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2-Gerbes bound by complexes of *gr*-stacks, and cohomology

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Abstract

We define 2-gerbes bound by complexes of braided group-like stacks. We prove a classification result in terms of hypercohomology groups with values in abelian crossed squares and cones of morphisms of complexes of length 3. We give an application to the geometric construction of certain elements in Hermitian Deligne cohomology groups.

MSC: 18D05; 18D30; 14F43

0. Introduction

The aim of the present work is to study in some detail gerbes and, mostly, 2-gerbes bound by complexes of groups and braided *gr*-stacks, respectively, and the cohomology groups determined by their equivalence classes.

0.1. Background and motivations

The idea of a gerbe bound by a complex is of course not new: it dates back to Debremaeker [1] in the form of a gerbe $\mathscr{G}$ on a site S bound by a crossed module $\delta\colon A \to B$. Milne [2] adopts the same idea in the special case of an *abelian* crossed module. It is observed in loc. cit. that the crossed module in fact reduces to a homomorphism of sheaves of abelian groups, and the whole structure simplifies to that of a gerbe $\mathscr{G}$ bound by the sheaf A and equipped with a functor $\mathscr{G} \to \mathrm{T\small{ORS}}(B)$ which is a δ-morphism, i.e. compatible with the homomorphism δ (see below for the precise definition).

Our starting point is the observation that this structure captures the differential geometric notion of "connective structure" on an abelian gerbe, introduced by Brylinski and McLaughlin[1] ([4–6], see also [3] for a version in the context of smooth manifolds). Briefly, by suitably generalizing the familiar concept of connection on an invertible sheaf on an analytic or algebraic manifold X, they defined a connective structure on an abelian gerbe bound by $\mathscr{O}_X^\times$ as a functor $x \rightsquigarrow \mathscr{C}o(x)$ associating with each local object x over an open U a Ω^1_U-torsor, subject to a certain list of

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[1] In [3] the concept is ascribed to Deligne.

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properties reviewed in Section 2.2. It turns out, and we show it explicitly in Section 2.2, that this is exactly the same thing as prescribing a structure of gerbe bound by the complex

$$\mathscr{O}_X^\times \xrightarrow{\mathrm{d}\log} \Omega_X^1.$$

More recently, we have similarly introduced the concept of Hermitian structure on an abelian gerbe bound by $\mathscr{O}_X^\times$, by modeling it on the corresponding familiar notion of invertible sheaf equipped with a fiber Hermitian metric [7]. In simplified terms, this structure is also of the type introduced above, namely we find that in this case it can be conveniently encoded in the structure of gerbe bound by the complex

$$\mathscr{O}_X^\times \xrightarrow{\log|\cdot|} \mathscr{E}_X^0,$$

where the latter denotes the sheaf of smooth real functions on X.

It is reasonable to expect that the list can be made longer with other interesting examples. However, we want to point out that the real interest of these construction lies in a different direction (or directions). On the one hand, there is the obvious interest of being able to generalize to the case of gerbes several structures of differential geometric interest. On the other hand, there is the fact that typical equivalence classes (suitably defined) of these structures turn out to be classified by interesting cohomology theories, and as a feedback we can get a geometric characterization of the elements of these groups. For instance, the cohomology groups relevant in the above examples are the Deligne cohomology group $\mathrm{H}^3_{\mathcal{D}}(X, \mathbf{Z}(2))$, and the Hermitian Deligne cohomology group $\widehat{\mathrm{H}}^3_{\mathcal{D}}(X, 1)$.

In fact, Brylinski and McLaughlin have shown that their constructions provide the adequate context for notable extensions of the tame symbol map in algebraic K-theory, where gerbes are useful in order to obtain a geometric picture for some regulator maps to Deligne–Beĭlinson cohomology (cf. [8]). More importantly, they extend their framework in two directions: (1) they consider the case of 2-gerbes as well, and (2) they define appropriate notions of curvature both for gerbes and 2-gerbes bound by $\mathscr{O}_X^\times$. Passing from gerbes to 2-gerbes corresponds to an increase in the degree of the involved cohomology groups, whereas introducing more levels of differential geometric structures corresponds to cohomology groups of higher *weights*. The geometric and the cohomological aspects are tied together very neatly in the following sense: the Deligne cohomology groups $\mathrm{H}^p_{\mathcal{D}}(X, A(k))$, where A is a subring of $\mathbf{R}$, can be regarded as somewhat pathological in the range $p > 2k$, where they cannot receive regulator maps from, say, absolute cohomology.[2] It is reassuring that the gerbes and 2-gerbes corresponding to the tame symbol maps and various related cup products turn out to naturally have a connective structure (and even curvatures), so that their classifying Deligne cohomology groups lie in the “safe” range $p \leq 2k$.[3]

A similar story was developed by the author in the case of Hermitian Deligne cohomology [7], motivated by the existence of certain natural Hermitian structures on tame symbols. As mentioned before, the cohomological counterpart is given by Hermitian Deligne cohomology, and there is a parallel for 2-gerbes as well. Namely, we have put forward a definition of Hermitian structure for 2-gerbes (to be reviewed and revised below) bound by $\mathscr{O}_X^\times$ and found that the corresponding equivalence classes are in one-to-one correspondence with the elements of the group $\widehat{\mathrm{H}}^4_{\mathcal{D}}(X, 1)$. In particular, the gerbes and 2-gerbes corresponding to the tame symbols studied by Brylinski and McLaughlin were found to naturally support a Hermitian structure as well. Moreover, it was found that these structures, namely the analytic (or algebraic) connective structure of Brylinski and McLaughlin and the Hermitian structure we introduced are compatible in the following sense: One of the byproducts of our work is that there is a natural notion of connective structure canonically associated with the Hermitian structure. It was found that this new connective structure agrees with the one of Brylinski–McLaughlin once they are mapped into an appropriate complex of smooth forms. (Part of this theory will be recalled and further clarified in the last part of the present paper.)

Not quite satisfying, as the reader will have no doubt noticed, is the fact that weights and degrees are precisely in what seems to be the bad range. However, a more interesting group $\widehat{\mathrm{H}}^4_{\mathcal{D}}(X, 2)$ does appear in the following way: in [7] we introduced a complex, denoted $\Gamma(2)^\bullet$ (defined in Section 7.1), and we (informally) argued that the hypercohomology group $\mathbf{H}^4(X, \Gamma(2)^\bullet)$ classifies 2-gerbes equipped with both a connective structure à la

2 The absolute cohomology groups in that range are zero.

3 There is of course an interest in knowing that, say, $\mathrm{H}^3_{\mathcal{D}}(X, \mathbf{Z}(1))$ classifies abelian gerbes bound by $\mathscr{O}_X^\times$, however the nice connection with regulators, etc. is lost.

Brylinski–McLaughlin and a Hermitian structure in our sense which are compatible as explained above. In loc. cit. we found that there is a surjection $\widehat{\mathrm{H}}^4_{\mathcal{D}}(X,2) \to \mathbf{H}^4(X,\Gamma(2)^\bullet)$, so classes of 2-gerbes can indeed be lifted to a more desirable group, but a truly geometric characterization was not provided. Let us remark that the interest of the group $\widehat{\mathrm{H}}^4_{\mathcal{D}}(X,2)$ lies in the fact that it is the receiving target of the cup product map

$$\widehat{\mathrm{Pic}}X \otimes \widehat{\mathrm{Pic}}X \longrightarrow \widehat{\mathrm{H}}^4_{\mathcal{D}}(X,2),$$

where $\widehat{\mathrm{Pic}}X \simeq \widehat{\mathrm{H}}^2_{\mathcal{D}}(X,1)$ is the group of isomorphism classes of metrized invertible sheaves. When X is a complete curve, this map gives a cohomological interpretation of Deligne's determinant of cohomology construction [9], which has been analyzed in various guises in [6,10,11] in the singular case.

The desire to remedy the above shortcoming and enhance the results of [7], as well as the desire to cast the results in the form expounded at the beginning of this introduction—suitably extended to include 2-gerbes—constitute our motivation for the present work. The framework we have found, that of 2-gerbes bound by a complex of braided *gr*-stacks, is quite more general than what would be minimally required for just solving the mentioned problems, and lends itself to possible generalizations to the non-abelian case, which we plan to address in part in a subsequent publication. We now proceed to describe the present results in the remaining part of this introduction.

0.2. Statement of the results

For the purpose of this introduction let us informally assume that X is a smooth base scheme, or an analytic manifold, and that C/X is an appropriate category of spaces "over" X with a Grothendieck topology, making it into a site.

To keep track of cohomology degrees, recall that Deligne cohomology and its variants have a built-in degree index shift. The convention we use in this introduction and the rest of the paper is to revert to standard cohomology degrees whenever we are not specifically dealing with one of these specific cohomology theories.

Our first result is a straightforward generalization of the concept of abelian gerbe bound by a homomorphism of sheaves of abelian groups to the case where we have a complex of abelian groups of the form:

$$A \xrightarrow{\delta} B \xrightarrow{\sigma} C.$$

We find that an abelian gerbe $\mathscr{G}$ bound by the above complex is conveniently defined as an A-gerbe $\mathscr{G}$ equipped with a functor

$$\mathscr{G} \longrightarrow \mathrm{T{\scriptsize ORS}}(B,C),$$

where the right-hand side denotes the gerbe of B-torsors with a section of the associated C-torsor obtained by the extension of the structure group from B to C. We then obtain through a simple Čech cohomology argument that equivalence classes of such gerbes are classified by the hypercohomology group

$$\mathbf{H}^2(X, A \to B \to C).$$

We show at the end of Section 3 that this is the appropriate general cadre for the notion of curvature: indeed we prove that Brylinski and McLaughlin's original definition of a gerbe with connective structure and "curving" can be cast as a gerbe bound by a complex of length 3, for an appropriate choice of the groups involved.

The extension of the idea of gerbe bound by a complex to the case of 2-gerbes is more involved, but quite interesting.

We want to consider abelian 2-gerbes, where of course the word "abelian" must be properly qualified. We adopt the point of view of [12] of calling "abelian" a 2-gerbe bound by a braided *gr*-stack in the following sense: It is known that the fibered category of automorphisms of an object x over $U \to X$ in a 2-gerbe is a *gr*-stack. Let $\mathscr{A}$ be a *gr*-stack over X. A 2-$\mathscr{A}$-gerbe $\mathfrak{G}$ is a 2-gerbe with the property that each local automorphism *gr*-stack is equivalent to (the restriction of) $\mathscr{A}$. As we know from [12], if this equivalence is natural in x, then $\mathscr{A}$ will be forced to be braided, i.e. its group law has a non-strict commutativity property. A special case is when $\mathscr{A} = \mathrm{T{\scriptsize ORS}}(A)$, that is, the *gr*-stack is the stack of torsors (in fact, a gerbe) over an abelian group A. Then we speak of a 2-gerbe bound by A, or 2-A-gerbe.

Note that it follows from [13,12] that for an abelian 2-$\mathscr{A}$-gerbe $\mathfrak{G}$ the stack of morphisms $\mathscr{A}ut_U(x,y)$ of two objects over $U \to X$ has the structure of $\mathscr{A}|_U$-torsor, and that $\mathfrak{G}$ determines a 1-cocycle, hence a cohomology set, with values in $\mathrm{T{\scriptstyle ORS}}(\mathscr{A})$. Note that for any *gr*-stack $\mathscr{A}$ this is a neutral 2-gerbe, see [14]. By suitably decomposing the torsors comprising this cocycle, we obtain a degree 2 cohomology set with values in $\mathscr{A}$ itself. This leads to the familiar degree 3 cohomology group with values in A in the case $\mathscr{A} = \mathrm{T{\scriptstyle ORS}}(A)$. We will find generalizations by studying the analogous constructions for complexes of *gr*-stacks, defined below.

Thus, given an additive functor $\lambda\colon \mathscr{A} \to \mathscr{B}$ of braided *gr*-stacks we define a 2-gerbe bound by this "complex" as a pair $(\mathfrak{G}, J)$, where $\mathfrak{G}$ is a 2-$\mathscr{A}$-gerbe and J is a cartesian 2-functor

$$J\colon \mathfrak{G} \longrightarrow \mathrm{T{\scriptstyle ORS}}(\mathscr{B})$$

which is a λ-morphism, see Section 5.3 for the precise definition. Once the notion of morphism and then of equivalence of such pairs are defined, we find that equivalence classes are in one-to-one correspondence with the elements of a cohomology set which we could provisionally write as:

$$\mathrm{H}^1(\mathrm{T{\scriptstyle ORS}}(\mathscr{A}) \to \mathrm{T{\scriptstyle ORS}}(\mathscr{B})).$$

Once again, by suitably decomposing the torsors comprising the 1-cocycle with values in the complex $\lambda_*\colon \mathrm{T{\scriptstyle ORS}}(\mathscr{A}) \to \mathrm{T{\scriptstyle ORS}}(\mathscr{B})$ determined by $\mathfrak{G}$, we obtain a degree 2 cohomology set with values in the complex $\lambda\colon \mathscr{A} \to \mathscr{B}$ itself.

In order to properly handle the Hermitian Deligne cohomology group we are ultimately interested in, we can further generalize this notion to that of a 2-gerbe bound by a *complex* of *gr*-stacks, that is a diagram of additive functors:

$$\mathscr{A} \xrightarrow{\lambda} \mathscr{B} \xrightarrow{\mu} \mathscr{C}, \qquad (+)$$

where the composition $\mu \circ \lambda$ is required to be isomorphic to the null functor sending $\mathscr{A}$ to the unit object of $\mathscr{C}$. Thus a 2-gerbe $\mathfrak{G}$ is bound by the above complex if there is a cartesian 2-functor

$$\tilde{J}\colon \mathfrak{G} \longrightarrow \mathrm{T{\scriptstyle ORS}}(\mathscr{B}, \mathscr{C}),$$

where the right-hand side denotes the 2-gerbe of $\mathscr{B}$-torsors which become equivalent to the trivial $\mathscr{C}$-torsor. Then we show that equivalence classes of such pairs $(\mathfrak{G}, \tilde{J})$ are classified by a cohomology set:

$$\mathrm{H}^1(\mathrm{T{\scriptstyle ORS}}(\mathscr{A}) \to \mathrm{T{\scriptstyle ORS}}(\mathscr{B}) \to \mathrm{T{\scriptstyle ORS}}(\mathscr{C})),$$

from which we can obtain a degree two cohomology set with coefficients in the *gr*-stack complex above. This is done in Sections 5 and 6, where the relevant theorems are stated and proven in full.

Along the way we get interesting byproducts shedding a new light on the notion of gerbe bound by a complex. In Section 5.4 we prove that for a strictly abelian (and not just braided) *gr*-stack $\mathscr{B}$, that is, one that arises from a homomorphism of sheaves of abelian groups, we have the equivalence

$$\mathrm{G{\scriptstyle ERBES}}(B, H) \xrightarrow{\sim} \mathrm{T{\scriptstyle ORS}}(\mathscr{B})$$

where $\mathscr{B} = \mathrm{T{\scriptstyle ORS}}(B, H)$. Then later in Section 6.1, we observe that $\mathrm{T{\scriptstyle ORS}}(\mathscr{B}, \mathscr{C})$ introduced above is equivalent, when $\mathscr{C} = \mathrm{T{\scriptstyle ORS}}(C, K)$ with the 2-gerbe of gerbes bound by $B \to H$ which become neutral as gerbes bound by $C \to K$.

These partial results are part of a general process whereby we make contact with ordinary hypercohomology by assuming that all the involved *gr*-stacks are strictly abelian. Concretely, if $\mathscr{A} = \mathrm{T{\scriptstyle ORS}}(A, G)$, $\mathscr{B} = \mathrm{T{\scriptstyle ORS}}(B, H)$, and $\mathscr{C} = \mathrm{T{\scriptstyle ORS}}(C, K)$ the complex of *gr*-stacks we have been considering reduces to the commutative diagram of (sheaves of) abelian groups:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} \\ G & \xrightarrow[u]{} & H & \xrightarrow[v]{} & K \end{array} \qquad (*)$$

The theorem we obtain in Section 6.4 is that equivalence classes of 2-gerbes bound by the complex (+) are classified by the standard hypercohomology group

$$\mathbf{H}^3(X, \text{(cone of (*))}[-1]).$$

As we will see in Section 7, this is exactly the kind of cohomology group we need in order to give a geometric construction of the elements of the Hermitian Deligne cohomology group $\widehat{\mathrm{H}}^4_{\mathcal{D}}(X, 2)$. In particular, in Section 7.3, we give a reasonably detailed construction of a 2-gerbe, denoted $\widehat{(\mathscr{L}, \mathscr{M}]}_{h.h.}$, whose class in $\widehat{\mathrm{H}}^4_{\mathcal{D}}(X, 2)$ is the cup product $[\mathscr{L}, \rho] \cup [\mathscr{M}, \sigma]$ of $[\mathscr{L}, \rho], [\mathscr{M}, \sigma] \in \widehat{\mathrm{Pic}} X$.

In Section 5, especially in Sections 5.4 and 5.5 we prove intermediate results for the case where there is no $\mathscr{C}$, so the diagram (∗) above reduces to the left square.

In all cases, when moving from cohomology sets with values in complexes of gerbes of torsors to (hyper)cohomology groups with values in cone of complexes, we compute explicit cocycles with respect to hypercovers, rather than ordinary covers. We find that even in the case of groups the cocycles so obtained present additional interesting terms.

0.3. Organization and contents of the paper

Overall we have adopted a mix of bottom-up and top-down approaches. We have refrained from starting from the most general statement and then working our way down. Instead we have adopted a sequence of successive generalizations.

Our treatment of cohomology deserves some explanations. At the beginning, where several proofs are standard, we have adopted a Čech point of view. In the latter part of the paper, where we deal with torsors over *gr*-stacks, we have found it worthwhile *not* to assume that decompositions with respect to Čech covers are sufficient. So we have actually computed cocycles using hypercovers, adopting the same point of view and formalism of [12]. Since we have dealt with hypercovers in a rather direct way, formulas acquire a substantial decoration of indices, which can be quite daunting. The usual advice is to ignore the hypercover indices on first parsing and reduce everything to the Čech formalism and replace (hyper)cohomology with its Čech counterpart.

A note about sites: When dealing with categorical matters, it comes at no additional cost to formulate everything, including cohomology sets, for sites. Thus usually we will assume that gerbes and 2-gerbes are fibered over a site S. This site will in fact be a category of objects over an object X, so that we will often use the notation C/X, assuming that the category C has been equipped with an appropriate Grothendieck topology. By thinking of X as the terminal object in C/X, we can conveniently denote cohomology sets as $\mathrm{H}^\bullet(X, -)$ or $\mathbf{H}^\bullet(X, -)$, depending on whether we wish to emphasize the "hyper" aspect.

As for gerbes and 2-gerbes, we have chosen to follow different approaches. When dealing with 2-gerbes and, perhaps, other less familiar objects such as torsors over *gr*-stacks in the second part of the paper, we have chosen to collect and provide a fair amount of details from the existing literature. Unfortunately we cannot make this paper completely self-contained without writing another book on 2-gerbes, therefore referring to the literature, especially [12], remains indispensable. For gerbes, on the other hand, we have tried to rely much more on the references without providing nearly as many details, on the grounds that the relevant notions will be more familiar to the reader. This includes the notion of gerbe bound by a length 2 complex, introduced in Section 2, for which we have decided to heavily rely on the abelian version given by Milne [2], rather than the original one by Debremaeker in Ref. [1], in order to avoid bringing in notions from non-abelian cohomology, which would have pushed our exposition a bit too far afield. We have, however, put a few hopefully clarifying remarks and pointers in Section 2 where appropriate.

In conclusion, this paper is organized as follows. In Section 1 we recall a few background notions, collect some notations, and we provide a quick overview of various Deligne-type cohomology theories needed in the rest of the paper. We introduce the concept of gerbe bound by a length 2 complex in Section 2, where we also review the pivotal example of connective structure in some detail. We then proceed in Section 3 to define and classify gerbes bound by a length 3 complex. Section 4 is dedicated to a quick review of 2-gerbes. Sections 5 and 6 then contain our main results, where we classify 2-gerbes bound by complexes of *gr*-stacks. Finally, in Section 7, we return to the realm complex algebraic manifolds, and give some applications to Hermitian Deligne cohomology.

1. Background notions

1.1. Assumptions and notations

In the following, X will be a smooth scheme or a complex analytic manifold. In the algebraic case, some results can be stated for X smooth over a base scheme S. Actually, in most of the applications we will be concerned with the case when X is an algebraic manifold,[4] hence $S = \operatorname{Spec}\mathbf{C}$. In this case the complex analytic manifold above will be X^{an}, the set of complex points of X with the analytic topology, but usually we will not explicitly mark this in the notation.

Gerbes "over X" are stacks in groupoids and, similarly, 2-gerbes are 2-categories fibered in (lax) 2-groupoids satisfying certain conditions to be explained below, over an appropriate site of "spaces" over X. As explained at the end of the introduction, whenever dealing with general categorical matters, the specific choice of this site will be somewhat immaterial. In order to fix ideas, and to revert in the end to specific cohomology theories, we will assume that we are given an appropriate category with fiber products C/X of spaces over X equipped with a Grothendieck topology. The main requirement will be that the various sheaves such as $\mathscr{O}_X$, $\Omega^\bullet_X$, etc. as defined with respect to C/X restrict to their usual counterparts under $U \to X$, whenever U is open in the ordinary – for the Zariski or Analytic topology – sense. More specifically, following Ref. [3], if X is a scheme we may as well consider the small étale site $X_{\acute{e}t}$, namely $\mathsf{C}/X = \mathsf{Et}/X$, where we denote by Et the class of étale maps over X, and covers are jointly surjective families of étale maps. It is useful to allow the same type of construction when $S = \operatorname{Spec}\mathbf{C}$, and we want to consider X^{an}. Namely we obtain a corresponding "analytic" site by mapping $U \to X$ from $X_{\acute{e}t}$ to $U^{an} \to X^{an}$. According to Ref. [15], this determines the same topology as the standard analytic one. In the latter case, that is if X is a complex manifold, C/X will be the small Top site. Similarly, when X is a scheme to be considered with its ordinary topology, we set $\mathsf{C}/X = X_{\mathrm{zar}}$, the small Zariski site of X whose covers are injective maps $V \to U$ with U open in X. Note that in general we will not be considering the corresponding "big" sites. However, the general categorical constructions which form the main body of this paper are going to work in that context too.[5]

In general we will refer to the topology on C/X simply as a topology on X, and accordingly we will simply speak of "open" sets for members $V \to U$ of a cover of $U \to X$. As it is well-known, fibered products take the place of intersections, and we will use the standard notation of denoting the various multiple "intersections" (i.e. fibered products) relative to a covering $\{U_i \to U\}_{i\in I}$ as: $U_{ij} = U_i \times_U U_j$, $U_{ijk} = U_i \times_U U_j \times_U U_k$, etc. Also in the relative case of X over a base S, C/X will be obtained by restriction from C/S. However, our notation will not always explicitly reflect this.

1.1.1. Often used notations

For a subring A of $\mathbf{R}$ and an integer p, the Tate twist of A is the A-module $A(p) = (2\pi\sqrt{-1})^p A$. The introduction of such a device allows a number of algebraic manipulations, complexes, etc. to become independent of the choice of the imaginary unit.[6] In particular, we can write $\mathbf{C} \simeq \mathbf{R}(p) \oplus \mathbf{R}(p-1)$, and $\mathbf{C}/\mathbf{R}(p) \simeq \mathbf{R}(p-1)$. The projections onto the "real" and "imaginary" parts, $\pi_p \colon \mathbf{C} \to \mathbf{R}(p)$, is given by $\pi_p(z) = \frac{1}{2}(z + (-1)^p \bar{z})$, for $z \in \mathbf{C}$ – and similarly for any other complex quantity. We identify $\mathbf{C}/\mathbf{Z}(p) \simeq \mathbf{C}^\times$ via the exponential map $z \mapsto \exp(z/(2\pi\sqrt{-1})^{p-1})$.

If E is a set (or group, ring, module, etc.), then E_X denotes the corresponding constant sheaf of sets (or groups, rings, modules, etc.).

If X is a scheme or complex manifold, $\Omega^\bullet_X$ denotes the corresponding (algebraic or analytic) de Rham complex. We set $\mathscr{O}_X \equiv \Omega^0_X$ as usual. $\mathscr{E}^\bullet_X$ denotes the de Rham complex of sheaves of $\mathbf{R}$-valued smooth forms on the underlying smooth manifold. Furthermore, $\mathscr{A}^\bullet_X = \mathscr{E}^\bullet_X \otimes_{\mathbf{R}} \mathbf{C}$, and is $\mathscr{E}^\bullet_X(p)$ the twist $\mathscr{E}^\bullet_X \otimes_{\mathbf{R}} \mathbf{R}(p)$. Also, $\mathscr{A}^{p,q}_X$ will denote the sheaf of smooth (p,q)-forms, and $\mathscr{A}^n_X = \bigoplus_{p+q=n} \mathscr{A}^{p,q}_X$, where the differential decomposes in the standard fashion,

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[4] By algebraic manifold we mean a smooth, separated scheme of finite type over $\mathbf{C}$.

[5] To be more specific one could consider sites such as $X_{\acute{E}t}$, the big étale site of X, if X is a scheme, namely $\mathsf{C}/X = \mathsf{Sch}/X$ equipped with the étale topology defined by the class Et of étale maps over X; correspondingly, $\mathsf{C}/X = \mathsf{Cmplx}/X$, with the topology given by standard open covers, or by analytification of étale covers as described above.

[6] For more details see [16,17] and also [18].

$\mathrm{d} = \partial + \bar{\partial}$, according to types. We also introduce the imaginary operator $\mathrm{d}^c = \partial - \bar{\partial}$ [7] and we have the rules

$$\mathrm{d}\,\pi_p(\omega) = \pi_p(\mathrm{d}\,\omega), \qquad \mathrm{d}^c\,\pi_p(\omega) = \pi_{p+1}(\mathrm{d}^c\,\omega),$$

for any complex form ω. Note that we have $2\partial\bar{\partial} = \mathrm{d}^c\,\mathrm{d}$.

The standard Hodge filtrations on $\Omega^\bullet_X$ and $\mathscr{A}^\bullet_X$ are as follows: $F^p\Omega^\bullet_X \equiv \sigma^p\Omega^\bullet_X$ is the sharp truncation in degree p:

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Omega^p_X \longrightarrow \cdots \longrightarrow \Omega^{\dim X}_X,$$

whereas $F^p\mathscr{A}^\bullet_X$ is the total complex of: $\bigoplus_{r\geq p}\mathscr{A}^{r,\bullet-r}_X$.

1.1.2. *Torsors*

For torsors, we adopt the notation $P\wedge^A B$ for the usual quotient of the action

$$\begin{aligned} &P\times B\times A \longrightarrow P\times B\\ &(u,b,a)\longmapsto (ua,a^{-1}b), \end{aligned}$$

where A is a sheaf of groups, P an A-torsor, and A acts on a sheaf B. In particular, if B is a sheaf of groups, and $\delta\colon A\to B$ is a homomorphism, then A acts on B by $(a,b)\mapsto\delta(a)b$. In this situation $P\wedge^A B$ is a B-torsor. Throughout the paper we will only consider the abelian situation, so that there will be no distinction between left or right torsors. Our notation will in general reflect a preference for right torsors.

1.2. *Various Deligne complexes and cohomologies*

Standard references on Deligne cohomology are: [16,17].

For a subring $A\subset\mathbf{R}$ and an integer p, the Deligne cohomology groups of weight p of X with values in A are the hypercohomology groups:

$$\mathrm{H}^\bullet_{\mathcal{D}}(X,A(p)) \overset{\text{def}}{=} \mathbf{H}^\bullet(X,A(p)^\bullet_{\mathcal{D},X}), \tag{1.2.1}$$

where $A(p)^\bullet_{\mathcal{D},X}$ is the complex

$$A(p)^\bullet_{\mathcal{D},X} = \quad \mathrm{Cone}\left(A(p)_X\oplus F^p\Omega^\bullet_X\longrightarrow\Omega^\bullet_X\right)[-1] \tag{1.2.2}$$

$$\xrightarrow{\simeq}\left(A(p)_X\xrightarrow{\iota}\mathscr{O}_X\xrightarrow{\mathrm{d}}\Omega^1_X\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}\Omega^{p-1}_X\right), \tag{1.2.3}$$

where the map in the cone is the difference of the two inclusions and $\xrightarrow{\simeq}$ denotes a quasi-isomorphism. The complex in (1.2.3) is the one we will normally use in what follows.

When $A=\mathbf{R}$, Deligne cohomology groups can be computed using other complexes quasi-isomorphic to (1.2.2) or (1.2.3), in particular:

$$\widetilde{\mathbf{R}(p)}^\bullet_{\mathcal{D}} = \mathrm{Cone}\left(F^p\mathscr{A}^\bullet_X\to\mathscr{E}^\bullet_X(p-1)\right)[-1]. \tag{1.2.4}$$

(See the references quoted above for a proof.)

The *Hermitian* variant of Deligne cohomology is obtained by considering the hypercohomology groups

$$\widehat{\mathrm{H}}^\bullet_{\mathcal{D}}(X,p)\overset{\text{def}}{=}\mathbf{H}^\bullet(X,C(p)^\bullet_X) \tag{1.2.5}$$

of the complex

$$C(p)^\bullet_X = \mathrm{Cone}\left(\mathbf{Z}(p)_X\bigoplus(F^p\mathscr{A}^\bullet_X\cap\sigma^{2p}\mathscr{E}^\bullet_X(p))\longrightarrow\mathscr{E}^\bullet_X(p)\right)[-1], \tag{1.2.6}$$

[7] We omit the customary factor $1/(4\pi\sqrt{-1})$.

introduced by Brylinski in [6]. We proved in [10] that it is quasi-isomorphic to the complex:

$$D_{h.h.}(p)^\bullet_X = \mathrm{Cone}\Big(\mathbf{Z}(p)^\bullet_{\mathcal{D}} \oplus (F^p\mathscr{A}^\bullet_X \cap \sigma^{2p}\mathscr{E}^\bullet_X(p)) \longrightarrow \widetilde{\mathbf{R}(p)}^\bullet_{\mathcal{D}}\Big)[-1]. \tag{1.2.7}$$

The interest of (1.2.7) lies in the fact that the second hypercohomology group of $D_{h.h.}(1)^\bullet_X$ provides a characterization of the *canonical connection* associated with a Hermitian line bundle [10,7]. We will also need a leaner version of the complex (1.2.7) introduced in [11], namely:

$$\mathfrak{D}_{h.h.}(p)^\bullet_X = \mathrm{Cone}\Big(\mathbf{Z}(p)^\bullet_{\mathcal{D}} \xrightarrow{\rho_p} \sigma^{<2p}\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)\Big)[-1]. \tag{1.2.8}$$

Here $\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)$ is the *Deligne Algebra* over the complex $\mathscr{A}^\bullet_X$, discussed in full in [19–21], and $\sigma^{<2p}$ denotes its sharp truncation in degrees above $2p$, so that:

$$\sigma^{<2p}\mathfrak{D}^n(\mathscr{A}^\bullet_X, p) = \begin{cases} 0 & n = 0, \\ \mathscr{E}^{n-1}_X(p-1) \bigcap \bigoplus_{\substack{p'+q'=n-1 \\ p'<p, q'<p}} \mathscr{A}^{p',q'}_X & n \le 2p-1. \end{cases} \tag{1.2.9}$$

The differential is $-\pi \circ \mathrm{d}$, where π is the projection that simply chops off the degrees falling outside the scope of (1.2.9). Using (1.2.3), the map ρ_p is:

$$\rho^n_p = \begin{cases} 0 & n = 0, \\ (-1)^n \pi_{p-1} & 1 \le n \le p. \end{cases}$$

Example 1.1. In the following we will be concerned almost exclusively with the complexes of weight $p = 1$ and $p = 2$. Explicitly, we have:

$$\mathfrak{D}_{h.h.}(1)^\bullet_X = \Big(\mathbf{Z}(1)_X \xrightarrow{\imath} \mathscr{O}_X \xrightarrow{\pi_0} \mathscr{E}^0_X\Big), \tag{1.2.10}$$

whereas the complex $\mathfrak{D}_{h.h.}(2)^\bullet_X$ is the cone (shifted by 1) of the map:

$$\begin{array}{ccccc} \mathbf{Z}(2)_X & \xrightarrow{\imath} & \mathscr{O}_X & \xrightarrow{\mathrm{d}} & \Omega^1_X & & \\ & & \big\downarrow{\scriptstyle -\pi_1} & & \big\downarrow{\scriptstyle \pi_1} & & \\ & & \mathscr{E}^0_X(1) & \xrightarrow{-\mathrm{d}} & \mathscr{E}^1_X(1) & \xrightarrow{-\pi\circ\mathrm{d}} & \mathscr{E}^2_X(1) \cap \mathscr{A}^{1,1}_X \end{array} \tag{1.2.11}$$

Remark 1.2. Using the complex (1.2.10), one shows that

$$\widehat{\mathrm{H}}^2_{\mathcal{D}}(X, 1) \simeq \widehat{\mathrm{Pic}}X,$$

the group of isomorphism classes of line bundles with Hermitian metric. This follows from an easy Čech argument, as in [22]. Thus the same type of argument, using the complex $D_{h.h.}(1)^\bullet_X$, implies the uniqueness of the canonical connection, see [7].

We conclude this review section by observing that all complexes introduced so far possess a product structure (or several mutually homotopic such structures), additive with respect to the weights, so that we have graded commutative cup products

$$\mathrm{H}^k_{\mathcal{D}}(X, A(p)) \otimes \mathrm{H}^l_{\mathcal{D}}(X, A(q)) \xrightarrow{\cup} \mathrm{H}^{k+l}_{\mathcal{D}}(X, A(p+q))$$

and

$$\widehat{\mathrm{H}}^k_{\mathcal{D}}(X, p) \otimes \widehat{\mathrm{H}}^l_{\mathcal{D}}(X, q) \xrightarrow{\cup} \widehat{\mathrm{H}}^{k+l}_{\mathcal{D}}(X, p+q).$$

The reader should refer to the literature cited in this section for more details and explicit formulas for the products.

2. Gerbes with abelian band

In the following we recall a few definitions about gerbes. The canonical reference is [23], whereas a detailed exposition adapted to spaces is [12]. We will need the abelian part of the whole theory, for which a readable account is to be found in [3].

Let C be a category with finite fibered products, equipped with a Grothendieck topology. C will be assumed to be as in Section 1.1, therefore we will have a terminal object X at our disposal, satisfying the hypotheses expounded in that section.

A *gerbe* $\mathscr{G}$ over C is a stack in groupoids $p\colon \mathscr{G} \to \mathsf{C}$ such that:

(1) $\mathscr{G}$ is *locally non-empty*, namely there exists a cover $U \to X$ such that $\mathrm{Ob}(\mathscr{G}_U)$ is non-empty;
(2) $\mathscr{G}$ is *locally connected*, that is, for each pair of objects of $\mathscr{G}$, there is a cover $\varphi\colon V \to U$ such that their inverse images are isomorphic. In other words if $x, y \in \mathrm{Ob}\,\mathscr{G}_U$, then $\mathrm{Hom}_U(\varphi^* x, \varphi^* y)$ is non-empty.

For an object $x \in \mathrm{Ob}\,\mathscr{G}_U$, the sheaf $\underline{\mathrm{Aut}}(x)$ is a sheaf of groups on C/U. (Recall that over $\varphi\colon V \to U$, we have $\underline{\mathrm{Aut}}(x)(V) = \mathrm{Aut}_V(\varphi^* x)$.) Now let A be a sheaf of groups on C: We say that $\mathscr{G}$ is an *A-gerbe* if for each object x with $\varphi(x) = U$ as above there is a *natural* isomorphism

$$a_x\colon \underline{\mathrm{Aut}}(x) \xrightarrow{\sim} A|_U.$$

The naturality in x will force the group A to be abelian, and in the following we will restrict our attention to this case. The sheaf A will be referred to as the *band* of the gerbe $\mathscr{G}$. We also say that $\mathscr{G}$ is *bound* by A. (In the general—non-abelian—case, the band $L(A)$ will have a more complicated definition, as the various sheaves $A|_U$ are glued along $U \times_X U$ only up to inner automorphisms. In the abelian case this is immaterial and we can abuse the language and call A the band of $\mathscr{G}$.) Note also that in this setting, given two objects $x, y \in \mathrm{Ob}\,\mathscr{G}_U$, the sheaf $\underline{\mathrm{Hom}}(x, y)$ of morphisms from x to y is simply an $A|_U$-torsor.

A morphism $\lambda : \mathscr{G} \to \mathscr{H}$ is a cartesian functor between the underlying fibered categories, and it is an equivalence if it is an equivalence of categories. Moreover, if $\mathscr{G}$ is an A-gerbe, and $\mathscr{H}$ is a B-gerbe, with a group homomorphism $f\colon A \to B$, then the morphism λ will have to satisfy the obvious commutative diagrams. Such a morphism is called an f-morphism.

An f-morphism for which f is an isomorphism is automatically an equivalence. So is, in particular, a morphism between two A-gerbes $\mathscr{G}$ and $\mathscr{G}'$. So if A is abelian, it follows from [23] that A classes of equivalences of A-gerbes are classified by $\mathrm{H}^2(X, A)$, the standard second cohomology group of X in the derived functor sense. See also, e.g. [3], for a proof in the Čech setting.

2.1. Gerbes bound by a complex

We are going to use the notion of gerbe bound by a length two complex $A \to B$ of sheaves of abelian groups over C/X, as in [2]. Let us review the formal definition:

Definition 2.1. Let A and B be two sheaves of abelian groups on C/X, and $\delta \in \mathrm{Hom}(A, B)$, so that $A \xrightarrow{\delta} B$ is a complex of length two. A gerbe $\mathscr{G}$ bound by $A \to B$ is an A-gerbe over C/X equipped with a δ-morphism of gerbes

$$\mu\colon \mathscr{G} \to \mathrm{T{\scriptsize ORS}}(B).$$

(Notice that $\mathrm{T{\scriptsize ORS}}(B)$ is a B-gerbe, so the notion of δ-morphism makes sense.)

One could think of a gerbe bound by $A \to B$ as an A-gerbe $\mathscr{G}$ which becomes neutral, i.e. equivalent to $\mathrm{T{\scriptsize ORS}}(B)$, as a B-gerbe. This can actually be made precise in the following way. Recall from Refs. [23,3] that if $\mathscr{G}$ is an abelian A-gerbe, and $\delta\colon A \to B$ is a homomorphism, we can "extend" the band (defined above) from A to B along δ to obtain a B-gerbe, denoted $\mathscr{G} \wedge^A B$, which is defined by requiring that, as a prestack, its fiber categories have the same objects as those of $\mathscr{G}$, and that the sheaf of isomorphisms in $\mathscr{G} \wedge^A B$ between two objects x, y be the $B|_U$-torsor

$$\underline{\mathrm{Hom}}_{\mathscr{G}_U}(x, y) \wedge^{A|_U} B|_U.$$

Then we have the easily verified

Lemma 2.2. *$\mathscr{G}$ is bound by $A \to B$ iff there is an equivalence of B-gerbes*

$$\mathscr{G} \wedge^A B \xrightarrow{\sim} \mathrm{T{\small ORS}}(B).$$

Remark 2.3. A better perspective on this matter will be gained later in the paper in Section 5.4, Theorem 5.8, when we interpret gerbes bound by $A \to B$ as torsors in a suitable way. Then, using the symmetry of the diagram at the end of Section 5.4, and Proposition 6.4, the idea of $\mathscr{G}$ as an A-gerbe which becomes trivial as a B-gerbe will become precise in the appropriate context.

Remark 2.4. Definition 2.1 can be seen as a particular case of the more general notion of a gerbe $\mathscr{G}$ bound by a sheaf of *crossed modules,* as per Debremaeker's original definition in Ref. [1]. This latter notion will not be used anywhere else, and it is briefly recalled here for the convenience of the interested reader.

If (A, B, δ) is a crossed module, where $\delta\colon A \to B$ is a group homomorphism, compatible with the action of B over A, a gerbe bound by it is a gerbe $\mathscr{G}$ with a δ-morphism μ as above, *plus* functorial isomorphisms

$$\jmath_x\colon \underline{\mathrm{Aut}}(x) \xrightarrow{\sim} \mu(x) \wedge^B A$$

for each object x of $\mathscr{G}$. The twisted group on the right-hand side above arises from the action of B on A. It is a group over U, if $x \in \mathrm{Ob}\,\mathscr{G}_U$. The isomorphisms $\jmath_x$ are required to map under μ to the canonical isomorphisms

$$\underline{\mathrm{Aut}}(\mu(x)) \simeq \mu(x) \wedge^B B,$$

as automorphisms of B-torsors. Now, if the crossed module is *abelian*, which means that both A and B are abelian, and the action of B over A is trivial, the crossed module becomes simply a complex, and everything reduces to the data in the previous definition. In particular, the isomorphisms $\jmath_x$ simply reduce to the isomorphisms a_x introduced above characterizing $\mathscr{G}$ as an A-gerbe.

We now return to the abelian situation. As usual, a morphism of complexes

$$(f, g)\colon (A, B, \delta) \to (A', B', \delta')$$

is a commutative diagram of group homomorphisms:

$$\begin{array}{ccc} A & \xrightarrow{\delta} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A' & \xrightarrow{\delta'} & B' \end{array}$$

If $\mathscr{G}$ and $\mathscr{G}'$ are bound by (A, B) and (A', B'), respectively, then we have a corresponding notion of (f, g)-morphism as follows:

Definition 2.5. An (f, g)-morphism from $\mathscr{G}$ to $\mathscr{G}'$ consists of:

(1) an f-morphism $\lambda\colon \mathscr{G} \to \mathscr{G}'$;
(2) a natural isomorphism of functors

$$\alpha\colon g_* \circ \mu \Longrightarrow \mu' \circ \lambda$$

from $\mathscr{G}$ to $\mathrm{T{\small ORS}}(B')$.

In the definition g_* is the g-morphism $\mathrm{T{\small ORS}}(A) \to \mathrm{T{\small ORS}}(B)$ induced by g in the obvious way.

For completeness, let us also mention that we also have the notion of morphism of morphisms, see [2]. Namely, let (λ_1, α_1) and (λ_2, α_2) be two morphisms $(\mathscr{G}, \mu) \to (\mathscr{G}', \mu')$. A morphism $m\colon (\lambda_1, \alpha_1) \to (\lambda_2, \alpha_2)$ is a natural transformation $m\colon \lambda_1 \Rightarrow \lambda_2$ such that the following is verified:

$$(\mu * m) \circ \alpha_1 = \alpha_2.$$

With these notions the gerbes bound by a complex of length 2 form a 2-category. In particular, when $A' = A$ and $B = B'$ we denote this 2-category by $\mathrm{G{\small ERBES}}(A, B)$.

2.1.1. Classification of (A, B)-gerbes

Once again, consider the special case $A' = A$ and $B' = B$, with f and g being the respective identity maps. Then we speak of an (A, B)-morphism, and in particular of an (A, B)-*equivalence* if the underlying functor $\lambda\colon \mathscr{G} \to \mathscr{G}'$ is an equivalence in the usual sense. (A, B)-*equivalence* is an equivalence relation, and the set of equivalence classes is $\mathbf{H}^2(X, A \to B)$. While this can be defined in general (see Ref. [1]) in the abelian case it turns out to coincide with the second hypercohomology group with values in the complex $A \to B$ in the standard sense (cf. [2]).

2.1.2. The canonical (f, g)-morphism

Given a commutative diagram of group homomorphisms as above, there is a canonical (f, g)-morphism

$$(f, g)_*\colon \mathrm{GERBES}(A, B) \longrightarrow \mathrm{GERBES}(A', B'),$$

given by the extension of the band. Namely, if $\mathscr{G}$ is an A-gerbe, there is a well-defined procedure, recalled above, giving an A'-gerbe which we may call $f_*(\mathscr{G})$. Since locally

$$\mathscr{G}_U \simeq \mathrm{TORS}(A|_U),$$

$f_*(\mathscr{G})_U$ is simply given by the standard extension of the structure group. Now, if $(\mathscr{G}, \mu)$ is an (A, B)-gerbe, then $(\mathscr{G}, g_* \circ \mu)$ is an (A, B')-gerbe and locally the functor $g_* \circ \mu$ will be isomorphic to $g_* \circ \delta_*$ (see in particular the proof of Theorem 5.8 below for more details[8]). The latter will be replaced, by commutativity induced from the commutative square of group homomorphisms, by $\delta'_* \circ f_*$, which glues back to a functor $\mu'\colon f_*(\mathscr{G}) \to \mathrm{TORS}(B')$.

This construction is universal in the sense that an (f, g)-morphism can be written by the composition of $(f, g)_*$ followed by a unique (up to equivalence) (A', B')-morphism.

An alternative characterization of (A, B)-gerbes will appear in Section 5.4, in particular Theorem 5.8, when we discuss 2-gerbes bound by complexes.

2.2. Examples

The following are few examples of Gerbes bound by complexes of length 2 which are relevant from the point of view of extending differential geometric structures to gerbes.

We will first review the definition of connection—or *connective structure*—on a $\mathscr{O}_X^\times$-gerbe according to Brylinski and McLaughlin (see, e.g. [4,5], or [3] for the smooth case).

Definition 2.6. Let $\mathscr{G}$ be a $\mathscr{O}_X^\times$-gerbe. A *connective structure* $\mathscr{C}o$ on $\mathscr{G}$ is the datum of a Ω^1_U-torsor $\mathscr{C}o(x)$ for any object $x \in \mathscr{G}_U$, where $U \subset X$, subject to the following conditions.

(1) For every isomorphism $f : x \to y$ in $\mathscr{G}_U$ there is an isomorphism

$$f_* \equiv \mathscr{C}o(f) : \mathscr{C}o(x) \longrightarrow \mathscr{C}o(y)$$

of Ω^1_U-torsors. In particular, if $f \in \mathrm{Aut}(x) \simeq \mathscr{O}_X^\times|_U$, we require:

$$\begin{aligned} f_* &: \mathscr{C}o(x) \longrightarrow \mathscr{C}o(x) \\ \nabla &\longmapsto \nabla + \mathrm{d}\log f, \end{aligned} \tag{2.2.1}$$

where ∇ is a section of $\mathscr{C}o(x)$.

(2) If $g : y \to z$ is another morphism in $\mathscr{G}_U$, then $(gf)_* \simeq g_* f_*$.

(3) The correspondence must be compatible with the restriction functors and natural transformations. Namely, if $\iota^* : \mathscr{G}_U \to \mathscr{G}_V$ is the restriction functor corresponding to the morphism $\iota : V \to U$ in C/X, then there is a natural

8 This construction will not be used until Section 6.1 and it is only dependent on the arguments of Section 5.4, in particular the proof of Theorem 5.8.

isomorphism $\alpha_\imath : \imath^* \circ \mathscr{C}o \Rightarrow \mathscr{C}o \circ \imath^*$ such that the diagram:

$$\begin{array}{ccc} \imath^*\mathscr{C}o(x) & \xrightarrow{\alpha_\imath(x)} & \mathscr{C}o(\imath^* x) \\ {\scriptstyle \imath^*(f_*)}\downarrow & & \downarrow{\scriptstyle (\imath^* f)_*} \\ \imath^*\mathscr{C}o(y) & \xrightarrow{\alpha_\imath(y)} & \mathscr{C}o(\imath^* y) \end{array}$$

commutes. Moreover given $\jmath : W \to V$ and the corresponding $\alpha_\jmath$, there must be the obvious pentagonal compatibility diagram with the natural transformations $\varphi_{\imath,\jmath} : \jmath^* \imath^* \to (\imath\jmath)^*$ arising from the structure of fibered category over X. That is, given the object x, we have the commutative diagram:

$$\begin{array}{ccccc} \mathscr{C}o_W(\jmath^*\imath^* x) & \xrightarrow{\alpha_\jmath} & \jmath^*\mathscr{C}o_V(\imath^* x) & \xrightarrow{\jmath^*\alpha_\imath} & \jmath^*\imath^*\mathscr{C}o_U(x) \\ {\scriptstyle \phi_{\imath,\jmath}(x)_*}\downarrow & & & & \downarrow{\scriptstyle \phi_{\imath,\jmath}(\mathscr{C}o_U(x))} \\ \mathscr{C}o_W((\imath\jmath)^* x) & & \xrightarrow{\alpha_{\imath\jmath}} & & (\imath\jmath)^*\mathscr{C}o_U(x) \end{array}$$

mapping to a corresponding one with y.

The following is a reformulation of the conditions in Definition 2.6:

Proposition 2.7. *A connective structure on the $\mathscr{O}_X^\times$-gerbe $\mathscr{G}$ amounts to the datum of a structure of gerbe bound by the complex*

$$\Gamma : \mathscr{O}_X^\times \xrightarrow{\mathrm{d}\log} \Omega_X^1.$$

Proof. That the various conditions in Definition 2.6 define a cartesian functor

$$\mathscr{C}o : \mathscr{G} \longrightarrow \mathrm{T{\small ORS}}(\Omega_X^1)$$

is just a matter of unraveling the definition of cartesian functor. Moreover, Eq. (2.2.1) implies that $\mathscr{C}o$ is in fact a d log-morphism. □

According to the general results, $\mathscr{O}_X^\times$-gerbes with connective structure are classified by the hypercohomology group

$$\mathbf{H}^2(X, \mathscr{O}_X^\times \xrightarrow{\mathrm{d}\log} \Omega_X^1).$$

Via the quasi-isomorphisms:

$$\left(\mathscr{O}_X^\times \xrightarrow{\mathrm{d}\log} \Omega_X^1\right)[-1] \xrightarrow{\simeq} \left(\mathbf{Z}(2) \longrightarrow \mathscr{O}_X \xrightarrow{\mathrm{d}} \Omega_X^1\right) \xrightarrow{\simeq} \mathbf{Z}(2)^\bullet_{\mathcal{D}},$$

where $\mathbf{Z}(k)^\bullet_{\mathcal{D}}$ is the weight k Deligne complex, we have that the classifying group is isomorphic to the *Deligne cohomology group*

$$\mathrm{H}^3_{\mathcal{D}}(X, \mathbf{Z}(2)).$$

2.3. *Further examples*

Several variations on the theme established in Definition 2.6 and Proposition 2.7 have been considered, typically by providing the necessary modifications in Definition 2.6. Following the idea embodied in Proposition 2.7 they can be restated in terms of gerbes bound by a complex.

In Ref. [7] we have introduced a notion of Hermitian structure and a variant of connective structure valued in the Hodge filtration. We consider these examples next.

2.3.1. Hermitian structures

Consider the complex:

$$\mathscr{O}_X^\times \xrightarrow{|\cdot|^2} \mathscr{E}_X^+,$$

where $\mathscr{E}_X^+$ is the sheaf of *smooth* functions valued in $\mathbf{R}_{>0}$, the connected component of 1 in $\mathbf{R}^\times$. A $\mathscr{O}_X^\times$-gerbe $\mathscr{G}$ is said to have a *Hermitian structure* (cf. Ref. [7, Definition 5.2.1]) if it has the structure of a gerbe bound by $(\mathscr{O}_X^\times, \mathscr{E}_X^+)$.

Classes of equivalences of $\mathscr{O}_X^\times$-gerbes equipped with Hermitian structures are therefore classified by the group

$$\mathbf{H}^2(X, \mathscr{O}_X^\times \xrightarrow{|\cdot|^2} \mathscr{E}_X^+) \simeq \widehat{\mathrm{H}}^3_{\mathcal{D}}(X, 1).$$

Recall that the latter is the third *Hermitian Deligne cohomology group* of weight 1, and the isomorphism is induced by the quasi-isomorphism

$$\left(\mathbf{Z}(1) \to \mathscr{O}_X \xrightarrow{\Re} \mathscr{E}_X^0\right) \xrightarrow{\simeq} \left(\mathscr{O}_X^\times \xrightarrow{|\cdot|^2} \mathscr{E}_X^+\right)[-1],$$

where the first is the corresponding Hermitian Deligne complex.

2.3.2. F^1-connections

A slight modification of the notion of connective structure recalled in Section 2.2 is to consider the length 2 complex [7]:

$$\mathscr{O}_X^\times \xrightarrow{\partial \log} F^1\mathscr{A}_X^1.$$

Note that $F^1\mathscr{A}_X^1 = \mathscr{A}_X^{1,0}$, so this is called a "type $(1,0)$ connective structure" in [7].

2.3.3. Compatibility

We have the obvious map $\partial \log \colon \mathscr{E}_X^+ \to F^1\mathscr{A}_X^1$ and the morphism of complexes

$$\begin{array}{ccc} \mathscr{O}_X^\times & \xrightarrow{|\cdot|^2} & \mathscr{E}_X^+ \\ \| & & \downarrow{\scriptstyle \partial \log} \\ \mathscr{O}_X^\times & \xrightarrow{\partial \log} & F^1\mathscr{A}_X^1 \end{array}$$

The notion of compatibility between a Hermitian and a type $(1,0)$ connective structures on $\mathscr{G}$ amounts to an $(\mathrm{id}, \partial \log)$-morphism. In fact, it is the canonical one in the sense of Section 2.1.2. The equivalence with [7, 5.3.2], is merely a question of unraveling Definition 2.5 for the case at hand. The classifying group was identified in [7] with $\widehat{\mathrm{H}}^3_{\mathcal{D}}(X, 1)$, computed using the complex $D_{h.h.}(1)^\bullet_X$.

Remark 2.8. It was found that the notion of connection compatible with a given Hermitian structure as defined in loc. cit. is not the same as the one used by Brylinski and others (see, e.g. [6, Proposition 6.9(1)]). Here we can further elucidate the remarks at the end of [7] by pinpointing the geometric difference: the notion of compatibility used by Brylinski involves solely the structure of $(\mathscr{E}_X^+, \mathscr{E}_X^1(1))$-gerbe, whereas the definition we put forward uses the notion of *morphism* of gerbes bound by a complex. The latter remembers, so to speak, the structure of $\mathscr{O}_X^\times$-gerbe.

3. Gerbes bound by complexes of length 3

3.1. (B, C)-torsors

First, recall that for a given complex $B \xrightarrow{\sigma} C$ of non-necessarily abelian groups, a (B, C)-*torsor* (see [24,14]) is a pair (P, s) where P is a B-torsor and s a section of $\sigma_*(P) \overset{\text{def}}{=} P \wedge^B C$. A morphism between two pairs (P, s) and

(P', s') is a morphism $f : P \to P'$ of B-torsors such that $\sigma_*(f)(s) = s'$. With these definitions the (B, C)-torsors form a category, in fact a gerbe, $\mathrm{TORS}(B, C)$, and we denote by $\mathbf{H}^1(X, B \to C)$ the set of isomorphism classes. There is an obvious forgetful functor $\mathrm{TORS}(B, C) \longrightarrow \mathrm{TORS}(B)$, and a corresponding map of cohomology sets $\mathbf{H}^1(X, B \to C) \longrightarrow \mathrm{H}^1(X, B)$.

When B and C are abelian, which is the case of interest here, the cohomology set classifying isomorphism classes of (B, C)-torsors is isomorphic to the standard hypercohomology group.

Suppose we are given a map of complexes

$$\begin{array}{ccc} B & \xrightarrow{\sigma} & C \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ B' & \xrightarrow{\sigma'} & C' \end{array}$$

then we obtain a functor

$$(g, h)_* : \mathrm{TORS}(B, C) \longrightarrow \mathrm{TORS}(B', C'),$$

which is defined as follows. To an object (P, s) of $\mathrm{TORS}(B, C)$ we associate the pair $(g_* P, h_*(s))$, where $g_* P = P \wedge^B B'$. This is well-defined, since $\sigma'_* g_* P \cong h_* \sigma_* P$. Then it is immediate to verify that morphisms $(P, s) \to (P', s')$ in $\mathrm{TORS}(B, C)$ are brought to morphisms in $\mathrm{TORS}(B', C')$.

The following alternative characterization will be useful in the following. Using [23, III.1.6.1], it is easily seen that the structure of (B, C)-torsor on P corresponds to the datum of a C-equivariant map:

$$\begin{aligned} \sigma_*(P) &\longrightarrow \underline{\mathrm{Hom}}_B(P, C) \\ t &\longmapsto [s \mapsto t^{-1} \sigma_* s] \end{aligned} \tag{3.1.1}$$

where $\underline{\mathrm{Hom}}_B$ denotes (right) B-equivariant maps, and C is considered as a right B-space via σ.

3.2. (A, B, C)-gerbes

Let $A \xrightarrow{\delta} B \xrightarrow{\sigma} C$ be a complex of abelian groups on C/X, and let $p : \mathscr{G} \to \mathsf{C}/X$ be a gerbe with band A.

Definition 3.1. We say that $\mathscr{G}$ is bound by the complex $A \to B \to C$, or that is an (A, B, C)-gerbe, if there is a morphism

$$\tilde{\mu} : \mathscr{G} \longrightarrow \mathrm{TORS}(B, C)$$

such that $\mathscr{G}$ is an (A, B)-gerbe for the δ-morphism defined by the composition of $\tilde{\mu}$ with the forgetful functor $\mathrm{TORS}(B, C) \to \mathrm{TORS}(B)$.

In other words, the structure of (A, B, C)-gerbe on $\mathscr{G}$ is a factorization of the morphism μ defining the structure of (A, B)-gerbe through $\mathrm{TORS}(B, C)$. For an object $x \in \mathrm{Ob}\, \mathscr{G}_U$, denote

$$\tilde{\mu}(x) = (\mu(x), \nu(x)),$$

where $\mu = \mathit{forget} \circ \tilde{\mu}$, and $\nu(x)$ is a section of $\sigma_*(\mu(x))$.

Next, we can consider the notion of morphism of two such gerbes along the same lines as for (A, B)-gerbes. Thus, let us be given a morphism of complexes of abelian sheaves over C/X:

$$\begin{array}{ccccc} A & \xrightarrow{\delta} & B & \xrightarrow{\sigma} & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ A' & \xrightarrow{\delta'} & B' & \xrightarrow{\sigma'} & C' \end{array}$$

Let $\mathscr{G}$ and $\mathscr{G}'$ be two gerbes bound by (A, B, C) and (A', B', C'), respectively.

Definition 3.2. An (f, g, h)-morphism from $\mathscr{G}$ to $\mathscr{G}'$ consists of:

(1) an f-morphism $\lambda \colon \mathscr{G} \longrightarrow \mathscr{G}'$;
(2) a natural isomorphism of functors

$$\tilde{\alpha} \colon (g, h)_* \circ \tilde{\mu} \Longrightarrow \tilde{\mu}' \circ \lambda$$

from $\mathscr{G}$ to $\mathrm{TORS}(B', C')$ such that the composition (=pasting) $F' * \tilde{\alpha}$ with the forgetful functor $F' \colon \mathrm{TORS}(B', C') \longrightarrow \mathrm{TORS}(B')$ is the natural isomorphism associated with an (f, g)-morphism as in Definition 2.5.

The second condition in the definition can be explained as follows. Consider the diagram

$$\begin{array}{ccccc}
\mathscr{G} & \xrightarrow{\tilde{\mu}} & \mathrm{TORS}(B, C) & \xrightarrow{F} & \mathrm{TORS}(B) \\
\lambda \big\downarrow & \swarrow_{\tilde{\alpha}} & \big\downarrow (g,h)_* & & \big\downarrow g_* \\
\mathscr{G}' & \xrightarrow{\tilde{\mu}'} & \mathrm{TORS}(B', C') & \xrightarrow{F'} & \mathrm{TORS}(B')
\end{array}$$

Pasting with F' gives

$$F' * \tilde{\alpha} \colon F' \circ (g, h)_* \circ \tilde{\mu} \Longrightarrow F' \circ \tilde{\mu}' \circ \lambda$$

that is,

$$F' * \tilde{\alpha} \colon g_* \circ F \circ \tilde{\mu} \Longrightarrow \mu' \circ \lambda.$$

We require this to coincide with the isomorphism α in Definition 2.5.

Again, we call this morphism an *equivalence*, or more precisely, an (f, g, h)-equivalence, if so is the underlying functor $\lambda \colon \mathscr{G} \longrightarrow \mathscr{G}'$. In particular, this is the case when $A' = A$, $B' = B$, $C' = C$ and f, g, and h are the identity maps, which we refer to as an (A, B, C)-equivalence. Being equivalent in this sense is an equivalence relation, and we have:

Proposition 3.3. *Classes of equivalences of (A, B, C)-gerbes are classified by the hypercohomology group*

$$\mathbf{H}^2(X, A \to B \to C).$$

Proof. We will just sketch how to obtain the class corresponding to a gerbe $\mathscr{G}$ on C/X bound by the complex $A \to B \to C$. Let us proceed under the assumption that working with Čech cohomology is sufficient. Thus, let $(U_i \to X)_{i \in I}$ be a cover for X and assume that $\mathscr{G}$ is decomposed [12] by the choice of objects $x_i \in \mathrm{Ob}\,\mathscr{G}_{U_i}$ and morphisms $\varphi_{ij} \colon x_j|_{U_{ij}} \to x_i|_{U_{ij}}$.

For each object x_i the functor $\tilde{\mu} \colon \mathscr{G} \longrightarrow \mathrm{TORS}(B, C)$ gives us a pair $\tilde{\mu}(x_i) = (\mu(x_i), \nu(x_i))$, where $\nu(x_i) \in \Gamma(\sigma_*(x_i))$. Then, from the morphism φ_{ij} we obtain the morphism of torsors

$$(\varphi_{ij})_* \equiv \mu(\varphi_{ij}) \colon \mu(x_j) \longrightarrow \mu(x_i)$$

so that

$$\nu(x_i) = \sigma_*((\varphi_{ij})_*)(\nu(x_j)). \tag{3.2.1}$$

The decomposition (x_i, φ_{ij}) of $\mathscr{G}$ gives a cocycle $(a_{ijk}) \in \mathrm{Z}^2((U_i \to X), A)$ in the usual way, [12], [23, IV.3.5.1]. Furthermore, let $(s_i)_{i \in I}$ be a collection where s_i is a section of the $B|_{U_i}$-torsor $\mu(x_i)$. It follows that a cochain (b_{ij}) with values in B is defined by

$$(\varphi_{ij})_*(s_j) = s_i b_{ij},$$

and the usual argument shows that

$$a_{ijk} = b_{ik}^{-1} b_{ij} b_{jk}. \tag{3.2.2}$$

Now, since $\tilde{\mu}(x_i)$ is a (B, C)-torsor, we have that

$$\sigma_*(s_i) = \nu(x_i)c_i,$$

for an appropriate section c_i of $C|_{U_i}$, for each $i \in I$. On the one hand, this gives:

$$\sigma_*((\varphi_{ij})_*(s_j)) = \nu(x_i)c_i\sigma(b_{ij}).$$

On the other hand, by functoriality we have

$$\begin{aligned}\sigma_*((\varphi_{ij})_*(s_j)) &= \sigma_*(\mu(\varphi_{ij}))(\sigma_*(s_j)) \\ &= \sigma_*((\varphi_{ij})_*)(\nu(x_j))c_j,\end{aligned}$$

and using (3.2.1) we finally obtain

$$c_i\sigma(b_{ij}) = c_j. \tag{3.2.3}$$

Then (3.2.2), (3.2.3), and the cocycle property for (a_{ijk}) give the desired 2-cocycle with values in the complex $A \to B \to C$. □

The alternative characterization of (B, C)-torsor at the end of Section 3.1, and the technique used in the proof of the proposition can be put together to provide the following alternative characterization of the notion of (A, B, C)-gerbe.

Let $A \xrightarrow{\delta} B \xrightarrow{\sigma} C$ be a complex of abelian groups over C/X.

Lemma 3.4. *The structure of (A, B, C)-gerbe on $\mathscr{G} \longrightarrow \mathsf{C}/X$ is equivalent to the following data:*

(1) $\mu\colon \mathscr{G} \longrightarrow \mathrm{T{\small ORS}}(B)$ *making $\mathscr{G}$ into an (A, B)-gerbe;*
(2) *for each object $x \in \mathrm{Ob}\,\mathscr{G}_U$ a map $\nu(x)\colon \mu(x) \longrightarrow C|_U$ such that:*
   (a) $\nu(x)(sb) = \nu(x)(s)\sigma(b)$ *for each section s of $\mu(x)$ and b of $B|_U$;*
   (b) *for each morphism $f\colon x \longrightarrow y$ in $\mathscr{G}_U$ a commutative diagram*

$$\begin{array}{ccc} \mu(x) & \xrightarrow{\ \mu(f)\ } & \mu(y) \\ & {\scriptstyle \nu(x)}\searrow \quad \swarrow{\scriptstyle \nu(y)} & \\ & C|_U & \end{array}$$

Proof. The existence of the map $\nu(x)$ is simply a consequence of the existence of a section $\nu(x)$ of $\sigma_*(\mu(x))$ in the structure of (B, C)-torsor of $\mu(x)$ determines a morphism $\mu(x) \longrightarrow C|_U$ according to (3.1.1).

The commutativity of the diagram follows then from the fact that the structure of (B, C)-torsor of $\mu(x)$ implies that $\nu(y) = \sigma_*\mu(f)(\nu(x))$. □

A different characterization of (A, B, C)-gerbes in terms of torsors over a morphism of *gr*-stacks will appear in Section 6.1, when we will be discussing 2-gerbes bound by complexes (of *gr*-stacks).

3.3. Examples: Curvings

The main example we want to consider, is that of a curving on a $\mathscr{O}_X^\times$-gerbe $\mathscr{G}$ equipped with a connective structure. The concept, introduced by Brylinski [3], but attributed to Deligne, is the analogous of the curvature of a connection on a line bundle.

$\mathscr{G}$ possesses a connective structure if it is a gerbe bound by $\mathscr{O}_X^\times \xrightarrow{\mathrm{d}\log} \Omega_X^1$. We can move one step forward and consider instead the longer complex:

$$\mathscr{O}_X^\times \xrightarrow{\mathrm{d}\log} \Omega_X^1 \xrightarrow{\mathrm{d}} \Omega_X^2. \tag{3.3.1}$$

Definition 3.5. A *curving* on $\mathscr{G}$ is the structure of gerbe bound by the complex (3.3.1).

According to Lemma 3.4, a curving on a gerbe $\mathscr{G}$ with connective structure $\mathscr{C}o$ will be given by a map

$$\mathscr{K}(x) : \mathscr{C}o(x) \longrightarrow \Omega^2_U$$

for each object $x \in \operatorname{Ob}\mathscr{G}_U$, and open $U \to X$, such that

$$\mathscr{K}(x)(\nabla + \alpha) = \mathscr{K}(x)(\nabla) + \mathrm{d}\,\alpha,$$

where ∇ is a section of $\mathscr{C}o(x)$ and α is a section of Ω^1_U. Moreover, if $f : x \longrightarrow y$ is a morphism in $\mathscr{G}_U$, then the commutative diagram in Lemma 3.4 translates into

$$\mathscr{K}(y)(f_*(\nabla)) = \mathscr{K}(x)(\nabla).$$

By direct comparison, we can see that these are exactly the properties of the curving listed in [3], hence our definition agrees with the one in loc. cit.

It follows from the classification result above that we have a gerbe $\mathscr{G}$ equipped with connective structure and curving defines a class in the hypercohomology group:

$$\mathbf{H}^2(X, \mathscr{O}^\times_X \xrightarrow{\mathrm{d}\log} \Omega^1_X \xrightarrow{\mathrm{d}} \Omega^2_X) \simeq \mathrm{H}^3_{\mathcal{D}}(X, \mathbf{Z}(3)).$$

The isomorphism with the Deligne cohomology group follows from the quasi-isomorphisms:

$$\left(\mathscr{O}^\times_X \xrightarrow{\mathrm{d}\log} \Omega^1_X \xrightarrow{\mathrm{d}} \Omega^2_X\right)[-1] \xrightarrow{\simeq} \left(\mathbf{Z}(3) \longrightarrow \mathscr{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \Omega^2_X\right),$$

the complex on the right-hand side being $\mathbf{Z}(3)^\bullet_{\mathcal{D}}$.

4. 2-Gerbes: Main definitions

In this section we review some basic definitions and relevant facts about 2-gerbes here. The standard reference is [12], which should be referred to for a complete treatment.

Recall that a 2-gerbe is a 2-stack, in particular a fibered 2-category, satisfying local non-emptiness and connectivity requirements generalizing those of a gerbe. The general definition of fibered 2-categories can be found in [25]. Analogously to loc. cit., we will assume that given a fibration $p : \mathfrak{G} \longrightarrow \mathfrak{S}$ of 2-categories, the base 2-category is in effect a category regarded as a *discrete* 2-category—namely, one with all 2-arrows being identities. In other words, $\mathfrak{S} = 2\text{—}\mathfrak{Cat}(\mathsf{S})$, where S is a category. To avoid overburdening our notation, we will simply write our fibrations as $p : \mathfrak{G} \longrightarrow \mathsf{S}$, without risk of confusion. In the following, the category S will in fact be the site C/X, with all our standing assumptions concerning C/X to be kept for 2-gerbes as well.

4.1. 2-Stacks

A 2-stack is a fibered 2-category $p : \mathfrak{G} \longrightarrow \mathsf{S}$ such that:

(1) 1-arrows and 2-arrows can be glued, a fact that can be succinctly stated by saying that for any two objects $x, y \in \operatorname{Ob}\mathfrak{G}_U$ over $U \in \operatorname{Ob}\mathsf{S}$, the fibered category $\mathscr{H}om_U(x, y)$ is stack over S/U;
(2) Objects can be glued, namely 2-descent on objects holds.

(A pre-2-stack is a fibered 2-category satisfying only the first condition above.)

Without entering into too many details, it is worthwhile making the gluing condition on objects more explicit. Thus, let U be an object of S, and let $(U_i \to U)$ be a cover as usual. The assignment of 2-descent data over U is the assignment of a collection of objects $x_i \in \operatorname{Ob}\mathfrak{G}_{U_i}$ such that there is a 1-arrow:

$$\varphi_{ij} : x_j \longrightarrow x_i$$

over U_{ij} and a 2-arrow (in fact, a 2-isomorphism):

$$\begin{array}{ccccc} & & x_j & & \\ & \overset{\phi_{jk}}{\nearrow} & \Downarrow \alpha_{ijk} & \overset{\phi_{ij}}{\searrow} & \\ x_k & & \xrightarrow[\phi_{ik}]{} & & x_i \end{array}$$

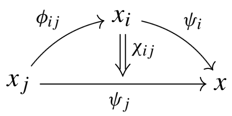
over U_{ijk} such that the following compatibility condition holds:

$$\alpha_{ikl} \circ (\alpha_{ijk} * \varphi_{kl}) = \alpha_{ijl} \circ (\varphi_{ij} * \alpha_{jkl}).$$

The assignment of the triple $(x_i, \varphi_{ij}, \alpha_{ijk})$, is called *2-Descent data.* Condition (2) above then means that there exists an object $x \in \operatorname{Ob} \mathfrak{G}_U$ with 1-arrows

$$\psi_i : x_i \longrightarrow x$$

and 2-isomorphisms



satisfying the now obvious compatibility conditions with the isomorphisms α_{ijk}. This is referred to by saying that the 2-descent data is effective.

4.2. 2-Gerbes

In words, a 2-gerbe $\mathfrak{G} \longrightarrow \mathsf{S}$ is a 2-stack in 2-groupoids which is locally non-empty and connected. A detailed account of several variants of this definition of a 2-gerbe is given in the text [12]. Following loc. cit., the properties characterizing a 2-Gerbe are the following:

(1) $\mathfrak{G}$ is *locally non-empty:* assuming that $\mathsf{S} = \mathsf{C}/X$, there exists a cover $U \to X$ such that $\operatorname{Ob} \mathfrak{G}_U$ is not empty.
(2) $\mathfrak{G}$ is *locally connected:* for each $x, y \in \operatorname{Ob} \mathfrak{G}_U$, for some object U of S, there exists a cover $\varphi : V \to U$ such that the set of arrows from x_V to y_V[9] is not empty.
(3) 1-*arrows are weakly invertible:* for any 1-arrow $f : x \to y$ in $\mathscr{G}_U$, $U \in \operatorname{Ob} \mathfrak{G}$, there is an inverse $g : y \to x$ up to two 2-arrows.
(4) 2-arrows are (strictly) invertible in $\mathfrak{G}_U$.

There are different equivalent forms of the last two axioms, as well as local versions of all four to be obtained by considering coverings of U, see [12] for more details. Here we only quote the fact that condition (3) above is equivalent (if condition (4) is also satisfied) to:

(3′) Given two 1-arrows $f : x \to y$ and $g : x \to z$ in $\mathfrak{G}_U$, there exists a 1-arrow $h : y \to z$ and a 2-arrow $\alpha : h \circ f \Rightarrow g$.

Finally, a note of caution: although the *stack* $\mathscr{H}om_U(x, y)$ is locally non-empty by condition (2), in general it will not be connected, so that condition (3) does not quite imply that $\mathscr{H}om_U(x, y)$ is a gerbe. This is the case when $x = y$ for fully *abelian* 2-gerbes, to be discussed below.

4.2.1. Gr-stacks of automorphisms

To conclude these remarks of preparatory nature, let us briefly discuss automorphisms of objects.

For any given object $x \in \operatorname{Ob} \mathfrak{G}_U$, the stack $\mathscr{A}ut_U(x)$ of self-arrows of x is a stack in groupoids equipped with a strictly associative *monoidal structure,* that is a functor $\mathscr{A}ut_U(x) \times \mathscr{A}ut_U(x) \longrightarrow \mathscr{A}ut_U(x)$ implementing a product law on $\mathscr{A}ut_U(x)$. It follows from the 2-gerbe axioms that $\mathscr{A}ut_U(x)$ admits a choice of inverses, compatible with descent, hence it is a *group-like* stack, or *gr-stack*, for short, cf. [13,12,26].

Analogously to the gerbe case, if $\mathscr{A}$ is a fixed *gr*-stack on S, we define a 2-$\mathscr{A}$-gerbe to be a 2-gerbe $\mathfrak{G}$ over S such that for every object $x \in \operatorname{Ob} \mathfrak{G}_U$ there is an equivalence

$$a_x : \mathscr{A}ut_U(x) \xrightarrow{\sim} \mathscr{A}|_U.$$

[9] Note that given $\varphi : V \to U$ and an object x above U thanks to the axioms of a fibered 2-category we can speak of "the" object x_V above V with an arrow $x_V \to x$ above φ up to 2-equivalence.

4.3. Abelian 2-gerbes

A 2-$\mathscr{A}$-gerbe to be *abelian* if the equivalences a_x introduced above are natural in the sense specified in [12, Definition 4.13]. As shown in loc. cit., this has the consequence that $\mathscr{A}$ is *braided,* that is, there is a commutativity functor for the monoidal structure.

An *additional* commutativity condition is to assume that

$$\mathscr{A} = \text{TORS}(A),$$

for a sheaf of abelian groups A over S. (Since A is abelian, this is a *gr*-stack under the standard contracted product of A-torsors.)

As explained in loc. cit., these two requirements have the consequence that the *gr*-stack $\mathscr{A}ut_U(x)$ is a gerbe over S/U, and in fact a neutral one, i.e. it is equivalent to $\text{TORS}(A|_U)$, since it has the global object id_x. Automorphisms of 1-arrows are then equivalent to sections of the sheaf of groups A, as in [4]. If both commutativity conditions hold, we commit a mild abuse of language and say that the 2-gerbe $\mathfrak{G}$ is *bound* by the sheaf of abelian groups A, or that it is a 2-A-gerbe, dropping the typographical reference to the *gr*-stack $\mathscr{A}$.

It is by now standard that the *fully* abelian 2-gerbes, or 2-A-gerbes, are classified up to equivalence by the ordinary cohomology group $\mathrm{H}^3(X, A)$.

In what follows we will limit our consideration to abelian 2-gerbes which are not, however, necessarily fully abelian.

4.3.1. Morphisms

As noted, a morphism between two 2-gerbes $\mathfrak{G}$ and $\mathfrak{H}$ is a cartesian 2-functor $F\colon \mathfrak{G} \longrightarrow \mathfrak{H}$ between the underlying 2-stacks.

Suppose that $\mathfrak{G}$ is a 2-$\mathscr{A}$-gerbe and $\mathfrak{H}$ is a 2-$\mathscr{B}$-gerbe, and $\lambda\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a morphism of *gr*-stacks, where we assume that both $\mathscr{A}$ and $\mathscr{B}$ are at least braided. By analogy with the case of gerbes, we will call F a λ-*morphism* if the obvious commutative diagrams (up to 2-isomorphism, this time) are satisfied. In particular, this means that F must be compatible with the morphisms a_x in the sense that we have the following diagram:

$$\begin{array}{ccc} \mathscr{A}ut(x) & \longrightarrow & \mathscr{A}ut(F(x)) \\ {\scriptstyle a_x}\downarrow & \overset{\nu_x}{\swarrow} & \downarrow{\scriptstyle b_x} \\ \mathscr{A}|_U & \xrightarrow[\lambda]{} & \mathscr{B}|_U \end{array}$$

for an appropriate isomorphism ν_x.

In particular, we are interested in the situation where a homomorphism $\delta\colon A \longrightarrow B$ of abelian groups is given, and $\lambda = \delta_*$ is simply the induced functor:

$$\delta_*\colon \text{TORS}(A) \longrightarrow \text{TORS}(B)$$

between the corresponding *gr*-stacks. In this case we will refer to F as a δ-morphism, with a mild abuse of language. The salient property of a δ-morphism in this sense is that if a section $a \in A|_U$ corresponds to an automorphism of a 1-arrow f of $\mathfrak{G}_U$, then the corresponding automorphism of $F(f)$ in $\mathfrak{H}_U$ will be $\delta(a) \in B|_U$.

4.3.2. Classification

As already mentioned, a 2-A-gerbe is classified by an element of the (ordinary) cohomology group $\mathrm{H}^3(X, A)$: Let us briefly recall here the well-known local calculation leading to the classification.

For simplicity, let us remain in the Čech setting, so let us once again consider a cover $(U_i \to X)_{i\in I}$ of X. Now, given a 2-gerbe $\mathfrak{G}$, let us choose a decomposition by selecting a collection of objects x_i in $\mathfrak{G}_{U_i}$. There is a 1-arrow

$$\varphi_{ij}\colon x_j \to x_i$$

between their restrictions to $\mathfrak{G}_{U_{ij}}$. Then axiom (3′) in Section 4.2, and the abelianness assumptions imply that there exist 2-arrows such that:

$$\alpha_{ijk}\colon \varphi_{ij} \circ \varphi_{jk} \Longrightarrow \varphi_{ik}.$$

Over a 4-fold intersection U_{ijkl}, we have two 1-arrows $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{kl} : x_l \to x_i$ and $\varphi_{il} : x_l \to x_i$ and between them *two* 2-arrows, namely $\alpha_{ijl} \circ (\mathrm{id}_{\varphi_{ij}} * \alpha_{jkl})$ and $\alpha_{ikl} \circ (\alpha_{ijk} * \mathrm{id}_{\varphi_{kl}})$. Since 2-arrows are strictly invertible, it follows again from the axioms that there exists a section a_{ijkl} of A over U_{ijkl} such that

$$\alpha_{ijl} \circ (\mathrm{id}_{\varphi_{ij}} * \alpha_{jkl}) = a_{ijkl} \circ \alpha_{ikl} \circ (\alpha_{ijk} * \mathrm{id}_{\varphi_{kl}}). \tag{4.3.1}$$

This section is a 3-cocycle and the assignment $\mathfrak{G} \mapsto [a]$ gives the classification isomorphism.

5. 2-Gerbes bound by a complex

5.1. $\mathscr{B}$-torsors

The notion of *torsor* under a *gr*-stack will play a significant role below. The definition has been given in full generality in [14, (6.1)], [13], so here we will confine ourselves to only recalling the main points.

Let $\mathscr{B}$ be a *gr*-stack on C/X. Briefly, a stack in groupoids $\mathscr{P}$ will be a (right) $\mathscr{B}$-torsor if there is a morphism of stacks

$$m : \mathscr{P} \times \mathscr{B} \longrightarrow \mathscr{P}$$

compatible with the group law of $\mathscr{B}$ in the sense specified in loc. cit., and such that the morphism

$$\tilde{m} = (\mathrm{pr}_1, m) : \mathscr{P} \times \mathscr{B} \longrightarrow \mathscr{P} \times \mathscr{P}$$

is an equivalence. As in loc. cit., there will be an associativity natural isomorphism:

$$\mu_{x,b,b'} : (x \cdot b) \cdot b' \xrightarrow{\sim} x \cdot (b \cdot b'),$$

where $x \cdot b$ stands for $m(x, b)$. This isomorphism will have to satisfy the standard pentagon diagram.

Having so far defined what ought to be called a *pseudo*-torsor, we need to complete the definition by adding the condition that there exists a cover $U \to X$ such that the fiber category $\mathscr{P}_U$ is non-empty.

There are a few constructions for $\mathscr{B}$-torsors that are generalizations of well-known ones for standard torsors which we are going to recall now: cocycles and contracted products.

5.1.1. Contracted product of torsors

The notion of contracted product for torsors over a *gr*-stack is introduced in [14, Section 6.7].

If $\mathscr{P}$ (resp. $\mathscr{Q}$) is a right (resp. left) $\mathscr{B}$-torsor, the contracted product $\mathscr{P} \wedge^{\mathscr{B}} \mathscr{Q}$ is defined as follows. The objects are pairs $(x, y) \in \mathrm{Ob}\, \mathscr{P} \times \mathscr{Q}$. A morphism $(x, y) \to (x', y')$, however, is an equivalence classes of triples (f, b, g), where $b \in \mathrm{Ob}\, \mathscr{B}$, and $f : x \cdot b \to x'$ and $g : y \to b \cdot y'$ are morphisms of $\mathscr{P}$ and $\mathscr{Q}$, respectively. Two triples (f, b, g) and (f', b', g') are equivalent if there is a morphism $\beta : b \to b'$ in $\mathscr{B}$ such that $f = f' \circ (x \cdot \beta)$ and $g' = (\beta \cdot y') \circ g$.

Properties analogous to the familiar ones for ordinary torsors hold. For example, one has the isomorphism

$$(x \cdot b, y) \xrightarrow{\sim} (x, b \cdot y),$$

given by the pair $(\mathrm{id}_{x \cdot b}, b, \mathrm{id}_{b \cdot y})$.

In the following we will be considering braided (and in fact, Picard) *gr*-stacks exclusively, hence the distinction between left and right torsor will not matter. In principle, by analogy with the case of standard torsors over an abelian group we could dispense with the notation for the contracted product and denote the product with the symbol $\mathscr{P} \otimes \mathscr{Q}$, instead. We will not do so, however.

5.1.2. Cocycles

A torsor $\mathscr{P}$ over a (not necessarily braided) *gr*-stack $\mathscr{B}$ can also be characterized by a cocycle with respect to a cover.

Given a cover $(U_i \to X)_{i \in I}$, the torsor $\mathscr{P}$ has non-empty fiber categories over it. Thus choose objects $x_i \in \mathrm{Ob}\, \mathscr{P}_{U_i}$. Since by definition $\mathscr{P}$ is locally (i.e. over the cover) equivalent to $\mathscr{B}$, it follows that we can obtain isomorphism $x_j \xrightarrow{\sim} x_i \cdot b_{ij}$, where b_{ij} is an object of $\mathscr{B}$ over U_{ij}, and the isomorphism takes place in $\mathscr{P}_{U_{ij}}$. (We are systematically ignoring the isomorphisms resulting from the pull-back functors.) By pulling back to U_{ijk} we obtain a 1-cocycle with values in $\mathscr{B}$:

$$\beta_{ijk} \colon b_{ij} \cdot b_{jk} \xrightarrow{\sim} b_{ik}. \tag{5.1.1}$$

The isomorphisms β_{ijk} in $\mathscr{B}|_{U_{ijk}}$ turn out to satisfy the obvious compatibility condition on quadruple intersections U_{ijkl}, which we do not explicitly write here. The pair (b_{ij}, β_{ijk}) is the 1-cocycle with values in the *gr*-stack $\mathscr{B}$ determined by $\mathscr{P}$.

5.1.3. $\mathscr{B}$-torsors and B-gerbes

It arises from the general classification theory of 2-gerbes that $\mathrm{TORS}(\mathscr{B})$ is a 2-$\mathscr{B}$-gerbe. Moreover, it follows from the general discussion in [14, Section 7.2 and Proposition 7.3] that if $\mathscr{B} = \mathrm{TORS}(B)$, then $\mathrm{TORS}(\mathscr{B})$ is equivalent to $\mathrm{GERBES}(B)$, the 2-gerbe of B-gerbes over X.

It is possible to see this via the 1-cocycle pair (5.1.1) as follows. Recall that $\mathscr{B} = \mathrm{TORS}(B)$ with B abelian, so we obtain a "torsor cocycle" in the sense of [27]. It follows that the groupoids $\mathrm{TORS}(B)|_{U_i}$ can be glued in the standard way to give a B-gerbe.

Remark 5.1. The argument just outlined is of course not specific to B being abelian. Upon replacing $\mathrm{TORS}(B)$ with $\mathrm{BITORS}(B)$ everything works in general.

Remark 5.2. The 1-cocycle written above coincides with Hitchin's notion of "gerbe data," [28]. The latter lacks the categorical input, however.

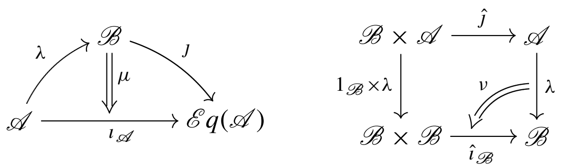
5.2. Crossed modules of gr-categories

It was observed above that the complex $\delta \colon A \longrightarrow B$ of abelian groups ought to be considered as an abelian crossed module, namely one where we impose strict commutativity on the associated *gr*-category. (That is, we demand it be strictly Picard.)

It turns out that a similar pattern holds in the case of a *crossed module of gr-categories* in the sense of [13, Définition 2.2.8]. It requires that there exist additive functors

$$\lambda \colon \mathscr{A} \longrightarrow \mathscr{B}, \qquad \jmath \colon \mathscr{B} \longrightarrow \mathscr{E}q(\mathscr{A})$$

such that the relations determined by the following diagrams hold:



where $\mathscr{E}q(\mathscr{A})$ denotes the *gr*-stack of self-equivalences of $\mathscr{A}$, $\imath_{\mathscr{A}}$ denotes the inner conjugation, and the top and bottom horizontal arrows in the diagram to the right are the actions of $\mathscr{B}$ on $\mathscr{A}$ and on itself induced by $\jmath$ and the inner conjugation.

Now observe that requiring the resulting group law on $\mathscr{A} \times \mathscr{B}$ to be commutative (up to natural isomorphism), entails that both $\mathscr{A}$ and $\mathscr{B}$ are braided, and that the action of $\mathscr{B}$ on $\mathscr{A}$ is trivial. Thus, an abelian crossed module of *gr*-categories will simply be an additive functor

$$\lambda \colon \mathscr{A} \longrightarrow \mathscr{B}, \tag{5.2.1}$$

between braided *gr*-categories. The same conclusions hold if we replace *gr*-categories with *gr*-stacks over C/X. We will also refer to (5.2.1) as a complex of (braided) *gr*-stacks.

If both $\mathscr{A}$ and $\mathscr{B}$ have strict group laws, then they are the *gr*-categories associated with crossed modules, so we obtain a "crossed module of crossed modules," namely a *crossed square*, see [29,13]. Thus (5.2.1) reduces to the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle\sigma} \\ G & \xrightarrow[u]{} & H \end{array} \tag{5.2.2}$$

where the vertical arrows are the crossed modules associated with $\mathscr{A}$ and $\mathscr{B}$, respectively, and the horizontal arrows, as well as the composite diagonal one, are also crossed modules. There are other axioms, for which we refer the reader to the treatment in loc. cit. We will not need them here, however, because if both $\mathscr{A}$ and $\mathscr{B}$ are strictly commutative, their associated crossed modules become complexes of abelian groups, so that (5.2.2) becomes a commutative square of homomorphisms of abelian groups, which is the situation we will be interested in. Thus "crossed square" will be meant as a synonym for a morphism of complexes of abelian groups.

5.3. 2-($\mathscr{A}$, $\mathscr{B}$)-gerbes

We are now going to consider the analog of Definition 2.1 for abelian 2-gerbes. We proceed by giving a direct generalization of Definition 2.1, where we replace the complex $A \to B$ with the length 2-complex (that is a morphism) of *gr*-stacks, which we assume braided, heeding to the principle that we climb the ladder of the higher algebraic structures by promoting the *coefficients* of cohomology from sheaves of (abelian) groups, to *gr*-stacks, etc.

Definition 5.3. A 2-gerbe bound by the complex (5.2.1) is a 2-$\mathscr{A}$-gerbe $\mathfrak{G}$ over C/X, equipped with a λ-morphism:

$$J : \mathfrak{G} \longrightarrow \mathrm{T{\small ORS}}(\mathscr{B}).$$

A 2-gerbe bound by the complex (5.2.1) will be called a 2-($\mathscr{A}$, $\mathscr{B}$)-gerbe. (Notice that $\mathrm{T{\small ORS}}(\mathscr{B})$ is a 2-$\mathscr{B}$-gerbe in an obvious way, hence the notion of λ-morphism makes sense.)

If $\mathfrak{G}$ is actually a 2-A-gerbe, and $\mathscr{B} = \mathrm{T{\small ORS}}(B)$, where B is a sheaf of abelian groups over C/X, with a homomorphism $\delta\colon A \longrightarrow B$, we call it a 2-$(A, B)$-gerbe, or a 2-gerbe bound by $A \to B$. (The morphism J in the definition is a $\lambda = \delta_*$-morphism.)

For a 2-(A, B)-gerbe, owing to the last remark in Section 5.1, Definition 5.3 can be recast in the form used in [7, Definition 5.5.1] (in a special case), which we state here as a lemma:

Lemma 5.4. *The datum of a* 2-(A, B)*-gerbe is equivalent to that of a Cartesian* 2*-functor*

$$J : \mathfrak{G} \longrightarrow \mathrm{G{\small ERBES}}(B)$$

which is a δ*-morphism of* 2*-gerbes.*

Morphisms of 2-gerbes bound by a complex of length 2 can be defined by promoting Definition 2.5 to using braided *gr*-stacks and then (for those coming from abelian groups) using Lemma 5.4. Specifically, analogously to what was done in Section 2.1, consider the square of *gr*-stacks:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\lambda} & \mathscr{B} \\ {\scriptstyle\phi}\big\downarrow & \overset{J}{\swarrow} & \big\downarrow{\scriptstyle\psi} \\ \mathscr{A}' & \xrightarrow[\lambda']{} & \mathscr{B}' \end{array} \tag{5.3.1}$$

Definition 5.5. A (φ, ψ)-morphism $(F, \mu)\colon (\mathfrak{G}, J) \longrightarrow (\mathfrak{G}', J')$ consists of:

(1) a φ-morphism $F : \mathfrak{G} \longrightarrow \mathfrak{G}'$;

(2) a natural transformation of 2-functors:

$$\mu\colon \psi_* \circ J \Longrightarrow J' \circ F\colon \mathfrak{G} \longrightarrow \mathrm{TORS}(\mathscr{B}'),$$

where $\psi_*\colon \mathrm{TORS}(\mathscr{B}) \longrightarrow \mathrm{TORS}(\mathscr{B}')$ is induced from ψ in the obvious way.

In particular, the special case where (5.3.1) is induced by the morphism of complexes

$$(f, g)\colon (A, B, \delta) \longrightarrow (A', B', \delta')$$

of abelian groups will be referred to as an (f, g)-morphism of the 2-(A, B)-gerbe $(\mathfrak{G}, J)$ to the 2-(A', B')-gerbe $(\mathfrak{G}', J')$. Using Lemma 5.4, condition (2) in Definition 5.3 says that we have a natural transformation of 2-functors

$$\mu\colon g_{**} \circ J \Longrightarrow J' \circ F\colon \mathfrak{G} \longrightarrow \mathrm{GERBES}(B'),$$

where $g_{**}\colon \mathrm{GERBES}(B) \longrightarrow \mathrm{GERBES}(B')$ is induced from $g : B \longrightarrow B'$.

We speak of a $(\mathscr{A}, \mathscr{B})$-morphism if $\mathscr{A}' = \mathscr{A}$ and $\mathscr{B}' = \mathscr{B}$ and both φ and ψ are identities. We shorten this to (A, B)-morphism if both *gr*-stacks arise from abelian groups A and B in the usual way. We speak of an equivalence if the underlying 2-functor F is an equivalence of 2-stacks.

5.4. Classification I

The classification of 2-(A, B)-gerbes follows the usual pattern. The following theorem generalizes previous results on connective and Hermitian structures on 2-gerbes, see [5–7].

Theorem 5.6. *Let $\delta\colon A \to B$ be a complex of abelian groups over C/X. Equivalence classes of 2-(A, B)-gerbes are classified by the elements of the (ordinary) hypercohomology group*

$$\mathbf{H}^3(X, A \to B).$$

Proof. We only need to sketch the proof, for the details can be lifted from the above quoted references and adapted to the present situation without difficulty. Therefore let us only indicate how to obtain the cocycle representing the class of a given 2-(A, B)-gerbe.

Let us work in the Čech setting, so let $(U_i \to X)_{i\in I}$ be a cover as usual. Let $(\mathfrak{G}, J)$ be a 2-(A, B)-gerbe over X, and let x_i, φ_{ij}, and α_{ijk} be objects, morphisms, and 2-morphisms providing a full decomposition of $\mathfrak{G}$ relative to the chosen cover as in Section 4.3.2. In addition, let us pick a decomposition of the *gerbes* $J(x_i)$ over U_i by choosing objects r_i and arrows $\xi_{ij}\colon J(\varphi_{ij})(r_j) \to r_i$.

Over U_{ijk} we obtain the following diagram in $J(x_i)|_{U_{ijk}}$:

$$\begin{array}{ccccc}
J(\varphi_{ij}) \circ J(\varphi_{jk})(r_k) & \xrightarrow{J(\varphi_{ij})(\xi_{jk})} & J(\varphi_{ij})(r_j) & \xrightarrow{\xi_{ij}} & r_i \\
\downarrow & & & & \downarrow{\scriptstyle b_{ijk}} \\
J(\varphi_{ij} \circ \varphi_{jk})(r_k) & \xrightarrow[J(\alpha_{ijk})(r_k)]{} & J(\varphi_{ik})(r_k) & \xrightarrow[\xi_{ik}]{} & r_i
\end{array}$$

which defines the section $b_{ijk} \in \underline{\mathrm{Aut}}(r_i) \simeq B|_{U_{ijk}}$. (The left vertical arrow comes from the natural transformation built from the definition of 2-functor [25].)

Pulling back to U_{ijkl} we obtain a cubical diagram determined by the objects $r_i, \ldots, r_l$ as follows: four faces are built from copies of the previous (commutative) diagram. The top face is:

$$\begin{array}{ccc}
J(\varphi_{ij}) \circ J(\varphi_{jk}) \circ J(\varphi_{kl})(r_l) & \longrightarrow & J(\varphi_{ij}) \circ J(\varphi_{jl})(r_l) \\
\downarrow & & \downarrow \\
J(\varphi_{ik}) \circ J(\varphi_{kl})(r_l) & \longrightarrow & J(\varphi_{il})(r_l)
\end{array} \tag{5.4.1}$$

Note that each arrow results from the composition of the 2-arrow inherent in the definition of 2-functor J and one of the 2-arrows $J(\alpha_{ijk})$, etc. Finally, the diagram will have a bottom face given by:

$$\begin{array}{ccc} r_i & \xrightarrow{b_{jkl}} & r_i \\ {\scriptstyle b_{ijk}}\downarrow & & \downarrow{\scriptstyle b_{ijl}} \\ r_i & \xrightarrow[b_{ikl}]{} & r_i \end{array}$$

where the top arrow results from the automorphism $r_j \xrightarrow{b_{jkl}} r_j$ pushed forward along $J(\varphi_{ij})$ to obtain an automorphism of r_i. Now, the top face (5.4.1) only commutes up to an automorphism of a 2-arrow given by relation (4.3.1), and using the fact that J is a δ-morphism, we have:

$$b_{jkl} b_{ikl}^{-1} b_{ijl} b_{ijk}^{-1} = \delta(a_{ijkl}).$$

Together with the cocycle relation satisfied by a_{ijkl} (consequence of (4.3.1)), it gives the desired cocycle relation for (a_{ijkl}, b_{ijk}).

To conclude, let us hint at how the procedure is reversed. The first step is to glue the local trivial 2-gerbes $\mathrm{GERBES}(A|_{U_i})$ via a_{ijkl}. This is standard, see [12,4,5]. Then we define a 2-functor J by assigning to each object x_i over U_i, i.e. an $A|_{U_i}$-gerbe, the trivial $B|_{U_i}$-gerbe $J(x_i) = \mathrm{TORS}(B|_{U_i})$. Over U_{ijk}, the section b_{ijk} is used as an automorphism of an object r_i of $J(x_i)$, and the cocycle condition above ensures compatibility. □

Using the results in Section 2.1 about (A, B)-gerbes we can informally reword the proof of the theorem by noticing that the representative cocycle of the 2-(A, B)-gerbe $\mathfrak{G}$ was given in terms of (A, B)-gerbes. We want to make this observation precise.

To this end, let us first observe that if $\delta \colon A \to B$ is a complex of sheaves of abelian groups, then $\mathscr{G} = \mathrm{TORS}(A, B)$, introduced in Section 3.1, is a *gr*-stack: the group law is given by the standard contracted product, so for two pairs (P, s) and (Q, t) we have $(P, s) \otimes (Q, t) = (P \otimes Q, st)$. In fact $\mathscr{G}$ is the *gr*-stack associated with the homomorphism $A \to B$ viewed as an abelian crossed module. Thus,

$$\mathscr{G} = \mathrm{TORS}(A, B) \simeq (A \xrightarrow{\delta} B)^{\tilde{}},$$

cf. [14,12].

The following intermediate results (in the next proposition and theorem), are also of independent interest, as they provide an alternative characterization of (A, B)-gerbes.

Proposition 5.7. *Equivalence classes of* $\mathscr{G} = \mathrm{TORS}(A, B)$*-torsors are classified by the hypercohomology group* $\mathbf{H}^2(X, A \to B)$.

Proof. Let $\mathscr{P}$ be a $\mathscr{G}$-torsor. According to Section 5.1.3 the choice of objects x_i in the fiber categories $\mathscr{P}_{U_i}$ with respect to a cover $(U_i \to X)_{i \in I}$, determines a pair (g_{ij}, γ_{ijk}) with values in $\mathscr{G}$ satisfying the cocycle identity (5.1.1).

Given the specific nature of $\mathscr{G}$, each g_{ij} is an $(A|_{U_{ij}}, B|_{U_{ij}})$-torsor, namely it corresponds to a pair (P_{ij}, t_{ij}), where P_{ij} is an A-torsor over U_{ij}, and t_{ij} is a section of $P_{ij} \wedge^A B$. Moreover, $\gamma_{ijk} \colon P_{ij} \otimes P_{jk} \xrightarrow{\sim} P_{ik}$ (suitably restricted to U_{ijk}), and $\delta_*(\gamma_{ijk})(t_{ij} t_{jk}) = t_{ik}$.

It is perhaps better *not* to assume at this point that the torsor P_{ij} is trivialized, but rather consider the full blown hypercover (U_{ij}^α, U_i), where $(U_{ij}^\alpha \to U_{ij})_{\alpha \in A_{ij}}$ is a cover, and assume that s_{ij}^α is a trivializing section of P_{ij} over U_{ij}^α. This choice gives rise to sections $a_{ijk}^{\alpha\beta\gamma}$ of $A|_{U_{ijk}^{\alpha\beta\gamma}}$ and b_{ij}^α of $B|_{U_{ij}^\alpha}$, in the usual way:

$$\gamma_{ijk}(s_{ij}^\alpha \otimes s_{jk}^\beta) = s_{ik}^\gamma a_{ijk}^{\alpha\beta\gamma}, \qquad t_{ij} = (s_{ij}^\alpha \wedge 1) b_{ij}^\alpha.$$

Then, using $s \cdot a \wedge 1 = s \wedge \delta(a) = (s \wedge 1) \cdot \delta(a)$, it is easily checked that $(a_{ijk}^{\alpha\beta\gamma}, b_{ij}^\alpha)$ satisfies the cocycle condition with values in the complex $A \to B$ with respect to the chosen (hyper)cover. The rest of the details (to check that this is well-defined on classes) are routine and left to the reader.

Conversely, given a cocycle with values in $A \to B$ with respect to the above hypercover, we can reconstruct $(A|_{U_{ij}}, B|_{U_{ij}})$-torsors (P_{ij}, t_{ij}) satisfying the cocycle condition. We can then glue the various $\mathscr{G}_{U_i}$ using this cocycle to obtain a $\mathscr{G}$-torsor on X. Details are again left to the reader. □

Now we consider the trivial 2-gerbe $\mathrm{TORS}(\mathscr{G})$ of torsors over the *gr*-stack $\mathscr{G}$. Also recall that $\mathrm{GERBES}(A, B)$ denotes the fibered 2-category of (A, B)-gerbes over X.

Proposition 5.7, and the fact that the same hypercohomology group classifies (A, B)-gerbes as well, suggest the following theorem, which is an extension, in the abelian context, of [14, Proposition 7.3]. To prepare the statement, observe that there is an action

$$\mathrm{TORS}(A) \times \mathrm{TORS}(A, B) \longrightarrow \mathrm{TORS}(A)$$

given on objects by

$$(Q, (P, t)) \longmapsto (P \otimes Q),$$

where (P, t) is an (A, B)-torsor, and Q is an A-torsor. Of course, since A is an abelian group, $\mathrm{TORS}(A)$ is itself a *gr*-stack. Also, by the local triviality of torsors, an A-torsor is locally isomorphic to an (A, B)-torsor, thereby making $\mathrm{TORS}(A)$ a $\mathrm{TORS}(A, B)$-torsor.

Theorem 5.8. *Let* $\mathscr{G} = \mathrm{TORS}(A, B)$. *There is an equivalence (of* 2*-stacks)*

$$F\colon \mathrm{TORS}(\mathscr{G}) \xrightarrow{\sim} \mathrm{GERBES}(A, B)$$

given by:

$$F\colon \mathscr{P} \longmapsto \mathrm{TORS}(A) \wedge^{\mathscr{G}} \mathscr{P}.$$

In fact, the equivalence in the proposition is an equivalence of neutral (or trivial) 2-gerbes bound by $\mathscr{G}$.

Proof. We will confine ourselves to give a description of the 2-functor F, as well as its quasi-inverse, following loc. cit., and leave the verification of the details to the reader.

Given a cover $U \to X$, by definition we have an equivalence

$$\mathscr{P}_U \xrightarrow{\sim} \mathscr{G}_U = \mathrm{TORS}(A|_U, B|_U).$$

Moreover, observe that for any *gr*-stack $\mathscr{G}$ and for any stack in groupoid with $\mathscr{G}$-action $\mathscr{P}$, we have an equivalence

$$\mathscr{P} \xrightarrow{\sim} \mathscr{P} \wedge^{\mathscr{G}} \mathscr{G} \qquad x \longmapsto (x, o_{\mathscr{G}}),$$

where $o_{\mathscr{G}}$ is the unit object in $\mathscr{G}$. By the same argument in the proof of [14, Proposition 7.3], we have the equivalence:

$$\mathrm{TORS}(A|_U) \xrightarrow{\sim} \mathrm{TORS}(A|_U) \wedge^{\mathscr{G}_U} \mathscr{G}_U \xrightarrow{\sim} \mathrm{TORS}(A|_U) \wedge^{\mathscr{G}|_U} \mathscr{P}_U,$$

showing that $\mathrm{TORS}(A) \wedge^{\mathscr{G}} \mathscr{P}$ is locally equivalent to $\mathrm{TORS}(A)$, hence it is an A-gerbe. We make it into an (A, B)-gerbe by defining

$$\mu \stackrel{\text{def}}{=} \delta_* \wedge 1\colon \mathrm{TORS}(A) \wedge^{\mathscr{G}} \mathscr{P} \longrightarrow \mathrm{TORS}(B).$$

This is well-defined, since locally the definition dictates $(Q, (P, t)) \mapsto \delta_*(Q)$ and, using the properties of the contracted product, we have

$$(Q, (P, t)) \xrightarrow{\sim} (Q \cdot (P, t), (A, 1)) = (P \otimes Q, (A, 1)),$$

so that

$$(Q, (P, t)) \longmapsto \delta_*(P \otimes Q) \simeq \delta_*(P) \otimes \delta_*(Q) \simeq \delta_*(Q),$$

since $\delta_*(P) \simeq B$, by definition of (A, B)-torsor. (The pair $(A, 1)$ represents the unit element in $\mathscr{G} = \mathrm{TORS}(A, B)$.)

Conversely, let $(\mathscr{Q}, \mu)$ be an (A, B)-gerbe. Since it is in particular an A-gerbe, there is an equivalence

$$\mathscr{Q}|_U \simeq \mathrm{TORS}(A|_U)$$

with respect to a cover $U \to X$, so that locally the structure of (A, B)-gerbe becomes

$$\mathrm{T\small{ORS}}(A|_U) \xrightarrow{\mu|_U} \mathrm{T\small{ORS}}(B|_U).$$

In turn this is isomorphic to δ_*, the "change of structure group" functor. To see this, consider the image $E = \mu(A)$ of the trivial torsor. Since μ commutes with the product of torsors (since $Q_1 \otimes Q_2 \simeq Q_1 \wedge^A Q_2$ for A abelian), it follows from $Q \simeq Q \otimes A$ that $E \simeq B$, the trivial B-torsor. By local triviality over U and the fact that μ is a δ-morphisms, it follows that $\mu(Q) \simeq \delta_*(Q)$.

A calculation identical to the one carried out to show that $\delta_* \wedge 1$ is well-defined, shows that if (P, t) is an (A, B)-torsor, then the morphism

$$P \otimes - \colon \mathrm{T\small{ORS}}(A|_U) \longrightarrow \mathrm{T\small{ORS}}(A|_U)$$

preserves the functor δ_*, namely the diagram

$$\begin{array}{ccc} \mathrm{T\small{ORS}}(A|_U) & \xrightarrow{P\otimes -} & \mathrm{T\small{ORS}}(A|_U) \\ & {\scriptstyle\delta_*}\searrow \quad \swarrow{\scriptstyle\delta_*} & \\ & \mathrm{T\small{ORS}}(B|_U) & \end{array}$$

commutes. In other words, tensoring with an (A, B)-torsor is locally a morphism of (A, B)-gerbes. Moreover, since *any* equivalence $\nu\colon \mathrm{T\small{ORS}}(A|_U) \to \mathrm{T\small{ORS}}(A|_U)$ can be realized as $Q \mapsto P_\nu \otimes Q$ for an appropriate torsor P_ν, compatibility with the previous diagram forces P_ν to be an (A, B)-torsor. Denoting by $\mathscr{E}q$ the stack of equivalences, the foregoing proves that the correspondence

$$\mathscr{Q} \longmapsto \mathscr{E}q(\mathrm{T\small{ORS}}(A), \mathscr{Q})$$

gives the required quasi-inverse equivalence to F. □

Remark 5.9. The theorem gives another perspective on the canonical morphism introduced in Section 2.1.2. Namely, if we have a morphism (5.2.1) of Picard *gr*-stacks coming from the crossed square (5.2.2), from the theorem we obtain a morphism

$$\mathrm{G\small{ERBES}}(A, G) \longrightarrow \mathrm{G\small{ERBES}}(B, H)$$

as the conjugate $F_{\mathscr{B}} \circ \lambda_* \circ F^*_{\mathscr{A}}$ of the induced morphism

$$\lambda_*\colon \mathrm{T\small{ORS}}(\mathscr{A}) \longrightarrow \mathrm{T\small{ORS}}(\mathscr{B}),$$

where $F_\bullet$ is the appropriate equivalence from Theorem 5.8 and $F^*_\bullet$ its quasi-inverse. It is immediately seen that this morphism corresponds to the canonical morphism $(f, u)_*$.

We return to 2-gerbes. The following proposition generalizes Section 4.3.2 and Theorem 5.6, and it can be considered as the analog of Proposition 5.7 to the case of 2-gerbes.

Proposition 5.10. *Let $\mathscr{G} = \mathrm{T\small{ORS}}(A, B)$. Equivalence classes of 2-$\mathscr{G}$-gerbes are classified by the hypercohomology group $\mathbf{H}^3(X, A \to B)$.*

Proof. Most of the ingredients of the proof can be extracted from the cocycle analysis in [12], c.f. in particular Section 4.7.

Let $\mathfrak{G}$ be a 2-$\mathscr{G}$-gerbe. Given a cover $(U_i \to X)_{i\in I}$, the choice of objects $x_i \in \mathfrak{G}_{U_i}$ determines, by analogy with Section 5.1.3, $\mathscr{G}$-torsors $\mathscr{E}_{ij} = \mathscr{H}om(x_j|_{U_{ij}}, x_i|_{U_{ij}})$ over U_{ij}. Note that $\mathscr{E}_{ij}$ is a $\mathscr{G}$-torsor, rather than a *bi*-torsor, thanks to the fact that $\mathscr{G}$ is braided. The torsors $\mathscr{E}_{ij}$ satisfy the following cocycle condition: we have equivalences

$$g_{ijk}\colon \mathscr{E}_{ij} \wedge^{\mathscr{G}} \mathscr{E}_{jk} \xrightarrow{\sim} \mathscr{E}_{ik} \tag{5.4.2a}$$

and natural transformations (isomorphisms):

$$\nu_{ijkl}\colon g_{ikl} \circ (g_{ijk} \wedge 1) \Longrightarrow g_{ijl} \circ (1 \wedge g_{jkl}) \tag{5.4.2b}$$

arising from the pentagonal 2-cell determined by starting at

$$\left(\mathscr{E}_{ij} \wedge^{\mathscr{G}} \mathscr{E}_{jk}\right) \wedge^{\mathscr{G}} \mathscr{E}_{kl},$$

and associating with the help of (5.4.2a). Moreover, the morphisms ν_{ijkl} satisfy the appropriate coherence condition extracted from (5.4.2b) over U_{ijklm}.

Notice that a section of, say, $\mathscr{E}_{ij}$ over $U^{\alpha}_{ij} \to U_{ij}$ is a 1-arrow $f^{\alpha}_{ij}\colon x_j|_{U^{\alpha}_{ij}} \to x_i|_{U^{\alpha}_{ij}}$, and similarly for the other indices. Therefore the restriction $g^{\alpha\beta\gamma}_{ijk}$ of g_{ijk} to $U^{\alpha\beta\gamma}_{ijk}$ can be identified with an object of $\mathscr{G}|_{U^{\alpha\beta\gamma}_{ijk}}$. The same reasoning leads to the identification of the restriction of ν_{ijkl}, with the appropriate decoration of upper indices, with an arrow of (a corresponding restriction of) $\mathscr{G}$. Finally, we note that the equivalence in Eq. (5.4.2a) is given by the composition of 1-arrows and 2-arrows in $\mathfrak{G}$. Thus Eqs. (5.4.2) can be interpreted as giving a cocycle condition for (g_{ijk}, ν_{ijkl}) with values in $\mathscr{G}$.

Now, since $\mathscr{G} = \mathrm{T{\small ORS}}(A, B)$, is the stack associated with the abelian crossed module (i.e. complex of abelian groups) $\delta\colon A \to B$, the corresponding sheaf of groupoids will be

$$A \times B \underset{t}{\overset{s}{\rightrightarrows}} B$$

with source and target maps given by $s(a, b) = b$ and $t(a, b) = \delta(a)b$, so that (neglecting the upper indices) the object g_{ijk} can be identified with a section b_{ijk} of B, and the morphism ν_{ijkl} with a section a_{ijkl} of A. Now (5.4.2b) reads:

$$\delta(a_{ijkl})b_{ijk}b_{ikl} = b_{jkl}b_{ijl}$$

which is the desired relation. Putting it together with the cocycle condition for a_{ijkl} determined by the coherence condition on the ν_{ijkl} alluded to above, provides the required 3-cocycle with values in the complex $A \to B$. □

Methods similar to the approach of the proof of Theorem 5.8 give the following theorem. We omit the proof.

Theorem 5.11. *Again let $\mathscr{G} = \mathrm{T{\small ORS}}(A, B)$. Then a 2-$\mathscr{G}$-gerbe is equivalent to a 2-(A, B)-gerbe, where the equivalence takes place in the appropriate 3-category.*

The upshot of the foregoing unfortunately rather lengthy discussion can be summarized as follows. Given a complex of abelian groups $\delta : A \to B$, the following two structures on a 2-A-gerbe $\mathfrak{G}$ are equivalent:

(1) 2-gerbe bound by $\delta : A \to B$, and:
(2) 2-gerbe bound by $\mathscr{G} = \mathrm{T{\small ORS}}(A, B)$.

They correspond to the following crossed squares of the type (5.2.2):

$$\text{item 1:}\quad \begin{array}{ccc} A & \xrightarrow{\delta} & B \\ \big\downarrow & & \big\downarrow \\ 1 & \longrightarrow & 1 \end{array} \qquad \text{item 2:}\quad \begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow \\ B & \longrightarrow & 1 \end{array}$$

where for case (1) we consider A and B as crossed modules $A \to 1$ and $B \to 1$, whereas case (2) corresponds to the crossed module $\lambda\colon \mathscr{G} \to \mathscr{H}$ where $\mathscr{H}$ is associated with $1 \to 1$. The equivalence can be traced to the symmetry of the crossed square.

Next, we are going to explore the case when the crossed square (5.2.2) is non-trivial.

5.5. Classification II

Our first step is to address the case of a 2-gerbe bound by a crossed module of braided *gr*-stacks (5.2.1) in greater generality than in the preceding sections. Note that there is an obvious induced map:

$$\lambda_*\colon \mathrm{T{\small ORS}}(\mathscr{A}) \longrightarrow \mathrm{T{\small ORS}}(\mathscr{B}), \tag{5.5.1}$$

given by $\mathscr{P} \to \mathscr{P} \wedge^{\mathscr{A}} \mathscr{B}$. It is convenient to have the following definition at hand:

Lemma–Definition 5.12. *Given a cover $\mathfrak{U}_X = (U_i \to X)_{i\in I}$, a 1-cocycle with values in (5.5.1) is the datum of $\mathscr{A}$-torsors $\mathscr{E}_{ij}$ over U_{ij} and $\mathscr{B}$-torsors $\mathscr{F}_i$ over U_i, such that the cocycle condition (5.4.2) holds for the $\mathscr{E}_{ij}$'s, and moreover there are equivalences of $\mathscr{B}$-torsors*

$$\xi_{ij} : \lambda_*(\mathscr{E}_{ij}) \wedge^{\mathscr{B}} \mathscr{F}_j \xrightarrow{\sim} \mathscr{F}_i \tag{5.5.2a}$$

and natural transformations (isomorphisms):

$$m_{ijk} : \xi_{ij} \circ (1 \wedge \xi_{jk}) \Longrightarrow \xi_{ik} \circ (\lambda_*(g_{ijk}) \wedge 1). \tag{5.5.2b}$$

The natural transformations m_{ijk} are subject to the following coherence condition:

$$\begin{aligned} &\xi_{il} * \lambda_*(\nu_{ijkl}) \circ m_{ijl} * (1 \wedge \lambda_*(g_{jkl}) \wedge 1) \circ \xi_{ij} * m_{jkl} \\ &\quad = m_{ikl} * (\lambda_*(g_{ijk}) \wedge 1 \wedge 1) \circ m_{ijk} * (1 \wedge 1 \wedge \xi_{kl}). \end{aligned} \tag{5.5.3}$$

Remark 5.13. An easier (but less precise) way of displaying (5.5.3) is to ignore the pastings with the identity 2-arrows, so that we have:

$$\lambda_*(\nu_{ijkl}) \circ m_{ijl} \circ m_{jkl} = m_{ikl} \circ m_{ijk}.$$

Proof. The calculations are tedious, but entirely straightforward. We will content ourselves to note that one has to form the standard cube of morphisms ξ_{ij}, etc. starting from

$$\lambda_*\left(\mathscr{E}_{ij} \wedge^{\mathscr{A}} (\mathscr{E}_{jk} \wedge^{\mathscr{A}} \mathscr{E}_{kl})\right) \wedge^{\mathscr{B}} \mathscr{F}_l \tag{5.5.4}$$

and ending at $\mathscr{F}_i$, *modulo* the association isomorphisms for the contracted product, which have been ignored in Eq. (5.5.3). Then (5.5.3) is the result of composing the *faces* of this cube. Note that in (5.5.3) there are five terms, since one of the faces will be strictly commutative, namely the one corresponding to contracting the first two, and the second two terms in (5.5.4). □

We complement the definition of 1-cocycle with the notion of equivalence as follows:

Definition 5.14. Two 1-cocycles $(\mathscr{E}_{ij}, \mathscr{F}_i)$ and $(\mathscr{E}'_{ij}, \mathscr{F}'_i)$ with values in (5.5.4) are *equivalent* if there exist $\mathscr{A}|_{U_i}$-torsors $\mathscr{Q}_i$ over U_i such that there are equivalences:

$$\mathscr{E}'_{ij} \wedge^{\mathscr{A}} \mathscr{Q}_j \xrightarrow{\sim} \mathscr{Q}_i \wedge^{\mathscr{A}} \mathscr{E}_{ij} \tag{5.5.5a}$$

$$\lambda_*(\mathscr{Q}_i) \wedge^{\mathscr{B}} \mathscr{F}_i \xrightarrow{\sim} \mathscr{F}'_i. \tag{5.5.5b}$$

The following is a mild extension of the statement in [13, 4.1.11] in the braided case.

Theorem 5.15. *Equivalence classes of 2-$(\mathscr{A}, \mathscr{B})$-gerbes are classified by the set*

$$\mathbf{H}^1(X, \mathrm{T{\small ORS}}(\mathscr{A}) \to \mathrm{T{\small ORS}}(\mathscr{B})),$$

namely the (pointed) set of equivalence classes of 1-cocycles in Lemma–Definition 5.12 under the equivalence of Definition 5.14.

Proof. Let $\mathfrak{G}$ be a 2-$(\mathscr{A}, \mathscr{B})$-gerbe. Since it is in particular a 2-$\mathscr{A}$-gerbe, the choice of objects $x_i \in \mathrm{Ob}\,\mathfrak{G}_{U_i}$ with respect to an open cover $\mathfrak{U}_X = (U_i \to X)_{i\in I}$ will generate a 1-cocycle $\{\mathscr{E}_{ij}\}$ with values in TORS($\mathscr{A}$), as in the proof of Proposition 5.10, Eqs. (5.4.2). This part and the rest of the cocycle analysis of the 2-gerbe $\mathfrak{G}$ is as in [12], especially Section 4.7, with the additional hypothesis that we are in the braided case (so that we are in the "decoupled" situation). Full details will be found in loc. cit.

The new part is the one related to the extra structure given by the 2-functor

$$J : \mathfrak{G} \longrightarrow \mathrm{T{\small ORS}}(\mathscr{B}),$$

as part of the definition of 2-$(\mathscr{A}, \mathscr{B})$-gerbe. Using J, for each object x_i we obtain a $\mathscr{B}$-torsor $\mathscr{F}_i \overset{\text{def}}{=} J(x_i)$. Now, recall that $\mathscr{E}_{ij} = \mathscr{H}om(x_j|_{U_{ij}}, x_i|_{U_{ij}})$. Objects and arrows of $\mathscr{E}_{ij}$ over $U^{\alpha}_{ij} \to U_{ij}$ correspond to 1-arrows between $x_j|_{U^{\alpha}_{ij}}$ and $x_i|_{U^{\alpha}_{ij}}$ and 2-arrows between them. Via J, we get equivalences and natural isomorphisms between the corresponding torsors $\mathscr{F}_j$ and $\mathscr{F}_i$. In short, there is an equivalence:

$$\mathscr{E}_{ij} \xrightarrow{\sim} \mathscr{H}om(\mathscr{F}_j, \mathscr{F}_i),$$

where the $\mathscr{H}om$ on the right-hand side denotes the category of morphisms of torsors (defined e.g. as in [14, Section 6]). That is, it is the $\mathscr{H}om$ in $\mathrm{T{\small ORS}}(\mathscr{B})$. In turn, this equivalence can be written in the form of Eq. (5.5.2a), using the correspondence

$$f^{\alpha}_{ij} \longmapsto [y \longmapsto \lambda_*(f^{\alpha}_{ij})(y)] \simeq \lambda_*(f^{\alpha}_{ij}) \wedge y \longmapsto \lambda_*(f^{\alpha}_{ij})(y),$$

where f^{α}_{ij} is an object of $\mathscr{E}_{ij}$, i.e. 1-morphism of $\mathfrak{G}$, over U^{α}_{ij}, and similarly for 2-arrows. Here we have also used the fact that J is a λ-morphism, therefore an $\mathscr{A}$-torsor $\mathscr{P}$ corresponds to $\lambda_*(\mathscr{P}) = \mathscr{P} \wedge^{\mathscr{A}} \mathscr{B}$.

The inverse correspondence is obtained by generalizing the standard gluing of local trivial 2-gerbes $\mathrm{T{\small ORS}}(\mathscr{A}|_{U_i})$ in a way analogous to the proof of Theorem 5.6. Namely, given a 1-cocycle $(\mathscr{E}_{ij}, \mathscr{F}_i)$, first we glue $\mathrm{T{\small ORS}}(\mathscr{A}|_{U_j})|_{U_{ij}}$ with $\mathrm{T{\small ORS}}(\mathscr{A}|_{U_i})|_{U_{ij}}$ via $\mathscr{E}_{ij}$ by

$$\mathscr{P} \longmapsto \mathscr{P} \wedge^{\mathscr{A}} \mathscr{E}_{ij},$$

and verify that this is coherent thanks to Eqs. (5.4.2). Thus we obtain a 2-$\mathscr{A}$-gerbe $\mathfrak{G}$, and, as a byproduct, this procedure gives a collection of objects x_i providing the labeling with respect to which the newly obtained 2-gerbe $\mathfrak{G}$ is represented by the cocycle $\mathscr{E}_{ij}$. We then define J as:

$$J|_{U_i} : \mathfrak{G}_{U_i} \simeq \mathrm{T{\small ORS}}(\mathscr{A}|_{U_i}) \longrightarrow \mathrm{T{\small ORS}}(\mathscr{B}|_{U_i})$$

by assigning to x_i the $\mathscr{B}$-torsor $\mathscr{F}_i$. More generally, to any object of $\mathfrak{G}_{U_i}$, i.e. to any $\mathscr{A}|_{U_i}$-torsor $\mathscr{P}$, we assign the $\mathscr{B}|_{U_i}$-torsor

$$\lambda_*(\mathscr{P}) \wedge^{\mathscr{B}} \mathscr{F}_i.$$

We leave to the reader the task to verify that the two constructions are the inverse of one another.

Finally, given a 2-$(\mathscr{A}, \mathscr{B})$-gerbe, a second collection of objects $\{y_i\}$ subordinated to the same cover determines a new cocycle $(\mathscr{E}'_{ij}, \mathscr{F}'_i)$. Moreover, for each $i \in I$ we have the $\mathscr{A}|_{U_i}$-torsor $\mathscr{Q}_i = \mathscr{H}om(x_i, y_i)$. It is easily verified that the collection $\{\mathscr{Q}_i\}$ satisfies both Eqs. (5.5.5). □

When the coefficient complexes of braided stacks come from complexes of abelian groups the previous theorem can be rephrased in terms of ordinary hypercohomology. More precisely, we have the following statement.

Theorem 5.16. *If the braided gr-stacks $\mathscr{A}$ and $\mathscr{B}$ are strict and correspond to abelian crossed modules $A \to G$ and $B \to H$, respectively, then equivalence classes of 2-$(\mathscr{A}, \mathscr{B})$-gerbes are classified by the (ordinary) hypercohomology group*

$$\mathbf{H}^3(X, A \to B \oplus G \to H),$$

namely the coefficient complex is the cone (shifted by 1) of the abelian crossed square (5.2.2).

Proof. We will need to show how to extract an ordinary cocycle with value in the cone of (5.2.2) from the abstract cocycle of Theorem 5.15.

Let $\mathscr{A} = \mathrm{T{\small ORS}}(A, G)$ and $\mathscr{B} = \mathrm{T{\small ORS}}(B, H)$ with complexes $\delta : A \to G$ and $\sigma : B \to H$ and homomorphisms $f : A \to B$ and $u : G \to H$ arranged to make the square (5.2.2). The corresponding (sheaf of) crossed module(s) is:

$$\begin{array}{ccc} A \times G & \xrightarrow{(f,u)} & B \times H \\ {\scriptstyle t}\downarrow\downarrow{\scriptstyle s} & & {\scriptstyle t}\downarrow\downarrow{\scriptstyle s} \\ G & \xrightarrow[u]{} & H \end{array}$$

where in both cases the source and target maps s and t are as in the proof of Proposition 5.10. Thus the additive functor $\lambda\colon \mathscr{A} \to \mathscr{B}$ is induced (after having taken the associate stack functor) by the pair (f, u).

After having gone through these recollections, let us consider a 2-$(\mathscr{A}, \mathscr{B})$-gerbe $\mathfrak{G}$, and let us once again choose a cover $\mathfrak{U}_X = (U_i \to X)$, and objects $x_i \in \operatorname{Ob}\mathfrak{G}_{U_i}$. By Theorem 5.15, we obtain a 1-cocycle $(\mathscr{E}_{ij}, \mathscr{F}_i)$ with values in the complex (5.5.1) satisfying Eqs. (5.4.2) and (5.5.2). Our first task is to complement the proof of Proposition 5.10, and obtain a 1-cocycle with values in the complex $\lambda\colon \mathscr{A} \longrightarrow \mathscr{B}$ itself.

To this end, we will need to decompose the torsors $\mathscr{E}_{ij}$ as well as $\mathscr{F}_i$ with respect to some choice of objects, and then apply the reasoning preceding Eq. (5.1.1). More precisely, consider objects f^α_{ij} and ${y_i}^\alpha{}_j$, of $\mathscr{E}_{ij}$ and $\mathscr{F}_i$, respectively, given $(U^\alpha_{ij} \to U_{ij})_{\alpha\in A^\alpha_{ij}}$. (Similarly, we denote by ${y_j}^\alpha{}_i$ an object of $\mathscr{F}_j$ over U^α_{ij}.) Then, since $\mathscr{F}_i$ is a $\mathscr{B}$-torsor, the morphism ξ_{ij} in Eq. (5.5.2a) translates into

$$(f^\alpha_{ij})_*({y_j}^\alpha{}_i) \simeq {y_i}^\alpha{}_j \cdot h^\alpha_{ij}, \tag{5.5.6}$$

where h^α_{ij} is an object of $\mathscr{B}$ over U^α_{ij}. (Here we have used the notation $(f^\alpha_{ij})_* = J(f^\alpha_{ij})$.) Moreover, ${y_i}^\alpha{}_j$ and ${y_i}^\beta{}_k$ are related by:

$${y_i}^\alpha{}_j \simeq {y_i}^\beta{}_k \cdot q^{\beta\alpha}_{kij}, \tag{5.5.7}$$

with $q^{\beta\alpha}_{kij}$ an object of $\mathscr{B}$ over $U^{\alpha\beta}_{ijk}$. It is easily seen that these new objects satisfy the identity (up to isomorphism):

$$q^{\beta\alpha}_{kij} \cdot q^{\alpha\gamma}_{jil} \simeq q^{\beta\gamma}_{kil}. \tag{5.5.8}$$

For the part of the cocycle involving the $\mathscr{E}_{ij}$'s alone, subject to Eqs. (5.4.2), our choice of objects determines an object $g^{\alpha\beta\gamma}_{ijk}$ of $\mathscr{A}$ obtained from Eq. (5.4.2a) in the standard way:

$$f^\alpha_{ij} \wedge f^\beta_{jk} \longmapsto f^\alpha_{ij} \circ f^\beta_{jk} \simeq g^{\alpha\beta\gamma}_{ijk} \circ f^\gamma_{ik}.$$

(Recall that the map g_{ijk} is just a composition of 1-arrows of $\mathfrak{G}$.) Moreover, still using the arguments in [12], starting from Eq. (5.4.2b) we arrive at the morphism in $\mathscr{A}$:

$$\nu^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}\colon g^{\alpha\beta\gamma}_{ijk} \cdot g^{\gamma\delta\varepsilon}_{ikl} \xrightarrow{\sim} g^{\beta\delta\eta}_{jkl} \cdot g^{\alpha\eta\varepsilon}_{ijl}. \tag{5.5.9a}$$

To translate Eq. (5.5.2b), compute the composition over $U^{\alpha\beta\gamma}_{ijk}$:

$$(f^\alpha_{ij} \circ f^\beta_{jk})_*({y_k}^\beta{}_j)$$

in the two possible ways. A standard calculation, where we use (5.5.6) and (5.5.7), yields the sought-after arrow in $\mathscr{B}$:

$$m^{\alpha\beta\gamma}_{ijk}\colon h^\alpha_{ij} q^{\alpha\beta}_{ijk} h^\beta_{jk} \xrightarrow{\sim} \lambda(g^{\alpha\beta\gamma}_{ijk}) q^{\alpha\gamma}_{jik} h^\gamma_{ik} q^{\gamma\beta}_{ikj}. \tag{5.5.9b}$$

This arrow in turn satisfies a cocycle condition, which is the translation of Eq. (5.5.3). We arrive at it by considering the expression

$$h^\alpha_{ij} q^{\alpha\beta}_{ijk} h^\beta_{jk} q^{\beta\delta}_{jkl} h^\delta_{kl},$$

which would correspond to $\xi_{ij}\circ(1\wedge\xi_{jk})\circ(1\wedge 1\wedge\xi_{kl})$, and computing it in the two possible obvious ways using (5.5.9b), the braiding of $\mathscr{B}$ – and the help of (5.5.8). The calculation itself proceeds according to the techniques expounded in [12], therefore we will not reproduce it here. The result is that the arrows $m^{\alpha\beta\gamma}_{ijk}$ satisfy the cocycle condition:

$$\lambda(\nu^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}) \circ m^{\gamma\delta\varepsilon}_{ikl} \circ m^{\alpha\beta\gamma}_{ijk} = m^{\alpha\eta\varepsilon}_{ijl} \circ m^{\beta\delta\eta}_{jkl}. \tag{5.5.9c}$$

Of course this identity holds *modulo* the obvious isomorphisms arising from the association and braiding functors in $\mathscr{B}$, which we have silently ignored, as well as the pull-back functors between different fiber categories.

The cocycle with values in the complex $\lambda\colon \mathscr{A} \to \mathscr{B}$ we have obtained comprises the quintuple:

$$\left(h^\alpha_{ij}, q^{\alpha\beta}_{ijk}, m^{\alpha\beta\gamma}_{ijk}, g^{\alpha\beta\gamma}_{ijk}, \nu^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}\right) \tag{5.5.10}$$

subject to Eqs. (5.5.9) plus the cocycle condition on the terms $\nu^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}$ arising from the coherence condition on the maps (5.4.2b). We refrain from displaying such conditions here.

Now let us use the fact that both the gr-stacks $\mathscr{A}$ and $\mathscr{B}$ are strict and in fact associated with crossed modules. From the recollections at the beginning we have that in the above quintuple $g^{\alpha\beta\gamma}_{ijk}$ will be a section of the abelian group sheaf G, h^{α}_{ij} and $q^{\alpha\beta}_{ijk}$ are both sections of H, whereas the arrows $m^{\alpha\beta\gamma}_{ijk}$ and $\nu^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}$ will correspond to sections of B and A, respectively denoted $b^{\alpha\beta\gamma}_{ijk}$ and $a^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}$, satisfying the (strict) identities:

$$\delta(a^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl})\cdot g^{\alpha\beta\gamma}_{ijk}\cdot g^{\gamma\delta\varepsilon}_{ikl} = g^{\beta\delta\eta}_{jkl}\cdot g^{\alpha\eta\varepsilon}_{ijl}, \tag{5.5.11a}$$

$$\sigma(b^{\alpha\beta\gamma}_{ijk})\cdot h^{\alpha}_{ij}\cdot q^{\alpha\beta}_{ijk}\cdot h^{\beta}_{jk} = u(g^{\alpha\beta\gamma}_{ijk})\cdot q^{\alpha\gamma}_{jik}\cdot h^{\gamma}_{ik}\cdot q^{\gamma\beta}_{ikj}, \tag{5.5.11b}$$

$$f(a^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl})\cdot b^{\gamma\delta\varepsilon}_{ikl}\cdot b^{\alpha\beta\gamma}_{ijk} = b^{\alpha\eta\varepsilon}_{ijl}\cdot b^{\beta\delta\eta}_{jkl}. \tag{5.5.11c}$$

To these equations we have to add the condition satisfied by the $a^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}$ as a consequence of the identity satisfied by the arrows $\nu^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}$.

It is just a matter of using the definition of the mapping cone of a complex to realize that (5.5.11) express the condition for the quintuple

$$\left(h^{\alpha}_{ij}, q^{\alpha\beta}_{ijk}, b^{\alpha\beta\gamma}_{ijk}, g^{\alpha\beta\gamma}_{ijk}, a^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}\right) \tag{5.5.12}$$

to define a cocycle of degree 3 with values in the complex

$$A \xrightarrow{(f,\delta)} B\oplus G \xrightarrow{\sigma\cdot u^{-1}} H, \tag{5.5.13}$$

with A placed in degree 0. This finishes the proof. □

Remark 5.17. Ignoring the intimidating upper indices relative to the hypercover used in the proof allows us to set $q^{\alpha\beta}_{ijk} = 1$ so that Eqs. (5.5.11), plus the cocycle identity on a_{ijkl}, will assume the standard form for a Čech cocycle of degree 3 with values in (5.5.13).

Remark 5.18. The *proof* of Theorem 5.16 actually gives slightly more, in that it gives the 3-cocycle with values in the complex $\lambda\colon \mathscr{A}\to\mathscr{B}$ corresponding to the torsor 1-cocycle with values in (5.5.1), regardless of whether the involved (braided) gr-stacks are associated with crossed modules.

Remark 5.19. The statement (but not the proof) of Theorem 5.16 subsumes those of Theorem 5.6 and Proposition 5.10.

Remark 5.20. The cocycle identities (5.5.11) satisfied by the quintuple (5.5.12) are symmetric under the exchange

$$b^{\alpha\beta\gamma}_{ijk} \longleftrightarrow g^{\alpha\beta\gamma}_{ijk},$$

and the corresponding exchanges $f\leftrightarrow\delta$ and $\sigma\leftrightarrow u$. This symmetry rests upon that of the crossed square (5.2.2) determined by the crossed module of strict gr-categories under consideration. Thus, calling $\mathcal{P}$ the crossed square (5.2.2), a 2-gerbe $\mathfrak{G}$ satisfying the hypotheses of Theorem 5.16 ought be more properly called a 2-$\mathcal{P}$-gerbe.

Let us also observe that the situation described by the hypotheses of Theorem 5.16 has another interesting subcase. Namely, we can consider a complex of length 3 as it was done is Section 3, and then define the notion of a 2-gerbe bound by this complex. This is clearly possible using Theorem 5.16 by setting $G = 1$ (or $B = 1$). Thus we can state the following definition, generalizing Definition 3.1.

Definition 5.21. Let $A \xrightarrow{\delta} B \xrightarrow{\sigma} C$ be a complex of (sheaves of) abelian groups on C/X. A 2-(A, B, C)-gerbe is a 2-A-gerbe $\mathfrak{G}$ equipped with a structure of 2-$(\mathscr{A},\mathscr{B})$-gerbe where $\mathscr{A} = \textsc{Tors}(A)$ and $\mathscr{B} = \textsc{Tors}(B, C)$.

In the previous definition $\mathscr{A}$ is the gr-stack associated with the abelian group A viewed as a crossed module $A\to 1$. The additive functor λ is thus determined by the pair $(\delta, 1)$. Of course, up to a trivial isomorphism on the resulting cohomology group, we could have chosen the combination $\mathscr{A} = \textsc{Tors}(A, B)$, $\mathscr{B} = \textsc{Tors}(1, C)$ due to the symmetry of the two resulting crossed squares.

In the end, one outcome of the material expounded in this section is that the theory of 2-$(\mathscr{A}, \mathscr{B})$-gerbes can account for 2-gerbes bound by complexes of abelian *groups* which are in fact of length 3. It is particularly relevant, as we will see in the applications to Hermitian Deligne cohomology further below, that hypercohomology groups with values in the cone of a square can naturally be obtained in this framework.

Two issues however suggest to push this circle of ideas a little further. On the one hand, it is natural to ask whether Definition 5.21 admits a "naive" generalization by simply replacing groups with *gr*-stacks. On the other hand, capturing the geometric meaning of the hypercohomology groups with values in the complex (1.2.11) requires that we have a theory of 2-gerbes bound by complexes of the appropriate length, which cannot be obtained from what we have right now.

We will address the issue in Section 6.

5.6. Examples

We review here a few fairly standard examples to illustrate the foregoing theory. In fact, the following examples are the 2-gerbe counterpart of the examples presented in Sections 2.2 and 2.3. The analysis of more interesting examples will be deferred until the last section dedicated to the interpretation of certain Deligne cohomology groups.

X is an algebraic manifold, and we work with the standard site determined by X^{an} (see above).

5.6.1. Connective structures (or "concept of connectivity")

This is the classical example due to Brylinski and McLaughlin (see [4–6]).

Let $\mathfrak{G}$ be a 2-gerbe over X. As expected, a connective structure (or "concept of connectivity" as it was originally called) on $\mathfrak{G}$ is a structure of 2-gerbe bound by the complex

$$\mathscr{O}_X^\times \xrightarrow{\mathrm{d}\log} \Omega_X^1$$

in the sense of Definition 5.3 and Lemma 5.4. Thus we retrieve Brylinski and McLaughlin's original definition, wherein the connective structure is seen as a 2-functor assigning to each local object of $\mathfrak{G}$ over U a corresponding Ω_U^1-gerbe. In light of Proposition 5.10 and Theorem 5.11 $\mathfrak{G}$ can just as well be considered as a 2-gerbe bound by the *gr*-stack of $(\mathscr{O}_X^\times, \Omega_X^1)$-torsors.

From the classification results (see loc. cit. for the original arguments) we have that 2-gerbes with this connective structure are classified by the hypercohomology group:

$$\mathbf{H}^3(X, \mathscr{O}_X^\times \xrightarrow{\mathrm{d}\log} \Omega_X^1) \simeq \mathrm{H}_{\mathcal{D}}^4(X, \mathbf{Z}(2)).$$

5.6.2. Hermitian structures

This version of the idea of Hermitian structure was introduced in [7] by analogy with the notion of connective structure in the above mentioned works by Brylinski and McLaughlin. Thus, a 2-$\mathscr{O}_X^\times$-gerbe $\mathfrak{G}$ over X with Hermitian structure is a 2-gerbe bound by the complex:

$$\mathscr{O}_X^\times \xrightarrow{|\cdot|^2} \mathscr{E}_X^+,$$

or, alternatively, by the *gr*-stack of $(\mathscr{O}_X^\times, \mathscr{E}_X^+)$-torsors. Equivalence classes of such 2-gerbes are classified by the Hermitian Deligne cohomology group of weight 1:

$$\mathbf{H}^3(X, \mathscr{O}_X^\times \xrightarrow{|\cdot|^2} \mathscr{E}_X^+) \simeq \widehat{\mathrm{H}}_{\mathcal{D}}^4(X, 1),$$

where we use the same quasi-isomorphism as in Section 2.3.1.

It is easy to continue the list of examples by promoting those of Section 2.3 to the realm of 2-gerbes. We will not do so here, and leave this task to the interested reader. We will examine finer examples of geometric structures on 2-gerbes in Section 7.

6. 2-Gerbes bound by complexes of higher degree

So far, we have outlined a theory of 2-gerbes bound (in the appropriate sense) by a two-step complex of braided *gr*-stacks. We have found that this theory is powerful enough to provide an interpretation in geometric terms of the elements of degree three hypercohomology groups with values in (cones of) crossed squares of abelian groups. However, as pointed out above, we need to address the case where the coefficient complexes have degree higher than 3, where the degree loosely corresponds to the length. We set out to accomplish this goal by generalizing the concept of (A, B, C)-gerbe, introduced in Section 3.2, to the case of 2-gerbes by promoting the coefficient groups to be *gr*-stacks instead. We will ultimately be interested in the case of *gr*-stacks associated with abelian crossed modules, therefore the general style for this section will be slightly more descriptive – and perhaps informal – compared to the preceding ones.

6.1. $(\mathscr{B}, \mathscr{C})$-torsors

Consider a complex (i.e. a morphism) of two (braided, as usual) *gr*-stacks $\mu\colon \mathscr{B} \longrightarrow \mathscr{C}$ on C/X. By analogy with Section 3.1, define a $(\mathscr{B}, \mathscr{C})$-torsor to be a pair $(\mathscr{P}, \sigma)$, where $\mathscr{P}$ is a $\mathscr{B}$-torsor, and σ is an equivalence:

$$\sigma\colon \mathscr{P} \wedge^{\mathscr{B}} \mathscr{C} \xrightarrow{\sim} \mathscr{C}$$

where on the right-hand side $\mathscr{C}$ is considered as a trivial torsor. Equivalently, we require that there be a morphism:

$$\sigma\colon \mathscr{P} \longrightarrow \mathscr{C},$$

namely a global object (over C/X) of the fibered category $\mathscr{H}om(\mathscr{P}, \mathscr{C})$. Yet another equivalent point of view is to regard σ as a global object of the torsor $\mathscr{P} \wedge^{\mathscr{B}} \mathscr{C}$. The latter point of view is useful to arrive at a description in terms of cocycles. Suppose indeed that $\mathscr{P}$ is decomposed as in Section 5.1.3, with associated 1-cocycle (b_{ij}, β_{ijk}) with values in $\mathscr{B}$ satisfying (5.1.1). By the stack condition, an object of $\mathscr{P} \wedge^{\mathscr{B}} \mathscr{C}$ is equivalent to a collection of pairs

$$(x_i, c_i) \in \mathrm{Ob}(\mathscr{P} \wedge^{\mathscr{B}} \mathscr{C})|_{U_i}$$

satisfying the descent condition on objects. Using the description of contracted product found in [14, Section 6.7], we find that the objects $c_i \in \mathrm{Ob}\,\mathscr{C}|_{U_i}$ satisfy the condition

$$\rho_{ij}\colon c_j \xrightarrow{\sim} \mu(b_{ij}^*) \cdot c_i \tag{6.1.1a}$$

(where b^* is a quasi-inverse of b). This essentially follows from the fact that a morphism $(x_j, c_j)|_{U_{ij}} \to (x_i, c_i)|_{U_{ij}}$ in $\mathscr{P} \wedge^{\mathscr{B}} \mathscr{C}$ corresponds to the triple

$$\Big(x_j \cdot b_{ji} \xrightarrow{\sim} x_i, b_{ji}, c_j \xrightarrow{\sim} \mu(b_{ji}) \cdot c_i\Big)$$

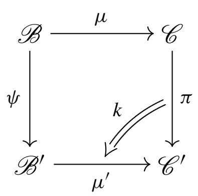
modulo an equivalence explained in loc. cit. The ρ_{ij} are morphisms in $\mathscr{C}_{U_{ij}}$ which then satisfy the coherence condition:

$$\mu(\beta_{ijk}) \circ \rho_{ij} \circ \rho_{jk} = \rho_{ik}. \tag{6.1.1b}$$

This and (5.1.1) ensure, via the above mentioned equivalence relation, that b_{ki} and $b_{kj} \cdot b_{ji}$ correspond to the same morphism, thereby ensuring that the cocycle condition in the descent condition is indeed satisfied.

Definition 6.1. The triple $(b_{ij}, \beta_{ijk}, \rho_{ij})$ satisfying Eqs. (6.1.1), plus (5.1.1) and the coherence condition on the β_{ijk} is a 1-*cocycle with values in the complex* $\mu\colon \mathscr{B} \longrightarrow \mathscr{C}$.

Given the square of *gr*-stacks



we obtain a morphism

$$(\psi, \pi)_* \colon \text{TORS}(\mathscr{B}, \mathscr{C}) \longrightarrow \text{TORS}(\mathscr{B}', \mathscr{C}') \tag{6.1.2}$$

by sending a $\mathscr{B}$-torsor $\mathscr{P}$ to $\mathscr{P} \wedge^{\mathscr{B}} \mathscr{B}'$ and the morphism σ to $\pi \circ \sigma$.

A morphism from a $(\mathscr{B}, \mathscr{C})$-torsor $(\mathscr{P}, \sigma)$ to a $(\mathscr{B}', \mathscr{C}')$-torsor $(\mathscr{P}', \sigma')$ consists of a square

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{\sigma} & \mathscr{C} \\ {\scriptstyle \xi}\downarrow & \overset{t}{\swarrow} & \downarrow{\scriptstyle \pi} \\ \mathscr{P}' & \xrightarrow[\sigma']{} & \mathscr{C}' \end{array} \tag{6.1.3}$$

In particular, for $\mathscr{B}' = \mathscr{B}$, $\mathscr{C}' = \mathscr{C}$, it reduces to a triangle

$$\begin{array}{ccc} \mathscr{P} & & \\ {\scriptstyle \xi}\downarrow & \overset{t}{\Downarrow} & \searrow{\scriptstyle \sigma} \\ \mathscr{P}' & \xrightarrow[\sigma']{} & \mathscr{C} \end{array} \tag{6.1.4}$$

Actually, any morphism (6.1.3) can be factored as the canonical morphism (6.1.2) followed by a morphism of $(\mathscr{B}', \mathscr{C}')$-torsors. A morphism will be called an equivalence if so is the underlying functor ξ.

In summary, a $(\mathscr{B}, \mathscr{C})$-torsor $\mathscr{P}$ determines (and it is determined by, up to equivalence) an equivalence class of 1-cocycles as in the definition. The equivalence relation being the obvious one, we obtain the following

Proposition 6.2. (1) *Equivalence classes of $(\mathscr{B}, \mathscr{C})$-torsors are classified by the cohomology set:*

$$\mathbf{H}^1(X, \mathscr{B} \longrightarrow \mathscr{C}).$$

(2) *Moreover, if $\mu \colon \mathscr{B} \to \mathscr{C}$ comes from the crossed square of abelian groups:*

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \tau} \\ H & \xrightarrow[v]{} & K \end{array}$$

then the above cohomology set can be identified with the hypercohomology group

$$\mathbf{H}^2(X, B \to C \oplus H \to K).$$

Proof. This repeats previous arguments, hence is omitted. □

Remark 6.3. We can use the statement in the above proposition to obtain another characterization of gerbes bound by length 3-complex, specifically, the cone of the above crossed square. This gives an alternative point of view for the discussion in Section 3.2.

Since by definition $(\mathscr{B}, \mathscr{C})$-torsors are $\mathscr{B}$-torsors which become trivial as $\mathscr{C}$-torsors, the following alternative characterization of $(\mathscr{B}, \mathscr{C})$-torsors coming from a crossed square as in Proposition 6.2(2) is an immediate consequence of Theorem 5.8:

Proposition 6.4. *Let $\mu \colon \mathscr{B} \to \mathscr{C}$ arise from a crossed square as in Proposition 6.2(2). The 2-functor F of Theorem 5.8 induces an equivalence*

$$\text{TORS}(\mathscr{B}, \mathscr{C}) \xrightarrow{\sim} \text{GERBES}(B, H)_{(\text{TORS}(C), \tau_*)}$$

where the right-hand side denotes the "fiber" of the canonical morphism

$$(g, v)_* : \mathrm{GERBES}(B, H) \to \mathrm{GERBES}(C, K)$$

over the neutral (C, K)*-gerbe, that is* $\tau_* : \mathrm{TORS}(C) \to \mathrm{TORS}(K)$.

Proof. If $\mathscr{P}$ is a $(\mathscr{B}, \mathscr{C})$-torsor, by definition there is a morphism $\sigma : \mathscr{P} \to \mathscr{C}$, and the diagram

$$\begin{array}{ccc} \mathrm{TORS}(\mathscr{B}) & \xrightarrow{F_{\mathscr{B}}} & \mathrm{GERBES}(B, H) \\ {\scriptstyle \mu_*}\downarrow & & \downarrow{\scriptstyle (g,v)_*} \\ \mathrm{TORS}(\mathscr{C}) & \xrightarrow[F_{\mathscr{C}}]{} & \mathrm{GERBES}(C, K) \end{array}$$

from Remark 5.9 gives

$$\begin{array}{ccc} \mathscr{P} & \longmapsto & \mathrm{TORS}(B) \wedge^{\mathscr{B}} \mathscr{P} \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle g_* \wedge \sigma} \\ \mathscr{C} & \longmapsto & \mathrm{TORS}(C) \end{array}$$

and the lower right corner gives the neutral (C, K)-gerbe. □

6.2. *Complexes of braided gr-stacks*

Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be braided *gr*-stacks over C/X, and let $\lambda : \mathscr{A} \to \mathscr{B}$ and $\mu : \mathscr{B} \to \mathscr{C}$ be additive functors. We define the composition

$$\mathscr{A} \xrightarrow{\lambda} \mathscr{B} \xrightarrow{\mu} \mathscr{C} \tag{6.2.1}$$

a *complex of gr-stacks* if $\mu \circ \lambda$ is isomorphic to the "null" functor $\mathscr{A} \longrightarrow \mathbf{1}$, to the punctual category determined by the unit object $o_{\mathscr{C}}$ of $\mathscr{C}$.

As before, a situation of particular interest for us will be when everything in sight is strict, and all the *gr*-stacks above are in fact associated with abelian crossed modules. Building on what we have already seen in Section 5.2, assume that the morphisms λ and μ are associated with the squares

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \sigma} \\ G & \xrightarrow[u]{} & H \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{g} & C \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \tau} \\ H & \xrightarrow[v]{} & K \end{array}$$

respectively, which we splice together to obtain the map of complexes:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} \\ G & \xrightarrow[u]{} & H & \xrightarrow[v]{} & K \end{array} \tag{6.2.2}$$

In all the above we have of course assumed $\mathscr{C}$ to be associated with the complex $\tau : C \to K$, the rest of the notations being as in Section 5.2.

6.3. *2-$(\mathscr{A}, \mathscr{B}, \mathscr{C})$-gerbes*

The main idea is to define 2-gerbes bound by the complex (6.2.1) of braided *gr*-stacks by analogy with what was done for gerbes in Section 3.2.

Definition 6.5. Let $\mathfrak{G}$ be a 2-gerbe over C/X. We say that $\mathfrak{G}$ is bound by the complex (6.2.1), or that is a 2-$(\mathscr{A}, \mathscr{B}, \mathscr{C})$-gerbe, for short, if there is a 2-functor

$$\tilde{J} : \mathfrak{G} \longrightarrow \textsc{Tors}(\mathscr{B}, \mathscr{C})$$

such that $\mathfrak{G}$ is a 2-$(\mathscr{A}, \mathscr{B})$-gerbe for the λ-morphism defined by the composition of $\tilde{J}$ with the obvious morphism $\textsc{Tors}(\mathscr{B}, \mathscr{C}) \to \textsc{Tors}(\mathscr{B})$.

Next, we can consider the diagram of *gr*-stacks:

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{\lambda} & \mathscr{B} & \xrightarrow{\mu} & \mathscr{C} \\ {\scriptstyle\phi}\downarrow & \overset{J}{\swarrow} & \downarrow{\scriptstyle\psi} & \overset{k}{\swarrow} & \downarrow{\scriptstyle\pi} \\ \mathscr{A}' & \xrightarrow[\lambda']{} & \mathscr{B}' & \xrightarrow[\mu']{} & \mathscr{C}' \end{array}$$

where the top and bottom rows are complexes in the sense specified above in Section 6.2. Still by analogy with Section 3.2, where the corresponding concept for gerbes was introduced, we define a morphism of a 2-$(\mathscr{A}, \mathscr{B}, \mathscr{C})$-gerbe $\mathfrak{G}$ to a 2-$(\mathscr{A}', \mathscr{B}', \mathscr{C}')$-gerbe $\mathfrak{G}'$ to be a cartesian 2-functor

$$F : \mathfrak{G} \longrightarrow \mathfrak{G}'$$

which is a φ-morphism, supplemented by a 2-natural transformation

$$\tilde{\alpha} : (\psi, \pi)_* \circ \tilde{J} \Longrightarrow \tilde{J}' \circ F : \mathfrak{G} \longrightarrow \textsc{Tors}(\mathscr{B}, \mathscr{C}).$$

We require that composing (pasting) this with the obvious morphism $\textsc{Tors}(\mathscr{B}, \mathscr{C}) \to \textsc{Tors}(\mathscr{B})$ gives (up to a modification) the natural morphism associated with the underlying (φ, ψ)-morphism.

6.4. *Classification III*

Given the complex (6.2.1), we obtain a corresponding "complex" of trivial 2-gerbes:

$$\textsc{Tors}(\mathscr{A}) \xrightarrow{\lambda_*} \textsc{Tors}(\mathscr{B}) \xrightarrow{\mu_*} \textsc{Tors}(\mathscr{C}) \tag{6.4.1}$$

where $\mu_* \circ \lambda_* \simeq (\mu \circ \lambda)_* \simeq \mathbf{1}$.

Lemma–Definition 6.6. *Given a cover $\mathfrak{U}_X = (U_i \to X)_{i \in I}$, a 1-cocycle with values in (6.4.1) is given by the same data as those for a 1-cocycle with values in (5.5.1) stated in Lemma–Definition 5.12, supplemented by the requirement that there exist morphisms*

$$\sigma_i : \mathscr{F}_i \longrightarrow \mathscr{C}|_{U_i} \tag{6.4.2}$$

such that given the morphism ξ_{ij} in (5.5.2a) there is a morphism of $(\mathscr{B}, \mathscr{C})$-torsors

$$(\xi_{ij}, t_{ij}) : (\mathscr{F}_j, \sigma_j)|_{U_{ij}} \longrightarrow (\mathscr{F}_i, \sigma_i)|_{U_{ij}} \tag{6.4.3}$$

satisfying a triangle analogous to (6.1.4), namely:

$$\begin{array}{ccc} \mathscr{F}_j & & \\ {\scriptstyle\xi_{ij}}\downarrow & \overset{t_{ij}}{\Downarrow} & \searrow{\scriptstyle\sigma_j} \\ \mathscr{F}_i & \xrightarrow[\sigma_i]{} & \mathscr{C} \end{array}$$

Proof. We need only observe that a morphism

$$\lambda_*(\mathscr{E}_{ij}) \wedge^{\mathscr{B}} \mathscr{F}_j \longrightarrow \mathscr{C}|_{U_{ij}}$$

can equivalently be seen as a morphism of $\mathscr{C}$-torsors:

$$(\lambda_*(\mathscr{E}_{ij}) \wedge^{\mathscr{B}} \mathscr{F}_j) \wedge^{\mathscr{B}} \mathscr{C}|_{U_{ij}} \longrightarrow \mathscr{C}|_{U_{ij}}.$$

But we have

$$(\lambda_*(\mathscr{E}_{ij}) \wedge^{\mathscr{B}} \mathscr{F}_j) \wedge^{\mathscr{B}} \mathscr{C}|_{U_{ij}} \simeq \lambda_*(\mathscr{E}_{ij}) \wedge^{\mathscr{B}} (\mathscr{F}_j \wedge^{\mathscr{B}} \mathscr{C}|_{U_{ij}}) \simeq \mathscr{F}_j \wedge^{\mathscr{B}} \mathscr{C}|_{U_{ij}}$$

since $\mu_* \circ \lambda_* \simeq (\mu \circ \lambda)_* \simeq \mathbf{1}$. □

The argument of the proof also implies that two 1-cocycles $(\mathscr{E}_{ij}, \mathscr{F}_i, \sigma_i)$ and $(\mathscr{E}'_{ij}, \mathscr{F}'_i, \sigma'_i)$ with values in (6.4.1) ought to be considered equivalent if the same conditions of Definition 5.14 are satisfied, with the additional requirement that the morphism (5.5.5b) induces a morphism of $(\mathscr{B}, \mathscr{C})$-torsors

$$(\mathscr{F}_i, \sigma_i) \longrightarrow (\mathscr{F}'_i, \sigma'_i).$$

We leave to the reader the task of spelling out the rest of the details.

The next results combines the generalizations of Theorems 5.15 and 5.16 to the present case. Large parts of the proof can be simply carried over, therefore we will be sketchy.

Theorem 6.7. (1) *Equivalence classes of* 2-$(\mathscr{A}, \mathscr{B}, \mathscr{C})$*-gerbes are classified by the (pointed) set*

$$\mathbf{H}^1(X, \mathrm{TORS}(\mathscr{A}) \to \mathrm{TORS}(\mathscr{B}) \to \mathrm{TORS}(\mathscr{C}))$$

of equivalence classes of 1*-cocycles with values in the complex* (6.4.1)*, according to the Lemma–Definition* 6.6*.*

(2) *If the braided gr-stacks are all strict and associated with abelian cross modules as in Section* 6.2*, then the above pointed set of equivalence classes is actually in one-to-one correspondence with the hypercohomology group*

$$\mathbf{H}^3(X, A \to B \oplus G \to C \oplus H \to K)$$

where we recognize the cone (shifted by 1*) of the morphism* (6.2.2)*.*

Proof. Let $(\mathfrak{G}, \tilde{J})$ be a 2-gerbe over C/X bound by the complex (6.2.1). Let us make the usual choice of a cover $\mathfrak{U}_X$, to be enhanced to a hypercover below. The proof of Part I rests upon the choice of a decomposition of $\mathfrak{G}$ with respect to a collection of objects $x_i \in \mathrm{Ob}\, \mathfrak{G}_{U_i}$. By applying $\tilde{J}$ we obtain $(\mathscr{B}, \mathscr{C})$-torsors $\tilde{J}(x_i) = \hat{\mathscr{F}}_i \equiv (\mathscr{F}_i, \sigma_i)$ and morphisms

$$\mathscr{E}_{ij} \longrightarrow \mathscr{H}om(\hat{\mathscr{F}}_j|_{U_{ij}}, \hat{\mathscr{F}}_i|_{U_{ij}}).$$

Forgetting the morphisms into $\mathscr{C}$ gives the underlying functor in $\mathrm{TORS}(\mathscr{B})$, therefore Part I follows from Theorem 5.15 (or rather, its proof) and the argument made in the proof of Lemma–Definition 6.6 to handle the extra morphisms into $\mathscr{C}$.

The proof of Part 2 is more laborious, but only computationally so. Fortunately everything that was done in the proof of Theorem 5.16 can be transported verbatim here, so that we only have to deal with the extra data ensuing from the $(\mathscr{B}, \mathscr{C})$-torsor.

Our first task is to rewrite the classifying 1-cocycle with values in (6.4.1) from Part I in terms of a cocycle with values in the complex of *gr*-stacks (6.2.1). As before, this is accomplished by decomposing the cocycle $(\mathscr{E}_{ij}, \mathscr{F}_i, \sigma_i)$ with respect to a choice of objects subordinated to a given hypercover. As in the proof of Theorem 5.16, we refine $\mathfrak{U}_X$ by $(U^{\alpha}_{ij} \to U_{ij})_{\alpha \in A^{\alpha}_{ij}}$. We also keep all the choices and notations made there.

Recall that we had obtained the quintuple (5.5.10) which we rewrite here for convenience:

$$\left(h^{\alpha}_{ij}, q^{\alpha\beta}_{ijk}, m^{\alpha\beta\gamma}_{ijk}, g^{\alpha\beta\gamma}_{ijk}, \nu^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}\right)$$

where $h^{\alpha}_{ij}, q^{\alpha\beta}_{ijk}$ are objects of $\mathscr{B}$, $m^{\alpha\beta\gamma}_{ijk}$ are morphisms of $\mathscr{B}$, and $g^{\alpha\beta\gamma}_{ijk}$ and $\nu^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}$ are objects and morphisms of $\mathscr{A}$, respectively. They satisfy the cocycle conditions given by the Eqs. (5.5.9) and (5.4.2b).

Since the morphism $\sigma_i : \mathscr{F}_i \to \mathscr{C}|_{U_i}$ are global over U_i, the arguments in Section 6.1 imply that there are objects ${z_i}^\alpha{}_j \in \operatorname{Ob}\mathscr{C}|_{U^\alpha_{ij}}$ and morphisms t^α_{ij} and $\rho^{\alpha\beta}_{jik}$ in $\mathscr{C}|_{U^\alpha_{ij}}$ and $\mathscr{C}|_{U^{\alpha\beta}_{ijk}}$ such that:

$$t^\alpha_{ij} : {z_j}^\alpha{}_i \xrightarrow{\sim} \mu(h^\alpha_{ij}) \cdot {z_i}^\alpha{}_j \tag{6.4.4a}$$

and

$$\rho^{\beta\alpha}_{kij} : {z_i}^\alpha{}_j \xrightarrow{\sim} \mu(q^{\alpha\beta}_{jik}) \cdot {z_i}^\beta{}_k. \tag{6.4.4b}$$

Both Eqs. (6.4.4) are obtained by applying the morphisms σ_i, σ_j, etc., namely the triangle right after Eq. (6.4.3), to Eqs. (5.5.6) and (5.5.7), respectively. We have used the relation $q^{\alpha\beta}_{jik} \simeq (q^{\beta\alpha}_{kij})^*$, easily derived from (5.5.7), where $(\cdot)^*$ denotes the quasi-inverse. The final piece of the cocycle condition is a relation for the morphisms t^α_{ij} and $\rho^{\beta\alpha}_{kij}$ which is computed by passing from ${z_k}^\gamma{}_i$ to ${z_i}^\alpha{}_j$ in two different ways. Either as:

$$\rho^{\alpha\gamma}_{jik} \circ t^\gamma_{ik} : {z_k}^\gamma{}_i \xrightarrow{\sim} \mu(h^\gamma_{ki}) \cdot \mu(q^{\gamma\alpha}_{kij}) \cdot {z_i}^\alpha{}_j, \tag{6.4.5}$$

or as:

$$t^\alpha_{ij} \circ \rho^{\alpha\beta}_{ijk} \circ t^\beta_{jk} \circ \rho^{\beta\gamma}_{jki} : {z_k}^\gamma{}_i \xrightarrow{\sim} \mu(q^{\gamma\beta}_{ikj}) \cdot \mu(h^\beta_{kj}) \cdot \mu(q^{\beta\alpha}_{kji}) \cdot \mu(h^\alpha_{ji}) \cdot {z_i}^\alpha{}_j, \tag{6.4.6}$$

where, as before, we are ignoring the various associator isomorphisms and natural transformations associated with μ.

If we replace the three middle terms in the right-hand side of (6.4.6) using (5.5.9b) and the relations $\mu \circ \lambda(g^{\beta\alpha\gamma}_{kji}) \simeq o_{\mathscr{C}}$ and $q^{\gamma\beta}_{ikj} \cdot q^{\beta\gamma}_{jki} \simeq o_{\mathscr{C}}$, where $o_{\mathscr{C}}$ is the unit element of $\mathscr{C}$, we find

$$\mu(m^{\beta\alpha\gamma}_{kji}) \circ t^\alpha_{ij} \circ \rho^{\alpha\beta}_{ijk} \circ t^\beta_{jk} \circ \rho^{\beta\gamma}_{jki} : {z_k}^\gamma{}_i \xrightarrow{\sim} \mu(h^\gamma_{ki}) \cdot \mu(q^{\gamma\alpha}_{kij}) \cdot {z_i}^\alpha{}_j.$$

Comparing with (6.4.5), we obtain the desired relation:

$$\mu(m^{\beta\alpha\gamma}_{kji}) \circ t^\alpha_{ij} \circ \rho^{\alpha\beta}_{ijk} \circ t^\beta_{jk} \circ \rho^{\beta\gamma}_{jki} = \rho^{\alpha\gamma}_{jik} \circ t^\gamma_{ik}. \tag{6.4.7}$$

Thus, starting from the cocycle $(\mathscr{E}_{ij}, \mathscr{F}_i, \sigma_i)$ with values in (6.4.1), the corresponding cocycle with values in the complex (6.2.1) is the 8-tuple

$$\left({z_i}^\alpha{}_j, t^\alpha_{ij}, \rho^{\alpha\beta}_{kij}, h^\alpha_{ij}, q^{\alpha\beta}_{ijk}, m^{\alpha\beta\gamma}_{ijk}, g^{\alpha\beta\gamma}_{ijk}, \nu^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}\right) \tag{6.4.8}$$

satisfying the conditions (5.5.9), (5.4.2b), (6.4.4) and (6.4.7).

To complete the proof, we need to specialize (6.4.8) and the relations it satisfies to the case where all the involved *gr*-stacks are Picard and associated with the abelian crossed modules introduced in Section 6.2. That is, we are going to operate under the same assumptions as those spelled out at the beginning of the proof of Theorem 5.16, so that for $\mathscr{A}$ and $\mathscr{B}$ the appropriate notations and relations can be taken directly from there, in particular Eqs. (5.5.11). Similarly, we set $\mathscr{C} = \text{TORS}(C, K)$, for the complex $\tau : C \to K$, and $\text{TORS}(C, K)$ can be realized as the *gr*-stack associated with the sheaf of groupoids $C \times K \rightrightarrows K$ determined by the crossed module $C \to K$, whose source and target maps are given by $(c, z) \to z$ and $(c, z) \to \tau(c)z$, respectively. In addition to the various sections of A, G, etc. introduced before Eqs. (5.5.11), the objects ${z_i}^\alpha{}_j$ will be identified with sections of the group $K|_{U^\alpha_{ij}}$, and we will introduce sections c^α_{ij} of $C|_{U^\alpha_{ij}}$ and $l^{\alpha\beta}_{ijk}$ of $C|_{U^{\alpha\beta}_{ijk}}$ to account for the morphisms t^α_{ij} and $\rho^{\alpha\beta}_{ijk}$, respectively. With these provisions, the 8-tuple (6.4.8) becomes

$$\left({z_i}^\alpha{}_j, c^\alpha_{ij}, l^{\alpha\beta}_{kij}, h^\alpha_{ij}, q^{\alpha\beta}_{ijk}, b^{\alpha\beta\gamma}_{ijk}, g^{\alpha\beta\gamma}_{ijk}, a^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}\right), \tag{6.4.9}$$

and (6.4.4) and (6.4.7) become

$$\tau(c^\alpha_{ij}) \cdot {z_j}^\alpha{}_i = \upsilon(h^\alpha_{ij}) \cdot {z_i}^\alpha{}_j \tag{6.4.10a}$$

$$\tau(l^{\beta\alpha}_{kij}) \cdot {z_i}^\alpha{}_j = \upsilon(q^{\alpha\beta}_{jik}) \cdot {z_i}^\beta{}_k \tag{6.4.10b}$$

$$g(b^{\beta\alpha\gamma}_{kji}) \cdot c^\alpha_{ij} \cdot l^{\alpha\beta}_{ijk} \cdot c^\beta_{jk} \cdot l^{\beta\gamma}_{jki} = l^{\alpha\gamma}_{jik} \cdot c^\gamma_{ik}. \tag{6.4.10c}$$

The full cocycle condition for the 8-tuple (6.4.9) is then given by Eqs. (6.4.10) *plus* Eqs. (5.5.11), which we rewrite here:

$$\begin{aligned}
&\delta(a^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl})\cdot g^{\alpha\beta\gamma}_{ijk}\cdot g^{\gamma\delta\varepsilon}_{ikl} = g^{\beta\delta\eta}_{jkl}\cdot g^{\alpha\eta\varepsilon}_{ijl},\\
&\sigma(b^{\alpha\beta\gamma}_{ijk})\cdot h^{\alpha}_{ij}\cdot q^{\alpha\beta}_{ijk}\cdot h^{\beta}_{jk} = u(g^{\alpha\beta\gamma}_{ijk})\cdot q^{\alpha\gamma}_{jik}\cdot h^{\gamma}_{ik}\cdot q^{\gamma\beta}_{ikj},\\
&f(a^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl})\cdot b^{\gamma\delta\varepsilon}_{ikl}\cdot b^{\alpha\beta\gamma}_{ijk} = b^{\alpha\eta\varepsilon}_{ijl}\cdot b^{\beta\delta\eta}_{jkl}.
\end{aligned}$$

Finally we need also to add the cocycle condition on the elements $a^{\alpha\beta\delta\gamma\eta\varepsilon}_{ijkl}$.

The amount of typographical decoration provided by the upper indices related to the hypercover can be quite daunting. Ignoring these indices (that is, reducing everything to the Čech case), although potentially less precise from the cohomological point of view (cf. the discussion in [12]) does shed some light on how the various parts are organized. Without upper indices we need to set $q^{\alpha\beta}_{ijk}=1$ and $l^{\alpha\beta}_{ijk}=1$ in the above formulas. Thus, the 8-tuple (6.4.9) becomes a sextuple

$$\big(z_i, c_{ij}, h_{ij}, b_{ijk}, g_{ijk}, a_{ijkl}\big)$$

satisfying the cocycle condition:

$$\begin{aligned}
&\tau(c_{ij})\cdot z_j = v(h_{ij})\cdot z_i\\
&g(b_{kji})\cdot c_{ij}\cdot c_{jk} = c_{ik}\\
&\delta(a_{ijkl})\cdot g_{ijk}\cdot g_{ikl} = g_{jkl}\cdot g_{ijl}\\
&\sigma(b_{ijk})\cdot h_{ij}\cdot h_{jk} = u(g_{ijk})\cdot h_{ik}\\
&f(a_{ijkl})\cdot b_{ikl}\cdot b_{ijk} = b_{ijl}\cdot b_{jkl}.
\end{aligned}$$

Now write the cone of the the morphism of complexes (6.2.2) in the form:

$$A \xrightarrow{\binom{f}{\delta}} B\oplus G \xrightarrow{\begin{pmatrix} g & 1\\ \sigma & u^{-1}\end{pmatrix}} C\oplus H \xrightarrow{(\tau\ \ v^{-1})} K$$

It can now be seen in a direct way that the 8-tuple (6.4.9) (or its simplified Čech version) indeed defines a 3-cocycle with values in the cone of (6.2.2). This is straightforward and left to the reader. We will also omit the verification that passing to an equivalent torsor 1-cocycle $(\mathscr{E}'_{ij}, \mathscr{F}'_i, \sigma'_i)$ representing $(\mathfrak{G}, \tilde{J})$, we obtain an equivalent 3-cocycle. □

An even more special case of Theorem 6.7(2) is when the diagram (6.2.2) reduces to the complex $A \xrightarrow{f} B \xrightarrow{g} C$. Let $(\mathfrak{G}, \tilde{J})$ be a 2-gerbe over C/X bound by $\textsc{Tors}(A)\to\textsc{Tors}(B)\to\textsc{Tors}(C)$. By comparing the classifying cocycles we immediately obtain the following

Corollary 6.8. $(\mathfrak{G}, \tilde{J})$ *is equivalent to a* 2-(A, B, C)-*gerbe in the sense of Definition* 5.21.

7. Applications

In this section we will address a few questions about the correspondence between certain Hermitian Deligne Cohomology groups and equivalence classes of 2-gerbes equipped with various geometric structures of the type described in the previous sections.

For consistency with the results of [7] and previous work in Deligne cohomology we will be placing $\mathbf{Z}(p)_X$ in degree zero, therefore all cohomology degrees will be shifted up in comparison with those appearing in the previous sections.

7.1. Truncated Hermitian Deligne complexes

Besides the Hermitian Deligne complexes recalled in Section 1.2, we need two more complexes that we introduced in [7], namely

$$\Gamma(2)^\bullet = \operatorname{Cone}\left(\begin{array}{ccccc} \mathbf{Z}(2) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega^1_X \\ & & \downarrow & & \downarrow \\ & & \mathcal{E}^0_X(1) & \longrightarrow & \mathcal{E}^1_X(1) \end{array}\right)[-1],$$

plus the truncation

$$\tilde{\Gamma}(2)^\bullet = \operatorname{Cone}\left(\begin{array}{ccccc} \mathbf{Z}(2) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega^1_X \\ & & \downarrow & & \downarrow \\ & & \mathcal{E}^0_X(1) & \longrightarrow & 0 \end{array}\right)[-1],$$

where the maps are the same as in the corresponding places in the diagram defining $\mathfrak{D}_{h.h.}(2)^\bullet_X$. (It is convenient to pass, from now on, to an additive notation.) Note that $\Gamma(2)^\bullet$ is an obvious truncation of the Hermitian Deligne complex $\mathfrak{D}_{h.h.}(2)^\bullet_X$, while $\tilde{\Gamma}(2)^\bullet$ is in turn a truncation of $\Gamma(2)^\bullet$. These two complexes were introduced as part of the effort to analyze the interplay and compatibility of different types of differential geometric structures on 2-gerbes. Indeed, it can be shown that $\Gamma(2)^\bullet$ arises from the diagram of complexes:

$$\mathbf{Z}(2)^\bullet_{\mathcal{D},X} \longrightarrow C(2)^\bullet \longleftarrow 2\pi\sqrt{-1} \otimes \mathfrak{D}_{h.h.}(1)^\bullet_X$$

in the sense of [30], namely as the cone of the difference of the two maps. Here $C(2)^\bullet$ is the complex

$$\mathbf{Z}(2)_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}^1_X(1).$$

Similarly, $\tilde{\Gamma}(2)^\bullet$ arises in the same way from the diagram:

$$\mathbf{Z}(2)^\bullet_{\mathcal{D},X} \longrightarrow \mathbf{Z}(1)^\bullet_{\mathcal{D},X} \longleftarrow 2\pi\sqrt{-1} \otimes \mathfrak{D}_{h.h.}(1)^\bullet_X,$$

where the two maps are just the forgetful maps. We have repeatedly seen how the complexes $\mathbf{Z}(2)^\bullet_{\mathcal{D},X}$ (resp. $\mathfrak{D}_{h.h.}(1)^\bullet_X$) intervene in the definition of connective (resp. Hermitian) structures. Note, however, that the above complexes and their geometric role were introduced rather informally in the context of [7]. The results of Section 7.2 provide a more rigorous footing.

We quote from [7] the following exact sequences. From the definitions we immediately have:

$$0 \longrightarrow \mathcal{E}^1_X(1)[-3] \longrightarrow \Gamma(2)^\bullet \longrightarrow \tilde{\Gamma}(2)^\bullet \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{E}^2_X(1) \cap \mathcal{A}^{1,1}_X[-4] \longrightarrow \mathfrak{D}_{h.h.}(2)^\bullet_X \longrightarrow \Gamma(2)^\bullet \longrightarrow 0.$$

Furthermore, using the standard arguments, as well as the softness of $\mathcal{E}^1_X(1)$, $\mathcal{E}^2_X(1)$, and $\mathcal{A}^{1,1}_X$, we obtain:

$$\cdots \longrightarrow E^1_X(1) \longrightarrow \mathbf{H}^3(X, \Gamma(2)^\bullet) \longrightarrow \mathbf{H}^3(X, \tilde{\Gamma}(2)^\bullet) \longrightarrow 0$$
$$\cdots \longrightarrow E^2_X(1) \cap A^{1,1}_X \longrightarrow \widehat{\mathrm{H}}^4_{\mathcal{D}}(X, 2) \longrightarrow \mathbf{H}^4(X, \Gamma(2)^\bullet) \longrightarrow 0$$

and the isomorphism

$$\mathbf{H}^k(X, \Gamma(2)^\bullet) \simeq \mathbf{H}^k(X, \tilde{\Gamma}(2)^\bullet), \quad k \geq 4.$$

7.2. Geometric interpretation of some cohomology groups

Observe that using $\mathcal{O}_X/\mathbf{Z}(2)_X \simeq \mathcal{O}^\times_X$, the complex $\Gamma(2)^\bullet$ can be identified (modulo the index shift) with the cone of the square

$$\begin{array}{ccc} \mathcal{O}_X/\mathbf{Z}(2)_X & \longrightarrow & \Omega^1_X \\ \downarrow & & \downarrow \\ \mathcal{E}^0_X(1) & \longrightarrow & \mathcal{E}^1_X(1) \end{array} \qquad (7.2.1)$$

and similarly for $\Gamma(2)^\bullet$ by replacing $\mathscr{E}^1_X(1)$ with 0:

$$\begin{array}{ccc} \mathscr{O}_X/\mathbf{Z}(2)_X & \longrightarrow & \Omega^1_X \\ \downarrow & & \downarrow \\ \mathscr{E}^0_X(1) & \longrightarrow & 0 \end{array} \tag{7.2.2}$$

Both cases correspond to the diagram (5.2.2).

To make contact with the contents of Section 5, let us set

$$\mathscr{A} = \mathrm{TORS}(\mathscr{O}_X/\mathbf{Z}(2)_X, \mathscr{E}^0_X(1)), \qquad \mathscr{B} = \mathrm{TORS}(\Omega^1_X, \mathscr{E}^1_X(1))$$

so that we have the equivalences

$$\mathrm{TORS}(\mathscr{A}) \xrightarrow{\sim} \mathrm{GERBES}(\mathscr{O}_X/\mathbf{Z}(2)_X, \mathscr{E}^0_X(1))$$

and

$$\mathrm{TORS}(\mathscr{B}) \xrightarrow{\sim} \mathrm{GERBES}(\Omega^1_X, \mathscr{E}^1_X(1)).$$

Using Theorem 5.8 and Proposition 6.4 we find the following alternative characterization of $\mathscr{O}^\times_X$-gerbes with compatible Hermitian and connective structure:

Corollary 7.1. *The group $\mathbf{H}^3(X, \Gamma(2)^\bullet)$ classifies equivalence classes of gerbes bound by $\mathscr{O}_X/\mathbf{Z}(2)_X \to \mathscr{E}^0_X(1)$, that is, Hermitian gerbes in the sense of Section 2.3.1, which become neutral as $(\Omega^1_X, \mathscr{E}^1_X(1))$-gerbes.*

Of course, the other possible but entirely equivalent statement would have been that the cohomology group under scrutiny classifies $(\mathscr{A}, \mathscr{B})$-torsors, where $\mathscr{A} = \mathrm{TORS}(\mathscr{O}_X/\mathbf{Z}(2)_X, \mathscr{E}^0_X(1))$ and $\mathscr{B} = \mathrm{TORS}(\Omega^1_X, \mathscr{E}^1_X(1))$. We leave to the reader the task of formulating a similar statement for the complex $\tilde{\Gamma}(2)^\bullet$.

Remark 7.2. A short remark is in order about other possible ways of interpreting the same cohomology group. As noted, we can take advantage of the symmetry of the square (7.2.1) in the sense explained in Remark 5.20, and modify things accordingly. This preserves the cone, namely $\Gamma(2)^\bullet$, and does not alter the classifying group. It *does* change the *gr*-stacks $\mathscr{A}$ and $\mathscr{B}$, but ultimately not the fact that we are dealing with $\mathscr{O}^\times_X$-gerbes.

Remark 7.3. The above characterization (and the general theory it descends from) provides a finer description of the corresponding geometric objects whose equivalence classes correspond to the group elements when the coefficient complex come from a cone. Had we just used the complex $\Gamma(2)^\bullet$ as it stands, we would have been in the rather awkward position of calling something with values in $\Omega^1_X \oplus \mathscr{E}^0_X(1)$ a "connective structure," a fact that does not seem to sit well with the degrees.

The corresponding result for 2-gerbes provides a similar interpretation for the group of equivalence classes of 2-$\mathscr{O}^\times_X$-gerbes with compatible Hermitian and connective structure defined in [7]. It is an immediate consequence of Theorem 5.16 as follows:

Corollary 7.4. *Elements of the hypercohomology group $\mathbf{H}^4(X, \Gamma(2)^\bullet)$ are in one-to-one correspondence with equivalence classes of 2-gerbes on X bound by the square (7.2.1) (in the sense of Remark 5.20). A similar conclusion holds by replacing $\Gamma(2)^\bullet$ with $\tilde{\Gamma}(2)^\bullet$.*

Note that a remark concerning the square similar to the one just made for gerbes holds in this case as well.

In a similar vein to what was just done for the complex $\Gamma(2)^\bullet$, we can identify $\mathfrak{D}_{h.h.}(2)^\bullet_X$ defined in Eq. (1.2.11) with the cone of

$$\begin{array}{ccccc} \mathscr{O}_X/\mathbf{Z}(2)_X & \longrightarrow & \Omega^1_X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{E}^0_X(1) & \longrightarrow & \mathscr{E}^1_X(1) & \longrightarrow & \mathscr{E}^2_X(1) \cap \mathscr{A}^{1,1}_X \end{array} \tag{7.2.3}$$

which will correspond to the diagram (6.2.2). We have explicitly written the last column as $0 \to \mathscr{E}^2_X(1) \cap \mathscr{A}^{1,1}_X$ in order to emphasize the correspondence. To take the point of view of Section 6, let us introduce the *discrete gr*-stack

$$\mathscr{C} = \mathrm{TORS}(0, \mathscr{E}^2_X(1) \cap \mathscr{A}^{1,1}_X) \simeq \mathscr{E}^2_X(1) \cap \mathscr{A}^{1,1}_X,$$

namely the only morphisms are the identity maps. Note that since $\mathscr{C}$ is discrete, the corresponding 2-gerbe is discrete as well, that is we have:

$$\mathrm{TORS}(\mathscr{C}) \simeq \mathrm{TORS}(\mathscr{E}^2_X(1) \cap \mathscr{A}^{1,1}_X).$$

In other words, it has only identity 2-arrows, and it corresponds to the neutral gerbe of torsors.

Now, as a consequence of Theorem 6.7 we obtain the following general geometric interpretation for the Hermitian Deligne cohomology group:

Corollary 7.5. *Elements of the Hermitian Deligne cohomology group $\widehat{\mathrm{H}}^4_{\mathcal{D}}(X, 2)$ are in one-to-one correspondence with equivalence classes of 2-gerbes on X bound by the diagram (7.2.3), that is, by the complex (6.2.1) of gr-stacks associated with the columns of (7.2.3).*

7.3. Geometric construction of some cup products

7.3.1

If $(\mathscr{L}, \rho)$ and $(\mathscr{M}, \sigma)$ are two metrized line bundles (invertible sheaves) over X, their isomorphism classes determine elements of $\widehat{\mathrm{H}}^2_{\mathcal{D}}(X, 1) \simeq \widehat{\mathrm{Pic}}X$. According to the last paragraph of Section 1.2, the cup product $[\mathscr{L}, \rho] \cup [\mathscr{M}, \sigma]$ in Hermitian Deligne cohomology will land in $\widehat{\mathrm{H}}^4_{\mathcal{D}}(X, 2)$.

It is known from the works of Brylinski and McLaughlin [4–6] that the corresponding problem in standard Deligne cohomology has a geometric interpretation: there is a 2-gerbe $(\mathscr{L}, \mathscr{M}]$ bound by $\mathbf{Z}(2)^\bullet_{\mathcal{D},X}$ whose class is the cup product $[\mathscr{L}]\cup[\mathscr{M}] \in \mathrm{H}^4_{\mathcal{D}}(X, \mathbf{Z}(2))$ of the elements in Pic X determined by $\mathscr{L}$ and $\mathscr{M}$. Similarly, in [7] we constructed a modified cup product

$$\mathrm{Pic}\, X \otimes \mathrm{Pic}\, X \longrightarrow \widehat{\mathrm{H}}^4_{\mathcal{D}}(X, 1)$$

and a corresponding "tame symbol", namely a 2-gerbe $(\mathscr{L}, \mathscr{M}]_{h.h.}$ bound by $\mathfrak{D}_{h.h.}(1)^\bullet_X$. It turns out that both symbols have the "same" (in the sense of equivalent) underlying 2-gerbe, obtained by applying a suitable forgetful functor to both sides. In other words we have a lift

$$\mathrm{Pic}\, X \otimes \mathrm{Pic}\, X \longrightarrow \mathbf{H}^4(X, \tilde{\Gamma}(2)^\bullet)$$

and it follows from the material recalled in Section 7.1 that at the level of cohomology the latter lift can be arranged to take values in $\mathbf{H}^4(X, \Gamma(2)^\bullet)$. Thus, from a pair of invertible sheaves $\mathscr{L}$ and $\mathscr{M}$ we obtain (canonically) a 2-gerbe bound by the square (7.2.2), and (non-canonically) by way of softness of one of the sheaves involved, a 2-gerbe bound by the square (7.2.1).

The cohomology exact sequences recalled in Section 7.1, and the fact that truncation will map the diagram (7.2.3) to the square (7.2.1), and then to the square (7.2.2), show that the 2-gerbe bound by (7.2.3) corresponding to the cup product $[\mathscr{L}, \rho] \cup [\mathscr{M}, \sigma]$ will provide the required lift.

7.3.2

We will denote by $\widehat{(\mathscr{L}, \mathscr{M}]}_{h.h.}$ the 2-gerbe bound by (7.2.3) corresponding to the cup product of the two metrized line bundles. Let us sketch the geometric construction of such 2-gerbe borrowing the corresponding constructions of [5,7].

If we work locally with respect to some cover $U \to X$ of X, any 2-$\mathscr{A}$-gerbe $\mathfrak{G}$ will be a 2-gerbe of torsors, namely there is an equivalence:

$$\mathfrak{G}_U \xrightarrow{\sim} \mathrm{TORS}(\mathscr{A}|_U) \xrightarrow{\sim} \mathrm{GERBES}(\mathscr{O}_X/\mathbf{Z}(2)_X|_U, \mathscr{E}^0_X(1)|_U),$$

where the latter equivalence follows from Theorem 5.8. Thus if $\mathfrak{G}$ is bound by the complex of gr-stacks determined by the diagram (7.2.3), with $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ as in Section 7.2, then locally it has the form

$$\mathrm{TORS}(\mathscr{A}|_U) \longrightarrow \mathrm{TORS}(\mathscr{B}|_U, \mathscr{C}|_U).$$

Note that, thanks to 6.2(2), Proposition 6.4, and to the fact that in the relevant diagram one of the group is zero, we have an equivalence:

$$\mathrm{TORS}(\mathscr{B}|_U, \mathscr{C}|_U) \xrightarrow{\sim} \mathrm{GERBES}(\Omega^1_X|_U, \mathscr{E}^1_X(1)|_U, \mathscr{E}^2_X(1) \cap \mathscr{A}^{1,1}_X|_U).$$

Let $\langle \mathscr{L}, \mathscr{M}]$ denote the underlying 2-gerbe of both $(\mathscr{L}, \mathscr{M}]$ and $(\mathscr{L}, \mathscr{M}]_{h.h.}$. The local objects of $\langle \mathscr{L}, \mathscr{M}]$ over U are in one-to-one correspondence with the non-vanishing sections of $\mathscr{L}|_U$. We may denote such a section s, which is thought of as an object, by $\langle s, \mathscr{M}]$.

The choice of s will determine an $\mathscr{A}|_U$-torsor as follows. Given any other non-vanishing section s', write $s = s' \cdot g$ where $g \in \mathscr{O}_X/\mathbf{Z}(2)$. The $\mathscr{A}|_U$-torsor $\mathscr{H}om(s, s')$ can be identified with the $(\mathscr{O}_X/\mathbf{Z}(2)_X|_U, \mathscr{E}^0_X(1)|_U)$-gerbe $(g, \mathscr{M}]_{h.h.}$ by the above equivalence. Let us denote by $\langle g, \mathscr{M}]$ the underlying $\mathscr{O}_X/\mathbf{Z}(2)_X$-gerbe. Recall from [5, 7] that its objects over U are in one-to-one correspondence with the non-vanishing sections t of $\mathscr{M}|_U$, denoted $\langle g, t]$, and that an arrow $\varphi\colon \langle g, t] \to \langle g, t']$ is identified with a section of Deligne's torsor $(g, g']$, where $t = t' \cdot g'$, for g' a section of $\mathscr{O}_X/\mathbf{Z}(2)_X$ over U, see [31]. We reserve the notation $(g, g']$ for the same torsor equipped with the connection defined in loc. cit., whereas the notation $(g, g']_{h.h.}$ denotes the same underlying torsor equipped with the Hermitian structure defined in [7].

To summarize, to define $\widehat{(\mathscr{L}, \mathscr{M}]}_{h.h.}$ we have to define a 2-functor $\tilde{J}_U$ from $\mathfrak{G}_U$ to the fibered 2-category of gerbes bound by

$$\Omega^1_X|_U \xrightarrow{\pi_1} \mathscr{E}^1_X(1)|_U \xrightarrow{\pi \circ \mathrm{d}} \mathscr{E}^2_X(1) \cap \mathscr{A}^{1,1}_X.$$

To begin with, let us define a 2-functor J_U to $\mathrm{GERBES}(\Omega^1_X|_U, \mathscr{E}^1_X(1)|_U)$ as follows. To an object $\langle s, \mathscr{M}]$ assign the trivial $\mathscr{B}|_U$-torsor $T(\mathscr{B}|_U) \simeq \mathrm{TORS}(\Omega^1_U, \mathscr{E}^1_U(1))$. To a 1-arrow

$$\langle g, t]\colon \langle s, \mathscr{M}] \longrightarrow \langle s', \mathscr{M}]$$

the functor $\langle g, t]_*\colon T(\mathscr{B}|_U) \to T(\mathscr{B}|_U)$ is defined as follows: an object of $T(\mathscr{B}|_U)$ is identified with an object (C, ξ) of $\mathrm{TORS}(\Omega^1_U, \mathscr{E}^1_U(1))$, where C is a Ω^1_U-torsor which becomes trivial as a $\mathscr{E}^1_U$-torsor by way of ξ, which in turn can be identified with a section of $\mathscr{E}^1_U$. Then we define $\langle g, t]_*$ by

$$\langle g, t]_*\colon (C, \xi) \longmapsto (C, \xi + \xi_t), \tag{7.3.1}$$

where the underlying map on $\mathrm{TORS}(\Omega^1_U)$ is the identity, and ξ_t is the imaginary 1-form:

$$\xi_t = -\frac{1}{2} \log|g| \cdot \mathrm{d}^c \log\sigma(t) + \frac{1}{2} \mathrm{d}^c \log|g| \cdot \log\sigma(t). \tag{7.3.2}$$

Here we have used the notation $\sigma(t) = |t|^2_\sigma$. It is straightforward to verify that this is compatible with morphisms in $T(\mathscr{B}|_U)$ and with the action of $\mathscr{B}|_U$: if (D, η) is an object of $\mathrm{TORS}(\Omega^1_U, \mathscr{E}^1_U(1))$, then

$$(C, \xi) \cdot (D, \eta) = (C \otimes D, \xi + \eta),$$

and obviously this commutes with (7.3.1), making it a morphism of torsors.

Now, if φ is a section of $\langle g, g']$, the corresponding object of $(g, g']_{h.h.}$ is $(\varphi, \|\varphi\|)$ where $\|\cdot\|$ is the Hermitian structure given in [7]. To it we assign the natural transformation given by the morphism in $T(\mathscr{B}|_U)$:

$$(\varphi, \|\varphi\|)_*\colon (C, \xi + \xi_t) \longrightarrow (C, \xi + \xi_{t'}), \tag{7.3.3}$$

which is defined by the underlying map

$$\begin{aligned} \varphi\colon C &\longrightarrow C \\ c &\longmapsto c + \varphi^{-1}\nabla\varphi, \end{aligned} \tag{7.3.4}$$

where ∇ is the connection on $(g, g']$. From [31] we have that locally it has the form $-\log g \,\mathrm{d}\log g'$. Therefore the section $\xi + \xi_t$ will map to $\xi + \xi_t + \pi_1(\varphi^{-1}\nabla\varphi)$ and notice that this differs from $\xi_{t'}$ by $2\pi\sqrt{-1}\,\mathrm{d}\log\|\varphi\|$, using the fact that locally $\|\cdot\|$ is given by $\pi_1(\log g)\log|g'|$. Note that the addition of $\mathrm{d}\log\|\varphi\|$ is just the action of $(\Omega^1_U, 2\pi\sqrt{-1}\,\mathrm{d}\log\|\varphi\|)$ as an object of $\mathscr{B}|_U$.

Finally, in order to get the functor $\tilde{J}_U$, we need one more prescription. Namely we define it by assigning to $\langle s, \mathscr{M}]$ the $(\mathscr{B}|_U, \mathscr{C}|_U)$-torsor defined as follows. It is the trivial $\mathscr{B}|_U$-torsor defined as above equipped with the morphism

$$\mathrm{TORS}(\Omega^1_U, \mathscr{E}^1_U(1)) \longrightarrow \mathscr{E}^2_U(1) \cap \mathscr{A}^{1,1}_U$$

defined by the assignment

$$(C, \xi) \longmapsto \pi(\mathrm{d}\,\xi) - \frac{1}{4}\log\rho(s)\,\mathrm{d}\,\mathrm{d}^c\log\sigma(t) \tag{7.3.5}$$

for every object (C, ξ) of $\mathrm{TORS}(\Omega^1_U, \mathscr{E}^1_U(1))$. Observe that $\mathrm{d}\,\mathrm{d}^c\log\sigma(t) = c_1(\mathscr{M})$, hence there is no dependence on t. Now, a calculation shows that

$$\pi(\mathrm{d}\,\xi_t) = -\frac{1}{2}\log|g|\,\mathrm{d}\,\mathrm{d}^c\log\sigma(t)$$

so that it is immediately verified that the assignment (7.3.5) commutes with the morphism (7.3.1).

With these provisions we have:

Theorem 7.6. *The class of the* 2*-gerbe* $\widehat{(\mathscr{L}, \mathscr{M}]}_{h.h.}$ *in the cohomology group* $\widehat{\mathrm{H}}^4_{\mathcal{D}}(X, 2)$ *is the cup product* $[\mathscr{L}, \rho] \cup [\mathscr{M}, \sigma]$ *in Hermitian Deligne cohomology.*

Proof. It follows immediately from Theorem 6.7, the form of the maps in diagrams (1.2.11) and (7.2.3), and the cup product map

$$\mathfrak{D}_{h.h.}(1)^\bullet_X \otimes \mathfrak{D}_{h.h.}(1)^\bullet_X \longrightarrow \mathfrak{D}_{h.h.}(2)^\bullet_X$$

given in [11], where the explicit cup product in Čech cohomology is computed. □

8. Conclusions

We have generalized the concept of "abelian gerbe bound by a complex" to the case of longer coefficient complexes, and to 2-gerbes, where we have used complexes of *gr*-stacks of length 3. We have verified that these 2-gerbes are classified by cohomology sets of degree 1 with values in the associated complexes of torsors over these *gr*-stacks. We have also obtained, by choosing appropriate decompositions and hypercovers, that in the strictly abelian situation the general classification reduces to degree 3 cohomology groups with values in cones of crossed squares, and other similar diagrams. In all cases we have obtained explicit cocycles, where we have given their expression in terms of hypercover, rather than simply in terms of Čech cocycles.

As an application, we have dealt with differential geometric structures on gerbes and 2-gerbes and questions of geometric constructions of certain cup products in Hermitian Deligne cohomology. In particular, we have put certain by now standard constructions of the concept of connection and curvature in the general context of gerbe (or 2-gerbe) bound by a complex. We have further clarified the reason why there seem to exist different possibilities in defining what a "Hermitian gerbe" should be (cf. Remark 2.8). Finally, in the last section we have geometrically constructed a 2-gerbe bound by the Hermitian Deligne complex $\mathfrak{D}_{h.h.}(2)^\bullet_X$ corresponding to the cup product of two metrized line bundles in Hermitian Deligne cohomology.

There are several possible extensions and generalizations of the work carried out in this paper. In the case of gerbes, it would be interesting to remove the abeliannes assumption and work in the same framework as [1] to study extended structures as coefficients, beyond crossed modules: crossed squares, 2-crossed complexes, etc. come to mind. In particular, it would be interesting to see whether the idea of phrasing the notion of connection and curvature in terms of gerbes bound by complexes extends to the non-abelian case, and how it compares with other existing approaches (see, e.g. [32]). In [1] a compelling motivation was to obtain a theory of non-abelian H^2 which behaved better than Giraud's with respect to group exact sequences. Pursuing some of these ideas in the case of 2-gerbes would also be quite interesting. We hope to return to some of these issues in future publications.

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