Positive almost periodic solution for a class of Lasota–Wazewska model with multiple time-varying delays

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A B S T R A C T

In this paper, we study the existence and global exponential convergence of positive almost periodic solutions for the generalized Lasota–Wazewska model with multiple time-varying delays. Under proper conditions, we employ a novel proof to establish some criteria to ensure that all solutions of this model converge exponentially to a positive almost periodic solution. Moreover, we give an example to illustrate our main results.

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1. Introduction

The variation of the environment plays an important role in many biological and ecological dynamical systems. As pointed out in [1,2], a periodically varying environment and an almost periodically varying environment are foundations for the theory of natural selection. Compared with periodic effects, almost periodic effects are more frequent. Hence, the effects of the almost periodic environment on the evolutionary theory have been the object of intensive analysis by numerous authors and some of these results can be found in [3–6].

In 1999, Gopalsamy and Trofimchuk [7] studied the existence of an almost periodic solution of the Lasota–Wazewska-type delay differential equation:

$$\dot{x}(t) = -\alpha(t)x(t) + \beta(t)e^{-\gamma x(t-\tau)}$$  \hspace{1cm} (1.1)

which was used by Wazewska-Czyzewska and Lasota [8] as a model for the survival of red blood cells in an animal. Assuming that $\alpha, \beta : \mathbb{R} \to (0, +\infty)$ are almost periodic functions, $\gamma > 0$ and $\tau > 0$ are constants, and employing some other additional assumptions on the boundedness for solutions of (1.1), the authors of [7] proved that (1.1) has a globally attractive almost periodic solution.

Recently, Stamov [9] considered the following generalized impulsive Lasota–Wazewska model:

$$\begin{cases}
\dot{x}(t) = -\alpha(t)x(t) + \sum_{j=1}^{m} \beta_j(t)e^{-\gamma_j x(t-\tau_j)}, & t \neq \tau_k, \\
\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = \alpha_k x(\tau_k) + \nu_k,
\end{cases}$$  \hspace{1cm} (1.2)

where $t \in \mathbb{R}, \alpha, \beta_j, \gamma_j : \mathbb{R} \to (0, +\infty)$ are almost periodic functions, $\tau > 0$ is a constant, and

$$0 < \sup_{t \in \mathbb{R}} |\beta_j(t)| < B_j, \quad 0 < \sup_{t \in \mathbb{R}} |\gamma_j(t)| < G_j, \quad \beta_j(0) = \gamma_j(0) = 0, \quad j = 1, 2, \ldots, m.$$  \hspace{1cm} (1.3)

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Under some other additional assumptions, Stamov obtained that (1.2) has a globally attractive almost periodic solution. As is well-known, only positive solutions are meaningful in the realistic biological models of (1.1) and (1.2). Unfortunately, [7, 9] cannot show that the almost periodic solution is positive. On the other hand, in the real world, the delays in differential equations of population and ecology problems are usually time-varying. Thus, it is worthwhile continuing to investigate the existence and convergence of positive almost periodic solutions of the Lasota–Wazewski model with time-varying delays.

In this paper we consider the following Lasota–Wazewska model with multiple time-varying delays:

\[ x'(t) = -\alpha(t)x(t) + \sum_{j=1}^{m} \beta_j(t)e^{-\gamma_j(t)x(t-\tau_j(t))}, \]  

(1.4)

where \( t \in \mathbb{R}, \alpha, \beta_j, \gamma_j, \tau_j : \mathbb{R} \rightarrow (0, +\infty) \) are almost periodic functions, and

\[ \alpha^- = \inf_{t \in \mathbb{R}} \alpha(t) > 0, \quad \alpha^+ = \sup_{t \in \mathbb{R}} \alpha(t), \quad \beta_j^- = \inf_{t \in \mathbb{R}} \beta_j(t) > 0, \quad \beta_j^+ = \sup_{t \in \mathbb{R}} \beta_j(t), \quad \gamma_j^- = \inf_{t \in \mathbb{R}} \gamma_j(t), \]  

(1.5)

\[ \gamma_j^+ = \sup_{t \in \mathbb{R}} \gamma_j(t), \quad r = \max \{ \sup_{t \in \mathbb{R}} \gamma_j(t) \} > 0, \quad \text{and} \quad \sum_{j=1}^{m} \beta_j^+ \gamma_j^+ < \alpha^-, \quad j = 1, 2, \ldots, m. \]  

(1.6)

Let \( C = C([-r, 0], \mathbb{R}) \) be the continuous functions space equipped with the usual supremum norm \( \| \cdot \| \), and let \( C_+ = C([-r, 0], \mathbb{R}_+) \). If \( x(t) \) is defined on \([-r + t_0, \sigma)\) with \( t_0, \sigma \in \mathbb{R} \), then, for \( t \in [t_0, \sigma) \), we define \( x_t \in C \) where \( x_t(\theta) = x(t + \theta) \) for all \( \theta \in [-r, 0] \).

Due to the biological interpretation of model (1.4), only positive solutions are meaningful and therefore admissible. Thus we just consider the admissible initial conditions

\[ x_{t_0} = \varphi, \quad \varphi \in C_+ \text{ and } \varphi(0) > 0. \]  

(1.7)

We write \( x_t(t_0, \varphi) = x(t; t_0, \varphi) \) for a admissible solution of the admissible initial value problems (1.4) and (1.7) with \( x_{t_0}(t_0, \varphi) = \varphi \in C_+ \) and \( t_0 \in \mathbb{R} \). Also, let \( [t_0, \sigma(\varphi)) \) be the maximal right-interval of existence of \( x_t(t_0, \varphi) \).

The remaining part of this paper is organized as follows. In Section 2, we shall give some notations and preliminary results. In Section 3, we shall derive new sufficient conditions for checking the existence, uniqueness and global exponential convergence of the positive almost periodic solution of (1.4). In Section 4, we shall give an example and a remark to illustrate our results obtained in the previous sections.

2. Preliminary results

In this section, some lemmas and definitions will be presented, which are of importance in proving our main results in Section 3.

**Definition 2.1** ([1,2]). Let \( u(t) : \mathbb{R} \rightarrow \mathbb{R}^d \) be continuous in \( t \). \( u(t) \) is said to be almost periodic on \( \mathbb{R} \) if, for any \( \varepsilon > 0 \), the set \( \{ T(u, \varepsilon) = \{ \delta : \| u(t + \delta) - u(t) \| < \varepsilon \text{ for all } t \in \mathbb{R} \} \} \) is relatively dense, i.e., for any \( \varepsilon > 0 \), it is possible to find a real number \( l = l(\varepsilon) > 0 \), such that for any interval with length \( l(\varepsilon) \), there exists a number \( \delta = \delta(\varepsilon) \) in this interval such that \( |u(t + \delta) - u(t)| < \varepsilon \), for all \( t \in \mathbb{R} \).

**Definition 2.2** ([1,2]). Let \( x \in \mathbb{R}^d \) and \( Q(t) \) be a \( n \times n \) continuous matrix defined on \( \mathbb{R} \). The linear system

\[ x'(t) = Q(t)x(t) \]  

(2.1)

is said to admit an exponential dichotomy on \( \mathbb{R} \) if there exist positive constants \( k, \alpha \), projection \( P \) and the fundamental solution matrix \( X(t) \) of (2.1) satisfying

\[ \| X(t)P X^{-1}(s) \| \leq ke^{-\alpha(t-s)} \text{ for } t \geq s, \]

\[ \| X(t)(I - P) X^{-1}(s) \| \leq ke^{-\alpha(s-t)} \text{ for } t \leq s. \]

Set

\[ B = \{ \varphi | \varphi \text{ is an almost periodic function on } \mathbb{R} \}. \]

For \( \forall \varphi \in B \), if we define the induced modulus \( \| \varphi \|_B = \sup_{t \in \mathbb{R}} |\varphi(t)| \), then \( B \) is a Banach space.

**Lemma 2.1** ([1,2]). If the linear system (2.1) admits an exponential dichotomy, then almost periodic system

\[ x'(t) = Q(t)x + g(t) \]  

(2.2)

has a unique almost periodic solution \( x(t) \), and

\[ x(t) = \int_{-\infty}^{t} X(t)P X^{-1}(s)g(s)ds - \int_{t}^{+\infty} X(t)(I - P) X^{-1}(s)g(s)ds. \]  

(2.3)
Lemma 2.2 (See [1,2]). Let \( c_i(t) \) be an almost periodic function on \( R \) and
\[
M[c_i] = \lim_{T \to +\infty} \frac{1}{T} \int_{t}^{t+T} c_i(s)ds > 0, \quad i = 1, 2, \ldots, n.
\]
Then the linear system
\[
x'(t) = \text{diag}(-c_1(t), -c_2(t), \ldots, -c_n(t))x(t)
\]
admits an exponential dichotomy on \( R \).

By the same approach used in the proof of [10, Lemma 2.3], we have

Lemma 2.3. Every solution \( x(t; t_0, \varphi) \) of (1.4) and (1.7) is positive and bounded on \([t_0, \eta(\varphi))\), and \( \eta(\varphi) = +\infty \).

Lemma 2.4 (See [11]). If \( u(t), g(t) : R \to R \) are almost periodic, then \( u(t - g(t)) \) is almost periodic.

3. Main results

Theorem 3.1. Let \( M_1 = \sum_{j=1}^{m} \beta_j^+ \) and \( M_2 = \sum_{j=1}^{m} \beta_j^- e^{-M_1 \gamma_j^+} \). Then, there exists a unique positive almost periodic solution of Eq. (1.4) in the region \( B^* = \{ \varphi | \varphi \in B, M_2 \leq \varphi(t) \leq M_1, \text{ for all } t \in R \} \).

Proof. For any \( \phi \in B \), we consider an auxiliary equation
\[
x'(t) = -\alpha(t)x(t) + \sum_{j=1}^{m} \beta_j(t) e^{-\gamma_j^+ t} \, \phi(t) - \tau_j(t).
\]
It follows from Lemma 2.4 that \( \phi(t - \tau_j(t)) \) is almost periodic. Notice that \( M[\alpha] > 0 \), it follows from Lemma 2.2 that the linear equation
\[
x'(t) = -\alpha(t)x(t),
\]
admits an exponential dichotomy on \( R \). Thus, by Lemma 2.1, we obtain that the system (3.1) has exactly one almost periodic solution:
\[
x^\phi(t) = \int_{-\infty}^{t} e^{-\int_{s}^{t} \alpha(u)du} \left[ \sum_{j=1}^{m} \beta_j(s) e^{-\gamma_j^+ s} \phi(t - \tau_j(s)) \right] ds.
\]
Define a mapping \( T : B \to B \) by setting
\[
T(\phi(t)) = x^\phi(t), \quad \forall \phi \in B.
\]
Since \( B^* = \{ \varphi | \varphi \in B, M_2 \leq \varphi(t) \leq M_1, \text{ for all } t \in R \} \), it is easy to see that \( B^* \) is a closed subset of \( B \). For any \( \varphi \in B^* \), from (3.3), we have
\[
x^\phi(t) \leq \int_{-\infty}^{t} e^{-\int_{s}^{t} \alpha(u)du} \sum_{j=1}^{m} \beta_j(s) ds \leq M_1 \quad \text{for all } t \in R,
\]
and
\[
x^\phi(t) \geq \int_{-\infty}^{t} e^{-\int_{s}^{t} \alpha(u)du} \sum_{j=1}^{m} \beta_j(s) e^{-\gamma_j^+ s} M_1 ds \geq \sum_{j=1}^{m} \beta_j^- e^{-M_1 \gamma_j^+} \alpha^+ M_1 = M_2 \quad \text{for all } t \in R.
\]
This implies that the mapping \( T \) is a self-mapping from \( B^* \) to \( B^* \). Now, we prove that the mapping \( T \) is a contraction mapping on \( B^* \). In fact, for \( \varphi, \psi \in B^* \), we get
\[
\|T(\varphi) - T(\psi)\|_B = \sup_{t \in R} |T(\varphi)(t) - T(\psi)(t)|
\]
\[
= \sup_{t \in R} \left| \int_{-\infty}^{t} e^{-\int_{s}^{t} \alpha(u)du} \sum_{j=1}^{m} \beta_j(s) [e^{-\gamma_j^+ s} \phi(t - \tau_j(s)) - e^{-\gamma_j^+ s} \psi(t - \tau_j(s))] ds \right|.
\]
In view of (1.5), (1.6) and (3.3)-(3.6), from the inequality
\[
|e^{-x} - e^{-y}| \leq |x - y| \quad \text{for all } x, y \in [0, +\infty),
\]
(3.7)
we have

\[ \| T(\varphi) - T(\psi) \|_B \leq \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{-\int_{t}^{\tau} \alpha(u)du} \left( \sum_{j=1}^{m} \beta_j(s) \gamma_j(s) |\varphi(s - \tau_j(s)) - \psi(s - \tau_j(s))| \right) ds \]

\[ \leq \| \varphi - \psi \|_B \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{-\int_{t}^{\tau} \alpha(u)du} \sum_{j=1}^{m} \beta_j(s) \gamma_j(s) ds \]

\[ \leq \frac{\sum_{j=1}^{m} \beta_j^+ \gamma_j^+}{\alpha^-} \| \varphi - \psi \|_B. \]

Noting that \( \sum_{j=1}^{m} \beta_j^+ \gamma_j^+ \leq 1 \), it is clear that the mapping \( T \) is a contraction on \( B^* \). Using Theorem 0.3.1 of [12], we obtain that the mapping \( T \) possesses a unique fixed point \( \varphi^* \in B^* \). \( T \varphi^* = \varphi^* \). By (3.1), \( \varphi^* \) satisfies (1.4). So \( \varphi^* \) is an almost periodic solution of (1.4) in \( B^* \). The proof of Theorem 3.1 is now complete. \( \square \)

**Theorem 3.2.** Let \( x^*(t) \) be the positive almost periodic solution of Eq. (1.4) in the region \( B^* \). Then, the solution \( x(t; t_0, \varphi) \) of (1.4) and (1.7) converges exponentially to \( x^*(t) \) as \( t \to +\infty \).

**Proof.** Set \( x(t) = x(t; t_0, \varphi) \) and \( y(t) = x(t) - x^*(t) \), where \( t \in [t_0 - r, +\infty) \). Then

\[ y'(t) = -\alpha(t)y(t) + \sum_{j=1}^{m} \beta_j(t)[e^{-\gamma_j(t)x(t - \tau_j(t))} - e^{-\gamma_j(t)x^*(t - \tau_j(t))}]. \]

(3.8)

Define a continuous function \( \Gamma(u) \) by setting

\[ \Gamma(u) = -(\alpha^- - u) + \sum_{j=1}^{m} \beta_j^+ \gamma_j^+ e^{\lambda t}, \quad u \in [0, 1]. \]

(3.9)

Then, we have

\[ \Gamma(0) = -\alpha^- + \sum_{j=1}^{m} \beta_j^+ \gamma_j^+ < 0, \]

which implies that there exist two constants \( \eta > 0 \) and \( \lambda \in (0, 1) \) such that

\[ \Gamma(\lambda) = -\alpha^- - \lambda + \sum_{j=1}^{m} \beta_j^+ \gamma_j^+ e^{\lambda t} < -\eta < 0. \]

(3.10)

We consider the Lyapunov functional

\[ V(t) = |y(t)|e^{\lambda t}. \]

(3.11)

Calculating the upper right derivative of \( V(t) \) along the solution \( y(t) \) of (3.8), we have

\[ D^+(V(t)) \leq -\alpha(t)|y(t)|e^{\lambda t} + \sum_{j=1}^{m} \beta_j(t)[e^{-\gamma_j(t)x(t - \tau_j(t))} - e^{-\gamma_j(t)x^*(t - \tau_j(t))} |e^{\lambda t} + \lambda |y(t)|e^{\lambda t} \]

\[ = \left[ (\lambda - \alpha(t))|y(t)| + \sum_{j=1}^{m} \beta_j(t)[e^{-\gamma_j(t)x(t - \tau_j(t))} - e^{-\gamma_j(t)x^*(t - \tau_j(t))} \right] e^{\lambda t}, \quad \text{for all } t > t_0. \]

(3.12)

We claim that

\[ V(t) = |y(t)|e^{\lambda t} < e^{\lambda t_0} \max_{t \in [t_0 - r, t_0]} |\varphi(t) - x^*(t)| + M_1 \]

(3.13)

for all \( t > t_0 \). Contrarily, there must exist \( t_* > t_0 \) such that

\[ V(t_*) = M_3 \quad \text{and} \quad V(t) < M_3 \quad \text{for all } t \in [t_0 - r, t_*), \]

(3.14)

which implies that

\[ V(t_*) - M_3 = 0 \quad \text{and} \quad V(t) - M_3 < 0 \quad \text{for all } t \in [t_0 - r, t_*). \]

(3.15)
Since \( x(t) \geq 0 \) and \( x^*(t) \geq 0 \) for all \( t \geq t_0 - r \). Together with (3.7), (3.12) and (3.15), we obtain

\[
0 \leq D^+ (V(t_s) - M_3) = D^+ (V(t_s)) \leq \left[ (\lambda - \alpha(t_s)) |y(t_s)| + \sum_{j=1}^{m} \beta_j(t_s) e^{-\gamma_j(t_s) \tau_j(t_s)} - e^{-\gamma_j(t_s) x^*(t_s - \tau_j(t_s))} \right] e^{\lambda t} \leq (\lambda - \alpha(t_s)) |y(t_s)| e^{\lambda t} + \sum_{j=1}^{m} \beta_j(t_s) \gamma_j(t_s) |y(t_s - \tau_j(t_s))| e^{\lambda(t_s - \tau_j(t_s))} e^{\lambda \tau_j(t_s)} \leq \left[ (\lambda - \alpha^+) + \sum_{j=1}^{m} \beta_j^+ \gamma_j^+ e^{\lambda t} \right] M_3. \tag{3.16}
\]

Thus,

\[
0 \leq (\lambda - \alpha^+) + \sum_{j=1}^{m} \beta_j^+ \gamma_j^+ e^{\lambda t},
\]

which contradicts with (3.10). Hence, (3.13) holds. It follows that

\[
|y(t)| < M_3 e^{-\lambda t} \quad \text{for all } t > t_0.
\tag{3.17}
\]

This completes the proof. \( \square \)

4. An example

In this section, we give an example to demonstrate the results obtained in previous sections.

**Example 4.1.** Consider the following Lasota–Wazewska model with multiple time-varying delays:

\[
x'(t) = -(18 + \cos^2 t) x(t) + (1 + |\sin \sqrt{2} t|) e^{-(2 + 1 + \sin \sqrt{3} t |x(t - e^2 + 1)|)} + (3 + |\sin \sqrt{5} t|) e^{-(3 + 1 + \sin \sqrt{7} t |x(t - e^2 + 1)|)}.
\]

(4.1)

Obviously,

\[
\alpha^- = 18, \quad \alpha^+ = 29, \quad \beta_j^- = \gamma_j^- = 1 > 0, \quad \beta_j^+ = \gamma_j^+ = 2, \quad i = 1, 2,
\]

(4.2)

\[
r = \max_{t \in [0, 2]} \{\sup_{t \in \mathbb{R}} \tau_j(t)\} = e^2 > 0, \quad \text{and} \quad \sum_{j=1}^{2} \beta_j^+ \gamma_j^+ = 8 < 18 = \alpha^-.
\]

(4.3)

This implies that the Lasota–Wazewska model (4.1) satisfies (1.5) and (1.6). Hence, from Theorems 3.1 and 3.2, Eq. (4.1) has a positive almost periodic solution \( x^*(t) \). Moreover, if \( \varphi \in \{\varphi \in C_+: \varphi(0) > 0\} \), then \( x(t; t_0, \varphi) \) converges exponentially to \( x^*(t) \) as \( t \to +\infty \).

**Remark 4.1.** It is clear that the delays in Eq. (4.1) are not constants. Therefore, all the results in [7,9,13,14] and the references therein cannot be applicable to prove that all the solutions of (4.1) with admissible initial conditions converge exponentially to the positive almost periodic solution. This implies that the results of this paper are new and they complement previously known results.

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