Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 153 (2008) 161–169
Journal of Approximation Theory
www.elsevier.com/locate/jat

Orthogonal exponentials on the generalized plane Sierpinski gasket

Jian-Lin Li

College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, PR China

Received 27 November 2007; accepted 28 January 2008

Communicated by Paul Nevai
Available online 15 March 2008

Abstract

The self-affine measure $\mu_{M_p,D}$ corresponding to

$$M_p = \begin{bmatrix} 2 & p \\ 0 & 2 \end{bmatrix} \quad (p \in \mathbb{Z}) \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is supported on the the generalized plane Sierpinski gasket $T(M_p, D)$. In the present paper we show that there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M_p,D})$, and the number 3 is the best. This generalizes several known results on the non-spectral self-affine measure problem.

MSC: 42C05; 28A80

Keywords: Iterated function system; Self-affine measure; Orthogonal exponentials; Plane Sierpinski gasket

1. Introduction

Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix, that is, all the eigenvalues of the integer matrix M have moduli > 1. Associated with a finite subset $D \subset \mathbb{Z}^n$, there exists a unique non-empty compact set $T := T(M, D)$ such that $MT = \bigcup_{d \in D}(T + d)$. More precisely, $T(M, D)$ is the attractor (or invariant set) of the iterated function system (IFS) $\{\phi_d(x) = M^{-1}(x + d)\}_{d \in D}$. Let $|D|$ be the cardinality of D. Relating to the IFS $\{\phi_d\}_{d \in D}$, there exists a unique probability

E-mail address: jllimath@yahoo.com.cn.

0021-9045/$ - see front matter
doi:10.1016/j.jat.2008.01.005

measure $\mu := \mu_{M,D}$ satisfying the self-affine identity

$$\mu = \frac{1}{|D|} \sum_{d \in D} \mu \circ \phi_d^{-1}. \tag{1.1}$$

Such a measure $\mu_{M,D}$ is supported on $T(M, D)$ (cf. [6]), and is called a *self-affine measure*.

For a probability measure μ of compact support on $\mathbb{R}^n$, we call μ a *spectral measure* if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that $E_\Lambda := \{e^{2\pi i \langle \lambda, x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis for $L^2(\mu)$. The set Λ is then called a *spectrum* for μ; we also say that (μ, Λ) is a *spectral pair*. Spectral measure is a natural generalization of spectral set introduced by Fuglede [4] whose famous conjecture and its related problems have received much attention in recent years (see [16,17]). The spectral self-affine measure problem at the present day consists in determining conditions under which $\mu_{M,D}$ is a spectral measure, and has been studied in the papers [1,2,13,15,17,19–21] (see also [22,23] for the main goal). The non-spectral self-affine measure problem originated from the Lebesgue measure case (see papers [4,5,7–11,14,18] where the conjecture that the disk has no more than three orthogonal exponentials is still unsolved) usually consists of the following two classes:

(I) There are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$, that is, $\mu_{M,D}$-orthogonal exponentials contains at most finite elements. The main questions here are to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them (see [3]).

(II) There are natural infinite families of orthogonal exponentials, but none of them forms an orthogonal basis in $L^2(\mu_{M,D})$. The questions concerning this class can be found in [12].

In the present paper we will consider the questions of the class (I) for the generalized plane Sierpinski gasket. We recall the following related conclusions.

(i) The familiar middle 3rd Cantor set $T(M, D)$ corresponding to $M = 3$ and $D = \{0, 2\}$, Jorgensen and Pedersen [13, Theorem 6.1] proved that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 2 elements.

(ii) The plane Sierpinski gasket $T(M, D)$ corresponding to

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \tag{1.2}$$

Li [17, p. 65; 20, Example 1], Dutkay and Jorgensen [3, Theorem 5.1(ii)] proved that $\mu_{M,D}$-orthogonal exponentials contains at most 3 elements and found such 3 elements orthogonal exponentials.

(iii) The generalized plane Sierpinski gasket $T(M, D)$ corresponding to

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \tag{1.3}$$

see Figure 3 and Example 3.1 in [3], by applying [3, Theorem 3.1], Dutkay and Jorgensen obtain that $\mu_{M,D}$-orthogonal exponentials contains at most 7 elements.

The main result of the present paper is the following.

Theorem. *For the self-affine measure $\mu_{M_p,D}$ corresponding to*

$$M_p = \begin{bmatrix} 2 & p \\ 0 & 2 \end{bmatrix} \ (p \in \mathbb{Z}) \quad \textit{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \tag{1.4}$$

there exist at most 3 *mutually orthogonal exponential functions in* $L^2(\mu_{M_p,D})$, *and the number* 3 *is the best*.

This generalizes the above mentioned results on the non-spectral self-affine measure problem.

2. Proof of Theorem

For the general expanding matrix $M \in M_n(\mathbb{Z})$ and finite subset $D \subset \mathbb{Z}^n$, the Fourier transform of the self-affine measure $\mu_{M,D}$ is

$$\hat{\mu}_{M,D}(\xi) = \int e^{2\pi i\langle \xi,t\rangle} d\mu_{M,D}(t) \quad (\xi \in \mathbb{R}^n).$$

From (1.1), we have

$$\hat{\mu}_{M,D}(\xi) = m_D(M^{*-1}\xi)\hat{\mu}_{M,D}(M^{*-1}\xi) \quad (\xi \in \mathbb{R}^n), \tag{2.1}$$

which yields

$$\hat{\mu}_{M,D}(\xi) = \prod_{j=1}^{\infty} m_D(M^{*-j}\xi), \tag{2.2}$$

by iteration, where

$$m_D(t) := \frac{1}{|D|}\sum_{d\in D} e^{2\pi i\langle d,t\rangle} \tag{2.3}$$

and M^* denotes the conjugate transpose of M, in fact $M^* = M^t$.

For any $\lambda_1, \lambda_2 \in \mathbb{R}^n$, $\lambda_1 \neq \lambda_2$, the orthogonality condition

$$\langle e^{2\pi i\langle\lambda_1,x\rangle}, e^{2\pi i\langle\lambda_2,x\rangle}\rangle_{L^2(\mu_{M,D})} = \int e^{2\pi i\langle\lambda_1-\lambda_2,x\rangle}\, d\mu_{M,D} = \hat{\mu}_{M,D}(\lambda_1-\lambda_2) = 0 \tag{2.4}$$

directly relates to the zero set $Z(\hat{\mu}_{M,D})$ of $\hat{\mu}_{M,D}$. From (2.2), we have

$$Z(\hat{\mu}_{M,D}) = \{\xi \in \mathbb{R}^n : \exists j \in \mathbb{N} \text{ such that } m_D(M^{*-j}\xi) = 0\}. \tag{2.5}$$

This set has a simple property that $\xi_0 \in Z(\hat{\mu}_{M,D}) \Leftrightarrow -\xi_0 \in Z(\hat{\mu}_{M,D})$.

In the following, we will restrict our discussion on the special M and D given by (1.4), and find out some characteristic properties on the set $Z(\hat{\mu}_{M,D})$ to finish the proof of Theorem.

For the given M_p and D in (1.4), we first have

$$m_D(M_p^{*-j}\xi) = \frac{1}{3}\left\{1 + e^{2\pi i\frac{\xi_1}{2^j}} + e^{2\pi i\frac{2\xi_2 - jp\xi_1}{2^{j+1}}}\right\}, \tag{2.6}$$

where $\xi = (\xi_1, \xi_2)^t \in \mathbb{R}^2$. Relating to the zero set of the function m_D, it is known that if $1 + w_1 + w_2 = 0$ and $|w_1| = |w_2| = 1$, then $\{w_1, w_2\} = \{e^{2\pi i/3}, e^{4\pi i/3}\}$. So it follows from (2.5) and (2.6) that

$$Z(\hat{\mu}_{M_p,D}) = \bigcup_{j=1}^{\infty}(Z_j \cup \tilde{Z}_j), \tag{2.7}$$

where

$$Z_j = \left\{ \begin{pmatrix} 2^j/3 \\ 2^{j-1}(1+jp)/3 \end{pmatrix} + \begin{pmatrix} 2^j k_1 \\ 2^{j-1}(1+2k_2+jpk_1) \end{pmatrix} : k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2 \tag{2.8}$$

and

$$\tilde{Z}_j = \left\{ \begin{pmatrix} 2^{j+1}/3 \\ 2^j(1+jp)/3 \end{pmatrix} + \begin{pmatrix} 2^j \tilde{k}_1 \\ 2^{j-1}(2\tilde{k}_2+jp\tilde{k}_1) \end{pmatrix} : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2. \tag{2.9}$$

Secondly, from (2.8) and (2.9), one can verify that

$$Z_{j+3} \subseteq \tilde{Z}_j \quad \text{and} \quad \tilde{Z}_{j+3} \subseteq Z_j \tag{2.10}$$

hold for all $j = 1, 2, \ldots$. Hence we further obtain from (2.7) that

$$Z(\hat{\mu}_{M_p,D}) = Z_1 \cup Z_2 \cup Z_3 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3. \tag{2.11}$$

We divide our discussion into the following two subsections.

2.1. The case $p = 3l (l \in \mathbb{Z})$

In the case when $p = 3l$ ($l \in \mathbb{Z}$), we can verify that

$$Z_2 \subseteq \tilde{Z}_1, \quad \tilde{Z}_2 \subseteq Z_1, \quad Z_3 \subseteq Z_1, \quad \tilde{Z}_3 \subseteq \tilde{Z}_1. \tag{2.12}$$

Hence it follows from (2.11) that

$$Z(\hat{\mu}_{M_p,D}) = Z_1 \cup \tilde{Z}_1 \tag{2.13}$$

with the properties that

(a) $(a, b)^t \in Z_1 \Leftrightarrow (-a, -b)^t \in \tilde{Z}_1$, that is, $Z_1 = -\tilde{Z}_1$ or $\tilde{Z}_1 = -Z_1$;
(b) $Z_1 - Z_1 \subseteq \mathbb{Z}^2$ and $\tilde{Z}_1 - \tilde{Z}_1 \subseteq \mathbb{Z}^2$;
(c) $Z_1 \cap \tilde{Z}_1 = \emptyset$ and $(Z_1 \cup \tilde{Z}_1) \cap \mathbb{Z}^2 = \emptyset$.

If λ_j $(j = 1, 2, 3, 4) \in \mathbb{R}^2$ are such that the exponential functions

$$e^{2\pi i\langle\lambda_1,x\rangle}, \quad e^{2\pi i\langle\lambda_2,x\rangle}, \quad e^{2\pi i\langle\lambda_3,x\rangle}, \quad e^{2\pi i\langle\lambda_4,x\rangle}$$

are mutually orthogonal in $L^2(\mu_{M_p,D})$, then the differences $\lambda_j - \lambda_k$ $(1 \leqslant j \neq k \leqslant 4)$ are in $Z(\hat{\mu}_{M_p,D})$. From (2.13), we have

$$\lambda_j - \lambda_k \in Z_1 \cup \tilde{Z}_1 \quad (1 \leqslant j \neq k \leqslant 4). \tag{2.14}$$

In particular, the following three differences:

$$\lambda_1 - \lambda_2, \quad \lambda_1 - \lambda_3, \quad \lambda_1 - \lambda_4 \tag{2.15}$$

are in $Z_1 \cup \tilde{Z}_1$. The well-known *pigeon hole principle*, combined with the properties (a)–(c) and (2.14), immediately deduces a contradiction, since any two of three differences in (2.15) cannot belong to the same set Z_1 or $\tilde{Z}_1$. For example, if $\lambda_1 - \lambda_2 \in Z_1$ and $\lambda_1 - \lambda_4 \in Z_1$, then, by the property (b),

$$\lambda_4 - \lambda_2 = (\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_4) \in Z_1 - Z_1 \subseteq \mathbb{Z}^2$$

which contradicts (2.14) and the property (c). Hence any set of $\mu_{M,D}$-orthogonal exponentials contains at most 3 elements. One can obtain many such orthogonal systems which contain 3 elements, for instance, E_S with S given by

$$S = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 + l + 1 \end{pmatrix}, \begin{pmatrix} 4/3 \\ 2/3 + 2l \end{pmatrix} \right\} \tag{2.16}$$

is a three elements orthogonal system in $L^2(\mu_{M_p,D})$. This shows that the number 3 is the best.

2.2. *The case $p = 3l + 1$ or $p = 3l + 2$ ($l \in \mathbb{Z}$)*

In the case when $p = 3l + 1$ or $p = 3l + 2$ ($l \in \mathbb{Z}$), we can further verify that the sets $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$ in (2.11) have the following properties:

(a) $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$ are mutually disjoint;
(b) $\bigcup_{j=1}^{3}(Z_j \bigcup \tilde{Z}_j) \bigcap \mathbb{Z}^2 = \emptyset$;
(c) $Z_j - Z_j \subseteq \mathbb{Z}^2$ and $\tilde{Z}_j - \tilde{Z}_j \subseteq \mathbb{Z}^2$ $(j = 1, 2, 3)$;
(d) $Z_j + Z_j \subseteq \tilde{Z}_j$ and $\tilde{Z}_j + \tilde{Z}_j \subseteq Z_j$ $(j = 1, 2, 3)$;
(e) $Z_j = -\tilde{Z}_j$ and $\tilde{Z}_j = -Z_j$ $(j = 1, 2, 3)$.

If λ_j $(j = 1, 2, 3, 4) \in \mathbb{R}^2$ are such that the exponential functions

$$e^{2\pi i\langle\lambda_1,x\rangle}, \quad e^{2\pi i\langle\lambda_2,x\rangle}, \quad e^{2\pi i\langle\lambda_3,x\rangle}, \quad e^{2\pi i\langle\lambda_4,x\rangle}$$

are mutually orthogonal in $L^2(\mu_{M_p,D})$, then the differences $\lambda_j - \lambda_k$ $(1 \leqslant j \neq k \leqslant 4)$ are in $Z(\hat{\mu}_{M_p,D})$. From (2.11), we have

$$\lambda_j - \lambda_k \in Z_1 \cup Z_2 \cup Z_3 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3 \quad (1 \leqslant j \neq k \leqslant 4). \tag{2.17}$$

We will use the above properties (a)–(e) to deduce a contradiction.

Observe that the following six differences:

$$\begin{array}{lll} \lambda_1 - \lambda_2, & \lambda_1 - \lambda_3, & \lambda_1 - \lambda_4, \\ & \lambda_2 - \lambda_3, & \lambda_2 - \lambda_4, \\ & & \lambda_3 - \lambda_4 \end{array} \tag{2.18}$$

belong to the six sets $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$. By the properties (a)–(c) and (2.17), the elements (or differences) in each row of (2.18) (except the final row where there is only one element $\lambda_3 - \lambda_4$) and the elements (or differences) in each column of (2.18) (except the first column where there is only one element $\lambda_1 - \lambda_2$) cannot belong to the same set. In particular, the following three elements:

$$\lambda_1 - \lambda_2, \quad \lambda_1 - \lambda_3, \quad \lambda_1 - \lambda_4$$

in the first row will be in the three different sets of the six sets $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$. There are 120 distribution methods. We only consider the following three typical cases:

Case 1: $\lambda_1 - \lambda_2 \in Z_1,\ \lambda_1 - \lambda_3 \in Z_2,\ \lambda_1 - \lambda_4 \in Z_3$.
Case 2: $\lambda_1 - \lambda_2 \in Z_2,\ \lambda_1 - \lambda_3 \in Z_3,\ \lambda_1 - \lambda_4 \in \tilde{Z}_1$.
Case 3: $\lambda_1 - \lambda_2 \in Z_2,\ \lambda_1 - \lambda_3 \in Z_3,\ \lambda_1 - \lambda_4 \in \tilde{Z}_3$.

The other cases (by applying the property (e)) can be proved in the same manner.

Case 1: By the property (e), we see that in this case, each set contains elements (or differences) in the following box.

Z_1	Z_2	Z_3	$\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$
$\lambda_1-\lambda_2$	$\lambda_1-\lambda_3$	$\lambda_1-\lambda_4$			
			$\lambda_2-\lambda_1$	$\lambda_3-\lambda_1$	$\lambda_4-\lambda_1$

The other elements in (2.18) are also in certain boxes. Firstly, we have the following fact that

$$\lambda_2-\lambda_3 \text{ cannot belong to the sets (or boxes) } Z_1,\ Z_2,\ \tilde{Z}_1,\ \tilde{Z}_2. \tag{2.19}$$

The reason is as follows. (i) If $\lambda_2-\lambda_3\in Z_1$, then, by the property (d),

$$\lambda_1-\lambda_3=(\lambda_1-\lambda_2)+(\lambda_2-\lambda_3)\in Z_1+Z_1\subseteq\tilde{Z}_1, \tag{2.20}$$

which contradicts the property (a) and $\lambda_1-\lambda_3\in Z_2$. The same reason shows that $\lambda_2-\lambda_3\notin\tilde{Z}_2$. (ii) If $\lambda_2-\lambda_3\in Z_2$, then, by the property (c),

$$\lambda_1-\lambda_2=(\lambda_1-\lambda_3)-(\lambda_2-\lambda_3)\in Z_2-Z_2\subseteq\mathbb{Z}^2 \tag{2.21}$$

which contradicts the property (b) and $\lambda_1-\lambda_2\in Z_1$. The same reason shows that $\lambda_2-\lambda_3\notin\tilde{Z}_1$.

Similarly, we have the following facts that

$$\lambda_2-\lambda_4 \text{ cannot belong to the sets (or boxes) } Z_1,\ Z_3,\ \tilde{Z}_1,\ \tilde{Z}_3, \tag{2.22}$$

$$\lambda_3-\lambda_4 \text{ cannot belong to the sets (or boxes) } Z_2,\ Z_3,\ \tilde{Z}_2,\ \tilde{Z}_3. \tag{2.23}$$

Hence, from (2.19), (2.22) and (2.23), we have

$$\lambda_2-\lambda_3\in Z_3 \text{ or } \tilde{Z}_3,\quad \lambda_2-\lambda_4\in Z_2 \text{ or } \tilde{Z}_2,\quad \lambda_3-\lambda_4\in Z_1 \text{ or } \tilde{Z}_1 \tag{2.24}$$

which is impossible. To see this, we only consider the following two typical cases:

(i′) If

$$\lambda_2-\lambda_3\in Z_3,\quad \lambda_2-\lambda_4\in Z_2,\quad \lambda_3-\lambda_4\in Z_1,$$

then, by the property (e), the above box becomes the following box.

Z_1	Z_2	Z_3	$\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$
$\lambda_1-\lambda_2$	$\lambda_1-\lambda_3$	$\lambda_1-\lambda_4$			
			$\lambda_2-\lambda_1$	$\lambda_3-\lambda_1$	$\lambda_4-\lambda_1$
$\lambda_3-\lambda_4$	$\lambda_2-\lambda_4$	$\lambda_2-\lambda_3$			
			$\lambda_4-\lambda_3$	$\lambda_4-\lambda_2$	$\lambda_3-\lambda_2$

By the property (d), the elements in Z_2 and Z_3 (or in $\tilde{Z}_2$ and $\tilde{Z}_3$) have the character that

$$(\lambda_1-\lambda_3)+(\lambda_2-\lambda_4)=(\lambda_1-\lambda_4)+(\lambda_2-\lambda_3)\in\tilde{Z}_2\cap\tilde{Z}_3,$$

which contradicts the property (a). Another way to deduce a contradiction is to apply the properties (c) and (d) on the sets Z_1 and Z_3 (or $\tilde{Z}_1$ and $\tilde{Z}_3$), respectively. Since

$$(\lambda_1-\lambda_2)+(\lambda_3-\lambda_4)=(\lambda_1-\lambda_4)-(\lambda_2-\lambda_3),$$

the left-hand side is in $Z_1+Z_1\subseteq\tilde{Z}_1$ and the right-hand side is in $Z_3-Z_3\subseteq\mathbb{Z}^2$, which also leads to a contradiction by the property (b).

(ii′) If

$$\lambda_2 - \lambda_3 \in \tilde{Z}_3, \quad \lambda_2 - \lambda_4 \in \tilde{Z}_2, \quad \lambda_3 - \lambda_4 \in Z_1,$$

then, by the property (e), we have the following box.

Z_1	Z_2	Z_3	$\widetilde{Z}_1$	$\widetilde{Z}_2$	$\widetilde{Z}_3$
$\lambda_1 - \lambda_2$	$\lambda_1 - \lambda_3$	$\lambda_1 - \lambda_4$			
			$\lambda_2 - \lambda_1$	$\lambda_3 - \lambda_1$	$\lambda_4 - \lambda_1$
$\lambda_3 - \lambda_4$				$\lambda_2 - \lambda_4$	$\lambda_2 - \lambda_3$
	$\lambda_4 - \lambda_2$	$\lambda_3 - \lambda_2$	$\lambda_4 - \lambda_3$		

By the property (d), the elements in Z_1 and Z_3 (or in $\tilde{Z}_1$ and $\tilde{Z}_3$) have the character that

$$(\lambda_1 - \lambda_2) + (\lambda_3 - \lambda_4) = (\lambda_1 - \lambda_4) + (\lambda_3 - \lambda_2) \in \tilde{Z}_1 \cap \tilde{Z}_3,$$

which contradicts the property (a). Another way to deduce a contradiction is to apply the properties (c) and (d) on the sets Z_1 and Z_2 (or $\tilde{Z}_1$ and $\tilde{Z}_2$), respectively. Since

$$(\lambda_1 - \lambda_2) - (\lambda_3 - \lambda_4) = (\lambda_1 - \lambda_3) + (\lambda_4 - \lambda_2),$$

the left-hand side is in $Z_1 - Z_1 \subseteq \mathbb{Z}^2$ and the right-hand side is in $Z_2 + Z_2 \subseteq \tilde{Z}_2$, which also leads to a contradiction by the property (b). This completes the proof of Case 1.

Case 2: By the property (e), we see that in this case, each set contains elements (or differences) in the following box.

Z_1	Z_2	Z_3	$\widetilde{Z}_1$	$\widetilde{Z}_2$	$\widetilde{Z}_3$
	$\lambda_1 - \lambda_2$	$\lambda_1 - \lambda_3$	$\lambda_1 - \lambda_4$		
$\lambda_4 - \lambda_1$				$\lambda_2 - \lambda_1$	$\lambda_3 - \lambda_1$

The other elements in (2.18) are also in certain boxes. As in Case 1, we have the following facts that

$$\lambda_2 - \lambda_3 \text{ cannot belong to the sets (or boxes) } Z_2,\ Z_3,\ \tilde{Z}_2,\ \tilde{Z}_3, \tag{2.25}$$

$$\lambda_2 - \lambda_4 \text{ cannot belong to the sets (or boxes) } Z_1,\ Z_2,\ \tilde{Z}_1,\ \tilde{Z}_2, \tag{2.26}$$

$$\lambda_3 - \lambda_4 \text{ cannot belong to the sets (or boxes) } Z_1,\ Z_3,\ \tilde{Z}_1,\ \tilde{Z}_3. \tag{2.27}$$

Hence, from (2.25), (2.26) and (2.27), we have

$$\lambda_2 - \lambda_3 \in Z_1 \text{ or } \tilde{Z}_1, \quad \lambda_2 - \lambda_4 \in Z_3 \text{ or } \tilde{Z}_3, \quad \lambda_3 - \lambda_4 \in Z_2 \text{ or } \tilde{Z}_2 \tag{2.28}$$

which is impossible. The reason is the same as Case 1. This completes the proof of Case 2.

Case 3: By the properties (d) and (e), we see that in this case, each set contains elements (or differences) in the following box.

Z_1	Z_2	Z_3	$\widetilde{Z}_1$	$\widetilde{Z}_2$	$\widetilde{Z}_3$
	$\lambda_1 - \lambda_2$	$\lambda_1 - \lambda_3$			$\lambda_1 - \lambda_4$
		$\lambda_4 - \lambda_1$		$\lambda_2 - \lambda_1$	$\lambda_3 - \lambda_1$
		$\lambda_3 - \lambda_4$			$\lambda_4 - \lambda_3$

The other elements in (2.18) are also in certain boxes. As in Case 1, we have the following facts that

$$\lambda_2 - \lambda_3 \text{ cannot belong to the sets(or boxes) } Z_2,\ Z_3,\ \tilde{Z}_2,\ \tilde{Z}_3; \tag{2.29}$$

$$\lambda_2 - \lambda_4 \text{ cannot belong to the sets (or boxes) } Z_2,\ Z_3,\ \tilde{Z}_2,\ \tilde{Z}_3. \tag{2.30}$$

Hence, from (2.29) and (2.30), we have

$$\lambda_2 - \lambda_3 \in Z_1 \text{ or } \tilde{Z}_1, \quad \lambda_2 - \lambda_4 \in Z_1 \text{ or } \tilde{Z}_1 \tag{2.31}$$

which is also impossible. The reason is the same as Case 1. This completes the proof of Case 3.

Hence any set of $\mu_{M_p,D}$-orthogonal exponentials contains at most three elements. One can obtain many such orthogonal systems which contain three elements, for instance, $E_{\tilde{S}}$ with $\tilde{S}$ given by

$$\tilde{S} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 2/3 + l + 1 \end{pmatrix}, \begin{pmatrix} 4/3 \\ 1/3 + 2l + 1 \end{pmatrix} \right\} \tag{2.32}$$

is a three elements orthogonal system in $L^2(\mu_{M_p,D})$ for $p = 3l + 1$, and $E_{\tilde{S}}$ with $\tilde{S}$ given by

$$\tilde{S} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4/3 \\ 4/3 \end{pmatrix}, \begin{pmatrix} 8/3 \\ 8/3 \end{pmatrix} \right\} \tag{2.33}$$

is a three elements orthogonal system in $L^2(\mu_{M_p,D})$ for $p = 3l + 2$. This shows that the number 3 is the best. The proof of theorem is complete.

3. A concluding remark

The proof above is based on the description of the zero set $Z(\hat{\mu}_{M_p,D})$ of the Fourier transform $\hat{\mu}_{M_p,D}$. This is based in turn on the zero set of the function m_D. Generally speaking, the non-spectral self-affine measure problems of the case (I) mentioned in the Introduction depend largely on the characterization of the zero set $Z(\hat{\mu}_{M,D})$. For the finite set $D \subset \mathbb{Z}^n$ (usually called the *digit set*) of cardinality $|D| = 3$ or 4, one can obtain certain expression for the set $Z(\hat{\mu}_{M,D})$ similar to (2.7). But it is not easy to obtain certain properties on this set. For example, the self-affine measure $\mu_{M,D}$ corresponding to

$$M = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} \ (p \text{ is odd}) \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

in $\mathbb{R}^3$ is not a spectral measure, Dutkay and Jorgensen [3, Theorem 5.1(iii)] proved that there are at most 256 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. The number 256 is certainly not the best one. The questions of the case (I) on this non-spectral measure $\mu_{M,D}$ are still open. The method here may be provide a way to deal with such questions.

Acknowledgment

The author would like to thank the anonymous referees for their valuable suggestions. The present research is partially supported by the Key Project of Chinese Ministry of Education (No. 108117).

References

[1] D.E. Dutkay, P.E.T. Jorgensen, Iterated function systems, Rulle operators, and invariant projective measures, Math. Comp. 75 (2006) 1931–1970.

[2] D.E. Dutkay, P.E.T. Jorgensen, Fourier frequencies in affine iterated function systems, J. Funct. Anal. 247 (2007) 110–137.

[3] D.E. Dutkay, P.E.T. Jorgensen, Analysis of orthogonality and of orbits in affine iterated function systems, Math. Z. 256 (2007) 801–823.

[4] B. Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, J. Funct. Anal. 16 (1974) 101–121.

[5] B. Fuglede, Orthogonal exponentials on the ball, Exposition. Math. 19 (2001) 267–272.

[6] J.E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981) 713–747.

[7] A. Iosevich, P. Jaming, Orthogonal exponentials, diffence sets, and arithmetic combinatorics, preprint (available on the home page of Alex Iosevich), 2006.

[8] A. Iosevich, P. Jaming, Distance sets that are a shift of the integers and Fourier basis for planar convex sets, available on: ⟨http://arxiv.org/abs/0709.4133v1⟩.

[9] A. Iosevich, N. Katz, S. Pedersen, Fourier basis and a distance problem of Erdös, Math. Res. Lett. 6 (1999) 251–255.

[10] A. Iosevich, N. Katz, T. Tao, Convex bodies with a point of curvature do not have Fourier bases, Amer. J. Math. 123 (2001) 115–120.

[11] A. Iosevich, M. Rudnev, A combinatorial approach to orthogonal exponentials, Internat. Math. Res. Notices 49 (2003) 1–12.

[12] P.E.T. Jorgensen, K.A. Kornelson, K. Shuman, Orthogonal exponentials for Bernoulli iterated function systems, available on: ⟨http://arxiv.org/abs/math.OA/0703385⟩.

[13] P.E.T. Jorgensen, S. Pedersen, Dense analytic subspaces in fractal L^2-spaces, J. Anal. Math. 75 (1998) 185–228.

[14] M.N. Kolountzakis, Non-symmetric convex domains have no basis of exponentials, Illinois J. Math. 44 (2000) 542–550.

[15] I. Łaba, Y. Wang, On spectral Cantor measures, J. Funct. Anal. 193 (2002) 409–420.

[16] I. Łaba, Y. Wang, Some properties of spectral measures, Appl. Comput. Harmon. Anal. 20 (2006) 149–157.

[17] J.-L. Li, Spectral sets and spectral self-affine measures, Ph.D. Thesis, The Chinese University of Hong Kong, November, 2004.

[18] J.-L. Li, On characterizations of spectra and tilings, J. Funct. Anal. 213 (2004) 31–44.

[19] J.-L. Li, Spectral self-affine measures in $\mathbb{R}^n$, Proc. Edinburgh Math. Soc. 50 (2007) 197–215.

[20] J.-L. Li, $\mu_{M,D}$-Orthogonality and compatible pair, J. Funct. Anal. 244 (2007) 628–638.

[21] R. Strichartz, Remarks on dense analytic subspaces in fractal L^2-spaces, J. Anal. Math. 75 (1998) 229–231.

[22] R. Strichartz, Mock Fourier series and transforms associated with certain Cantor measures, J. Anal. Math. 81 (2000) 209–238.

[23] R. Strichartz, Convergence of mock Fourier series, J. Anal. Math. 99 (2006) 333–353.