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PROJECTIVE MODULES OVER SOME NON-NOETHERIAN POLYNOMIAL RINGS

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The aim of this note is to prove a non-Noetherian generalization of the Serre Conjecture. However, since we also show that for all zero-dimensional commutative rings R, finitely generated projective $R[X_1, \ldots, X_n]$-modules are extended, the paper represents our contribution to the seemingly difficult problem of determining the rings for which some form of the Serre Conjecture is valid.

As ever, all rings are commutative with 1 and, we hope, most notation is standard. A noteworthy exception is that if R is a ring with X an indeterminate, then we shall denote by $R(X)$ the localization of $R[X]$ at the multiplicative set of all monic polynomials of $R[X]$. We realize that this notation is at variance with the usual meaning of "$R(X)$", but those familiar with [6] will recognize its origin.

1.

Quillen and Suslin recently proved that every finitely generated projective module over $k[X_1, \ldots, X_n]$ is free, where k is a principal ideal domain. We show here that this remarkable theorem admits a non-Noetherian generalization. Namely, the theorem remains true if we require only that k be a Bezout domain of Krull dimension one. (A Bezout domain is an integral domain in which finitely generated ideals are principal.) The idea of the proof is to use the corollary to Theorem 2 in [6], which says that if a finitely generated projective $R[X]$ module is free when tensored by $R(X)$, then it is free. If $\mathfrak{R}$ is a class of rings closed under the formation of $R(X)$ and such that, over any R in $\mathfrak{R}$, finitely generated projective modules are free, then by the aforementioned corollary finitely generated projectives are free over $R[X]$, for any R in $\mathfrak{R}$. An easy induction then yields that finitely generated projectives over $R[X_1, \ldots, X_n]$ are free, for R in $\mathfrak{R}$.

Bezout domains are characterized by the property that finitely generated submodules of free modules are free, whence finitely generated projectives are free over Bezout domains. Thus, our task is to show that if R is a one-dimensional Bezout domain, then $R(X)$ is also.

Lemma 1. (L. LeRiche has independently obtained the same result.) $\dim R(X) = \dim R[X] - 1$.

Proof. We may assume that $\dim R$ is finite.

Let Q be a maximal ideal of $R[X]$, and let $P = Q \cap R$. Then $Q \neq P[X]$, so by [3, Lemma 1], $\operatorname{ht} Q = \operatorname{ht} P[X] + 1$. As $P[X]$ survives in $R(X)$, $\operatorname{ht} Q \leq \dim R(X) + 1$. Thus $\dim R[X] \leq \dim R(X) + 1$, giving us $\dim R[X] - 1 \leq \dim R(X) \leq \dim R[X]$. It now suffices to show that no prime ideal of $R[X]$ of maximal height survives in $R(X)$.

Let Q be a prime ideal of $R[X]$ with $\operatorname{ht} Q = \dim R[X]$, and let $P = Q \cap R$. We must have $Q \neq P[X]$, since Q is maximal. Hence $\operatorname{ht} Q = \operatorname{ht} P[X] + 1$. If P were not maximal in R we could find a prime ideal P' of R such that $P \subsetneq P'$, and a prime ideal Q' of $R[X]$ with $Q' \cap R = P'$ and $Q' \neq P'[X]$. Then

$$\operatorname{ht} Q' = \operatorname{ht} P'[X] + 1 > \operatorname{ht} P[X] + 1 = \dim R[X],$$

which is absurd. P is therefore a maximal ideal of R, from which it follows easily that Q contains a monic polynomial. So Q does not survive in $R(X)$.

It is obvious from Lemma 1 that $\dim R(X) = \dim R$ if and only if $\dim R[X] = \dim R + 1$. In particular, $\dim R(X) = \dim R$ if R is a Noetherian ring or a Prüfer domain.

Theorem 1. *Let D be an integral domain, not a field.*

(i) *$D(X)$ is a Prüfer domain if and only if D is a one-dimensional Prüfer domain. In this case,* $\dim D(X) = 1$.

(ii) *$D(X)$ is a Bezout domain if and only if D is a one-dimensional Bezout domain. In this case,* $\dim D(X) = 1$.

Proof. Let K be the quotient field of D.

(i) Suppose that D is a one-dimensional Prüfer domain. Let Q be a prime ideal of $D[X]$ which survives in $D(X)$. Then $D(X)_{QD(X)} = D[X]_Q$, so we must show that $D[X]_Q$ is a valuation ring. If $Q \cap D = (0)$, this is clear, since $D[X]_Q = K[X]_{QK[X]}$. Otherwise, $Q \cap D$ is a maximal ideal of D and $Q = (Q \cap D)[X]$, since Q contains no monics. Then $D[X]_Q = D[X]_{(Q \cap D)[X]}$ is a valuation ring by [4, Proposition 18.7].

Conversely, suppose that $D(X)$ is a Prüfer domain. Let M be a maximal ideal of D. Then $M[X]$ is prime in $D[X]$ and survives in $D(X)$, so $D[X]_{M[X]}$ is a valuation ring of $K(X)$. Thus $D_M = D[X]_{M[X]} \cap K$ is a valuation ring. This shows that D is a Prüfer domain.

If $\dim D > 1$, let $(0) \subsetneq P_1 \subsetneq P_2$ be a chain of prime ideals of D. Choose $a \in P_2 \backslash P_1$ and consider $Q = (P_1[X], aX - 1)$. Examination of $Q/P_1[X]$ shows that Q is a prime ideal of $D[X]$ which contains no monic polynomial. Hence, $D[X]_Q = D(X)_Q$ is an essential valuation ring of $D[X]$. By [4, Exercise 12, p. 221], Q must have height one or be the extension of a prime ideal of D. As neither of these is true, $\dim D = 1$.

(ii) If D is a one-dimensional Bezout domain, then by part (i) $D(X)$ is a Prüfer domain. Since D is a *GCD* domain, $D[X]$, and hence $D(X)$, is also a *GCD* domain. But a Prüfer *GCD* domain is Bezout.

Conversely, suppose that $D(X)$ is Bezout. Being Prüfer then, we get from part (i) that $\dim D = 1$. To see that D is Bezout, let $a, b \in D$. Choose a polynomial $r(X) \in D[X]$ such that $(a, b)D(X) = r(X)D(X)$. Then there are polynomials $p(X)$, $q(X)$, $f(X)$, $g(X)$ and monic polynomials $m_1(X)$, $m_2(X)$, $m_3(X)$, $m_4(X)$, such that

$$m_1(X)m_2(X)r(X) = ap(X)m_2(X) + bq(X)m_1(X),$$

$$am_3(X) = r(X)f(X) \quad \text{and} \quad bm_4(X) = r(X)g(X).$$

Let c be the leading coefficient of $r(X)$. Since the $m_i(X)$'s are monic, the equations above show that $(a, b)D = cD$.

The fact that $\dim D(X) = 1$ in each case follows from Lemma 1.

Corollary 1. *If D is a one-dimensional Bezout domain, then finitely generated projective $D[X_1, \ldots, X_n]$-modules are free. If D is a one-dimensional Prüfer domain, then finitely generated projective $D[X_1, \ldots, X_n]$-modules are extended.*

Proof. For the first assertion, observe that by Theorem 1, $D(X_1)$ is a Bezout domain, so by the corollary to Theorem 2 in [6], finitely generated projective $D[X_1]$-modules are free. Proceeding by induction on n, assume that finitely generated projective $D[X_1, \ldots, X_n]$-modules are free for any one-dimensional Bezout domain D. Let E be a finitely generated projective $D[X_1, \ldots, X_n, X_{n+1}]$-module. Then by the inductive hypothesis, $E \otimes D(X_{n+1})[X_1, \ldots, X_n]$ is free. Since $D[X_1, \ldots, X_n](X_{n+1})$ is a localization of $D(X_{n+1})[X_1, \ldots, X_n]$, $E \otimes D[X_1, \ldots, X_n](X_{n+1})$ is free and so E is also, by Quillen's result.

For the second assertion, observe that for each maximal ideal P of D, D_P is a one-dimensional Bezout domain. Hence finitely generated projective $D_P[X_1, \ldots, X_n]$-modules are extended, by the first assertion. The proof is now complete upon invoking [6, Theorem 1'].

Corollary 1 is proved in the noetherian case in Quillen's paper and also for the one variable case in an earlier paper of Bass [2, Theorem 2.4].

We do not know whether Corollary 1 is true for Bezout domains of dimension greater than one. In the one variable case, using Bass's generalized version of

Seshadri's theorem [1, Corollary, 6.2, p. 212], it is easy to see that if V is a discrete valuation ring of finite rank, then finitely generated projective $V[X]$-modules are free. Using the same technique and Corollary 1, one also gets that finitely generated projectives are free over $V[X]$, where V is a rank two valuation ring with principal maximal ideal.

Concluding this section on a more positive note, we indicate ways of obtaining one-dimensional non-noetherian Bezout domains. The domain of all algebraic integers is such a domain [5, Theorem 102] as is any rank one valuation ring which is not a DVR. A generalization of the latter idea may be obtained from the Krull–Jaffard–Ohn Theorem, which asserts that any lattice-ordered abelian group is the divisibility group of some Bezout domain [4, Theorem 19.6].

2.

Now we shall extend the Quillen–Suslin result in a different way by proving that for all zero-dimensional commutative rings R, finitely generated projective $R[X_1, \dots, X_n]$-modules are extended. Although this result is essentially contained in [7], we include it here because our proof uses a lemma (Lemma 2) which seems to be new, and which we believe holds promise for shedding light on the problem of determining which rings R have the property that finitely generated projective $R[X_1, \dots, X_n]$-modules are extended.

Lemma 2. *Let R be a quasi-local ring with maximal ideal M. Suppose that E is a finitely generated projective $R[X_1, \dots, X_n]$-module and that I is an ideal of R such that E/IE is free over $(R/I)[X_1, \dots, X_n]$. Then there is a free submodule $F \subseteq E$ and a polynomial $f \in 1 + I[X_1, \dots, X_n]$ such that $fE \subseteq F$.*

Proof. Choose elements $e_1, \dots, e_r \in E$ whose images in E/IE form a basis. Let F be the submodule of E generated by $e_1, \dots, e_r$. Then $E = F + IE$, so $E/F = I(E/F)$. The existence of the polynomial f now follows.

It remains to show that F is free. For each positive integer m,

$$R[X_1, \dots, X_n]/(X_1, \dots, X_n)^m = A_m$$

is a quasi-local ring with maximal ideal $(M, X_1, \dots, X_n)/(X_1, \dots, X_n)^m$. Thus the finitely generated projective module $E_m = E \otimes A_m$ is free. Now E/IE is free, so

$$r = \mathrm{rank}_{(R/I)[X_1, \dots, X_n]}(E/IE) = \mathrm{rank}_{R/M}(E/(M, X_1, \dots, X_n)E) = \mathrm{rank}_{A_m} E_m.$$

Also, since the images of $e_1, \dots, e_r$ in $E/M(X_1, \dots, X_n)E$ are a basis, so are their images in E_m. Then the $(X_1, \dots, X_n)$-adic completion of E, $\hat{E} = \varprojlim E_m$, is free on $e_1, \dots, e_r$ over $R[[X_1, \dots, X_n]]$. Since $e_1, \dots, e_r$ are linearly independent over $R[[X_1, \dots, X_n]]$, they are over $R[X_1, \dots, X_n]$ as well, i.e., F is free.

Theorem 2. *Let R be a ring with nilradical N. If E is a finitely generated projective $R[X_1, \ldots, X_n]$-module such that E/NE is extended, then E is extended.*

Proof. For each prime ideal P of R, E_P/N_PE_P is free. Applying Lemma 2 to E_P with $I = N_P$, and using the fact that $1 + N_P[X_1, \ldots, X_n]$ consists of units, shows that E_P is free over $R_P[X_1, \ldots, X_n]$. Then by [6, Theorem 1'], E is extended.

Corollary 2. *Let R be a zero-dimensional ring. Every finitely generated projective $R[X_1, \ldots, X_n]$-module is extended. Every finitely generated projective $R[X_1, \ldots, X_n]$-module is free if and only if R is quasi-local.*

Proof. By [6, Theorem 1'] the first assertion follows from the second. So assume R is quasi-local. Then the nilradical of R is its maximal ideal, so by Theorem 2 and the Serre conjecture finitely generated projective $R[X_1, \ldots, X_n]$-modules are free.

Conversely, if R is not quasi-local it has non-trivial idempotents. These generate non-free projective ideals in $R[X_1, \ldots, X_n]$.

An arithmetical ring is a ring R such that for each maximal ideal M of R the ideals of R_M are linearly ordered by inclusion. Any homomorphic image of a Prüfer domain is arithmetical

Corollary 3. *If R is a one-dimensional arithmetical ring, then finitely generated projective $R[X_1, \ldots, X_n]$-modules are extended.*

Proof. It is sufficient to assume that R is quasi-local. Then the nilradical N of R is a prime ideal and R/N is a one-dimensional valuation ring or a field. By Corollary 1 and Theorem 2, finitely generated projectives are free over $R[X_1, \ldots, X_n]$.

It follows from Corollary 3 that if V is a two-dimensional valuation ring, I a non-zero ideal of V, and E a finitely generated projective $V[X_1, \ldots, X_n]$-module, then E/IE is free.

3.

We next use Lemma 2 to prove a result which, together with Corollary 1, characterizes finitely generated projective modules over $V[X_1, \ldots, X_n]$ where V is a two-dimensional valuation ring.

Theorem 3. *Let V be a valuation ring of finite dimension r with maximal ideal M. Suppose that finitely generated projectives are free over $A[X_1, \ldots, X_n]$ if A is a valuation ring of dimension $r-1$. Let P be the prime ideal of V of height either 1 or $r-1$. Choose $m \in M \setminus P$ and $f \in 1 + P[X_1, \ldots, X_n]$. Let F be a finite rank free*

$V[X_1, \ldots, X_n]$-module, and let G be any free submodule of F such that $mfF \subseteq G \subseteq F$. Then $G+fF$ is a finitely generated projective $V[X_1, \ldots, X_n]$-module. Moreover, every finitely generated projective $V[X_1, \ldots, X_n]$-module arises in this way.

Proof. Choose m, f, F, G as indicated and let $E = G+fF$. Since $m \in M \backslash P$, $P \subseteq mV$, so $f-1 \in mV[X_1, \ldots, X_n]$. Thus

$$(m, f)V[X_1, \ldots, X_n] = V[X_1, \ldots, X_n].$$

Hence if Q is a prime ideal of $V[X_1, \ldots, X_n]$, either m or f is a unit in $V[X_1, \ldots, X_n]_Q$. If m is a unit, $E_Q = G_Q$; while if f is a unit, $E_Q = F_Q$. E is therefore flat and being finitely generated over the integral domain $V[X_1, \ldots, X_n]$, is projective.

On the other hand, suppose E is a finitely generated projective $V[X_1, \ldots, X_n]$-module. If E is free, let m be any element of $M \backslash P$, $f = 1$, and $F = G = E$, to get $E = G+fF$. Thus we may suppose E is not free. $E \otimes V_P[X_1, \ldots, X_n]$ is free either by assumption or by Corollary 1. Hence there is an $m \in V \backslash P$ and a free module $F_1 \subseteq E$ such that $mE \subseteq F_1$. Since E is not free, $m \in M \backslash P$. Now $E \otimes (V/P) \times [X_1, \ldots, X_n]$ is also free, so by Lemma 2 there is an $f \in 1+P[X_1, \ldots, X_n]$ and a free module $F_2 \subseteq E$ such that $fE \subseteq F_2$. Then $E = (m, f)E \subseteq F_1 + F_2 \subseteq E$, so $E = F_1 + F_2$. Let $F = f^{-1}F_2$ and $G = F_1$. Then

$$mfF = mF_2 \subseteq mE \subseteq F_1 = G \subseteq E \subseteq F,$$

and $E = F_1 + F_2 = G + fF$.

Note that if one could show that every module of the form $G+fF$ given in Theorem 3 were free, it would follow by induction that finitely generated projective $V[X_1, \ldots, X_n]$-modules are free if V is a valuation ring of finite dimension. This in turn would imply that for any finite dimensional Prüfer domain D, finitely generated projective $D[X_1, \ldots, X_n]$-modules are extended. We have been unable to show that modules of the above form are free, and we do not know whether these conjectures are true. As indicated in section one, they are true in the following cases: $\dim V = 1$; $n = 1$ and V is discrete; $n = 1$, $\dim V = 2$ and V has principal maximal ideal. Finally, we note that modules of the form $G+fF$ are easily seen to be free in each of the following cases: $fF \subseteq G$, $G \subseteq fF$, $PF \subseteq G$.

Note added in proof

Y. Lequain and A. Simis have shown, in a forthcoming article, that finitely generated projectives over $D[X_1, \ldots, X_n]$ are extended if D is a Prüfer domain.

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