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On the spectral radius of weighted trees with fixed diameter and weight set ☆

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A R T I C L E I N F O

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A B S T R A C T

The spectrum of weighted graphs are often used to solve the problems in the design of networks and electronic circuits. We first give some perturbational results on the spectral radius of weighted graphs when some weights of edges are modified, then we derive the weighted tree with the largest spectral radius in the set of all weighted trees with fixed diameter and weight set. Furthermore, an open problem of spectral radius on weighted paths is solved [H.Z. Yang, G.Z. Hu, Y. Hong, Bounds of spectral radii of weighted tree, Tsinghua Sci. Technol. 8 (2003) 517–520].

1. Introduction

In this paper, we only consider simple weighted graphs on positive weight set. Let G be a weighted graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, edge set $E(G) \neq \emptyset$ and weight set $W(G) = \{w_j > 0 : j = 1, 2, \ldots, |E(G)|\}$. The function $w_G : E(G) \to W(G)$ is called a weight function of G. It is obvious that each weighted graph corresponds to a weight function. The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = w_G(v_iv_j)$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$ otherwise. The characteristic polynomial of $A(G)$ is said to be the characteristic polynomial of G, denoted by $\phi(G, \lambda)$ or $\phi(G)$. Since $A(G)$ is a nonnegative symmetric matrix, its eigenvalues are all real numbers and its largest eigenvalue is a positive number. The largest eigenvalue of $A(G)$ is called the spectral radius of G, denoted by $\rho(G)$.

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Since $A(G)$ is nonnegative, there is a nonnegative eigenvector corresponding to $\rho(G)$. In particular, when G is connected, $A(G)$ is irreducible and by the Perron–Frobenius Theorem (e.g. [1]), $\rho(G)$ is simple and there is a unique positive unit eigenvector. We shall refer to such an eigenvector as the Perron vector of G. Let B^t denote the transpose of a matrix B. All other notations and definitions not given in the paper are standard terminology of graph theory (e.g. [2]).

Since graphs of the design of networks and electronic circuits are usually weighted, the spectrum of weighted graphs is often used to solve problems. On the other hand, graphs may be regarded as weighted graphs whose edges have weight 1. Therefore, it is significant and necessary to investigate the spectrum of weighted graphs. M. Fiedler had introduced the following question: What is the optimal distribution of nonnegative weights (with total sum 1) among the edges of a given graph, so that the spectral radius of the resulting matrix is minimum? He himself shown that the optimum solution is achieved and S. Poljak presented a polynomial time algorithm which finds such optimum solution [3]. Yang et al. had obtained an upper bound of spectral radius of weighted trees with fixed order and weight set [4] and proposed the following open problem: are there better bounds of spectral radius for all weighted paths with order n? Yuan and Shu had given the second largest value of spectral radius of weighted trees with fixed order and weight set [5]. Tan had determined an upper bound of spectral radius of weighted trees with fixed order, edge independence number and weight set [6].

The remainder of the paper is organized as follows. In Section 2 we will give some perturbational results on the spectral radius of weighted graphs when some weights of edges are modified. In Section 3 we will determine the weighted tree with the largest spectral radius in the set of all weighted trees with fixed diameter and weight set. Furthermore, an open problem of spectral radius on weighted paths proposed in [4] is solved.

2. Some perturbational results on spectral radius

Let G be a weighted graph with a positive weight set. Let $N_G(v)$ denote the set of vertices adjacent to the vertex v in G. For convenience, we define $w_G(uv) = 0$ if $uv \notin E(G)$. So G may be regarded as a weighted graph with a nonnegative weight set, where $uv \in E(G)$ if and only if $w_G(uv) > 0$.

Lemma 2.1 [1]. *Let A be a Hermitian matrix and let $\rho(A)$ be the largest eigenvalue of A. Then $\rho(A) = \max_{\|x\|=1, x\in R^n} x^tAx$, and $\rho(A) = x^tAx$ if x is a unit eigenvector corresponding to $\rho(A)$.*

Lemma 2.2 [7]. *Let A be a nonnegative symmetric matrix and x be a unit vector of R^n. If $\rho(A) = x^tAx$, then $Ax = \rho(A)x$.*

Any modification of a weighted graph gives rise to perturbations of its eigenvalues. In literature, this topic is mostly investigated for the largest eigenvalue of graphs. In the following we present some perturbational results on the spectral radius of weighted graphs, which are useful and more ordinary than those of graphs.

Theorem 2.3. *Let a, b, u, v be four vertices of a weighted graph G and let $x = (x_1, x_2, \dots, x_n)^t$ be a nonnegative unit eigenvector corresponding to $\rho(G)$, where x_i corresponds to the vertex v_i of G. For $0 < \delta \leqslant w_G(uv)$, let G^1 be the weighted graph obtained from G such that*

$$w_{G^1}(uv) = w_G(uv) - \delta, \quad w_{G^1}(ab) = w_G(ab) + \delta,$$
$$w_{G^1}(e) = w_G(e), \quad e \in E(G) - \{ab, uv\}.$$

If $x_ux_v \leqslant x_ax_b$, then $\rho(G) \leqslant \rho\left(G^1\right)$. In addition, if $x_ux_v < x_ax_b$ or $x_ux_v \leqslant x_ax_b$ and G is connected, then $\rho(G) < \rho\left(G^1\right)$.

Proof. From Lemma 2.1, we have that

$$\begin{aligned}\rho\left(G^1\right) - \rho(G) &= \max_{\|y\|=1} y^tA\left(G^1\right)y - x^tA(G)x \geqslant x^t\left(A\left(G^1\right) - A(G)\right)x \\ &= 2\delta(x_ax_b - x_ux_v) \geqslant 0. \qquad (1)\end{aligned}$$

Suppose that $x_ux_v < x_ax_b$. By Eq. (1), it is obvious that $\rho(G) < \rho\left(G^1\right)$.

Suppose that $x_ux_v \leqslant x_ax_b$ and G is connected. By the Perron–Frobenius Theorem, x is a positive unit eigenvector. Assume that $\rho(G) = \rho\left(G^1\right)$. Then from Eq. (1), we get that $\rho\left(G^1\right) = x^tA\left(G^1\right)x$. Again by Lemma 2.2, we obtain $A\left(G^1\right)x = \rho\left(G^1\right)x$. Without loss of generality, assume that $a \notin \{u, v\}$. Then

$$\begin{aligned}\rho\left(G^1\right)x_a &= w_{G^1}(ab)x_b + \sum_{z\in N_{G^1}(a)-\{b\}} w_{G^1}(za)x_z \\ &= \delta x_b + w_G(ab)x_b + \sum_{z\in N_G(a)-\{b\}} w_G(za)x_z \\ &= \delta x_b + \sum_{z\in N_G(a)} w_G(za)x_z.\end{aligned}$$

Also from $A(G)x = \rho(G)x$, we have that $\rho(G)x_a = \sum_{z\in N_G(a)} w_G(za)x_z$. So we have $\left(\rho\left(G^1\right) - \rho(G)\right)x_a = \delta x_b = 0$. This implies that $x_b = 0$, a contradiction with $x_b > 0$. Therefore, $\rho(G) < \rho\left(G^1\right)$. □

Corollary 2.4. *Let G and G^1 be the two weighted graphs defined in Theorem 2.3. Let $x = (x_1, x_2, \ldots, x_n)^t$ and $x^1 = (x_1^1, x_2^1, \ldots, x_n^1)^t$ be two nonnegative unit eigenvectors corresponding to $\rho(G)$ and $\rho(G^1)$, where x_i and x_i^1 correspond to the vertex v_i of G and G^1, respectively. If $x_ux_v \leqslant x_ax_b$, then $x_u^1x_v^1 \leqslant x_a^1x_b^1$. In addition, if $x_ux_v < x_ax_b$ or $x_ux_v \leqslant x_ax_b$ and G is connected, then $x_u^1x_v^1 < x_a^1x_b^1$.*

Proof. We first prove that $x_u^1x_v^1 \leqslant x_a^1x_b^1$. Assume $x_u^1x_v^1 > x_a^1x_b^1$. It is easy to see that G can be obtained from G^1 in the following way:

$$\begin{aligned}&w_G(uv) = w_{G^1}(uv) + \delta, \quad w_G(ab) = w_{G^1}(ab) - \delta,\\ &w_G(e) = w_{G^1}(e), \quad e \in E\left(G^1\right) - \{ab, uv\}.\end{aligned}$$

Since $x_a^1x_b^1 < x_u^1x_v^1$, by the additional claim of Theorem 2.3, we have $\rho\left(G^1\right) < \rho(G)$. On the other hand, since $x_ux_v \leqslant x_ax_b$, again by Theorem 2.3, we get that $\rho(G) \leqslant \rho\left(G^1\right)$, a contradiction. Therefore, $x_u^1x_v^1 \leqslant x_a^1x_b^1$.

We next prove the additional claim. Assume $x_u^1x_v^1 \geqslant x_a^1x_b^1$. On the one hand, since $x_ux_v < x_ax_b$ or $x_ux_v \leqslant x_ax_b$ and G is connected, by the additional claim of Theorem 2.3, we have that $\rho(G) < \rho\left(G^1\right)$. On the other hand, since $x_a^1x_b^1 \leqslant x_u^1x_v^1$, again by Theorem 2.3, we get that $\rho\left(G^1\right) \leqslant \rho(G)$, a contradiction. Therefore, $x_u^1x_v^1 < x_a^1x_b^1$. □

Corollary 2.5 [5]. *Let u, v be two distinct vertices of a connected weighted graph G and let $u_1, u_2, \ldots, u_s (u_i \neq v, s \neq 0)$ be some vertices of $N_G(u) - N_G(v)$. Let $x = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector of G, where x_i corresponds to the vertex v_i of G. Let G' be the weighted graph obtained from G by deleting the edges uu_j and adding the edges vu_j such that*

$$w_{G'}(vu_j) = w_G(uu_j), \quad w_{G'}(e) = w_G(e), \quad e \neq uu_j, \ j = 1, 2, \ldots, s.$$

If $x_v \geqslant x_u$, then $\rho(G) < \rho\left(G'\right)$.

Proof. Put $H_0 = G$. For $j = 1, 2, \ldots, s$, let H_j be the weighted graph obtained from H_{j-1} by deleting the edge uu_j and adding the edge vu_j such that

$$w_{H_j}(vu_j) = w_{H_{j-1}}(uu_j), \quad w_{H_j}(e) = w_{H_{j-1}}(e), \quad e \in E(H_{j-1}) - \{uu_j\}.$$

Since $w_{H_{j-1}}(uu_j) > w_{H_{j-1}}(vu_j) = 0$ for $j = 1, 2, \ldots, s$, set $\delta_j = w_{H_{j-1}}(uu_j)$, then H_j can be obtained from H_{j-1} in the following way:

$$w_{H_j}(vu_j) = w_{H_{j-1}}(vu_j) + \delta_j, \quad w_{H_j}(uu_j) = w_{H_{j-1}}(uu_j) - \delta_j,$$
$$w_{H_j}(e) = w_{H_{j-1}}(e), \quad e \in E(H_{j-1}) - \{vu_j, uu_j\}.$$

Let $x^j = \left(x_1^j, x_2^j, \ldots, x_n^j\right)^t$ be a nonnegative unit eigenvector corresponding to $\rho(H_j)$, where $x^0 = x$ and x_i^j corresponds to the vertex v_i of H_j. Then $x_{u_1}^0 x_v^0 \geqslant x_{u_1}^0 x_u^0$, and by the additional claim of Corollary 2.4, we have that

$$x_{u_{j+1}}^j x_v^j > x_{u_{j+1}}^j x_u^j, \quad j = 1, 2, \ldots, s-1.$$

Since $H_s = G'$, by the additional claim of Theorem 2.3, we get that

$$\rho(G) = \rho(H_0) < \rho(H_1) < \cdots < \rho(H_s) = \rho\left(G'\right). \quad \square$$

Corollary 2.6. *Let G be a connected weighted graph with total weight sum c. Then $\rho(G) \leqslant c$, with equality if and only if G is a star with two vertices and weight c.*

Proof. Let $x^H = \left(x_1^H, x_2^H, \ldots, x_n^H\right)^t$ denote a nonnegative unit eigenvector corresponding to $\rho(H)$ of a weighted graph H, where x_i^H corresponds to the vertex v_i of H. Let ab and uv be two distinct edges of G. Without loss of generality, assume $x_a^G x_b^G \geqslant x_u^G x_v^G$. Let G_1 be the weighted graph obtained from G such that

$$w_{G_1}(ab) = w_G(ab) + w_G(uv), \quad w_{G_1}(uv) = 0,$$
$$w_{G_1}(e) = w_G(e), \quad e \neq ab, uv.$$

By the additional claim of Theorem 2.3, we have that $\rho(G) < \rho(G_1)$. Again let pq and gh be two distinct edges of G_1. Without loss of generality, assume $x_p^{G_1} x_q^{G_1} \geqslant x_g^{G_1} x_h^{G_1}$. Let G_2 be the weighted graph obtained from G_1 such that

$$w_{G_2}(pq) = w_{G_1}(pq) + w_{G_1}(gh), \quad w_{G_2}(gh) = 0,$$
$$w_{G_2}(e) = w_{G_1}(e), \quad e \neq pq, gh.$$

By Theorem 2.3 (Note that G_1 may not be connected), we get $\rho(G_1) \leqslant \rho(G_2)$. To G_2, repeat the above procedure until we arrive at a weighted graph G_s with a unique edge. So we get weighted graphs $G, G_1, \ldots, G_s$ such that they have the weight sum c and

$$\rho(G) < \rho(G_1) \leqslant \cdots \leqslant \rho(G_s) = c.$$

It is obvious that $\rho(G) = c$ if and only if G only has an edge, i.e., G is a star with two vertices and weight c. $\square$

Theorem 2.7. *Let a, b, u, v be four distinct vertices of a connected weighted graph G and let $x = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector of G, where x_i corresponds to the vertex v_i of G. For $0 < \delta \leqslant w_G(uv)$ and $0 < \theta \leqslant w_G(ab)$, let G^2 be the weighted graph obtained from G such that*

$$w_{G^2}(uv) = w_G(uv) - \delta, \quad w_{G^2}(ub) = w_G(ub) + \delta, \quad w_{G^2}(ab) = w_G(ab) - \theta,$$
$$w_{G^2}(av) = w_G(av) + \theta, \quad w_{G^2}(e) = w_G(e), \quad e \in E(G) - \{ab, uv, ub, av\}.$$

If $(x_b - x_v)(\delta x_u - \theta x_a) \geqslant 0$, then $\rho(G) \leqslant \rho\left(G^2\right)$, and with equality $\rho(G) = \rho\left(G^2\right)$ if and only if $x_b = x_v$ and $\delta x_u = \theta x_a$.

Proof. From Lemma 2.1, we have that

$$\rho\left(G^2\right) - \rho(G) = \max_{\|y\|=1} y^t A\left(G^2\right) y - x^t A(G) x \geqslant x^t \left(A\left(G^2\right) - A(G)\right) x$$
$$= 2(x_b - x_v)(\delta x_u - \theta x_a) \geqslant 0. \quad (2)$$

Assume $\rho(G) = \rho\left(G^2\right)$. By Eq. (2), we have that $\rho\left(G^2\right) = x^t A\left(G^2\right) x$. Again from Lemma 2.2, we have that $A\left(G^2\right) x = \rho\left(G^2\right) x$. Thus

$$\begin{aligned}\rho\left(G^2\right) x_u &= w_{G^2}(uv)x_v + w_{G^2}(ub)x_b + \sum_{z \in N_{G^2}(u)-\{v,b\}} w_{G^2}(zu)x_z \\ &= (w_G(uv) - \delta)x_v + (w_G(ub) + \delta)x_b + \sum_{z \in N_G(u)-\{v,b\}} w_G(zu)x_z \\ &= \delta(x_b - x_v) + \sum_{z \in N_G(u)} w_G(zu)x_z \\ &= \delta(x_b - x_v) + \rho(G)x_u.\end{aligned}$$

So we obtain $x_b = x_v$. In the similar way, we can get that

$$\rho\left(G^2\right) x_v = -(\delta x_u - \theta x_a) + \sum_{z \in N_G(v)} w_G(zv)x_z = -(\delta x_u - \theta x_a) + \rho(G)x_v.$$

Therefore, we have $\delta x_u = \theta x_a$.

Assume $x_b = x_v$ and $\delta x_u = \theta x_a$. In the similar procedure above, we easily get that

$$\begin{aligned}\sum_{z \in N_{G^2}(u)} w_{G^2}(zu)x_z &= \delta(x_b - x_v) + \sum_{z \in N_G(u)} w_G(zu)x_z = \rho(G)x_u, \\ \sum_{z \in N_{G^2}(a)} w_{G^2}(za)x_z &= \theta(x_v - x_b) + \sum_{z \in N_G(a)} w_G(za)x_z = \rho(G)x_a, \\ \sum_{z \in N_{G^2}(v)} w_{G^2}(zv)x_z &= \theta x_a - \delta x_u + \sum_{z \in N_G(v)} w_G(zv)x_z = \rho(G)x_v, \\ \sum_{z \in N_{G^2}(b)} w_{G^2}(zb)x_z &= \delta x_u - \theta x_a + \sum_{z \in N_G(b)} w_G(zb)x_z = \rho(G)x_b.\end{aligned}$$

It is obvious that, for $p \in V(G) - \{a, b, u, v\}$, we have that

$$\sum_{z \in N_{G^2}(p)} w_{G^2}(zp)x_z = \sum_{z \in N_G(p)} w_G(zp)x_z = \rho(G)x_p.$$

Thus $A\left(G^2\right) x = \rho(G)x$. Since x is the Perron vector, by the Perron–Frobenius Theorem, we have $\rho\left(G^2\right) = \rho(G)$. □

Remark 1. Theorems 2.3 and 2.7 are the main tools used in Section 3. There we will apply them and the idea from [8,9] to determine the weighted tree with the largest spectral radius in the set of all weighted trees with fixed diameter and weight set.

3. On the weighted trees with the largest spectral radius

Let G be a weighted graph. The weight of vertex v of G, denoted by $w_G(v)$, is the weight sum of edges incident to v in G. The distance of vertex subsets A and B of G is the minimum number in $\{d(a, b) : a \in A, b \in B\}$, where $d(a, b)$ is the distance of vertices a and b. Let H be another weighted graph. G and H are called isomorphic, denoted by $G = H$, if there exists a bijection f from $V(G)$ to $V(H)$ such that $ab \in E(G)$ if and only if $f(a)f(b) \in E(H)$, and $w_G(ab) = w_H(f(a)f(b))$ for each $ab \in E(G)$. If H is a subgraph of G and $w_H(e) = w_G(e)$ for each $e \in E(H)$, then H is called a weighted subgraph of G. If H is a weighted subgraph of G, $V(H) = V(G)$ and $E(H) \neq E(G)$, then H is called a weighted proper spanning subgraph of G.

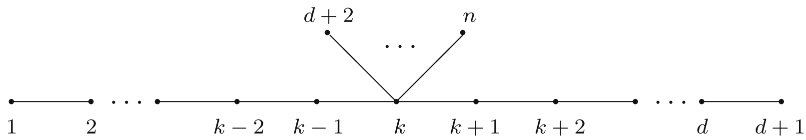

Fig. 1. The tree $\widetilde{T}_k$.

Let $\Gamma(d, m_1, m_2, \ldots, m_{n-1})$ denote the set of all weighted trees with n vertices, diameter d and positive weight set $\{m_1, m_2, \ldots, m_{n-1}\}$. Let $\Theta(n, d)$ be the subset of $\Gamma(d, m_1, m_2, \ldots, m_{n-1})$ such that each of $\Theta(n, d)$ can be obtained from some weighted path P_{d+1} by adding $n - d - 1$ pendent weighted edges at some non-pendent vertex of P_{d+1}.

Lemma 3.1. *Let $T \in \Gamma(d, m_1, m_2, \ldots, m_{n-1}) - \Theta(n, d)$. Then there exists a weighted tree $\widetilde{T} \in \Theta(n, d)$ such that $\rho(T) < \rho(\widetilde{T})$.*

Proof. Let $P_{d+1} = v_1 v_2 \cdots v_d v_{d+1}$ be a path in T. Then T can be obtained from P_{d+1} by attaching a proper weighted tree to the vertex v_i for each $i = 2, 3, \ldots, d$. Let $(x_1, x_2, \ldots, x_n)^t$ be the Perron vector of T, where x_i corresponds to the vertex v_i of T.

Case 1: Suppose that T is a Caterpillar.

Since $T \notin \Theta(n, d)$, there are two vertices v_i and $v_j (2 \leqslant i \neq j \leqslant d)$ of degree greater than 2. Without loss of generality, assume $x_{v_i} \geqslant x_{v_j}$. Let $v_{j-1}, v_{j+1}, z_1, z_2, \ldots, z_s$ be all adjacent vertices of v_j in T. Let T' be the weighted tree obtained from T by deleting the edges $v_j z_k$ and adding the edges $v_i z_k$ such that

$$w_{T'}(v_i z_k) = w_T(v_j z_k), \quad w_{T'}(e) = w_T(e), \quad e \neq v_j z_k, \quad k = 1, 2, \ldots, s.$$

By Corollary 2.5, we have that $\rho(T) < \rho\left(T'\right)$. To T' repeat the above procedure until we arrive at a weighted tree $\widetilde{T} \in \Theta(n, d)$. So by Corollary 2.5, we get that

$$\rho(T) < \rho\left(T'\right) < \cdots < \rho(\widetilde{T}).$$

Case 2: Suppose that T is not a Caterpillar.

It is obvious that T has at least a non-pendent edge uv not in P_{d+1}. Without loss of generality, assume $x_v \geqslant x_u$. Let $N_T(u) = \{v, u_1, u_2, \ldots, u_s\}$ and let T_1 be the weighted tree obtained from T by deleting the edges uu_j and adding the edges vu_j such that

$$w_{T_1}(vu_j) = w_T(uu_j), \quad w_{T_1}(e) = w_T(e), \quad e \neq uu_j, \ j = 1, 2, \cdots, s.$$

By Corollary 2.5, we have $\rho(T) < \rho(T_1)$. To T_1 repeat the above procedure until we arrive at a weighted tree T_l such that each edge of T_l not in P_{d+1} is a pendent weighted edge, i.e., T_l is a caterpillar tree. So by Corollary 2.5 and Case 1, the result holds. □

Let $\widetilde{T}_k$ be the tree shown in Fig. 1. Let T_M be the weighted tree in $\Theta(n, d)$ with the largest spectral radius and let P_M be a path of length d in T_M. Then there are a $k (2 \leqslant k \leqslant d)$ and a disposition of weights in all edges of $\widetilde{T}_k$ such that $T_M = \widetilde{T}_k$. Let $x = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector of T_M, where x_i corresponds to the vertex i of T_M. Without loss of generality, we also assume $d \geqslant 3$ (otherwise, the problem is trivial). Next we will investigate some spectral and structural properties of T_M. For convenience, write $a_1 = d + 2, a_2 = d + 3, \ldots, a_s = n$, and without loss of generality, assume $k \leqslant \frac{d+2}{2}$.

Lemma 3.2. *Let ab, uv be two distinct edges of P_M.*

(1) *If $x_a x_b \geqslant x_u x_v$, then $w_{T_M}(ab) \geqslant w_{T_M}(uv)$.*
(2) *If $w_{T_M}(ab) > w_{T_M}(uv)$, then $x_a x_b > x_u x_v$.*
(3) *If $x_a x_b = x_u x_v$, then $w_{T_M}(ab) = w_{T_M}(uv)$.*

Proof. Note that (2) and (3) can be immediately deduced from (1). So we only give the proof of (1). Assume $w_{T_M}(ab) < w_{T_M}(uv)$. Put $\delta = w_{T_M}(uv) - w_{T_M}(ab)$ and let T' be the weighted tree obtained from T_M such that

$$w_{T'}(ab) = w_{T_M}(ab) + \delta, \quad w_{T'}(uv) = w_{T_M}(uv) - \delta, \quad w_{T'}(e) = w_{T_M}(e), \quad e \neq ab, uv,$$

i.e., T' is the weighted tree obtained from T_M by exchanging the weights of edges ab and uv while making the weights of other edges not changed. Then $T' \in \Theta(n,d)$, and by the additional claim of Theorem 2.3, $\rho(T') > \rho(T_M)$, a contradiction with the assumption of T_M. □

Lemma 3.3. *Let a, b, u, v be four distinct vertices of P_M from the left to the right and ab, uv be two edges of P_M. Then*
$(x_u - x_b)(w_{T_M}(ab)x_a - w_{T_M}(uv)x_v) \leqslant 0$, *and* $x_b = x_u$ *if and only if* $w_{T_M}(ab)x_a = w_{T_M}(uv)x_v$.
$(x_v - x_a)(w_{T_M}(ab)x_b - w_{T_M}(uv)x_u) \leqslant 0$, *and* $x_a = x_v$ *if and only if* $w_{T_M}(ab)x_b = w_{T_M}(uv)x_u$.

Proof. We only give the proof of the first result. Assume the contrary, that is

$$(x_u - x_b)(w_{T_M}(ab)x_a - w_{T_M}(uv)x_v) > 0,$$

or that only one between $x_b = x_u$ and $w_{T_M}(ab)x_a = w_{T_M}(uv)x_v$ holds. Take $\delta = w_{T_M}(ab)$ and $\theta = w_{T_M}(uv)$. Let T' be the weighted tree obtained from T_M such that

$$w_{T'}(ab) = w_{T_M}(ab) - \delta, \quad w_{T'}(au) = w_{T_M}(au) + \delta, \quad w_{T'}(uv) = w_{T_M}(uv) - \theta,$$
$$w_{T'}(vb) = w_{T_M}(vb) + \theta, \quad w_{T'}(e) = w_{T_M}(e), \quad e \neq ab, uv,$$

i.e., T' is the weighted tree obtained from T_M by deleting the edges ab, uv and adding the edges au, vb such that $w_{T'}(au) = w_{T_M}(ab), w_{T'}(vb) = w_{T_M}(uv)$. Then $T' \in \Theta(n,d)$, and by Theorem 2.7, $\rho(T') > \rho(T_M)$, a contradiction with the assumption of T_M. □

Lemma 3.4. *Let p, q be two distinct vertices of P_M.*

(1) *If $w_{T_M}(p) > w_{T_M}(q)$, then $x_p > x_q$.*
(2) *If $x_p > x_q$, then $w_{T_M}(p) \geqslant w_{T_M}(q)$.*
(3) *If $x_p = x_q$, then $w_{T_M}(p) = w_{T_M}(q)$.*

Proof. We only prove (1). Suppose that $x_p \leqslant x_q$, and without loss of generality, assume $p < q$. We will get a contradiction by distinguishing the following three cases.

Case 1: Assume $p = 1$.

If $q = 2$, then 1 and 3 are the two adjacent vertices of q in P_M. Thus we have that $w_{T_M}(p) = w_{T_M}(12) < w_{T_M}(12) + w_{T_M}(23) \leqslant w_{T_M}(q)$, a contradiction.

If $q = 3$, then 2 and 4 are the two adjacent vertices of q in P_M. Since $w_{T_M}(p2) = w_{T_M}(p) > w_{T_M}(q) > w_{T_M}(2q)$, by Lemma 3.2(2), we have $x_p x_2 > x_2 x_q$. This implies that $x_p > x_q$, a contradiction.

If $q \geqslant 4$, then we consider the distinct vertices $a = p, b = p+1, u = q-1, v = q$. From $w_{T_M}(ab) > w_{T_M}(uv)$ and Lemma 3.2(2), we have $x_a x_b > x_u x_v$. Hence $x_b > x_u$. Also from Lemma 3.3, we have $(x_v - x_a)(w_{T_M}(ab)x_b - w_{T_M}(uv)x_u) \leqslant 0$. Therefore, combining $x_a \leqslant x_v$ and $w_{T_M}(ab) > w_{T_M}(uv)$, we get $x_b < x_u$, a contradiction.

Case 2: Assume $2 \leqslant p < q < d+1$.

Case 2.1: Assume $p = k$ and $n > d+1$.

Let T' be the weighted tree obtained from T_M by deleting the edges $pa_1, pa_2, \ldots, pa_s$ and adding the edges $qa_1, qa_2, \ldots, qa_s$ such that

$$w_{T'}(qa_i) = w_{T_M}(pa_i), \quad w_{T'}(e) = w_{T_M}(e), \quad e \neq pa_i, \ i = 1, 2, \ldots, s.$$

By Corollary 2.5, we have $T' \in \Theta(n,d)$ and $\rho(T') > \rho(T_M)$, a contradiction.

Case 2.2: Assume $p \neq k$ or $p = k$ and $n = d + 1$. Set $a = p - 1, b = p + 1, u = q - 1, v = q + 1$.

First assume $w_{T_M}(ap) > w_{T_M}(qv)$. By Lemma 3.2(2), we have $x_a x_p > x_q x_v$. Combining $x_p \leqslant x_q$, we get $x_a > x_v$. Thus

$$(x_q - x_p)(w_{T_M}(ap)x_a - w_{T_M}(qv)x_v) \geqslant 0, \quad w_{T_M}(ap)x_a > w_{T_M}(qv)x_v.$$

These contradict to the first results of Lemma 3.3.

Next assume $w_{T_M}(ap) \leqslant w_{T_M}(qv)$. If $q - p = 1$, i.e., that p is adjacent to q, then

$$w_{T_M}(p) = w_{T_M}(ap) + w_{T_M}(pq) \leqslant w_{T_M}(qv) + w_{T_M}(pq) \leqslant w_{T_M}(q),$$

a contradiction. Now assume $q - p \geqslant 2$. Then $w_{T_M}(pb) > w_{T_M}(uq)$. By Lemma 3.2(2), we have that $x_p x_b > x_u x_q$. If $q - p = 2$, then $b = u$. Therefore, we have that $x_p > x_q$, a contradiction. If $q - p \geqslant 3$, then $b \neq u$. Combining $x_p \leqslant x_q$, we get $x_b > x_u$. Thus

$$(x_q - x_p)(w_{T_M}(pb)x_b - w_{T_M}(uq)x_u) \geqslant 0, \quad w_{T_M}(pb)x_b > w_{T_M}(uq)x_u.$$

These contradict to the second results of Lemma 3.3.

Case 3: Assume $2 \leqslant p < q = d + 1$. Set $a_0 = p - 1$.

When $p \neq k$ or $p = k$ and $n = d + 1$, let T' be the weighted tree obtained from T_M by deleting the edge pa_0 and adding the edge qa_0 such that

$$w_{T'}(qa_0) = w_{T_M}(pa_0), \quad w_{T'}(e) = w_{T_M}(e), \quad e \in E(T_M) - \{pa_0\}.$$

While $p = k$ and $n > d + 1$, let T' be the weighted tree obtained from T_M by deleting the edges $pa_0, pa_1, \ldots, pa_s$ and adding the edges $qa_0, qa_1, \ldots, qa_s$ such that

$$w_{T'}(qa_i) = w_{T_M}(pa_i), \quad w_{T'}(e) = w_{T_M}(e), \quad e \neq pa_i, \ i = 0, 1, \ldots, s.$$

By Corollary 2.5, we have $T' \in \Theta(n, d)$ and $\rho(T') > \rho(T_M)$, a contradiction. □

Lemma 3.5. *Suppose that the vertices of P_M are relabeled by $v_1, \ldots, v_d, v_{d+1}$ so that $x_{v_1} \geqslant \cdots \geqslant x_{v_d} \geqslant x_{v_{d+1}}$. Then $\{v_d, v_{d+1}\} = \{1, d + 1\}$ and $v_1 = k$ if $n > d + 1$.*

Proof. First, suppose that $\{v_d, v_{d+1}\} \neq \{1, d + 1\}$. There exists a $l (l \leqslant d - 1)$ such that $1 = v_l$ or $d + 1 = v_l$. Without loss generality, assume $d + 1 = v_l$. Then there is a $p (2 \leqslant p \leqslant d)$ such that $p = v_d$ or $p = v_{d+1}$. Set $q = d + 1$. Then $2 \leqslant p < q = d + 1$ and $x_p \leqslant x_q$. In the similar way to Case 3 of Lemma 3.4, we will get a contradiction.

Next, let $n > d + 1$ and assume that $v_1 \neq k$. Then there exists a $p (p \neq 1, d + 1, k)$ such that $v_1 = p$. Let T' be the weighted tree obtained from T_M by deleting the edges $ka_1, ka_2, \ldots, ka_s$ and adding the edges $pa_1, pa_2, \ldots, pa_s$ such that

$$w_{T'}(pa_i) = w_{T_M}(ka_i), \quad w_{T'}(e) = w_{T_M}(e), \quad e \neq ka_i, \ i = 1, 2, \ldots, s.$$

Then $T' \in \Theta(n, d)$, and from $x_k \leqslant x_p$ and Corollary 2.5, we have that $\rho(T') > \rho(T_M)$, a contradiction. □

Let a, b be two vertices of P_M. An interval $[a, b]$ in P_M is the set of vertices of P_M between a and b, including a and b. In particular, $[a, a] = \{a\}$. Let $\tilde{w}_1 > \tilde{w}_2 > \cdots > \tilde{w}_l$ be the distinct weights of vertices from P_M in T_M. Set

$$V_i = \{j : w_{T_M}(j) = \tilde{w}_i, j = 1, 2, \ldots, d + 1\}, \widetilde{V}_i = \bigcup_{j \leqslant i} V_j, \quad i = 1, 2, \ldots, l.$$

Let $P = 12 \cdots s$ be a path of a weighted tree T. Write $c(P) = \left\{\frac{s}{2}, \frac{s+2}{2}\right\}$ if s is even, and $c(P) = \left\{\frac{s+1}{2}\right\}$ otherwise. We call $c(P)$ the center of P. Let e_i denote the edge $i(i + 1)$ of P, namely $e_i = i(i + 1)$ for $i = 1, 2, \ldots, s - 1$. If for each $i \left(1 \leqslant i \leqslant \frac{s-1}{2}\right)$, $w_T(e_i) = w_T(e_{s-i})$, i.e., any two symmetric edges of P with respect to its center have the same weights, then P is called symmetric in edge weights. When $s = 2r + 1$, if

$$w_T(e_r) \geqslant w_T(e_{r+1}) \geqslant w_T(e_{r-1}) \geqslant w_T(e_{r+2}) \geqslant \cdots \geqslant w_T(e_1) \geqslant w_T(e_{2r}),$$

and when $s = 2r$, if

$$w_T(e_r) \geqslant w_T(e_{r-1}) \geqslant w_T(e_{r+1}) \geqslant w_T(e_{r-2}) \geqslant w_T(e_{r+2}) \geqslant \cdots \geqslant w_T(e_1) \geqslant w_T(e_{2r-1}),$$

then P (any weighted path isomorphic to it) is called an alternating weighted path in edge weights.

Lemma 3.6. *If at least two components of x corresponding to the vertices of P_M are equal, then P_M is symmetric in edge weights.*

Proof. Let r and t be two vertices of P_M with $x_r = x_t$. Without loss of generality, let $r < t$. We will prove $x_{r+1} = x_{t-1}$. Assume the contrary, then, without loss of generality, $x_{r+1} > x_{t-1}$. Let $a = r, b = r+1, u = t-1, v = t$. Then $x_a x_b > x_u x_v$. By Lemma 3.2(1), we have that $w_{T_M}(ab) \geqslant w_{T_M}(uv)$. If $w_{T_M}(ab) = w_{T_M}(uv)$, then $x_b > x_u$ and $w_{T_M}(ab)x_a = w_{T_M}(uv)x_v$. If $w_{T_M}(ab) > w_{T_M}(uv)$, then $x_a = x_v$ and $w_{T_M}(ab)x_b > w_{T_M}(uv)x_u$. The above results contradict with Lemma 3.3. Therefore, $x_{r+1} = x_{t-1}$. By proceeding in this way, we can show that $x_{r+i} = x_{t-i}$ for each $1 \leqslant i \leqslant \frac{t-r}{2}$, and in the similar way we can also show that $x_{r-i} = x_{t+i}$ for each $1 \leqslant i \leqslant \min\{r-1, d+1-t\}$.

Assume $r-1 \neq d+1-t$, and without loss of generality, assume $r-1 > d+1-t$. Set $p = r+t-d-1, q = d+1$. Then $2 \leqslant p < q = d+1$ and $x_p = x_q$. In the similar way to Case 3 of Lemma 3.4, we will get a contradiction. Therefore, $r-1 = d+1-t$.

For each $1 \leqslant i < \frac{d+2}{2}$, let $a = i, b = i+1, u = d+1-i, v = d+2-i$, i.e., that ab and uv are two symmetric edges of P_M with respect to its center. By the above result, we have $x_a = x_v, x_b = x_u$. So $x_a x_b = x_u x_v$. From Lemma 3.2(3), we have $w_{T_M}(ab) = w_{T_M}(uv)$. This indicates that P_M is symmetric in edge weights. □

Lemma 3.7. *P_M is an alternating weighted path in edge weights.*

Proof. Relabel the vertices of P_M by $v_1, v_2, \ldots, v_{d+1}$ so that $x_{v_1} \geqslant x_{v_2} \geqslant \cdots \geqslant x_{v_{d+1}}$. By Lemma 3.5, we have that $v_1 = k, 2 \leqslant k \leqslant d$. We will distinguish the following two cases depending on $x_{v_1}, x_{v_2}, \ldots, x_{v_{d+1}}$.

Case 1: Assume $x_{v_1} > x_{v_2} > \cdots > x_{v_{d+1}}$.

Let $v_2 = i$, and assume $i \notin \{k-1, k+1\}$. If $i < k$, we consider the distinct vertices: $a = i, b = i+1, u = k, v = k+1$. If $i > k$, we consider the distinct vertices: $a = i, b = i-1, v = k-1, u = k$. By Lemma 3.3, we have that

$$(x_u - x_b)(w_{T_M}(ab)x_a - w_{T_M}(uv)x_v) \leqslant 0, \tag{3}$$

$$(x_v - x_a)(w_{T_M}(ab)x_b - w_{T_M}(uv)x_u) \leqslant 0. \tag{4}$$

Note that

$$x_a > x_v, \quad x_u > x_b. \tag{5}$$

So from Eqs. (3)–(5), we get that

$$w_{T_M}(ab) \leqslant w_{T_M}(uv) \cdot \frac{x_v}{x_a} < w_{T_M}(uv)$$

and

$$w_{T_M}(ab) \geqslant w_{T_M}(uv) \cdot \frac{x_u}{x_b} > w_{T_M}(uv),$$

a contradiction. Thus $i = k-1$ or $i = k+1$, i.e., that $v_2 = k-1$ or $v_2 = k+1$.

Case 1.1: Assume $v_2 = k+1$.

Set $S_1 = \tilde{e}_1 = k(k+1)$. Let $v_3 = i$, and assume $i \neq k-1$. If $i < k$, we consider the four distinct vertices: $a = i, b = i+1, u = k+1, v = k+2$. If $i > k+1$, we consider the four distinct vertices: $a = i, b = i-1, v = k-1, u = k$. Then Eqs. (3)–(5) hold. So we will get a contradiction. Therefore, $i = k-1$. So $v_3 = k-1$, and we now put $S_2 = \tilde{e}_2 S_1 (= \tilde{e}_2\tilde{e}_1 = (k-1)k(k+1))$.

Next let $v_4 = i$, and assume $i \neq k+2$. If $i < k-1$, we consider the four distinct vertices: $a = i, b = i+1, u = k+1, v = k+2$. If $i > k+1$, we consider the four distinct vertices: $a = i, b = i-1, v = k-2, u = k-1$. Then Eqs. (3)–(5) also hold. So we again get a contradiction. Therefore, $i = k+2$. So $v_4 = k+2$, and we now put $S_3 = S_2\tilde{e}_3 (= \tilde{e}_2\tilde{e}_1\tilde{e}_3 = (k-1)k(k+1)(k+2))$.

Suppose that the edges $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{l-1} (4 \leqslant l \leqslant d-1)$ are already placed in the string S_{l-1} and assume that $S_{l-1} = p \cdots (k-1)k(k+1) \cdots q$. Then at least two vertices of P_M are not in S_{l-1}. By Lemma 3.5, we have that $p \neq 1$ and $q \neq d+1$. Next, let $v_{l+1} = i$, and assume that l is odd (for l even the proof is quite analogous). Then $v_l = p$ and $x_p < x_q$. Assume that $i \neq q+1$. If $i < p$, we consider the four distinct vertices: $a = i, b = i+1, u = q, v = q+1$. If $i > q$, we consider the four distinct vertices: $a = i, b = i-1, v = p-1, u = p$. Then Eqs. (3)–(5) hold. These will yield a contradiction. Thus we have $i = q+1$. So $v_{l+1} = q+1$, and we now put $S_l = S_{l-1}\tilde{e}_l = p \cdots (k-1)k(k+1) \cdots q(q+1)$. Repeat the above procedure until we get S_r such that $1 \in S_r$ (where, when $v_1, v_2, \ldots, v_d \in S_{r-1}$, directly let $\tilde{e}_d = v_{d+1}2 = 12$ and $S_r = \tilde{e}_d S_{r-1}$). Since $1 \in S_r$, by Lemma 3.5, we must have $v_d \in S_r$. If S_r has included all vertices of P_M, then let $S = S_r$ (In this case, d is even and $k = \frac{d+2}{2}$). Otherwise, $v_{d+1} = d+1 \notin S_r$, let $\tilde{e}_d = d(d+1)$ and let $S = S_r\tilde{e}_d$ (In this case, d is odd and $k = \frac{d+1}{2}$).

The sequence of edges in S forms the path P_M. Let $\tilde{e}_s = ab$ and $\tilde{e}_t = uv$ be two edges in S, and assume $s < t$. By the structure of S, we have $x_a x_b > x_u x_v$. Again by Lemma 3.2(1), we have $w_{T_M}(\tilde{e}_s) \geqslant w_{T_M}(\tilde{e}_t)$. This indicates that P_M is an alternating weighted path in edge weights.

Case 1.2: Assume $v_2 = k-1$.

If $k = 2$, then $v_2 = 1$. From Lemma 3.5, we get $d = 1$ or $d = 2$, a contradiction with $d \geqslant 3$. Therefore, $k \geqslant 3$. The rest of this proof is similar to Case 1.1.

Case 2: At least two of $x_{v_1}, x_{v_2}, \ldots, x_{v_{d+1}}$ are equal.

We will show that each $\widetilde{V}_i$ is an interval for $i = 1, 2, \ldots, l-1$. Assume the contrary, and let s be the smallest number such that $\widetilde{V}_s$ is not an interval. Then there are two subintervals of $\widetilde{V}_s$, say U and V, whose distance is at least 2. Let $a \in U$ and $b \notin \widetilde{V}_s$ be the vertices of P_M chosen so that a is adjacent to b and a is on the right side of U. If $d+1 \in \widetilde{V}_s$, then $w_{T_M}(d+1) > w_{T_M}(b)$. By Lemma 3.4(1), $x_{d+1} > x_b$. Set $p = b$ and $q = d+1$. Then $2 \leqslant p < q = d+1$ and $x_p \leqslant x_q$. In the similar way to Case 3 of Lemma 3.4, we will get a contradiction. Therefore, $d+1 \notin \widetilde{V}_s$. So there are two vertices $u \in V$ and $v \notin \widetilde{V}_s$ such that u is adjacent to v and u is on the right side of V. Then Eqs. (3) and (4) hold. Since $w_{T_M}(a) > w_{T_M}(v)$ and $w_{T_M}(u) > w_{T_M}(b)$, by Lemma 3.4(1), we have $x_a > x_v$ and $x_u > x_b$. Thus we get a contradiction by Eqs. (3)–(5).

The above results also imply that $\widetilde{V}_1 \subseteq \widetilde{V}_2 \subseteq \cdots \subseteq \widetilde{V}_{l-1}$, and by Lemma 3.6, P_M is symmetric in edge weights. Therefore, P_M is a symmetric alternating weighted path in edge weights. □

Lemma 3.8 [6]. *Let G be the weighted graph obtained from two weighted graphs G_1 and G_2 by joining a vertex u of G_1 to a vertex v of G_2 with a new edge uv. Then*

$$\phi(G, \lambda) = \phi(G_1, \lambda)\phi(G_2, \lambda) - w_G^2(uv)\phi(G_1 - u, \lambda)\phi(G_2 - v, \lambda).$$

Lemma 3.9 [6]. *Let H be a weighted proper spanning subgraph of a weighted tree T. Then for $\lambda \geqslant \rho(T)$, we have $\phi(H, \lambda) > \phi(T, \lambda)$.*

Lemma 3.10. *Assume $s \neq 0$, i.e., that $n > d+1$.*

(1) *If $d = 3$, then for $i = 1, 2, \ldots, s$,*

$$\min\{w_{T_M}(k1), w_{T_M}(k3)\} \geqslant w_{T_M}(ka_i) \geqslant w_{T_M}(34).$$

(2) *If $d \geqslant 4$, set $a = k-2, b = k-1, u = k+1, v = k+2$, then for $i = 1, 2, \ldots, s$,*

$$\min\{w_{T_M}(kb), w_{T_M}(ku)\} \geqslant w_{T_M}(ka_i) \geqslant \max\{w_{T_M}(ab), w_{T_M}(uv)\}.$$

Proof. Without loss of generality, assume $w_{T_M}(ka_1) \geqslant w_{T_M}(ka_2) \geqslant \cdots \geqslant w_{T_M}(ka_s)$.

(1) It is obvious that $k = 2$. Write $a_0 = 1$. Since the edges $ka_0, ka_1, \ldots, ka_s$ are symmetric in their positions, without loss of generality, assume $w_{T_M}(ka_0) \geqslant w_{T_M}(ka_1)$. Now we need show that $w_{T_M}(k3) \geqslant w_{T_M}(ka_1), w_{T_M}(ka_s) \geqslant w_{T_M}(34)$. By Lemma 3.8, we have that

$$\phi(T_M,\lambda)=\lambda^s\left(\lambda^2-w_{T_M}^2(34)\right)\left[\lambda^2-\sum_{j=0}^{s}w_{T_M}^2(ka_j)\right]-\lambda^{s+2}w_{T_M}^2(k3).$$

We will first prove $w_{T_M}(k3)\geqslant w_{T_M}(ka_1)$. Assume $w_{T_M}(k3)<w_{T_M}(ka_1)$. Let T' be the weighted tree obtained from T_M by exchanging the weights of edges $k3$ and ka_1 while making the weights of other edges fixed. Then for $\lambda\geqslant\rho\left(T'\right)$, we have that

$$\phi(T_M,\lambda)-\phi\left(T',\lambda\right)=\lambda^s\left[w_{T_M}^2(ka_1)-w_{T_M}^2(k3)\right]w_{T_M}^2(34)>0.$$

This indicates that $\rho\left(T'\right)>\rho(T_M)$, a contradiction.

We will next prove $w_{T_M}(ka_s)\geqslant w_{T_M}(34)$. Assume $w_{T_M}(ka_s)<w_{T_M}(34)$. Let T'' be the weighted tree obtained from T_M by exchanging the weights of edges 34 and ka_s while making the weights of other edges fixed. Then for $\lambda\geqslant\rho\left(T''\right)$, we have that

$$\phi(T_M,\lambda)-\phi(T'',\lambda)=\lambda^s\left[w_{T_M}^2(34)-w_{T_M}^2(ka_s)\right]\cdot\sum_{j=0}^{s-1}w_{T_M}^2(ka_j)>0.$$

This indicates that $\rho\left(T''\right)>\rho(T_M)$, a contradiction.

(2) Let $P(i,j)$ denote the subpath between the vertex i and the vertex j in P_M, including i and j. In particular, $P(i,i)$ is an isolated vertex P_1. Set $q=d+1$ and $\phi(P(1,0))=1$. By Lemma 3.8, we have

$$\begin{aligned}\phi(T_M,\lambda)&=\phi(T_M-ka_1-kb)-w_{T_M}^2(kb)\lambda^s\phi(P(1,a))\phi(P(u,q))\\&\quad-w_{T_M}^2(ka_1)\lambda^{s-1}\phi(P(1,b))\phi(P(u,q)),\qquad(6)\\ \phi(T_M,\lambda)&=\phi(T_M-ka_s-ab)-w_{T_M}^2(ka_s)\lambda^s\phi(P(1,a))\phi(P(u,q))\\&\quad-w_{T_M}^2(ab)\phi(P(1,a-1))\left[\lambda\phi(G)-w_{T_M}^2(ka_s)\lambda^{s-1}\phi(P(u,q))\right],\qquad(7)\end{aligned}$$

where G is the weighted graph obtained from T_M by deleting vertices $1,2,\ldots,k-1,a_s$ together with the edges incident to them. By Lemma 3.8, we have

$$\phi(G)=\lambda^{s-1}\phi(P(k,q))-\lambda^{s-2}\phi(P(u,q))\sum_{j=1}^{s-1}w_{T_M}^2(ka_j),$$

where when $s=1$, we define $\sum_{j=1}^{s-1}w_{T_M}^2(ka_j)=0$.

We will first prove $\min\{w_{T_M}(kb),w_{T_M}(ku)\}\geqslant w_{T_M}(ka_1)$. Assume the contrary, and without loss of generality, assume $w_{T_M}(kb)<w_{T_M}(ka_1)$. Let T' be the weighted tree obtained from T_M by exchanging the weights of edges kb and ka_1 while keeping the weights of other edges not changed. Then $T'\in\Theta(n,d)$ and $T_M-ka_1-kb=T'-ka_1-kb$. By Eq. (6), we have

$$\frac{\phi(T_M,\lambda)-\phi\left(T',\lambda\right)}{w_{T_M}^2(ka_1)-w_{T_M}^2(kb)}=\lambda^{s-1}\phi(P(u,q))\cdot[\phi(P_1\cup P(1,a))-\phi(P(1,b))].$$

Since $P_1\cup P(1,a)$ is a weighted proper spanning subgraph of $P(1,b)$, by Lemma 3.9, for $\lambda\geqslant\rho(P(1,b))$, $\phi(P_1\cup P(1,a))>\phi(P(1,b))$. But $P(1,b)$ and $P(u,q)$ are two proper subgraphs of T', by the Perron–Frobenius Theorem, $\rho\left(T'\right)>\max\{\rho(P(1,b)),\rho(P(u,q))\}$. So for $\lambda\geqslant\rho\left(T'\right)$, $\phi(T_M,\lambda)>\phi\left(T',\lambda\right)$. This indicates that $\rho\left(T'\right)>\rho(T_M)$, a contradiction.

We will next show $w_{T_M}(ka_s)\geqslant\max\{w_{T_M}(ab),w_{T_M}(uv)\}$. Assume the contrary, and without loss of generality, assume $w_{T_M}(ka_s)<w_{T_M}(ab)$. Let T'' be the weighted tree obtained from T_M by exchanging the weights of edges ab and ka_s while keeping the weights of other edges fixed. Then $T''\in\Theta(n,d)$ and $T_M-ka_s-ab=T''-ka_s-ab$. By Eq. (7), we have

$$\frac{\phi(T_M,\lambda)-\phi(T'',\lambda)}{\lambda^s\left[w_{T_M}^2(ab)-w_{T_M}^2(ka_s)\right]}=\varDelta_0+\frac{1}{\lambda}\phi(P(1,k-3))\phi(P(k+1,q))\sum_{j=1}^{s-1}w_{T_M}^2(ka_j),$$

where

$$\Delta_j = \phi(P(1,k-2-j))\phi(P(k+1+j,q)) - \phi(P(1,k-3-j))\phi(P(k+j,q)).$$

Now assume $\lambda \geqslant \rho(T'')$. By Lemma 3.8, we have

$$\Delta_0 = w_{T_M}^2(e_k)\phi(P(1,k-3))\phi(P(k+2,q)) - w_{T_M}^2(e_{k-3})\phi(P(1,k-4))\phi(P(k+1,q)),$$

where e_i denotes the edge $i(i+1)$ of P_M. Since $k = \left\lfloor \frac{d+2}{2} \right\rfloor$ by Lemma 3.7, the distance of e_{k-3} and the center of P_M is greater than that of e_k and the center of P_M. But P_M is an alternating weighted path in edge weights, so $w_{T_M}(e_k) \geqslant w_{T_M}(e_{k-3})$. Hence we get $\Delta_0 \geqslant w_{T_M}^2(e_{k-3})\Delta_1$. Repeat the above procedure, we have

$$\begin{aligned}\Delta_0 &\geqslant w_{T_M}^2(e_{k-3})w_{T_M}^2(e_{k-4})\Delta_2 \geqslant \cdots \geqslant \Delta_{k-3} \cdot \prod_{j=1}^{k-3} w_{T_M}^2(e_j)\\ &= [\phi(P_1 \cup P(2k-2,q)) - \phi(P(2k-3,q))] \cdot \prod_{j=1}^{k-3} w_{T_M}^2(e_j).\end{aligned}$$

Since $P_1 \cup P(2k-2,q)$ is a weighted proper spanning subgraph of $P(2k-3,q)$, by Lemma 3.9, for $\lambda \geqslant \rho(P(2k-3,q))$, we have $\phi(P_1 \cup P(2k-2,q)) > \phi(P(2k-3,q))$, i.e., $\Delta_0 > 0$. But $P(2k-3,q)$ is a proper subgraphs of T'', by the Perron–Frobenius Theorem, $\rho(T'') > \rho(P(2k-3,q))$. Hence for $\lambda \geqslant \rho(T'')$, $\phi(T_M,\lambda) > \phi(T'',\lambda)$. This implies that $\rho(T'') > \rho(T_M)$, a contradiction. □

By Lemmas 3.7 and 3.10, we see, if $d = 2r$, that

$$\begin{aligned}w_{T_M}(e_r) &\geqslant w_{T_M}(e_{r+1}) \geqslant w_{T_M}((r+1)a_1) \geqslant w_{T_M}((r+1)a_2) \geqslant \cdots \geqslant w_{T_M}((r+1)a_s)\\ &\geqslant w_{T_M}(e_{r-1}) \geqslant w_{T_M}(e_{r+2}) \geqslant w_{T_M}(e_{r-2}) \geqslant \cdots \geqslant w_{T_M}(e_1) \geqslant w_{T_M}(e_{2r}),\end{aligned}$$

and if $d = 2r-1$, that

$$\begin{aligned}w_{T_M}(e_r) &\geqslant w_{T_M}(e_{r-1}) \geqslant w_{T_M}(ra_1) \geqslant w_{T_M}(ra_2) \geqslant \cdots \geqslant w_{T_M}(ra_s) \geqslant w_{T_M}(e_{r+1})\\ &\geqslant w_{T_M}(e_{r-2}) \geqslant w_P(e_{r+2}) \geqslant w_P(e_{r-3}) \geqslant \cdots \geqslant w_{T_M}(e_1) \geqslant w_{T_M}(e_{2r-1}).\end{aligned}$$

This indicates that, for given parameters $m_1, m_2, \ldots, m_{n-1}, T_M$ is uniquely determined. Therefore, by Lemmas 3.1 and 3.7 and the definition of T_M, we immediately get the following main result.

Theorem 3.11. *For $n \geqslant 2, T_M$ is the unique weighted tree in $\Gamma(d, m_1, m_2, \ldots, m_{n-1})$ having the largest spectral radius.*

Example. In Fig. 2, two weighted trees are displayed, where the numbers on the edges denote the weights of edges. The first has the largest spectral radius in

$$\Gamma(8, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2),$$

while the second has the largest spectral radius in

$$\Gamma(7, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2).$$

Remark 2. Theorem 3.11 indicates that the alternating weighted path P_M with n vertices is the unique weighted path having the largest spectral radius in the set of all weighted paths with n vertices and positive weight set $\{m_1, m_2, \ldots, m_{n-1}\}$. Suppose that at least two of $m_1, m_2, \ldots, m_{n-1}$ are distinct and $m = \max\{m_1, m_2, \ldots, m_{n-1}\}$. It is obvious that $\rho(P_M) < 2m \cos \frac{\pi}{n+1}$. This indicates that $\rho(P_M)$ is a better upper bound than $2m \cos \frac{\pi}{n+1}$, which gives an answer of an open problem proposed in [4].

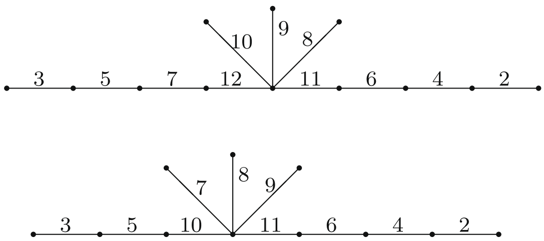

Fig. 2. Examples of two weighted trees with the largest spectral radius.

Write $T_M^d = T_M$. Let T' be the weighted tree obtained from T_M by deleting the edge $k(k+1)$, identifying the vertices k and $k+1$ and adding the pendent edge ka_{s+1} such that $w_{T'}(ka_{s+1}) = w_{T_M}(k(k+1))$. Then $T' \in \Gamma(d-1, m_1, m_2, \ldots, m_{n-1})$, and by Corollary 2.5, we have $\rho(T_M^d) < \rho(T') \leqslant \rho(T_M^{d-1})$. Thus we get the following results.

Corollary 3.12 [4]. *Let T be a weighted tree with n vertices and positive weight set. Then $\rho(T) \leqslant \rho(T_M^2)$, with equality if and only if $T = T_M^2 (= K_{1,n-1})$.*

Corollary 3.13 [5]. *Let $T \neq T_M^2$ be a weighted tree with n vertices and positive weight set. Then $\rho(T) \leqslant \rho(T_M^3)$, with equality if and only if $T = T_M^3$.*

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