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Convergence to a Constant of the Solutions of Differential Equations with Impulse Effect

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This paper examines the necessary and sufficient conditions for which the solutions of a system with impulse effect tend exponentially to a constant as $t \to \infty$. © 1988 Academic Press, Inc.

1. INTRODUCTION

In connection with a number of applications in the recent years, the interest towards the systems with impulse effect of the kind

$$\begin{aligned} \frac{dx}{dt} &= f(t, x), \qquad t \neq \tau_k, \\ \Delta x \,|_{t=\tau_k} &= I_k(x), \qquad k = 1, 2, \ldots \end{aligned} \tag{1}$$

has increased.

Some problems of the qualitative theory for the systems of the kind (1) are considered in [1–8].

We shall note that many papers are devoted to differential equations with impulses, introduced with a Dirac function. Some of these papers are given in the reference of Pandit and Deo's monography [9].

In this paper, with the help of partially continuous analogues of Liapunov's functions [8], necessary and sufficient conditions are found for which the solutions of the systems with impulse effect (1) tend exponentially to a constant as $t \to \infty$.

Some investigations are performed [10–14] for convergence of solutions of different classes of differential equations without impulses to a constant as $t \to \infty$.

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2. Preliminary Notes

The systems with impulse effect are characterized such that for $t \in I$, $t \neq \tau_k$ the solution of (1) is defined by the differential system $dx/dt = f(t, x)$. In the moments $t = \tau_k$, the mapping point (t, x) undergoing short period forces (a hit, an impulse, etc.) moves from position $(\tau_k, x(\tau_k))$ to position $(\tau_k, x(\tau_k) + I_k(x(\tau_k)))$ instantaneously. We assume, that the solutions of system (1) are left continuous in the moments of a jump, i.e., $x(\tau_k - 0) = x(\tau_k)$, $\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k) = I_k(x(\tau_k))$. Further on, we shall use the following notation: $|x|$, the norm of $x \in R^n$; $I = [0, \infty)$; E, the unit $n \times n$ matrix; $|A| = \sup_{|x|=1} |Ax|$, the norm of the matrix $A = (a_{ij})_1^n$; $i(t, s)$, the number of the points τ_k lying in the interval (t, s); $B_\Delta = \{x \in R^n: |x| < \Delta\}$; $G_k = (\tau_{k-1}, \tau_k) \times B_\rho$; and $D_k = (\tau_{k-1}, \tau_k] \times B_\rho$, $k = 1, 2, \ldots$.

We shall say that the conditions (H) hold if the following conditions are fulfilled:

H1. The functions $f: I \times B_\rho \to R^n$ and $I_k: B_\rho \to R^n$ $(k = 1, 2, \ldots)$ are continuous in their domains.

H2. $f(t, 0) = 0$, $I_k(0) = 0$ $(t \in I,\ k = 1, 2, \ldots)$.

H3. $\tau_{k+1} - \tau_k \geqslant \theta > 0$ $(k = 1, 2, \ldots)$.

H4. There exists a constant $L > 0$ such that

$$|f(t, x) - f(t, y)| \leqslant L\, |x - y| \qquad (x, y \in B_\rho,\ t \in I),$$

$$|I_k(x) - I_k(y)| \leqslant L\, |x - y| \qquad (x, y \in B_\rho,\ k = 1, 2, \ldots).$$

H5. There exists a constant $l > 0$ such that

$$|f(t, x) - f(t, y)| \leqslant l\, |x|\, |t - s| \qquad (x \in B_\rho;\ t, s \in I).$$

Let $t_0 \in I$ and $x_0 \in B_\rho$. We denote $x(t; t_0, x_0)$ as the solution of system (1), for which $x(t_0 + 0; t_0, x_0) = x_0$, and $J^+(t_0, x_0)$ as the maximum interval of the kind (t_0, ω), where this solution is right continuable.

Definition 1. We shall say that the system with impulse effect (1) has a property (A) if there exist constants M, Δ, and $\alpha (M > 0,\ 0 < \Delta \leqslant \rho,\ \alpha > 0)$ such that for every $(t_0, x_0) \in I \times B_\Delta$ there exists $K(t_0, x_0) \in R^n$, for which

$$|x(t; t_0, x_0) - K(t_0, x_0)| \leqslant M\, |x_0|\, e^{-\alpha(t - t_0)} \qquad (t > t_0). \tag{2}$$

Remark 1. If system (1) has a property (A) then

$$|x(t; t_0, x_0)| \leqslant (2M + 1)\, |x_0| \qquad (t > t_0 \geqslant 0,\ x_0 \in B_\Delta). \tag{3}$$

The proof of (3) is analogous to the proof of the respective inequality from [14].

Let the function $h: I \times B_\rho \to I$ be continuous in its domain. Following [15, 16], we shall give a definition for h-exponential stability of the trivial solution of the system with impulse effect (1).

DEFINITION 2. The trivial solution of the system with impulse effect (1) is called h-exponentially stable if there exist constants $M, \Delta, \alpha (M > 0, 0 < \Delta \leqslant \rho, \alpha > 0)$ such that for every $(t_0, x_0) \in I \times B_\Delta$

$$h(t, x(t; t_0, x_0)) \leqslant M\, |x_0|\, e^{-\alpha}(t - t_0) \qquad (t > t_0).$$

Let the functions $h_k: B_\rho \to I$, $k = 1, 2, \ldots$ be continuous in their domains.

DEFINITION 3. The trivial solution of (1) is called $(h; h_k)$-exponentially stable if there exist constants $M, \Delta, \alpha (M > 0, 0 < \Delta \leqslant \rho, \alpha > 0)$ such that for every $(t_0, x_0) \in I \times B_\Delta$

$$h(t; x(t; t_0, x_0)) \leqslant M\, |x_0|\, e^{-\alpha(t - t_0)} \qquad (t > t_0),$$

$$h_k(x(\tau_k; t_0, x_0)) \leqslant M\, |x_0|\, e^{-\alpha(\tau_k - t_0)} \qquad (\tau_k > t_0).$$

DEFINITION 4. We shall say that the function $V: I \times B_\rho \to I$ belongs to the class $\mathscr{V}_0$ if:

1. $V(t, x)$ is continuous in each of the sets D_k $(k = 1, 2, \ldots)$ and $V(t, 0) = 0$ $(t \in I)$.

2. For every $k = 1, 2, \ldots$ and $x \in B_\rho$ there exists a finite limit

$$\lim_{\substack{(t, y) \to (\tau_k, x) \\ t > \tau_k}} V(t, y) = V(\tau_k + 0, x).$$

Let $V \in \mathscr{V}_0$. For $(t, x) \in G_k$, $k = 1, 2, \ldots$ we define the upper right derivative $D^+ V_{(1)}(t, x)$ of the function V with respect to system (1):

$$D^+ V_{(1)}(t, x) = \limsup_{\tau \to 0_+} \left(V(t + \tau, x + \tau f(t, x)) - V(t, x) \right) / \tau.$$

We shall note that if $x(t)$ is a solution of system (1) and the function V is locally Lipschitzian in x, then for the upper right Dini derivative of the function $v(t) = V(t, x(t))$ the equality

$$D^+ v(t) = D^+ V_{(1)}(t, x(t)) \qquad \left((t, x(t)) \in \bigcup_{k=1}^{\infty} G_k \right)$$

holds [17].

When proving the main results we shall use the following lemmas:

LEMMA 1 [8]. *Let the function $v: I \to I$ be continuous for $t \neq \tau_k$, have a first kind discontinuity at the points τ_k, $k = 1, 2, \ldots$, and be left continuous. Also, let the following inequalities be fulfilled*:

$$D^+ v(t) \leqslant \kappa v(t) \qquad (t > t_0 \geqslant 0,\ t \neq \tau_k),$$
$$v(\tau_k + 0) \leqslant d_k v(\tau_k) \qquad (\tau_k > t_0),$$
$$v(t_0 + 0) \leqslant c,$$

where κ, $d_k \geqslant 0$, and $c \geqslant 0$ are constants. Then for $t > t_0$ the inequality

$$v(t) \leqslant c \prod_{t_0 < \tau_k < t} d_k e^{\kappa(t - t_0)}$$

holds.

LEMMA 2. *Let the function $u: I \to I$ be continuous for $t \neq \tau_k$, have a first kind discontinuity at the points τ_k, $k = 1, 2, \ldots$, and satisfy the inequality*

$$u(t) \leqslant u_0 + \int_{t_0}^{t} \alpha u(s)\, ds + \sum_{t_0 < \tau_k < t} \beta_k u(\tau_k) \qquad (t > t_0),$$

where $\alpha \geqslant 0$, $\beta_k \geqslant 0$, and $u_0 \geqslant 0$ are constants.

Then for $t > t_0$ the inequality

$$u(t) \leqslant u_0 \prod_{t_0 < \tau_k < t} (1 + \beta_k)\, e^{\alpha(t - t_0)}$$

holds.

Proof. The function $v(t) = u_0 + \int_{t_0}^{t} \alpha u(s)\, ds + \sum_{t_0 < \tau_k < t} \beta_k u(\tau_k)$ satisfies the conditions of Lemma 1 with $\kappa = d$, $d_k = 1 + \beta_k$, and $c = u_0$. We apply Lemma 1 and obtain the assertion of Lemma 2.

3. MAIN RESULTS

THEOREM 1. *Let the conditions* (H) *hold. Then system* (1) *has property* (A) *if and only if the trivial solution of system* (1) *is* $(|f|; |I_k|)$*-exponentially stable.*

Proof. *Sufficiency*. Let the trivial solution of system (1) be $(|f|; |I_k|)$-

exponentially stable; i.e., there exist constants $M, \Delta, \alpha (M>0,\ 0<\Delta \leqslant \rho,\ \alpha>0)$ such that for every $(t_0, x_0) \in I \times B_\Delta$

$$|f(t, x(t; t_0, x_0))| \leqslant M\, |x_0|\, e^{-\alpha(t-t_0)} \qquad (t>t_0), \tag{4}$$

$$|I_k(x(\tau_k; t_0, x_0))| \leqslant M\, |x_0|\, e^{-\alpha(\tau_k-t_0)} \qquad (k\colon \tau_k>t_0). \tag{5}$$

Then the integral $\int_t^\infty f(s, x(s; t_0, x_0))\, ds$ and the series $\sum_{\tau_k \geqslant t} I_k(x(\tau_k; t_0, x_0))$ are absolutely convergent for every $t>t_0$ and

$$\left|\int_t^\infty f(s, x(s; t_0, x_0))\, ds\right| \leqslant \int_t^\infty M\, |x_0|\, e^{-\alpha(s-t_0)}\, ds = \frac{M\, |x_0|}{\alpha}\, e^{-\alpha(t-t_0)},$$

$$\left|\sum_{\tau_k \geqslant t} I_k(x(t_k; t_0, x_0))\right| \leqslant \sum_{\tau_k \geqslant t} M\, |x_0|\, e^{-\alpha(\tau_k-t_0)} \leqslant \frac{M\, |x_0|}{1-e^{-\alpha\theta}}\, e^{-\alpha(t-t_0)}.$$

Moreover, for $t>t_0$

$$\left|x(t; t_0, x_0) - x_0 - \int_{t_0}^\infty f(s, x(s; t_0, x_0))\, ds - \sum_{\tau_k>t_0} I_k(x(\tau_k; t_0, x_0))\right|$$
$$= \left|\int_t^\infty f(s, x(s; t_0, x_0))\, ds + \sum_{\tau_k \geqslant t} I_k(x(\tau_k; t_0, x_0))\right|$$
$$\leqslant M\left(\frac{1}{\alpha} + \frac{1}{1-e^{-\alpha\theta}}\right) |x_0|\, e^{-\alpha(t-t_0)};$$

i.e., system (1) has property (A) and

$$K(t_0, x_0) = x_0 + \int_{t_0}^\infty f(s, x(s; t_0, x_0))\, ds$$
$$+ \sum_{\tau_k>t_0} I_k(x(\tau_k; t_0, x_0)).$$

Necessity. Let system (1) have property (A); i.e., there exist constants $M>0$, $\Delta>0$, and $\alpha>0$ such that for every $(t_0, x_0) \in I \times B_\Delta$ the inequality (2) is fulfilled.

For the function $x(t) = x(t; t_0, x_0)$ from the conditions H2, H4 and (3), it follows that

$$|f(t, x(t; t_0, x_0))| \leqslant L(2M+1)\, |x_0| \qquad (t>t_0 \geqslant 0,\ x_0 \in B_\Delta)$$

Let $\tau_{i-1} < t_0 \leqslant \tau_i$. Then for $\tau_i \leqslant \tau_k < t_1 \leqslant t_2 \leqslant \tau_{k+1}$ we have

$$|x(t_2) - x(t_1)| \leqslant \int_{t_1}^{t_2} |f(s, x(s))| \, ds \leqslant L(2M+1)\,|x_0|\,(t_2 - t_1),$$

$$\begin{aligned} |\dot{x}(t_2) - \dot{x}(t_1)| &= |f(t_2, x(t_2)) - f(t_1, x(t_1))| \\ &\leqslant |f(t_2, x(t_2)) - f(t_1, x(t_2))| \\ &\quad + |f(t_1, x(t_2)) - f(t_1, x(t_1))| \\ &\leqslant l\,|x(t_2)|\,(t_2 - t_1) + L\,|x(t_2) - x(t_1)| \\ &\leqslant (l + L^2)(2M+1)\,|x_0|\,(t_2 - t_1). \end{aligned} \tag{6}$$

We introduce the function $g(t) = x(t) - K(t_0, x_0)$ for $t > t_0$. From (6) and (2) for $g(t)$ we obtain

$$|g(t)| \leqslant N\,|x_0|\,e^{-\alpha(t-t_0)} \qquad (t > t_0), \tag{7}$$

$$|\dot{g}(t_2) - \dot{g}(t_1)| \leqslant N\,|x_0|\,(t_2 - t_1) \qquad (\tau_i \leqslant \tau_k < t_1 \leqslant t_2 < \tau_{k+1}), \tag{8}$$

where $N = \max(M, (l + L^2)(2M+1))$.

We choose successively the numbers $\lambda \in (0, 1)$ and $\beta > 0$ such that $(1-\lambda)/\lambda > 4(1 + e^{\alpha\theta/2})/\theta^2$, $\beta = \theta/2\lambda$. Then

$$\beta^2\lambda(1-\lambda) - e^{-\alpha\beta\lambda} > \beta^2\lambda(1-\lambda) - e^{\alpha\beta\lambda} > 1. \tag{9}$$

We shall prove that for this choice of β for $t > \tau_i$ the estimate

$$|\dot{g}(t)| \leqslant \sqrt{n}\,\beta N\,|x_0|\,e^{-(\alpha/2)(t-t_0)} \tag{10}$$

holds.

Let us assume that there exist j $(1 \leqslant j \leqslant n)$ and T $(\tau_i \leqslant \tau_k < T \leqslant \tau_{k+1})$ such that for the jth component $g_j(t)$ of $g(t)$ the inequality

$$|g_j(T)| > \beta N\,|x_0|\,e^{-(\alpha/2)(T-t_0)} \equiv \delta N\,|x_0|, \tag{11}$$

is fulfilled, where $\delta = \beta e^{-(\alpha/2)(T-t_0)} < \beta$.

(a) Let $\tau_k < T \leqslant \frac{1}{2}(\tau_k + \tau_{k+1})$. Then, according to condition H3 and the choice of β, the number $T_1 = T + \lambda\delta$ is smaller than τ_{k+1}, and for $t \in [T, T_1]$ we have

$$\begin{aligned} |\dot{g}_j(t)| &\geqslant |\dot{g}_j(T)| - |\dot{g}_j(T) - \dot{g}(t)| \geqslant \delta N\,|x_0| - \lambda\delta N\,|x_0| \\ &= (1-\lambda)\delta N\,|x_0|. \end{aligned} \tag{12}$$

Using (11), (12), (7), (8), and (9), we get

$$N|x_0|\,e^{-\alpha(T-t_0)} \geqslant |g_j(T)| \geqslant |g_j(T)-g_j(T_1)| - |g_j(T_1)|$$

$$\geqslant \int_T^{T_1} |\dot{g}_j(s)|\,ds - N|x_0|\,e^{-\alpha(T_1-t_0)}$$

$$\geqslant \int_T^{T_1} (1-\lambda)\delta N|x_0|\,ds - N|x_0|\,e^{-\alpha(T_1-t_0)}$$

$$\geqslant \lambda(1-\lambda)\,\delta^2 N|x_0| - N|x_0|\,e^{-\alpha(T_1-t_0)}$$

$$= N|x_0|\,e^{-\alpha(T-t_0)}(\beta^2\lambda(1-\lambda)-e^{-\alpha\lambda\delta})$$

$$> N|x_0|\,e^{-\alpha(T-t_0)},$$

which is a contradiction.

(b) Let $\frac{1}{2}(\tau_k+\tau_{k+1}) \leqslant T < \tau_{k+1}$. Then $T_1 \doteq T-\lambda\delta > \tau_k$ and for $t \in [T_1, T]$ we have

$$|\dot{g}_j(t)| \geqslant (1-\lambda)\delta N|x_0|,$$

after which we come to another contradiction:

$$N|x_0|\,e^{-\alpha(T-t_0)} \geqslant |g_j(T)| \geqslant |g_j(T_1)-g_j(T)| - |g_j(T_1)|$$

$$\geqslant \int_{T_1}^{T} |\dot{g}_j(s)|\,ds - N|x_0|\,e^{-\alpha(T_1-t_0)}$$

$$\geqslant N|x_0|\,e^{-\alpha(T-t_0)}(\beta^2\lambda(1-\lambda)-e^{\alpha\delta\lambda})$$

$$\geqslant N|x_0|\,e^{-\alpha(T-t_0)}.$$

Therefore, the assumption is not true and for every $i=1, ..., n$ and $t>\tau_i$ the inequality $|\dot{g}_j(t)| \leqslant \beta N e^{-(\alpha/2)(t-t_0)}|x_0|$ is fulfilled from where (10) follows. If $\tau_i - t_0 > \theta$, then similar to cases (a) and (b) we prove that (10) holds for $t \in (t_0, \tau_i]$. If $\tau_i - t_0 \leqslant \theta$, then for $t \in (t_0, \tau_i]$ the estimate

$$|\dot{g}(t)| = |f(t, x(t; t_0, x_0))| \leqslant L(2M+1)\,|x_0|$$

$$\leqslant L(2M+1)e^{\alpha\theta/2}e^{-\alpha\theta/2}\,|x_0|$$

$$\leqslant L(2M+1)e^{\alpha\theta/2}e^{-(\alpha/2)(t-t_0)}\,|x_0|$$

holds; i.e., for $t > t_0$

$$|f(t, x(t; t_0, x_0))| \leqslant Q\,|x_0|\,e^{-(\alpha/2)(t-t_0)},$$

where

$$Q = \max(M, (l+L^2)(2M+1), L(2M+1)e^{\alpha\theta/2}).$$

Moreover, for $\tau_k > t_0$

$$|I_k(x(\tau_k; t_0, x_0))| = |x(\tau_k + 0) - x(\tau_k)| = |g(\tau_k + 0) - g(\tau_k)| \leqslant 2M\,|x_0|\, e^{-\alpha(\tau_k - t_0)} \tag{13}$$

thus proving that the trivial solution of system (1) is $(|f|; |I_k|)$-exponentially stable.

THEOREM 2. *Let the conditions* H1–H4 *hold. Then the following assertion hold*:

1. *Let there exist constants* $M, \Delta, \alpha (M > 0,\ 0 < \Delta \leqslant \rho,\ \alpha > 0)$ *such that for every* $(t_0, x_0) \in I \times B_\Delta$ *there exist* $K(t_0, x_0) \in B_\rho$ *such that for every* $t > t_0$ *the estimates*

$$|x(t; t_0, x_0) - K(t_0, x_0)| \leqslant M\,|x_0|\, e^{-\alpha(t - t_0)} \tag{14}$$

$$|f(t, K(t_0, x_0))| \leqslant M\,|x_0|\, e^{-\alpha(t - t_0)} \tag{15}$$

are fulfilled.

Then the trivial solution of system (1) *is* $(|f|; |I_k|)$*-exponentially stable.*

2. *Let the trivial solution of system* (1) *be* $(|f|; |I_k|)$*-exponentially stable. Then there exist constants* $M, \Delta, \alpha (M > 0,\ 0 < \Delta \leqslant \rho,\ \alpha > 0)$ *such that for every* $(t_0, x_0) \in I \times B_\Delta$ *there exists* $K(t_0, x_0) \in R^n$ *such that* (14) *holds for* $t > t_0$. *If, additionally,* $K(t_0, x_0) \in B_\rho$ *then estimate* (15) *holds.*

The proof of Theorem 2 is carried out by applying the inequality of the triangle.

Let system (1) have property (A) and for $(t_0, x_0) \in I \times B_\Delta$, $PK(t_0, x_0) \equiv 0$ is fulfilled, where P is an $n \times n$-matrix. Then

$$|Px(t; t_0, x_0)| \leqslant |P|\, |x(t; t_0, x_0) - K(t_0, x_0)| + |PK(t_0, x_0)| \leqslant |P|\, M\,|x_0|\, e^{-\alpha(t - t_0)};$$

i.e., the trivial solution of system (1) is $(|P|; |I_k|)$-exponentially stable. If additionally, condition H5 holds, then the trivial solution of system (1) is $(|P| + |f|; |I_k|)$-exponentially stable.

THEOREM 3. *Let the conditions* (H) *hold and let there exist constants* $M, \delta, \alpha (M > 0,\ 0 < \delta < \rho L^{-1},\ \alpha > 0)$ *such that if* $x(t; t_0, x_0)$ *is a solution of system* (1) *and* $x(t; t_0, x_0) \in B_\delta$ *for* $t_0 < t < \omega$, *then the inequalities*

$$|Px(t; t_0, x_0)| + |(E-P) f(t, x(t; t_0, x_0))|$$
$$\leqslant M\,|x_0|\, e^{-\alpha(t-t_0)} \qquad (t_0 < t < \omega), \qquad (16)$$
$$|(E-P)\, I_k(x(\tau_k; t_0, x_0)) \leqslant M\,|x_0|\, e^{-\alpha(\tau_k - t_0)} \qquad (t_0 < \tau_k < \omega) \qquad (17)$$

hold, where P is an $n \times n$*-matrix.*

Then the trivial solution of system (1) *is* $(|P| + |(E-P) f|; |(E-P) I_k|)$*-exponentially stable and system* (1) *has property* (A) *with*

$$K(t_0, x_0) = (E-P)\left(x_0 + \int_{t_0}^{\infty} f(s, x(s; t_0, x_0))\, ds + \sum_{\tau_k > t_0} I_k(x(\tau_k; t_0, x_0))\right).$$

Proof. Let $0 < \Delta < \delta(M + M\alpha^{-1} + M(1-e^{-\alpha\theta})^{-1} + |E-P|)^{-1}(1+L)^{-1}$ and $(t_0, x_0) \in I \times B_\Delta$. Let (t_0, ω) be the maximum interval, such that $x(t; t_0, x_0) \in B_\delta$ for $t \in (t_0, \omega)$. Let us assume $\omega < \infty$. From (16) and (17) it follows that the solution $x(t) = x(t; t_0, x_0)$ satisfies the inequalities

$$|(E-P)(x(t) - x_0)|$$
$$= \left|(E-P)\int_{t_0}^{t} f(s, x(s))\, ds + (E-P) \sum_{t_0 < \tau_k < t} I_k(x(\tau_k))\right|$$
$$\leqslant \int_{t_0}^{t} M\,|x_0|\, e^{-\alpha(s-t_0)}\, ds + \sum_{t_0 < \tau_k < t} M\,|x_0|\, e^{-\alpha(\tau_k - t_0)}$$
$$\leqslant \left(\frac{M}{\alpha} + \frac{M}{1-e^{-\alpha\theta}}\right) |x_0|, \qquad (18)$$
$$|Px(t)| \leqslant M\,|x_0|\, e^{-\alpha(t-t_0)} \leqslant M\,|x_0|,$$
$$|x(t)| \leqslant |(E-P)x(t)| + |Px(t)|$$
$$\leqslant |(E-P)(x(t)-x_0)| + |(E-P)x_0| + |Px(t)|$$
$$\leqslant \left(\frac{M}{\alpha} + \frac{M}{1-e^{-\alpha\theta}} + M + |E-P|\right) |x_0| < \frac{\delta}{1+L} < \delta$$
$$(t_0 < t < \omega). \qquad (19)$$

In particular, $|x(\omega)| = |x(\omega - 0)| < \delta/(1+L)$ and recalling H2 and H4 we get $|x(\omega+0)| \leqslant (1+L)\,|x(\omega)| < \delta$. Then there exists $\omega_1 > \omega$ such that $|x(t)| < \delta$ for $t \in (t_0, \omega_1) \supset (t_0, \omega)$, a contradiction. Hence, $\omega = \infty$ and estimates (16)–(19) hold for $t > t_0$.

Thus, the trivial solution of system (1) is $(|P| + |(E-P) f|; |(E-P) I_k|)$-

exponentially stable. Besides, from estimate (18) it follows that the integral $\int_t^\infty (E-P)f(s,x(s))\,ds$ and the series $\Sigma_{\tau_k \geqslant t}(E-P)I_k(x(\tau_k))$ are absolutely convergent for $t > t_0$ and then

$$\begin{aligned}
&\left| x(t)-(E-P)\left(x_0+\int_{t_0}^{\infty} f(s,x(s))\,ds+\sum_{\tau_k>t_0} I_k(x(\tau_k))\right)\right| \\
&\quad\leqslant |Px(t)|+\left|(E-P)\int_t^{\infty} f(s,x(s))\,ds+(E-P)\sum_{\tau_k>t} I_k(x(\tau_k))\right| \\
&\quad\leqslant M\,|x_0|\,e^{-\alpha(t-t_0)}+M\alpha^{-1}\,|x_0|\,e^{-\alpha(t-t_0)} \\
&\qquad+M(1-e^{-\alpha\theta})^{-1}\,|x_0|\,e^{-\alpha(t-t_0)} \\
&\quad\leqslant (M+M\alpha^{-1}+M(1-e^{-\alpha\theta})^{-1})\,|x_0|\,e^{-\alpha(t-t_0)}.
\end{aligned}$$

Thus Theorem 3 is proved.

THEOREM 4. *Let the conditions* (H) *hold.*

Then the trivial solution of system (1) *is* $(|f|; |I_k|)$*-exponentially stable if and only if there exists a function* $V \in \mathscr{V}_0$, $V: I \times B_\Delta \to I$ *and positive constants* m, C_i $(i=1,2,3)$ *such that the following conditions hold*:

$$|f(t,x)|^m \leqslant V(t,x) \leqslant C_1\,|x|^m \qquad (t\in I,\ x\in B_\Delta), \tag{20}$$

$$|I_k(x)|^m \leqslant V(\tau_k,x) \qquad (x\in B_\Delta,\ k=1,2,...), \tag{21}$$

$$|V(t,x)-V(t,y)| \leqslant C_2\,|x-y| \qquad (t\in I;\ x,y\in B_\Delta), \tag{22}$$

$$D^+V_{(1)}(t,x) \leqslant -C_3 V(t,x) \qquad \left((t,x)\in \bigcup_{k=1}^{\infty} G_k\right), \tag{23}$$

$$V(\tau_k+0,\ x+I_k(x)) \leqslant V(\tau_k,x) \qquad (x\in B_\Delta,\ k=1,2,...) \tag{24}$$

Proof. Sufficiency. Let the conditions (20)–(24) be fulfilled, let $(t_0,x_0)\in I\times B_\Delta$, and let $x(t)=x(t;t_0,x_0)$ be a solution of system (1), defined in $J^+=(t_0,\omega)$. Then for the function $v(t)=V(t,x(t))$ from (23) and (24) we have

$$D^+v(t) \leqslant -C_3 v(t) \qquad (t\in J^+,\ t\neq\tau_k),$$

$$v(\tau_k+0) \leqslant v(\tau_k) \qquad (\tau_k\in J^+).$$

We apply Lemma 1 and get $v(t)\leqslant v(t_0+0)e^{-C_3(t-t_0)}$ for $t\in J^+$. Then from (20) and (21) it follows that

$$|f(t,x(t;t_0,x_0))| \leqslant C_1^{1/m}\,|x_0|\,e^{-(C_3/m)(t-t_0)} \qquad (t_0<t<\omega),$$

$$|I_k(x(\tau_k;t_0,x_0))| \leqslant C_1^{1/m}\,|x_0|\,e^{-(C_3/m)(\tau_k-t_0)} \qquad (t_0<\tau_k<\omega).$$

Applying Theorem 3 (with $P=0$) we obtain that the trivial solution of system (1) is $(|f|; |I_k|)$-exponentially stable.

Necessity. Let the trivial solution of system (1) be $(|f|; |I_k|)$-exponentially stable and the estimates (4) and (5) with constants $M, \Delta, \alpha (M>0,\quad 0<\Delta \leqslant \rho,\quad \alpha>0)$ be fulfilled. Let $\beta>0$ and $m=1+(L+\theta^{-1}\ln(1+L)+\beta)\alpha^{-1}$. For $(t, x)\in I\times B_\Delta$ we define

$$V(t, x)=W(t, x)+U(t, x), \tag{25}$$

where

$$\begin{aligned} W(t, x) &= \sup_{s>0} |f(t+s, x(t+s; t, x))|^m e^{\beta s} \qquad (t\neq \tau_k), \\ U(t, x) &= \sup_{k: \tau_k>t} |I_k(x(\tau_k; t, x))|^m e^{\beta(\tau_k-t)} \qquad (t\neq\tau_k), \end{aligned} \tag{26}$$

$$W(\tau_k, x)=W(\tau_k-0, x), \qquad U(\tau_k, x)=U(\tau_k-0, x). \tag{27}$$

From (26) and (27) it immediately follows that

$$W(t, x)\geqslant |f(t, x)|^m \qquad (t\geqslant 0) \tag{28}$$

and from (4) and (26) it follows that

$$W(t, x)\leqslant \sup_{s>0} (M\,|x|\,e^{-\alpha s})^m e^{\beta s}\leqslant M^m\,|x|^m\equiv C_1\,|x|^m. \tag{29}$$

Using conditions H3 and H4 and applying Lemma 2 we get the estimate

$$\begin{aligned} &|x(t; t_0, x_0)-x(t; t_0, y_0)| \\ &\quad\leqslant |x_0-y_0|\,(1+L)^{i(t_0,t)}e^{L(t-t_0)} \\ &\quad\leqslant |x_0-y_0|\,(1+L)e^{(L+(1/\theta)\ln(1+L))(t-t_0)} \qquad (t>t_0). \end{aligned} \tag{30}$$

Then from the inequality

$$|a^m-b^m|\leqslant m\,|a-b|\,(\max(a, b))^{m-1} \qquad (a\geqslant 0, b\geqslant 0)$$

and from extimate (30) it follows that for $x_0, y_0\in B_\Delta$ and $t>t_0$

$$\begin{aligned} &||f(t, x(t; t_0, x_0))|^m-|f(t, x(t; t_0, y_0))|^m| \\ &\quad\leqslant m\,|f(t, x(t; t_0, x_0))-f(t, x(t; t_0, y_0))| \\ &\qquad\times (M\max(|x_0|, |y_0|)e^{-\alpha(t-t_0)})^{m-1} \\ &\quad\leqslant mL(1+L)(M\Delta)^{m-1}\,|x_0-y_0|\,e^{(L+(1/\theta)\ln(1+L)-\alpha(m-1))(t-t_0)}. \end{aligned}$$

Therefore, according to the choice of m, we shall have

$$\begin{aligned}
&|W(t_0, x_0) - W(t_0, y_0)| \\
&\quad = |\sup_{s>0} | f(t_0+s, x(t_0+s; t_0, x_0))|^m e^{\beta s} \\
&\quad\quad - \sup_{s>0} |f(t_0+s, x(t_0+s; t_0, y))|^m e^{\beta s}| \\
&\quad \leqslant \sup_{s>0} ||f(t_0+s, x(t_0+s; t_0, x_0))|^m \\
&\quad\quad - |f(t_0+s, x(t_0+s; t_0, y_0))|^m | e^{\beta s} \\
&\quad \leqslant mL(1+L)(M\Delta)^{m-1} |x_0-y_0| \equiv \frac{C_2}{2} |x_0-y_0|. \qquad (31)
\end{aligned}$$

Let $x' = x(t+\tau; t, x)$. Then, successively we get

$$\begin{aligned}
W(t+\tau, x') &= \sup_{s>0} f(t+\tau+s, x(t+\tau+s; t+\tau, x'))|^m e^{\beta s} \\
&= \sup_{\sigma>\tau} |f(t+\sigma, x(t+\sigma; t, x))|^m e^{\beta\sigma} e^{-\beta\tau} \leqslant W(t, x) e^{-\beta\tau};
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\tau} (W(t+\tau, x(t+\tau; t, x)) - W(t, x)) \leqslant W(t, x) \frac{e^{-\beta\tau}-1}{\tau}; \\
&\qquad D^+ W_{(1)}(t, x) \leqslant -\beta W(t, x).
\end{aligned} \qquad (32)$$

Let $x \in B_\rho$, $x + I_k(x) \in B_\rho$, and $\xi(t)$, $\eta(t)$ be the solutions of the initial value problems

$$\frac{d\xi}{dt} = f(t, \xi), \qquad \xi(\tau_k) = x, \quad t \leqslant \tau_k;$$

$$\frac{d\eta}{dt} = f(t, \eta), \qquad \eta(\tau_k) = x + I_k(x), \quad t \geqslant \tau_k.$$

Let $\tau_{k-1} < t < \tau \leqslant \tau_k$, $y \in B_\rho$, $z \in B_\rho$. Then

$$\begin{aligned}
&|W(t, y) - W(\tau, z)| \\
&\quad \leqslant |W(t, y) - W(t, \xi(t))| + |W(t, \xi(t)) - W(\tau, \xi(\tau))| \\
&\quad\quad + |W(\tau, \xi(\tau)) - W(\tau, z)| \qquad (33)
\end{aligned}$$

and having in mind (31) and condition H4 we get

$$|W(t, y) - W(t, \xi(t))| \leqslant C_2 |y - \xi(t)| \leqslant C_2 |y - x| + C_2 |x - \xi(t)| \leqslant C_2 |y - x| + C_2 L\rho |t - \tau_k|; \quad (34)$$

$$|W(\tau, \xi(\tau)) - W(\tau, z)| \leqslant C_2 |x - z| + C_2 L_\rho |\tau - \tau_k|. \quad (35)$$

Moreover,

$$W(t, \xi(t)) = \sup_{s > t} |f(s, x(s; t, \xi(t)))|^m e^{\beta(s-t)}$$
$$= \max \left[\sup_{t < s \leqslant \tau} |f(s, x(s; t, \xi(t)))|^m e^{\beta(s-t)}, \sup_{s > \tau} |f(s, x(s; t, \xi(t)))|^m e^{\beta(s-t)} \right].$$

But since $x(s; t, \xi(t)) = x(s; \tau, \xi(\tau))$ for $s > \tau$ then

$$\sup_{s > \tau} |f(s, x(s; t, \xi(t)))|^m e^{\beta(s-t)}$$
$$= \sup_{s > \tau} |f(s, x(s; \tau, \xi(\tau)))|^m e^{\beta(s-\tau)} \cdot e^{\beta(\tau - t)} = W(\tau, \xi(\tau)) e^{\beta(\tau - t)}$$

and

$$\sup_{t < s \leqslant \tau} |f(s, x(s; t, \xi(t)))|^m e^{\beta(s-t)}$$
$$\leqslant |f(\tau, \xi(\tau))|^m e^{\beta(\tau - t)} + \sup_{\tau < s \leqslant t} ||f(s, \xi(s))|^m - |f(\tau, \xi(\tau))|^m| e^{\beta(\tau - t)}$$
$$\leqslant W(\tau, \xi(\tau)) e^{\beta(\tau - t)} + \alpha(t, \tau),$$

where

$$\alpha(t, \tau) = \sup_{\tau < s \leqslant t} ||f(s, \xi(s))|^m - |f(\tau, \xi(\tau))|^m| e^{\beta(\tau - t)} \leqslant \text{const } |t - \tau|.$$

Therefore,

$$|W(t, \xi(t)) - W(\tau, \xi(\tau)) e^{\beta(\tau - t)}| \leqslant \alpha(t, \tau) \leqslant \text{const } |t - \tau|. \quad (36)$$

From (33)–(36) it follows that the limit $W(\tau_k - 0, x)$ exists. The existence of the limits $W(\tau_k + 0, x)$ and $W(\tau_k + 0, x + I_k(x))$ is proved similarly.

Let $\tau_{k-1} < \lambda < \tau_k < \mu < \tau_{k+1}$, $y \in B_\rho$, $z \in B_\rho$.

Since $x(s; \lambda, \xi(\lambda)) = x(s; \mu, \eta(\mu))$ for $s > \mu$ then

$$W(\mu, \eta(\mu)) \leqslant W(\lambda, \xi(\lambda)). \tag{37}$$

Recalling that $\lim_{\lambda \to \tau_k - 0} |\xi(\lambda) - x| = 0$ and $\lim_{\mu \to \tau_k + 0} |\eta(\mu) - x - I_k(x)| = 0$ after a passage to the limit in (37) we obtain

$$W(\tau_k + 0, x + I_k(x)) \leqslant W(\tau_k - 0, x) = W(\tau_k, x). \tag{38}$$

With similar reasonings it is proved that for the function U the inequalities

$$U(t, x) \leqslant M^m |x|^m \equiv \frac{C_1}{2} |x|^m, \tag{39}$$

$$U(\tau_k, x) = U(\tau_k - 0, x) \geqslant |I_k(x)|^m, \tag{40}$$

$$|U(t, x) - U(t, y)| \leqslant \frac{C_2}{2} |x - y|, \tag{41}$$

$$D^+ U_{(1)}(t, x) \leqslant -\beta U(t, x), \tag{42}$$

$$U(\tau_k + 0, x + I_k(x)) \leqslant U(\tau_k, x) \tag{43}$$

hold.

From (25), (28), (29), (31), (32), (38)–(43) it follows that the function V satisfies the conditions (20)–(24).

Let P be an arbitrary $n \times n$-matrix.

THEOREM 5. *Let the conditions* (H) *hold. Then*:

1. *If there exists a function* $V \in \mathscr{V}_0$, $V: I \times B_\Delta \to I (0 < \Delta \leqslant \rho)$, *and positive constants* m, C_i $(i = 1, 2, 3)$ *such that the conditions* (22)–(24) *and the conditions*

$$|Px|^m + |(E-P) f(t, x)|^m \leqslant V(t, x) \leqslant C_1 |x|^m \qquad (t \in I, x \in B_\Delta), \tag{44}$$

$$|(E-P) I_k(x)|^m \leqslant V(\tau_k, x) \qquad (x \in B_\Delta, k = 1, 2, ...) \tag{45}$$

are fulfilled, then the trivial solution of system (1) *is* $(|P| + |(E-P) f|;$ $|(E-P) I_k|)$*-exponentially stable.*

2. *If the trivial solution of system* (1) *is* $(|P| + |(E-P) f|;$ $|(E-P) I_k|)$*-exponentially stable, then there exists a function* $V \in \mathscr{V}_0$, *which satisfies conditions* (22)–(24) *and conditions* (44), (45) *with appropriate positive constants* m, Δ, C_i $(i = 1, 2, 3)$.

3. *If the trivial solution of system* (1) *is* $(|P| + |f|; |I_k|)$*-exponentially*

stable, then there exists a function $V \in \mathscr{V}_0$, *which satisfies conditions* (22)–(24) *and conditions*

$$|Px|^m + |f(t, x)|^m \leqslant V(t, x) \leqslant C_1 |x|^m,$$
$$|I_k(x)|^m \leqslant V(\tau_k, x)$$

with appropriate positive constants m, Δ, C_i $(i = 1, 2, 3)$.

Proof. The proof of Theorem 5 is based on Theorem 3 and is analogous to the proof of Theorem 4 using respectively the functions

$$\begin{aligned} V(t, x) = \sup_{s>0} |Px(t+s; t, x)|^m e^{\beta s} \\ + \sup_{s>0} |(E-P) f(t+s, x(t+s; t, x))|^m e^{\beta s} \\ + \sup_{k:\tau_k > t} |(E-P) I_k(x(\tau_k; t, x))|^m e^{\beta(\tau_k - t)} \end{aligned} \tag{46}$$

and

$$\begin{aligned} V(t, x) = \sup_{s>0} |Px(t+s; t, x)|^m e^{\beta s} \\ + \sup_{s>0} |f(t+s, x(t+s; t, x))|^m e^{\beta s} \\ + \sup_{k.\tau_k > t} |I_k(x(\tau_k; t, x))|^m e^{\beta(\tau_k - t)}. \end{aligned} \tag{47}$$

Remark 2. The function V from assertion 3 of Theorem 5 satisfies condition (22) with $C_2 = m(1+L)(M\Delta)^{m-1} (2L + |P|)$.

Let us consider the linear system

$$\begin{aligned} \frac{dx}{dt} = A(t)x, \qquad t \neq \tau_k, \\ \Delta x|_{t=\tau_k} = B_k x, \qquad k = 1, 2, \ldots \end{aligned} \tag{48}$$

and its respective perturbed system

$$\begin{aligned} \frac{dx}{dt} = A(t)x + f(t, x), \qquad t \neq \tau_k, \\ \Delta x|_{t=\tau_k} = B_k x + I_k(x), \qquad k = 1, 2, \ldots, \end{aligned} \tag{49}$$

where $A(t)$ and B_k satisfy the following conditions (C):

C1. $A(t)$ and B_k are $n \times n$-matrices, $A(t)$ is continuous in I, and there exists a constant $L > 0$ such that $|A(t)| \leqslant L$, $|B_k| \leqslant L$ $(t \in I, k = 1, 2, \ldots)$.

C2. There exists a constant $K>0$ such that $|A(t_1)-A(t_2)| \leqslant K|t_1-t_2|$ $(t_1, t_2 \in I)$.

Remark 3. Since system (48) is linear, then for every $\lambda \in R$, $x_0, y_0 \in R^n$, and $t>t_0 \geqslant 0$

$$x(t; t_0, \lambda x_0) = \lambda x(t; t_0, x_0), \tag{50}$$

$$x(t; t_0, x_0) - x(t; t_0, y_0) = x(t; t_0, x_0 - y_0) \tag{51}$$

is fulfilled. If the trivial solution of system (48) is $(|A|; |B_k|)$-exponentially stable, then from (50) and Theorem 1 it follows that there exist positive constants M and α such that for every $(t_0, x_0) \in I \times R^n$

$$|A(t)x(t; t_0, x_0)| \leqslant M |x_0| e^{-\alpha(t-t_0)} \qquad (t>t_0),$$

$$|B_k x(\tau_k; t_0, x_0)| \leqslant M |x_0| e^{-\alpha(\tau_k - t_0)} \qquad (\tau_k > t_0)$$

is fulfilled.

In this case, following the proof of Theorem 4 and having in mind (51), we obtain that there exists a function $V \in \mathscr{V}_0$ for which the conditions

$$|A(t)x| \leqslant V(t, x) \leqslant 2M|x| \qquad (t \in I, x \in R^n),$$

$$|B_k x| \leqslant V(\tau_k, x) \qquad (x \in R^n, k=1, 2, ...), \tag{52}$$

$$|V(t,x) - V(t,y)| \leqslant 2M|x-y| \qquad (x, y \in R^n, t \in I), \tag{53}$$

$$D^+ V_{(48)}(t, x) \leqslant -\alpha V(t, x) \qquad (t \in I, x \in R^n), \tag{54}$$

$$V(\tau_k + 0, x + B_k x) \leqslant V(\tau_k, x) \qquad (x \in R^n, k = 1, 2, ...) \tag{55}$$

are fulfilled.

We shall also note that in the linear case, the functions (46) and (47) from Theorem 5 satisfy the inequality (54) and in Theorem 5 we can choose $m=1$.

Let P be an arbitrary $n \times n$-matrix.

THEOREM 6. *Let the conditions* (H) *and* (C) *be fulfilled, let system* (48) *have property* (A) *and its trivial solution be* $|P|$*-exponentially stable, and let functions* f *and* I_k *satisfy the inequalities*

$$|f(t,x)| \leqslant N(|Px|^{1+\gamma} + |A(t)x|^{1+\gamma}) \qquad (t \in I, x \in B_\rho), \tag{56}$$

$$|I_k(x)| \leqslant N(|Px|^{1+\gamma} + |B_k x|^{1+\gamma}) \qquad (x \in B_\rho, k=1, 2, ...), \tag{57}$$

where $N>0$ *and* $\gamma>0$ *are constants.*

Then system (49) *has property* (A) *and its trivial solution is* $|P|$*-exponentially stable.*

Proof. From Theorems 2 and 4, and according to Remark 3, it follows that there exists a function $V \in \mathscr{V}_0$, $V: I \times B_\rho \to I$ for which conditions (52)–(55) and condition

$$|Px| + |A(t)x| \leqslant V(t, x) \leqslant 2M\, |x| \tag{58}$$

are fulfilled. From (56), (57), (58), (52), (53), and (55) the estimates

$$\begin{aligned} |Px| + |A(t)x + f(t, x)| &\leqslant |Px| + |A(t)x| + |f(t, x)| \\ &\leqslant V(t, x) + N(|Px| + |A(t)x|)(|P|^\gamma + L^\gamma)\Delta^\gamma \\ &\leqslant (1 + \delta)\, V(t, x), \end{aligned} \tag{59}$$

$$|B_k x + I_k(x)| \leqslant (1 + \delta)\, V(\tau_k, x) \tag{60}$$

follow, where $\delta = N(|P|^\gamma + L^\gamma)\Delta^\gamma$. Also,

$$\begin{aligned} &V(\tau_k + 0, x + B_k x + I_k(x)) \\ &\quad = V(\tau_k + 0, x + B_k x) + V(\tau_k + 0, x + B_k x + I_k(x)) \\ &\qquad - V(\tau_k + 0, x + B_k x) \leqslant V(\tau_k, x) + 2M\, |I_k(x)| \\ &\quad \leqslant (1 + 2M\delta)\, V(\tau_k, x). \end{aligned} \tag{61}$$

For $D^+ V_{(49)}(t, x)$ the estimate

$$D^+ V_{(49)}(t, x) \leqslant D^+ V_{(48)}(t, x) + 2M\, |f(t, x)|$$

holds [18] or

$$D^+ V_{(49)}(t, x) \leqslant (-\alpha + 2M\delta)\, V(t, x). \tag{62}$$

We choose $\Delta > 0$ such that $\beta = \alpha - 2M\delta - \theta^{-1} \ln(1 + 2M\delta) > 0$. Let $(t_0, x_0) \in I \times B_\Delta$. Then from (62), (61), and Lemma 1, it follows that for $t > t_0$ the estimate

$$V(t, x(t; t_0, x_0)) \leqslant V(t_0, x_0)(1 + 2M\delta) e^{-\beta(t - t_0)} \tag{63}$$

is fulfilled and (63), (59), (60), and Theorem 1 yield that system (49) has property (A) and its trivial solution is $|P|$-exponentially stable.

Let us consider the system

$$\begin{aligned} \frac{dx}{dt} &= a(t, x), \qquad t \neq \tau_k, \\ \Delta x\, |_{t = \tau_k} &= b_k(x), \qquad k = 1, 2, \ldots \end{aligned} \tag{64}$$

and its perturbed system

$$\frac{dx}{dt} = a(t, x) + f(t, x), \qquad t \neq \tau_k,$$
$$\Delta x\,|_{t=\tau_k} = b_k(x) + I_k(x), \qquad k = 1, 2, \ldots. \tag{65}$$

We introduce the following condition:

D. The functions $a\colon I \times B_\rho \to R^n$ and $b_k\colon B_\rho \to R^n$ are continuous in their domains and for every $\eta > 0$ there exist real numbers $\sigma > 0$, $T \geqslant 0$, and $v > 0$, and an integer $k_0 \geqslant 1$ such that for every $t, s \in [T, \infty)$; $x, y \in B_\sigma$, and $k \geqslant k_0$ the inequalities

$$|a(t, x) - a(t, y)| \leqslant \eta\, |x - y|, \tag{66}$$

$$|b_k(x) - b_k(y)| \leqslant \eta\, |x - y|, \tag{67}$$

$$|a(t, x) - a(s, x)| \leqslant v\, |t - s| \tag{68}$$

are fulfilled and $a(t, 0) = 0$, $b_k(0) = 0$ $(t \in I, k = 1, 2, \ldots)$.

THEOREM 7. *Let the following conditions be fulfilled*:

1. *The conditions* (H) *and* (D) *hold.*
2. *For* $(t, x) \in I \times B_\rho$ *the inequalities*

$$|f(t, x) \leqslant N(|Px|^{1+\gamma} + |a(t, x)|^{1+\gamma}), \tag{69}$$

$$|I_k(x)| \leqslant N(|Px|^{1+\gamma} + |b_k(x)|^{1+\gamma}) \tag{70}$$

hold, where $N > 0$ *and* $\gamma > 0$ *are constants.*

3. *The system* (64) *has property* (A) *and its trivial solution is* $|P|$*-exponentially stable.*

Then system (65) *has property* (A) *and its trivial solution is* $|P|$*-exponentially stable.*

Proof. From the conditions 1 and 3 of Theorem 7 and according to Theorem 1, it follows that the trivial solution of system (64) is $(|P| + |a|; |b_k|)$-exponentially stable; i.e., there exist constants $M, \Delta, \alpha (M \geqslant 1, 0 < \Delta \leqslant \rho, \alpha > 0)$ such that for $(t_0, x_0) \in I \times B_\Delta$ the estimates

$$|Px(t; t_0, x_0)| + |a(t, x(t; t_0, x_0))| \leqslant M\, |x_0|\, e^{-\alpha(t - t_0)} \qquad (t > t_0),$$

$$|b_k(x(\tau_k; t_0, x_0))| \leqslant M\, |x_0|\, e^{-\alpha(\tau_k - t_0)} \qquad (\tau_k > t_0)$$

hold.

Let $0<\varepsilon<\gamma$. Successively we choose the positive constants β, η, and Δ such that:

$$\beta=\frac{\varepsilon\alpha}{2},$$

$$\eta\colon \frac{1}{\alpha}\left(\eta+\frac{1}{\theta}\ln(1+\eta)+\beta\right)=\varepsilon, \tag{71}$$

$$\Delta\colon \kappa\equiv\beta-\frac{1}{\theta}\ln(1+K(\Delta)C(\Delta))-K(\Delta)C(\Delta)>0, \tag{72}$$

$$(1+\varepsilon)(2r(\Delta))^{\varepsilon}<1, \tag{73}$$

where

$$\begin{aligned} K(\Delta)&=(1+\varepsilon)(2\eta+|P|)(1+\eta)(M\Delta)^{\varepsilon},\\ C(\Delta)&=M(|P|^{\gamma-\varepsilon}+\eta^{\gamma-\varepsilon})\Delta^{\gamma-\varepsilon}\\ r(\Delta)&=\eta\Delta+N(|P|^{1+\gamma}+\eta^{1+\gamma})\Delta^{1+\gamma}. \end{aligned} \tag{74}$$

We choose $\sigma(0<\sigma<\Delta)$, $T\geqslant 0$, $\nu>0$, and $k_0\geqslant 1$ such that the inequalities (66), (67), and (68) are fulfilled.

Then according to Theorem 5 there exists a function $V\in\mathscr{V}_0$, $V\colon [T,\infty)\times B_\sigma\to I$ such that

$$|Px|^{1+\varepsilon}+|a(t,x)|^{1+\varepsilon}\leqslant V(t,x)\leqslant C_1\,|x|^{1+\varepsilon}, \tag{75}$$

$$|b_k(x)|^{1+\varepsilon}\leqslant V(\tau_k,x), \tag{76}$$

$$|V(t,x)-V(t,y)|\leqslant K(\sigma)\,|x-y|, \tag{77}$$

$$D^+V_{(64)}(t,x)\leqslant -\beta V(t,x), \tag{78}$$

$$V(\tau_k+0,x+b_k(x))\leqslant V(\tau_k,x). \tag{79}$$

Using (69), (70), (75), and (76) we obtain successively

$$\begin{aligned} |f(t,x)|&\leqslant N(|Px|^{1+\varepsilon}\,|Px|^{\gamma-\varepsilon}+|a(t,x)|^{1+\varepsilon}\,|a(t,x)|^{\gamma-\varepsilon})\\ &\leqslant N(|Px|^{1+\varepsilon}+|a(t,x)|^{1+\varepsilon})(|P|^{\gamma-\varepsilon}+\eta^{\gamma-\varepsilon})\sigma^{\gamma-\varepsilon}\\ &\leqslant C(\sigma)\,V(t,x), \end{aligned} \tag{80}$$

$$|I_k(x)|\leqslant C(\sigma)\,V(\tau_k,x). \tag{81}$$

Recalling (77)–(81) we find that

$$\begin{aligned} D^+V_{(65)}(t,x)&\leqslant D^+V_{(64)}(t,x)+K(\sigma)\,|f(t,x)|\\ &\leqslant(-\beta+K(\sigma)C(\sigma))\,V(t,x),\\ V(\tau_k+0,x+b_k(x)+I_k(x))&\leqslant(1+K(\sigma)C(\sigma))\,V(\tau_k,x). \end{aligned}$$

Let $(t_0, x_0) \in [T, \infty) x B_\sigma$ and $v(t) = V(t, x(t; t_0, x_0))$. Then

$$D^+ v(t) \leqslant (-\beta + K(\sigma) C(\sigma)) v(t) \qquad (t \in J^+(t_0, x_0)),$$
$$v(\tau_k + 0) \leqslant (1 + K(\sigma) C(\sigma)) v(\tau_k) \qquad (\tau_k \in J^+(t_0, x_0)),$$

and applying Lemma 1 we get

$$V(t) \leqslant V(t_0 + 0)(1 + K(\sigma) \times C(\sigma)) e^{-(\beta - K(\sigma) C(\sigma) - (1/\theta) \ln(1 + K(\sigma) C(\sigma)))(t - t_0)}$$

or

$$V(t, x(t)) \leqslant V(t_0, x_0)(1 + K(\sigma) C(\sigma)) e^{-\kappa(t - t_0)} \leqslant C_1 (1 + K(\sigma) C(\sigma)) |x_0|^{1+\varepsilon} e^{-\kappa(t - t_0)}. \tag{82}$$

Since condition (73) is fulfilled then for the $u, v \in [0, r(\Delta))$ inequality

$$(u + v)^{1+\varepsilon} \leqslant v^{1+\varepsilon} + u \tag{83}$$

holds.

But for $|x| < \sigma$, according to (74), (66), (67), (69), and (70), we have that $|f(t, x)|$, $|a(t, x)|$, $|I_k(x)|$, and $|b_k(x)|$ do not exceed $r(\Delta)$. Therefore, from (83), (75), (76), (71), and (72) we obtain the estimate

$$|Px|^{1+\varepsilon} + |a(t, x) + f(t, x)|^{1+\varepsilon} \leqslant |Px|^{1+\varepsilon} + |a(t, x)|^{1+\varepsilon} + |f(t, x)| \leqslant (1 + C(\sigma)) V(t, x), \tag{84}$$

$$|b_k(x) + I_k(x)|^{1+\varepsilon} \leqslant |b_k(x)|^{1+\varepsilon} + |I_k(x)| \leqslant (1 + C(\sigma)) V(\tau_k, x). \tag{85}$$

From (84), (85), and (82) it follows that system (65) has property (A) and its trivial solution is $|P|$-exponentially stable.

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