1, \dots, x_n)$ denote by $\mathbf{x}^+$ the vector $\mathbf{x}^+ \equiv (x_1^+, \dots, x_n^+)$, where $x^+ = 0$ if $x < 0$ and $x^+ = x$ if $x \geqslant 0$. Now given a history h_t, we consider the random vector, $[(\mathbf{T} - t\mathbf{e})^+ | h_t]$, of residual lifetimes of the components of $\mathbf{T}$ given h_t, where, for any event A the notation $[X|A]$ stands for any random variable whose distribution is the conditional distribution of X given A. This concept extends to a multivariate setting the concept of residual lifetime of a random variable T given by $[T - t | T > t]$.

Next, we recall the definitions of some multivariate partial orders, which will be used to state multivariate aging notions (see [34]).

Definition 2.1. Let $\mathbf{T}$ and $\mathbf{S}$ be two n-dimensional random vectors with density functions $f_{\mathbf{T}}$ and $f_{\mathbf{S}}$, respectively. We say that $\mathbf{T}$ is less than $\mathbf{S}$ in the multivariate likelihood ratio order (denoted by $\mathbf{T} \leqslant_{\mathrm{lr}} \mathbf{S}$), if (below $\wedge$ and $\vee$ denote, respectively, the minimum and the maximum operations)

$$f_{\mathbf{T}}(x_1, x_2, \dots, x_n) f_{\mathbf{S}}(y_1, y_2, \dots, y_n) \leqslant f_{\mathbf{T}}(x_1 \wedge y_1, x_2 \wedge y_2, \dots, x_n \wedge y_n) f_{\mathbf{S}}(x_1 \vee y_1, x_2 \vee y_2, \dots, x_n \vee y_n)$$

for all $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$.

Now we proceed to give the definition of the multivariate hazard rate order. Given the history h_t, as above, let $i \in \bar{I}$, its multivariate conditional hazard rate, at time t, is defined as follows:

$$\lambda_i(t|\boldsymbol{t}_I) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} P[t < T_i \leqslant t + \Delta t | h_t].$$

It is clear that $\lambda_i(t|\boldsymbol{t}_I)$ is the probability of instant failure of component i, given the history h_t.

Definition 2.2. Let $\mathbf{T}$ and $\mathbf{S}$ be two n-dimensional random vectors with hazard rate functions $\eta_\cdot(\cdot|\cdot)$ and $\lambda_\cdot(\cdot|\cdot)$, respectively. We say that $\mathbf{T}$ is less than $\mathbf{S}$ in the multivariate hazard rate order (denoted by $\mathbf{T} \leqslant_{\mathrm{hr}} \mathbf{S}$), if

$$\eta_i(u|\mathbf{s}_{I \cup J}) \geqslant \lambda_i(u|\mathbf{t}_I)$$

whenever $I \cap J = \emptyset$, $\mathbf{0} \leqslant \mathbf{s}_I \leqslant \mathbf{t}_I \leqslant u\mathbf{e}$, and $\mathbf{0} \leqslant \mathbf{s}_J \leqslant u\mathbf{e}$, where $i \in \overline{I \cup J}$.

In the univariate case given two nonnegative random variables X and Y with distribution functions F and G, respectively, then X is said to be less than Y in the hazard rate order (denoted by $X \leqslant_{\rm hr} Y$ or $F \leqslant_{\rm hr} G$) if $\bar{F}(t)\bar{G}(s) \leqslant \bar{F}(s)\bar{G}(t)$ for all $s<t$.

As can be seen in [34] the multivariate likelihood ratio order is stronger than the multivariate hazard rate order, that is,

$$\mathbf{T} \leqslant_{\rm lr} \mathbf{S} \Rightarrow \mathbf{T} \leqslant_{\rm hr} \mathbf{S}. \tag{2.1}$$

Now we proceed to recall the definition of the multivariate aging classes that will be studied in this paper. The definition of these multivariate notions, is also motivated by some characterizations of the corresponding notions in the univariate case (see [34]).

First, we give the multivariate extension of univariate PF_2 distributions given by Shaked and Shanthikumar [33].

Definition 2.3. Let $\mathbf{T}$ be a nonnegative random vector. We say that $\mathbf{T}$ is multivariate Polya frequency of order 2 (denoted by $\mathbf{T}$ is MPF_2) if

$$[(\mathbf{T} - t\mathbf{e})^+ | h_t] \leqslant_{\rm lr} \mathbf{T}$$

for any history h_t, $t \geqslant 0$.

In the univariate case, a random variable T, with support $(0, +\infty)$, is MPF_2 (denoted by T is PF_2), if

$$\frac{f(x+y)}{f(y)} \text{ is decreasing in } y, \text{ for any } x \geqslant 0,$$

where f is the density of T. Clearly, from Definition 2.3, if $\mathbf{T}$ is MPF_2 then $\mathbf{T} \leqslant_{\rm lr} \mathbf{T}$, that is, $\mathbf{T}$ is multivariate totally positive of order 2 (MTP_2) (see [20]). Some interesting consequences of MTP_2 property are that $T_1, \ldots, T_n$ are associated in the sense of Esary et al. [14] and the conditional monotone regression endowment by Lehman [22].

Next, we give the multivariate extension of increasing failure rate (IFR) distributions given by Shaked and Shanthikumar [33].

Definition 2.4. Let $\mathbf{T}$ be a nonnegative random vector. We say that $\mathbf{T}$ is multivariate increasing failure rate (denoted by $\mathbf{T}$ is *MIFR*) if

$$[(\mathbf{T} - t\mathbf{e})^+ | h_t] \leqslant_{\rm hr} \mathbf{T} \tag{2.2}$$

for any history h_t, $t \geqslant 0$.

Condition (2.2) can be written in a different way under the following notation. For $s<t$ let $h_{[s,t]}$ denote an event which describes which components are alive at time s, and the components which failed during the time interval $[s,t]$ and their failure times. Therefore h_t can be written as $h_{[0,t]}$. Now for $s<t$ and $u \geqslant -s$ denote by $\theta_u h_{[s,t]}$ the history on the time interval $[s+u, t+u]$ which is obtained from $h_{[s,t]}$ by adding u to

each failure time in $h_{[s,t]}$, but without changing the identities of the failed components and without adding or deleting any failure.

We will need also to compare the 'severity' of two histories of the same length over different time intervals. First given two histories $h_{[s,t]}$ and $h'_{[s,t]}$ we will write $h_{[s,t]} \leqslant h'_{[s,t]}$ if every component which is alive at time s in $h'_{[s,t]}$ is also alive at time s in $h_{[s,t]}$, and if each component which failed in $h_{[s,t]}$ also failed in $h'_{[s,t]}$ and in $h'_{[s,t]}$ failed earlier than in $h_{[s,t]}$. Let $h_{[s,t]}$ and $h'_{[s+u,t+u]}$ be two histories we will write $h_{[s,t]} \leqslant h'_{[s+u,t+u]}$ if $\theta_u h_{[s,t]} \leqslant h'_{[s+u,t+u]}$.

Finally, for $0 \leqslant s < t < u$, let $h_{[s,t]}$ and $h'_{[t,u]}$ be two histories such that the set of components which are alive at time t in $h_{[s,t]}$ is the same as the set of components which are alive at time t in $h'_{[t,u]}$. Then by $h_{[s,t]} \oplus h'_{[t,u]}$ we denote the history which has at time s the same set of live components as $h_{[s,t]}$ has at time s, and which describes failures of components and their identities on $[s,t]$ as $h_{[s,t]}$ and on $[t,u]$ as $h'_{[t,u]}$.

With this notation condition (2.2) can be written (see [33]) as follows: For $s \geqslant 0,\ t' \geqslant 0,\ u \geqslant 0$,

$$\lambda_k(s+u|h'_{[0,s]}) \leqslant \lambda_k(t'+s+u|h_{[0,t']} \oplus h''_{[t',t'+s]}) \tag{2.3}$$

whenever $h'_{[0,s]} < h''_{[t',t'+s]}$ and whenever k is a component which is alive at time s in $h'_{[0,s]}$ and at time $t'+s$ in $h_{[0,t']} \oplus h''_{[t',t'+s]}$.

In the univariate case, T is MIFR (denoted by T is IFR) if the failure rate $r(x) = f(x)/\bar{F}(x)$ is increasing in x, where f and $\bar{F}$ are the density and survival functions of T. Clearly if $\mathbf{X}$ is MIFR, then $\mathbf{X} \leqslant_{\mathrm{hr}} \mathbf{X}$, that is $\mathbf{X}$ has the positive dependence property of "hazard increasing upon failures" (HIF) given by Shaked and Shanthikumar [32]. The HIF property is a sufficient condition for other notions of dependence as the "supportive lifetimes" (SL) and "weakened by failure" (WBF) notions (see [2,25,32]).

Arjas [1] proposed other possible multivariate extension of the IFR property, given by the condition

$$[(\mathbf{T} - t'\mathbf{e})^+ | h_{t'}] \leqslant_{\mathrm{st}} [(\mathbf{T} - t\mathbf{e})^+ | h_t]$$

for all $t \leqslant t'$ and whenever h_t and $h_{t'}$ coincide on $[0,t)$. As can be seen in [33] condition (2.3) is stronger than the MIFR notion by Arjas [1].

From the definitions, (2.1) and the previous comments we get the following relationships between the multivariate notions defined above:

$$\begin{array}{ccc} \mathbf{T} \text{ is MPF}_2 & \Rightarrow & \mathbf{T} \text{ is MIFR} \\ \Downarrow & & \Downarrow \\ \mathbf{T} \text{ is MTP}_2 & \Rightarrow & \mathbf{T} \text{ is HIF} \end{array}$$

The main objective of this paper is to study the previous notions when the components of the random vector are the first n epoch times of certain nonhomogeneous processes. These properties describe the effect of passing time over the occurrence of events. Some other consequences about association properties are given through the paper. First, we will consider epoch times of nonhomogeneous Poisson processes and then the results will be extended to the more general case of nonhomogeneous pure birth processes. Next, we describe such processes.

A counting process $\{N(t),\ t\geqslant 0\}$ is a *nonhomogeneous Poisson process* with intensity (or rate) function $r\geqslant 0$ if

(a) $\{N(t),\ t\geqslant 0\}$ has the Markov property;
(b) $P\{N(t+\Delta t)=n+1|N(t)=n\}=r(t)\Delta t+o(\Delta t),\ n\geqslant 1$;
(c) $P\{N(t+\Delta t)>n+1|N(t)=n\}=o(\Delta t),\ n\geqslant 1$.

We assume that

$$\int_t^\infty r(u)\,du=\infty \quad \text{for all } t\geqslant 0, \tag{2.4}$$

this ensures that with probability 1 the process has a jump after any time point t. For convenience, if $r(t_0)=\infty$ for some t_0 then we define $r(t)=\infty$ for $t\geqslant t_0$.

A nonnegative function r which satisfies (2.4) can be interpreted as the hazard rate function of a lifetime of an item. More explicitly, if r satisfies (2.4) and we define f by

$$f(t)=r(t)e^{-\int_0^t r(u)\,du}=r(t)e^{-R(t)}, \quad t\geqslant 0,$$

where $R(t)\equiv\int_0^t r(u)\,du$, then f is a probability density function of a lifetime X; in fact, f is the probability density function of the time of the first epoch of the underlying nonhomogeneous Poisson process. Let X be a random variable with density f, and let its distribution be called the underlying distribution of the nonhomogenous Poisson process with rate $r(t)$. As we have mentioned in Section 1 epoch times of nonhomogeneous Poisson processes are the repair times of a unit which is continuously being minimal repaired. Therefore it is of interest to study stochastic properties of the vector of repair times.

Let $0\equiv T_0\leqslant T_1\leqslant T_2\leqslant\cdots$ be the epoch times of the nonhomogeneous Poisson process. Then the density function of $\mathbf{T}=(T_1,\ldots,T_n)$ is given by

$$h_n(x_1,x_2,\ldots,x_n)=r(x_1)r(x_2)\cdots r(x_{n-1})f(x_n) \quad \text{for } x_1\leqslant x_2\leqslant\cdots\leqslant x_n. \tag{2.5}$$

It is noted also that

$$[T_{i+1}|T_1=t_1,\ldots,T_i=t_i]=_{\text{st}}[X|X>t_i], \quad i\geqslant 1. \tag{2.6}$$

The following extension of the nonhomogeneous Poisson process will also be studied in this paper. Let $r_n,\ n\geqslant 1$, be nonnegative functions that satisfy (2.4). A counting process $\{N(t),\ t\geqslant 0\}$ is a *nonhomogeneous pure birth process* with intensity (or rate) functions $r_n\geqslant 0$ if

(a) $\{N(t),\ t\geqslant 0\}$ has the Markov property;
(b) $P\{N(t+\Delta t)=n+1|N(t)=n\}=r_n(t)\Delta t+o(\Delta t),\ n\geqslant 1$;
(c) $P\{N(t+\Delta t)>n+1|N(t)=n\}=o(\Delta t),\ n\geqslant 1$.

Nonhomogeneous pure birth processes are called 'relevation counting processes' in [26], where some applications of them in reliability theory are described. When all the

r_n's are identical, a nonhomogeneous pure birth process reduces to a nonhomogeneous Poisson process. These epoch times can be applied in different contexts, apart from reliability theory. When $r_n(t) = (n+1)\lambda$, where $\lambda > 0$, we have a Yule birth process, which can be generalized to the case $r_n(t) = (n+1)\lambda(t)$. A generalized Yule birth process may model the spread of a disease where n is the number of infectives, and $\lambda(t)$ is the rate in which infectives pass the disease to new individuals at time t; this rate, in general, may depends on the calendar time t (see [4]). Another application is in load-sharing models (see [30]). See [6] for a discussion of these applications.

Let $0 \equiv T_0 \leqslant T_1 \leqslant T_2 \leqslant \cdots$ be the epoch times of the above nonhomogeneous pure birth process. We will describe now a useful stochastic representation of these epoch times. Consider a set of independent absolutely continuous nonnegative random variables $\{Y_n,\ n \geqslant 1\}$, with corresponding hazard rate functions $r_n,\ n \geqslant 1$. Define, recursively,

$$\hat{T}_1 = Y_1,$$

$$\hat{T}_n = [Y_n | Y_n > \hat{T}_{n-1}], \quad n \geqslant 2,$$

then it is not difficult to verify that the joint distribution of $(\hat{T}_1, \ldots, \hat{T}_n)$ is the same as the joint distribution of $(T_1, \ldots, T_n)$. Now denote by $\bar{F}_n$ and f_n the survival and the density functions of Y_n, then the joint density of $\mathbf{T} = (T_1, \ldots, T_n)$ is given by

$$h(x_1, x_2, \ldots, x_n) = \prod_{j=1}^{n-1} \frac{f_j(x_j)}{\bar{F}_{j+1}(x_j)} f_n(x_n) \quad \text{for } x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n. \tag{2.7}$$

Next, in Section 3, we describe conditions on the parameters of nonhomogeneous (Poisson and pure birth) processes to classify the random vector of the first n epoch times in some of the previous multivariate aging classes.

3. Multivariate aging properties of epoch times

In this section we proceed to give conditions on the parameters of nonhomogeneous Poisson and pure birth processes, to get that the random vector of the first n epoch times has some of the multivariate stochastic properties given in the previous section. We will distinguish epoch times of nonhomogeneous Poisson processes and nonhomogeneous pure birth processes.

3.1. Epoch times of nonhomogeneous Poisson processes

In this section given a nonhomogeneous Poisson process with epoch times $T_n,\ n \leqslant 1$, we give conditions on the underlying distribution of the process, for $\mathbf{T} = (T_1, \ldots, T_n)$ to be in some of the previous multivariate classes. First, we describe the structure of a typical history up to time t for $\mathbf{T} = (T_1, \ldots, T_n)$. Since $T_1 \leqslant T_2 \leqslant \cdots \leqslant T_n$ a.s., we see that the set I of failed components in $\mathbf{T}$ should be of the form $I = \{1, 2, \ldots, m\}$. Therefore

$$h_t = [T_1 = t_1, \ldots, T_m = t_m, \mathbf{T}_{\{m+1,\ldots,n\}} > t\mathbf{e}],$$

where $t_1 < \cdots < t_m < t$. Next, we give a theorem in which we state conditions under which the random vector of the first n epoch times is MIFR.

Theorem 3.5. *Let $T_n,\ n \geqslant 1$ be the epoch times of a nonhomogeneous Poisson process, with underlying distribution F as above. If X is random variable with distribution F and X is* IFR *then* $\mathbf{T} = (T_1, \ldots, T_n)$ *is* MIFR, *for all $n \geqslant 1$.*

Proof. Let us suppose that X is IFR. To prove that $\mathbf{T} = (T_1, \ldots, T_n)$ is MIFR we will prove condition (2.3). We see first how to describe the histories $h'_{[0,s]}$ and $h_{[0,t']} \oplus h''_{[t',t'+s]}$.

As stated above $h'_{[0,s]}$ is of the form

$$h'_{[0,s]} = \{\mathbf{T}_I = \mathbf{s}_I, \mathbf{T}_{\bar{I}} > s\mathbf{e}\}, \text{ where } I = \{1, 2, \ldots, m\}.$$

On the other hand we have in (2.3) $h'_{[0,s]} \leqslant h''_{[t',t'+s]}$, i.e., $\theta_{t'} h'_{[0,s]} \leqslant h''_{[t',t'+s]}$, where

$$\theta_{t'} h'_{[0,s]} = \{\mathbf{T}_I = \mathbf{s}_I + t'\mathbf{e}, \mathbf{T}_{\bar{I}} > (s + t')\mathbf{e}\}.$$

Since $h'_{[0,s]} \leqslant h''_{[t',t'+s]}$, every component which is alive at time t' in $h''_{[t',t'+s]}$ is also alive at time t' in $\theta_{t'} h'_{[0,s]}$, and every component which has failed in $\theta_{t'} h'_{[0,s]}$ also failed in $h''_{[t',t'+s]}$ and in $h''_{[t',t'+s]}$ it failed earlier than in $\theta_{t'} h'_{[0,s]}$, i.e., $h''_{[t',t'+s]}$ must be of the form

$$h''_{[t',t'+s]} = \{\mathbf{T}_J = \mathbf{t}'_J, \mathbf{T}_{\bar{J}} > (s + t')\mathbf{e}\}, \text{ where } J = \{1, 2, \ldots, m'\}$$

and $m \leqslant m'$ and $\mathbf{t}'_I \leqslant \mathbf{s}_I + t'\mathbf{e}$.

It is also seen that in $h_{[0,t']}$ there is no failure.

Therefore, by the previous considerations and (2.6),

$$\lambda_k(s + u | h'_{[0,s]}) = \begin{cases} r_F(s + u) & \text{if } k = m + 1, \\ 0 & \text{if } k > m + 1 \end{cases}$$

and

$$\lambda_k(t' + s + u | h_{[0,t']} \oplus h''_{[t',t'+s]}) = \begin{cases} r_F(t' + s + u) & \text{if } k = m' + 1, \\ 0 & \text{if } k > m' + 1. \end{cases}$$

To prove (2.3), we observe that k must satisfy $k \geqslant m' + 1 (\geqslant m + 1)$.

Let us suppose $m' > m$. If $k = m' + 1$, then

$$\lambda_k(t' + s + u | h_{[0,t']} \oplus h''_{[t',t'+s]}) = r_F(t' + s + u) \geqslant 0 = \lambda_k(s + u | h'_{[0,s]})$$

and, if $k > m' + 1$,

$$\lambda_k(t' + s + u | h_{[0,t']} \oplus h''_{[t',t'+s]}) = 0 = \lambda_k(s + u | h'_{[0,s]})$$

so (2.3) holds.

Let us suppose $m' = m$. If $k = m' + 1$, then

$$\lambda_k(t' + s + u | h_{[0,t']} \oplus h''_{[t',t'+s]}) = r_F(t' + s + u) \geqslant r_F(s + u) = \lambda_k(s + u | h'_{[0,s]}),$$

and, if $k > m' + 1$,

$$\lambda_k(t' + s + u | h_{[0,t']} \oplus h''_{[t',t'+s]}) = 0 = \lambda_k(s + u | h'_{[0,s]}),$$

where the inequality follows since X is IFR, so (2.3) holds. □

Now we proceed to give conditions for the MPF_2 property. To obtain the result we make the stronger assumption of the logconcavity of the intensity function. This property has been studied in [26], where properties and examples of distributions with that property are given. One of the properties that they proved is that for a random variable X, with density f and failure rate r, the following implication holds

$$r \text{ is logconcave} \Rightarrow f \text{ is logconcave.} \tag{3.8}$$

Theorem 3.6. *Let $T_n,\ n \geqslant 1$ be the epoch times of a nonhomogeneous Poisson process, with intensity r. If r is logconcave then $\mathbf{T} = (T_1, \ldots, T_n)$ is MPF_2, for all $n \geqslant 1$.*

Proof. Let us suppose that r is logconcave. Given a history $h_t = \{T_I = \mathbf{t}_I, \mathbf{T}_{\bar{I}} > t\mathbf{e}\}$, where $I = (1, \ldots, m)$ we want to prove $[(\mathbf{T} - t\mathbf{e})^+ | h_t] \leqslant_{\mathrm{lr}} \mathbf{T}$. Let us observe that the density of $[(\mathbf{T}_{\{m+1,\ldots,n\}} - t\mathbf{e})^+ | h_t]$ is given by

$$f(x_{m+1}, \ldots, x_n) = \frac{1}{\bar{F}(t)} r(x_{m+1} + t) \cdots r(x_{n-1} + t) f(x_n + t), \tag{3.9}$$

for $x_{m+1} \leqslant \cdots \leqslant x_n$.

In this case we compare two random vectors of different dimension in the likelihood ratio order, and this case will be treated as in [34, p. 132].

Considering $x_1 \leqslant \cdots \leqslant x_n$ and $y_{m+1} \leqslant \cdots \leqslant y_n$ and $t \geqslant 0$, by (2.5) and (3.9) we want to prove that

$$\begin{aligned} & r(x_1) \cdots r(x_{n-1}) f(x_n) \frac{1}{\bar{F}(t)} r(y_{m+1} + t) \cdots r(y_{n-1} + t) f(y_n + t) \\ & \quad \leqslant r(x_1) \cdots r(x_m) r(x_{m+1} \vee y_{m+1}) \cdots r(x_{n-1} \vee y_{n-1}) f(x_n \vee y_n) \\ & \qquad \times \frac{1}{\bar{F}(t)} r((x_{m+1} \wedge y_{m+1}) + t) \cdots r((x_{n-1} \wedge y_{n-1}) + t) f((x_n \wedge y_n) + t) \end{aligned} \tag{3.10}$$

which is equivalent to prove that

$$\begin{aligned} & r(x_{m+1}) \cdots r(x_{n-1}) f(x_n) \frac{1}{\bar{F}(t)} r(y_{m+1} + t) \cdots r(y_{n-1} + t) f(y_n + t) \\ & \quad \leqslant r(x_{m+1} \vee y_{m+1}) \cdots r(x_{n-1} \vee y_{n-1}) f(x_n \vee y_n) \\ & \qquad \times \frac{1}{\bar{F}(t)} r((x_{m+1} \wedge y_{m+1}) + t) \cdots r((x_{n-1} \wedge y_{n-1}) + t) f((x_n \wedge y_n) + t). \end{aligned} \tag{3.11}$$

Let $i = m+1, \dots, n-1$, next we prove that

$$r(x_i)r(y_i+t) \leqslant r(x_i \vee y_i) r((x_i \wedge y_i)+t). \tag{3.12}$$

If $x_i \geqslant y_i$, then clearly (3.12) holds. If $x_i < y_i$ then (3.12) is equivalent to

$$r(x_i)r(y_i+t) \leqslant r(y_i)r(x_i+t),$$

which holds since r is logconcave.

Now we prove that

$$f(x_n)f(y_n+t) \leqslant f(x_n \vee y_n) f((x_n \wedge y_n)+t). \tag{3.13}$$

Let us suppose that $x_n \geqslant y_n$ then (3.13) holds trivially. On the other hand if $x_n < y_n$ then (3.13) is equivalent to

$$f(x_n)f(y_n+t) \leqslant f(y_n)f(x_n+t)$$

which holds since, by (3.8), X is PF_2. Now by (3.12) and (3.13), inequality (3.11) is true. Therefore (3.10) holds. □

As we have mentioned in Section 2 any MPF_2 random vector is also MTP_2 and any MIFR random vector is also HIF. As mentioned by Shaked and Shanthikumar [32] these are sufficient conditions for the WBF and SL notions which, roughly speaking, mean that upon occurrence of an event there is a stochastic decrease in the residual time until the next event. The MTP_2 and HIF properties are also sufficient conditions for the association property (see [14,32]).

Definition 3.7. Given a random vector $\mathbf{X} = (X_1, \dots, X_n)$, we say that $X_1, \dots, X_n$ are associated if

$$Cov[f(\mathbf{X}), g(\mathbf{X})] \geqslant 0$$

for all nondecreasing functions f and g for which $E[f(\mathbf{X})]$, $E[g(\mathbf{X})]$, $E[f(\mathbf{X})g(\mathbf{X})]$ exist.

Therefore the logconcavity or the increasingness of the intensity, r, are sufficient conditions for the association of the first n epoch times of the corresponding nonhomogeneous Poisson process. However this result holds with no assumption on the intensity. The argument is the following. If we consider two processes with the same intensity then by Theorems 3.3 and 3.6 in [6] we obtain the HIF and MTP_2 properties of the first n epoch times of NHPP. In case $T_1, \dots, T_n$ are associated a useful consequence is that (see [14])

$$P[T_1 > t_1, \dots, T_n > t_n] \geqslant \prod_{i=1}^{n} P[T_i > t_i].$$

This is of interest because in some situations $P[T_i > t_i]$ can be bounded by the survival function of some well-known models. For example if $\bar{F}(t) \geqslant e^{-\lambda t}$, from Shaked and Szekli [36] and Belzunce et al. [6], we get that $T_n \geqslant_{\text{st}} S_n$ for all $n \geqslant 1$ where S_n follows a gamma distribution with shape parameter $1/\lambda$ and scale parameter n. Therefore $P[T_1 > t_1, \dots, T_n > t_n] \geqslant \prod_{i=1}^{n} P[S_i > t_i]$. A situation where $\bar{F}(t) \geqslant e^{-\lambda t}$ is the case in which the intensity of the nonhomogeneous Poisson process is bounded from above by λ.

We mention that Pellerey et al. [26] have proved Theorem 3.6 in the univariate case.

As we have mentioned in the introduction the epoch times of nonhomogeneous Poisson processes have been applied in different contexts. Examples of these applications can be seen in [3,13]. More recently, Kuo and Yang [21] show that epoch times of nonhomogeneous Poisson processes are appropriate to model the failure times in software testing. The most common parametric model is the Weibull distribution with intensity $r(t) = \beta t^{\alpha}$ (see [13]). Musa and Okumoto [24] consider the process with rate $r(t) = \alpha/(t+\beta)$ (Pareto distribution). Cox and Lewis [12] process has rate $r(t) = \exp(\alpha + \beta t)$ (extreme value distribution). All these models have a logconcave intensity as can be seen in [26], where other examples are shown.

3.2. Epoch times of nonhomogeneous pure birth processes

In this section we extend the previous results to the epoch times of nonhomogeneous pure birth processes. Since in most cases the proofs are similar we just outline the proof of some of them.

Theorem 3.8. *Let T_n, $n \geqslant 1$ be the epoch times of a nonhomogeneous pure birth process, with intensities $\{r_n\}_{n=1}^{\infty}$. If r_n is increasing for all $n \geqslant 1$ then $\mathbf{T} = (T_1, \ldots, T_n)$ is* MIFR.

Proof. The proof is similar to the proof of Theorem 3.5. To prove that $\mathbf{T} = (T_1, \ldots, T_n)$ is MIFR we will prove condition (2.3). As in Theorem 3.5 we have that

$$\lambda_k(s+u|h'_{[0,s]}) = \begin{cases} r_{m+1}(s+u) & \text{if } k = m+1, \\ 0 & \text{if } k > m+1 \end{cases}$$

and

$$\lambda_k(t'+s+u|h_{[0,t']} \oplus h''_{[t',t'+s]}) = \begin{cases} r_k(t'+s+u) & \text{if } k = m'+1, \\ 0 & \text{if } k > m'+1. \end{cases}$$

The proof follows similar steps to the proof of Theorem 3.5. $\square$

Theorem 3.9. *Let T_n, $n \geqslant 1$ be the epoch times of a nonhomogeneous pure birth process, with intensities $\{r_n\}_{n=1}^{\infty}$. If r_j is logconcave, and if*

$$r_{j+1}(x) - r_j(x) \text{ is decreasing in } x \geqslant 0 \tag{3.14}$$

for $j \geqslant 1$, then $\mathbf{T} = (T_1, \ldots, T_n)$ is MPF$_2$.

Proof. Following similar steps to Theorem 3.6 we have to prove that

$$\frac{1}{\bar{F}_{m+1}(t)}\prod_{j=1}^{n-1} r_j(x_j)\frac{\bar{F}_j(x_j)}{\bar{F}_{j+1}(x_j)}f_n(x_n)\prod_{j=m+1}^{n-1} r_j(y_j+t)\frac{\bar{F}_j(y_j+t)}{\bar{F}_{j+1}(y_j+t)}f_n(y_n+t)$$
$$\geqslant \frac{1}{\bar{F}_{m+1}(t)}\prod_{j=1}^{n-1} r_j(x_j\vee y_j)\frac{\bar{F}_j(x_j\vee y_j)}{\bar{F}_{j+1}(x_j\vee y_j)}\prod_{j=m+1}^{n-1} r_j((x_j\wedge y_j)+t)$$
$$\times\frac{\bar{F}_j((x_j\wedge y_j)+t)}{\bar{F}_{j+1}((x_j\wedge y_j)+t)}f_n((x_n\wedge y_n)+t) \quad (3.15)$$

which is equivalent to prove that

$$\prod_{j=m+1}^{n-1} r_j(x_j)\frac{\bar{F}_j(x_j)}{\bar{F}_{j+1}(x_j)}f_n(x_n)\prod_{j=m+1}^{n-1} r_j(y_j+t)\frac{\bar{F}_j(y_j+t)}{\bar{F}_{j+1}(y_j+t)}f_n(y_n+t)$$
$$\geqslant \prod_{j=m+1}^{n-1} r_j(x_j\vee y_j)\frac{\bar{F}_j(x_j\vee y_j)}{\bar{F}_{j+1}(x_j\vee y_j)}\prod_{j=m+1}^{n-1} r_j((x_j\wedge y_j)+t)$$
$$\times\frac{\bar{F}_j((x_j\wedge y_j)+t)}{\bar{F}_{j+1}((x_j\wedge y_j)+t)}f_n((x_n\wedge y_n)+t). \quad (3.16)$$

Note that condition (3.14) is equivalent to

$$\frac{\dfrac{\bar{F}_j(x)}{\bar{F}_{j+1}(x)}}{\dfrac{\bar{F}_j(x+t)}{\bar{F}_{j+1}(x+t)}} \text{ is increasing in } t\geqslant 0. \quad (3.17)$$

Now we prove that for $j: m+1,\dots,n-1$

$$r_j(x_j)\frac{\bar{F}_j(x_j)}{\bar{F}_{j+1}(x_j)}r_j(y_j+t)\frac{\bar{F}_j(y_j+t)}{\bar{F}_{j+1}(y_j+t)}f_n(y_n+t)$$
$$\geqslant r_j(x_j\vee y_j)\frac{\bar{F}_j(x_j\vee y_j)}{\bar{F}_{j+1}(x_j\vee y_j)}r_j((x_j\wedge y_j)+t)\frac{\bar{F}_j((x_j\wedge y_j)+t)}{\bar{F}_{j+1}((x_j\wedge y_j)+t)}. \quad (3.18)$$

If $x_j\geqslant y_j$ the result is trivial and for $x_j<y_j$ then (3.18) holds by the logconcavity of r_j and by (3.17). The proof of

$$f_n(x_n)f_n(y_n+t)\geqslant f_n(x_n\vee y_n)f_n((x_n\wedge y_n)+t),$$

follows as in Theorem 3.6. Therefore inequality (3.16) is true and (3.15) holds. □

4. Applications

In this section we mention two applications of previous results for l_∞-spherical densities and generalized order statistics. Some other applications for generalized Yule birth processes and load sharing models are easy to obtain (see [6]).

4.1. l_∞-spherical densities

Recently, Shaked et al. [35] highlight the relationship between l_∞-spherical densities of the form

$$g(x_1, x_2, \ldots, x_n) = \begin{cases} \psi(x_n) & \text{if } 0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n, \\ 0 & \text{otherwise}, \end{cases} \tag{4.19}$$

for some nonnegative function ψ, and nonhomogeneous pure birth processes.

They prove (Theorem 4.2) that given a random vector $(X_1, X_2, \ldots, X_n)$ with a joint density of the form (4.19) then there exists a nonhomogeneous pure birth process whose first epoch times $(T_1, T_2, \ldots, T_n)$ satisfy

$$(T_1, T_2, \ldots, T_n) =_{\text{st}} (X_1, X_2, \ldots, X_n).$$

Also they prove (Theorem 4.3) that in such case if f_i and $\bar{F}_i$, are the density and survival functions corresponding to the intensities r_i, $i = 1, 2, \ldots, n$ of the nonhomogeneous pure birth process, then

$$f_i(x) = \frac{\bar{F}_{i+1}(x)}{m_{i+1}}, \quad x \geqslant 0, \ \ i = 1, 2, \ldots, n-1,$$

where $m_{i+1} = \int_0^\infty \bar{F}_{i+1}(x)\,dx$, $i = 1, 2, \ldots, n$.

Next we see whether the conditions of Theorems 3.8 and 3.9 holds, for this nonhomogeneous pure birth process.

In Theorem 3.8 we need the intensities to be increasing. In this case the intensities are given by

$$r_i(x) = \frac{\bar{F}_{i+1}(x)}{m_{i+1}\bar{F}_i(x)}, \quad x \geqslant 0, \ \ i = 1, 2, \ldots, n-1$$

and

$$r_n(x) = \frac{f_n(x)}{\bar{F}_n(x)}, \quad x \geqslant 0.$$

Therefore we need r_n to be increasing and $F_i \leqslant_{\text{hr}} F_{i+1}$, for $i = 1, 2, \ldots, n-1$.

In Theorem 3.9 we need r_i, $i = 1, 2, \ldots, n$ to be logconcave. This is equivalent to $r_{i+1} - r_i$ be decreasing for $i = 1, 2, \ldots, n-1$ and to r_n be logconcave. Under these conditions we get the MPF_2 property of $g(x_1, x_2, \ldots, x_n)$.

4.2. Generalized order statistics

Kamps [19] introduces the concept of generalized order statistics, to provide a unified approach to several models of random vectors with ordered components.

Definition 4.10. Let $n \in \mathbb{N}$, $k \geqslant 1$, $m_1, \ldots, m_{n-1} \in \mathbb{R}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leqslant r \leqslant n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geqslant 1$ for all $r \in 1, \ldots, n-1$, and let $\tilde{m} = (m_1, \ldots, m_{n-1})$, if $n \geqslant 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary, if $n = 1$). We call uniform generalized order statistics to the random vector $(U_{(1,n,\tilde{m},k)}, \ldots, U_{(n,n,\tilde{m},k)})$ with joint density function

$$h(u_1, \dots, u_n) = k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left(\prod_{j=1}^{n-1} (1-u_j)^{m_j}\right)(1-u_n)^{k-1}$$

on the cone $0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant 1$. Now given a distribution function F we call generalized order statistics based on F to the random vector

$$(X_{(1,n,\tilde{m},k)}, \dots, X_{(n,n,\tilde{m},k)}) \equiv (F^{-1}(U_{(1,n,\tilde{m},k)}), \dots, F^{-1}(U_{(n,n,\tilde{m},k)})).$$

If F is an absolutely continuous distribution with density f, the joint density function of $(X_{(1,n,\tilde{m},k)}, \dots, X_{(n,n,\tilde{m},k)})$ is given by

$$f(x_1, \dots, x_n) = k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left(\prod_{j=1}^{n-1} \bar{F}(x_j)^{m_j} f(x_j)\right)\bar{F}(x_n)^{k-1} f(x_n) \tag{4.20}$$

on the cone $F^{-1}(0) \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant F^{-1}(1)$.

Several models of ordered random variables are included in this model.

Taking $m_i = 0$ for all $i = 1, \dots, n-1$ and $k = 1$ we get order statistics from a distribution F and taking $m_i = -1$ for all $i = 1, \dots, n-1$ and $k \in \mathbb{Z}_+$ we get first n k-record values from a sequence of random variables with distribution F. Another interesting model contained in the model of generalized order statistics, is the model of order statistics under multivariate imperfect repair (see [7,16,31]).

In a distributional theoretical sense generalized order statistics are contained in the model of epoch times of a NHPB process. Consider generalized order statistics based on F with failure rate r and parameters k, n and M_r, $r = 1, \dots, n-1$ then

$$(X_{(1,n,\tilde{m},k)}, \dots, X_{(n,n,\tilde{m},k)}) =_{\text{st}} (T_1, \dots, T_n),$$

where T_i are the epoch times of a NHPB process with intensities $r_i = (k + n - i + M_i)r$, for $i : 1, \dots, n$.

Through this relationship and from Theorems 3.8 and 3.9 it is possible to get the following result.

Theorem 4.11. *Let X be a random variable with hazard rate r and distribution function F, and let $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \dots, X_{(n,n,\tilde{m},k)})$ be a random vector of generalized order statistics based on F. If*

(a) *r is increasing then $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \dots, X_{(n,n,\tilde{m},k)})$ is MIFR.*
(b) *r is logconcave then $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \dots, X_{(n,n,\tilde{m},k)})$ is MPF_2.*

Proof. Condition (a) is obvious from Theorem 3.8.

Condition (b) follows from Theorem 3.9 and from the fact that the logconcavity of r implies the logconcavity of the density function of X and therefore r is increasing. □

Result (b) can be improved when all the m_i are nonnegative. In fact it is not difficult to prove the following theorem.

Theorem 4.12. *Let X be an absolutely continuous random variable with density function f and distribution function F, and let $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ be a random vector of generalized order statistics based on F. If $m_i \geqslant 0$ for all $i = 1, \ldots, n-1$, and f is logconcave then $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ is MPF_2.*

Remark 4.13. From Theorems 4.11(a) and 4.12, we get as a particular case Theorems 3.1 and 3.2 by Belzunce et al. [9].

References

[1] E. Arjas, A stochastic process approach to multivariate reliability systems: notions based on conditional stochastic order, Math. Oper. Res. 6 (1981) 263–276.

[2] E. Arjas, I. Norros, Life lengths and association: a dynamic approach, Math. Oper. Res. 9 (1984) 151–158.

[3] H. Ascher, H. Feingold, Repairable Systems Reliability, Decker, New York, 1984.

[4] N.T.J. Bailey, The Mathematical Theory of Infectious Diseases and its Applications, Griffin, London, 1975.

[5] F. Belzunce, T. Hu, T. Khaledi, Dispersion-type variability orders, Probab. Eng. Inform. Sci., 2003, to appear.

[6] F. Belzunce, R. Lillo, J.M. Ruiz, M. Shaked, Stochastic comparisons of nonhomogeneous processes, Probab. Eng. Inform. Sci. 15 (2001) 199–224.

[7] F. Belzunce, J.A. Mercader, J.M. Ruiz, Stochastic comparisons of generalized order statistics, Probab. Eng. Inform. Sci., 2003, to appear.

[8] F. Belzunce, J.M. Ruiz, Multivariate dispersive ordering of epoch times of nonhomogeneous Poisson processes, J. Appl. Probab. 39 (2002) 637–643.

[9] F. Belzunce, J.M. Ruiz, M.C. Ruiz, Multivariate properties of random vectors of order statistics, J. Statist. Plann. Inference, 2002, to appear.

[10] F. Belzunce, M. Shaked, Stochastic comparisons of mixtures of convexly ordered distributions with applications in reliability theory, Statist. Probab. Lett. 53 (2001) 363–372.

[11] H.W. Block, T.H. Savits, Multivariate classes of life distributions in reliability theory, Math. Oper. Res. 6 (1981) 453–461.

[12] D.R. Cox, P.A. Lewis, Statistical Analysis of Series of Events, Methuen, London, 1966.

[13] L.H. Crow, Reliability analysis for complex, repairable systems, in: F. Proschan, R.J. Serfling (Eds.), Reliability and Biometry, SIAM, Philadelphia, 1974, pp. 379–410.

[14] J.D. Esary, F. Proschan, D.W. Walkup, Association of random variables with applications, Ann. Math. Statist. 38 (1967) 1466–1474.

[15] R.C. Gupta, S.N.U.A. Kirmani, Closure and monotonicity properties of nonhomogeneous Poisson processes and record values, Probab. Eng. Inform. Sci. 2 (1988) 475–484.

[16] T. Hu, Stochastic comparisons of order statistics under multivariate imperfect repair, J. Appl. Probab. 33 (1996) 156–163.

[17] T. Hu, X. Pan, Preservation of multivariate dependence under multivariate claim models, Insurance Math. Econom. 25 (1999) 171–179.

[18] N.L. Johnson, S. Kotz, A vector multivariate hazard rate, J. Multivariate Anal. 5 (1975) 53–66.

[19] U. Kamps, A concept of generalized order statistics, J Statist Plann Inference 48 (1995) 1–23.

[20] S. Karlin, Y. Rinot, Classes of ordering measures and related correlation inequalities. I. Multivariate totally positive distributions, J. Multivariate Anal. 10 (1980) 467–498.

[21] L. Kuo, T.Y. Yang, Bayesian computation for non-homogeneous Poisson processes in software reliability, Amer. Statist. Assoc. 91 (1996) 763–773.

[22] E.L. Lehmann, Some concepts of dependence, Ann. Math. Statist. 37 (1966) 1137–1153.

[23] A.W. Marshall, M. Shaked, Multivariate new better than used distributions: a survey, Scand. J. Statist. 13 (1986) 277–290.

[24] J.D. Musa, K. Okumoto, A logarithmic Poisson execution time model for software reliability measurement, in: Proceedings of the Seventh International Conference on Software Engineering, 1984, pp. 230–238.
[25] I. Norros, Systems weakened by failures, Stochastic Process. Appl. 20 (1985) 181–196.
[26] F. Pellerey, M. Shaked, J. Zinn, Non-homogeneous Poisson processes and logconcavity, Probab. Eng. Inform. Sci. 15 (2000) 353–373.
[27] D. Roy, Classification of life distributions in multivariate models, IEEE Trans. Reliability 43 (1994) 224–229.
[28] T.H. Savits, Multivariate life classes and inequalities, in: Y.L. Tong (Ed.), Inequalities in Statistics and Probability, Institute of Mathematics and Statistics. Lectures Notes-Monographs Series, Vol. 5, Institute of Mathematical Statistics, Hayward, CA, 1984, pp. 193–198.
[29] T.H. Savits, A multivariate IFR class, J. Appl. Probab. 22 (1985) 197–204.
[30] Z. Schechner, A load-sharing model: the linear breakdown rule, Naval Res. Logist. Quart. 31 (1984) 137–144.
[31] M. Shaked, Shanthikumar, Multivariate imperfect repair, Oper. Res. 34 (1986) 437–448.
[32] M. Shaked, J.G. Shanthikumar, Multivariate stochastic orderings and positive dependence in reliability theory, Math. Oper. Res. 15 (1990) 545–552.
[33] M. Shaked, J.G. Shanthikumar, Dynamic multivariate aging notions in reliability theory, Stochastic Process. Appl. 38 (1991) 85–97.
[34] M. Shaked, J.G. Shanthikumar, Stochastic Orders and their Applications, Academic Press, San Diego, 1994.
[35] M. Shaked, F. Spizzichino, F. Suter, Nonhomogeneous birth processes and l_∞-spherical densities, with applications in reliability theory, Probab. Eng. Inform. Sci. 16 (2002) 271–288.
[36] M. Shaked, R. Szekli, Comparison of replacement policies via point processes, Adv. in Appl. Probab. 27 (1995) 1079–1103.