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Symbols of truncated Toeplitz operators ☆

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Abstract

We consider three topics connected with coinvariant subspaces of the backward shift operator in Hardy spaces H^p:

– properties of truncated Toeplitz operators;
– Carleson-type embedding theorems for the coinvariant subspaces;
– factorizations of pseudocontinuable functions from H^1.

These problems turn out to be closely connected and even, in a sense, equivalent. The new approach based on the factorizations allows us to answer a number of challenging questions about truncated Toeplitz operators posed by D. Sarason.

Keywords: Model space; Inner function; Toeplitz operator; Carleson measure

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1. Introduction

Let H^p, $1 \leqslant p \leqslant \infty$, denote the Hardy space in the unit disk $\mathbb{D}$, and let $H^p_- = \overline{zH^p}$. As usual, we identify the functions from H^p in the disk and their nontangential boundary values on the unit circle $\mathbb{T}$.

A function θ which is analytic and bounded in $\mathbb{D}$ is said to be *inner* if $|\theta| = 1$ m-almost everywhere on $\mathbb{T}$ in the sense of nontangential boundary values; by m we denote the Lebesgue measure on $\mathbb{T}$ normalized so that $m\mathbb{T} = 1$. With each inner function θ we associate the subspace

$$K^p_\theta = H^p \cap \theta H^p_-$$

of H^p. Equivalently, one can define K^p_θ, $1 \leqslant p < \infty$, as the set of all functions from H^p such that $\langle f, \theta g\rangle = \int_{\mathbb{T}} f\overline{\theta g}\,dm = 0$ for any $g \in H^q$, $1/p + 1/q = 1$. In particular,

$$K_\theta := K^2_\theta = H^2 \ominus \theta H^2$$

(in what follows we often omit the exponent 2). The norm in K^p_θ, that is, the norm of the space $L^p = L^p(\mathbb{T}, m)$, will be denoted by $\|\cdot\|_p$. It is well known that any closed subspace of H^p, $1 \leqslant p < \infty$, which is invariant with respect to the backward shift operator S^*, $(S^*f)(z) = \frac{f(z)-f(0)}{z}$, is of the form K^p_θ for some inner function θ (see [22, Chapter II] or [14]). Subspaces K^p_θ are often called *star-invariant subspaces*. These subspaces play an outstanding role both in function and operator theory (a detailed exposition of their theory may be found in N. Nikolski's books [27,28]) and, in particular, in the Sz.-Nagy–Foiaş function model theory for contractions on a Hilbert space; therefore they are sometimes referred to as *model subspaces*.

An important property of elements of the spaces K^p_θ is the existence of a *pseudocontinuation* outside the unit disk: if $f \in K^p_\theta$, then there exists a function g, which is meromorphic and of Nevanlinna class in $\{z\colon |z| > 1\}$, such that $g = f$ almost everywhere on $\mathbb{T}$ in the sense of nontangential boundary values.

Now we discuss in detail the three main themes of the paper, as indicated in the abstract.

1.1. Truncated Toeplitz operators on K_θ

Recall that the classical Toeplitz operator on H^2 with symbol $\varphi \in L^\infty(\mathbb{T})$ is defined by $T_\varphi f = P_+(\varphi f)$, $f \in H^2$, where P_+ stands for the orthogonal projection from L^2 onto H^2.

Now let $\varphi \in L^2$. We define the *truncated Toeplitz operator* A_φ on bounded functions from K_θ by the formula

$$A_\varphi f = P_\theta(\varphi f), \quad f \in K_\theta \cap L^\infty(\mathbb{T}),$$

where P_θ is the orthogonal projection onto K_θ, $P_\theta f = P_+ f - \theta P_+(\bar\theta f)$. In contrast to the Toeplitz operators on H^2 (which satisfy $\|T_\varphi\| = \|\varphi\|_\infty$), the operator A_φ may be extended to a bounded operator on K_θ even for some unbounded symbols φ. The class of all bounded truncated Toeplitz operators on K_θ will be denoted by $\mathcal{T}(\theta)$.

Certain special cases of truncated Toeplitz operators are well known and play a prominent role in the operator theory. If $\varphi(z) = z$, then $A_\varphi = S_\theta$ is the so-called compressed shift operator, the scalar model operator from the Sz.-Nagy–Foiaş theory. If $\varphi \in H^\infty$, then $A_\varphi = \varphi(S_\theta)$. Truncated Toeplitz operators include all finite Toeplitz matrices (corresponding to the case $\theta(z) = z^n$) and the Wiener–Hopf convolution operators on an interval, which are unitarily equivalent to the truncated Toeplitz operators on the space generated by the singular inner functions associated with mass points on the circle,

$$\theta_a(z) = \exp\left(a\frac{z+1}{z-1}\right), \quad a > 0 \tag{1}$$

(for a more detailed discussion see [9]). Bercovici, Foiaş and Tannenbaum studied truncated (or *skew*) Toeplitz operators (mainly, with symbols which are rational functions with pole at zero) in connection with control theory (see [11,12]). However, a systematic study of truncated Toeplitz operators with symbols from L^2 was started recently by Sarason in [31]. This paper laid the basis of the theory and inspired much of the subsequent activity in the field [9,15,16,21].

Unlike standard Toeplitz operators on H^2, the symbol of a truncated Toeplitz operator is not unique. The set of all symbols of an arbitrary operator A_φ is exactly the set $\varphi + \theta H^2 + \overline{\theta H^2}$, see [31]. Clearly, any bounded function $\varphi \in L^\infty$ determines the bounded operator A_φ with norm $\|A_\varphi\| \leqslant \|\varphi\|_\infty$. The first basic question on truncated Toeplitz operators posed in [31] is *whether every bounded operator A_φ has a bounded symbol*, i.e., is a restriction of a bounded Toeplitz operator on H^2. Note that if a truncated Toeplitz operator with symbol $\varphi \in H^2$ is bounded, then, as a consequence of the commutant lifting theorem, it admits a symbol from H^∞, see [31, Section 4]. On the other hand, by Rochberg's results [30] (proved in the context of the Wiener–Hopf operators and the Paley–Wiener spaces) any operator in $\mathcal{T}(\theta_a)$ has a bounded symbol. However, in general the answer to this question is negative: in [9] inner functions θ are constructed for which there exist operators in $\mathcal{T}(\theta)$ (even of rank one) that have no bounded symbols.

Thus, a natural question appears: *in which spaces K_θ does any bounded truncated Toeplitz operator admit a bounded symbol*? In this paper we obtain a description of such inner functions. In particular, we show that this is true for the interesting class of *one-component inner functions* introduced by Cohn in [18]: these are functions θ such that the sublevel set

$$\left\{z \in \mathbb{D}\colon \left|\theta(z)\right| < \varepsilon\right\}$$

is connected for some $\varepsilon \in (0, 1)$. This statement was conjectured in [9]. A basic example of a one-component inner function is the function θ_a given by (1).

1.2. Embeddings of the spaces K_θ^p

Let μ be a finite positive Borel measure in the closed unit disk $\overline{\mathbb{D}}$. We are interested in the class of measures such that the Carleson-type embedding $K_\theta^p \hookrightarrow L^p(\mu)$ is bounded. Since the functions from K_θ^p are well-defined only m-almost everywhere on $\mathbb{T}$, one should be careful when dealing with the restriction of μ to $\mathbb{T}$. Recall that, by Aleksandrov's theorem, functions from K_θ^p that are continuous in the closed disk $\overline{\mathbb{D}}$ are dense in K_θ^p ($1 \leqslant p < \infty$), see [2] or [13]. (While this statement is trivial for the Blaschke products, there is no constructive way to prove the statement in the general case.) This allows one to define the embedding on the dense set of all continuous functions from K_θ in a natural way and then ask if it admits a bounded continuation to the whole space K_θ^p. However, this extension may always be viewed as an embedding operator due to the following theorem by Aleksandrov.

Theorem 1.1. *(See [4, Theorem 2].) Let θ be an inner function, let μ be a positive Borel measure on $\mathbb{T}$, and let $1 \leqslant p < \infty$. Assume that for any continuous function $f \in K_\theta^p$ we have*

$$\|f\|_{L^p(\mu)} \leqslant C\|f\|_p. \tag{2}$$

Then all functions from K_θ^p possess angular boundary values μ-almost everywhere, and for any $f \in K_\theta^p$ relation (2) *holds, in which the left-hand side is defined via the boundary values.*

The angular convergence μ-almost everywhere gives us a nice illustration of how the embedding acts. This approach, essentially based on results of Poltoratski's paper [29], uses deep analytic techniques. For our purposes we will need the L^2-convergence, which can be established more simply. To make the exposition more self-contained, we present the corresponding arguments in Section 3.

Denote by $\mathcal{D}_p(\theta)$ the class of all finite complex Borel measures μ on the closed unit disk $\overline{\mathbb{D}}$, for which the embedding $K_\theta^p \hookrightarrow L^p(|\mu|)$ is continuous; for a complex measure μ, by $|\mu|$ we denote its total variation. The class of positive measures from $\mathcal{D}_p(\theta)$ is denoted by $\mathcal{D}_p^+(\theta)$. The classes $\mathcal{D}_p^+(\theta)$ contain all Carleson measures, i.e., measures for which the embedding $H^p \hookrightarrow L^p(\mu)$ is a bounded operator (for some, and hence for all $p > 0$). However, the class $\mathcal{D}_p(\theta)$ is usually much wider due to additional analyticity (pseudocontinuability) of the elements of K_θ^p on the boundary. The problem of description of the class $\mathcal{D}_p(\theta)$ for general θ was posed by Cohn in 1982; it is still open. Many partial results may be found in [18,19,33,6,26,7,8]. In particular, the classes $\mathcal{D}_p(\theta)$ are described if θ is a one-component inner function; in this case there exists a nice geometric description analogous to the classical Carleson embedding theorem [33,3,6]. Moreover, Aleksandrov [6] has shown that θ is one-component if and only if all classes $\mathcal{D}_p(\theta)$, $p > 0$, coincide.

In what follows we denote by $\mathcal{C}_p(\theta)$ the set of all finite complex Borel measures μ on the unit circle $\mathbb{T}$ such that $|\mu| \in \mathcal{D}_p(\theta)$; the class of positive measures from $\mathcal{C}_p(\theta)$ will be denoted by $\mathcal{C}_p^+(\theta)$.

If $\mu \in \mathcal{C}_2(\theta)$, we may define the bounded operator A_μ on K_θ by the formula

$$(A_\mu f, g) = \int f\bar{g}\,d\mu. \tag{3}$$

It is shown in [31] that $A_\mu \in \mathcal{T}(\theta)$. This follows immediately from the following characteristic property of truncated Toeplitz operators.

Theorem 1.2. *(See [31, Theorem 8.1].) A bounded operator A on K_θ is a truncated Toeplitz operator if and only if the condition $f, zf \in K_\theta$ yields $(Af, f) = (Azf, zf)$.*

Sarason asked in [31] whether *every bounded truncated Toeplitz operator A coincides with A_μ for some $\mu \in \mathcal{C}_2(\theta)$*. Below we answer this question in the affirmative. Moreover, we show that nonnegative bounded truncated Toeplitz operators are of the form A_μ with $\mu \in \mathcal{C}_2^+(\theta)$. We also prove that truncated Toeplitz operators with bounded symbols correspond to complex measures from the subclass $\mathcal{C}_1(\theta^2)$ of $\mathcal{C}_2(\theta^2) = \mathcal{C}_2(\theta)$, and, finally, that every bounded truncated Toeplitz operator has a bounded symbol if and only if $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2)$.

1.3. Factorizations

Now we consider a factorization problem for pseudocontinuable functions from H^1, which will be proved to have an equivalent reformulation in terms of truncated Toeplitz operators.

It is well known that any function $f \in H^1$ can be represented as the product of two functions $g, h \in H^2$ with $\|f\|_1 = \|g\|_2 \cdot \|h\|_2$. By the definition of the spaces K_θ^p, there is a natural involution on K_θ:

$$f \mapsto \tilde{f} = \bar{z}\theta \bar{f} \in K_\theta, \quad f \in K_\theta. \tag{4}$$

Hence, if $f, g \in K_\theta$, then $fg \in H^1$ and $\bar{z}^2\theta^2 \bar{f}\bar{g} \in H^1$. Thus,

$$fg \in H^1 \cap \bar{z}^2\theta^2\overline{H^1} = H^1 \cap \bar{z}\theta^2 H^1_-.$$

If $\theta(0) = 0$, then θ^2/z is an inner function and the expression on the right-hand side coincides with $K^1_{\theta^2/z}$.

It is not difficult to show that linear combinations of products of pairs of functions from K_θ form a dense subset of $H^1 \cap \bar{z}\theta^2 H^1_-$. We are interested in a stronger property:

For which θ may any function $f \in H^1 \cap \bar{z}\theta^2 H^1_-$ be represented in the form

$$f = \sum_k g_k h_k, \quad g_k, h_k \in K_\theta, \qquad \sum_k \|g_k\|_2 \cdot \|h_k\|_2 < \infty? \tag{5}$$

We still use the term *factorization* for the representations of the form (5), by analogy with the usual row-column product.

Below we will see that, for functions $f \in H^1 \cap \bar{z}\theta^2 H^1_-$, not only a usual factorization $f = g \cdot h$, $g, h \in K_\theta$, but even a weaker factorization (5) may be impossible. It will be proved that this problem is equivalent to the problem of existence of bounded symbols for all bounded truncated Toeplitz operators on K_θ.

Let us consider two special cases of the problem. Take $\theta(z) = z^{n+1}$. The spaces K_θ and $K^1_{\theta^2/z}$ consist of polynomials of degrees at most n and $2n$, respectively, and then, obviously, $K^1_{\theta^2/z} = K_\theta \cdot K_\theta$. However, it is not known if a norm controlled factorization is possible, i.e., if for any polynomial p of degree at most $2n$ there exist polynomials q, r of degree at most n such that $p = q \cdot r$ and $\|q\|_2 \cdot \|r\|_2 \leqslant C\|p\|_1$, where C is an absolute constant independent on n. On the other hand, it is shown in [32] that there exists a representation $p = \sum_{k=1}^4 q_k r_k$ with $\sum_{k=1}^4 \|q_k\|_2 \cdot \|r_k\|_2 \leqslant C\|p\|_1$.

For $\theta = \theta_a$ defined by (1), the corresponding model subspaces K_θ^p are natural analogs of the Paley–Wiener spaces $\mathcal{PW}_a^p$ of entire functions. The space $\mathcal{PW}_a^p$ consists of all entire functions of exponential type at most a, whose restrictions to $\mathbb{R}$ are in L^p. It follows from our results (and may be proved directly, see Section 7) that every entire function $f \in \mathcal{PW}^1_{2a}$ of exponential type at most $2a$ and summable on the real line $\mathbb{R}$ admits a representation $f = \sum_{k=1}^4 g_k h_k$ with $f_k, g_k \in \mathcal{PW}_a^2$, $\sum_{k=1}^4 \|g_k\|_2 \cdot \|h_k\|_2 \leqslant C\|f\|_1$.

The paper is organized as follows. In Section 2 we state the main results of the paper: on the representation of bounded truncated Toeplitz operators via Carleson measures for K_θ (Theorem 2.1), on the description of the space predual to $\mathcal{T}(\theta)$ (Theorem 2.3), and on the description of those model spaces where each operator from $\mathcal{T}(\theta)$ has a bounded symbol (Theorem 2.4).

Also we discuss the relationship between embeddings of K_θ into $L^2(\mu)$ and the radial convergence in $L^2(\mu)$. In Section 4 the space X is studied in more detail. Theorems 2.1–2.4 are proved in Sections 5 and 6. Finally, in Section 7 we give a direct proof of the existence of factorizations of the form (5) in the case of one-component inner functions.

2. The main results

Our first theorem answers Sarason's question about representability of bounded truncated Toeplitz operators via Carleson-type measures for K_θ. Recall that for $\mu \in \mathcal{C}_2(\theta)$, the operator A_μ is determined by (3), i.e., $(A_\mu f, g) = \int f\bar{g}\,d\mu$, $f, g \in K_\theta$.

Theorem 2.1. 1) *Every nonnegative bounded truncated Toeplitz operator on K_θ coincides with an operator A_μ for some $\mu \in \mathcal{C}_2^+(\theta)$.*

2) *For every bounded truncated Toeplitz operator A on K_θ there exists a complex measure $\mu \in \mathcal{C}_2(\theta)$ such that $A = A_\mu$.*

In assertion 1) of the theorem, μ cannot in general be chosen absolutely continuous, i.e., bounded nonnegative truncated Toeplitz operators may have no nonnegative symbols. Let δ be the Dirac measure at a point of $\mathbb{T}$, for which the reproducing kernel belongs to K_θ. Then the operator A_δ cannot be realized by a nonnegative symbol unless the dimension of K_θ is 1. Indeed, if μ is a positive absolutely continuous measure, then the embedding $K_\theta \hookrightarrow L^2(\mu)$ must have trivial kernel, while in our example it is a rank-one operator.

The next theorem characterizes operators from $\mathcal{T}(\theta)$ that have bounded symbols.

Theorem 2.2. *A bounded truncated Toeplitz operator A admits a bounded symbol if and only if $A = A_\mu$ for some $\mu \in \mathcal{C}_1(\theta^2)$.*

In the proofs of these results the key role is played by the Banach space X of functions on $\mathbb{T}$ defined m-almost everywhere,

$$X = \left\{ \sum_k x_k \bar{y}_k\colon\ x_k, y_k \in K_\theta,\ \sum_k \|x_k\|_2 \cdot \|y_k\|_2 < \infty \right\}. \tag{6}$$

The norm in X is defined as the infimum of $\sum \|x_k\|_2 \cdot \|y_k\|_2$ over all representations of the element in the form $\sum x_k \bar{y}_k$.

The space of all bounded operators on K_θ is the dual of the space of all trace class operators. By standard duality theory, the space of all truncated Toeplitz operators is the dual of the quotient space, namely, of the trace class on K_θ factored by the subclass of all trace class operators that annihilate all truncated Toeplitz operators. A trace class operator of the form $\sum(\cdot, y_k)x_k$ with $\sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty$ annihilates an operator A if $\sum(Ax_k, y_k) = 0$. If $A = A_\varphi$ is a truncated Toeplitz operator with bounded symbol φ, we obtain

$$\sum(Ax_k, y_k) = \sum \int \varphi \cdot x_k \bar{y}_k\,dm = \int \varphi \cdot \left(\sum x_k \bar{y}_k\right) dm.$$

Therefore, the fact that the operator $\sum(\cdot, y_k)x_k$ annihilates all truncated Toeplitz operators with bounded symbols implies that $\sum x_k \bar{y}_k = 0$ almost everywhere on the unit circle. It is natural to suppose that the weak closure of operators with bounded symbols coincides with the set $\mathcal{T}(\theta)$ of all truncated Toeplitz operators. This would allow us to identify the predual space of $\mathcal{T}(\theta)$ with the space X. This turns out to be possible, although the authors were not able to find a simple proof of this fact. The problems appear with the correctness of definition of the functional (7) for the truncated Toeplitz operators that have no bounded symbol, for which, in place of their symbols, we use the representation via measures from $\mathcal{C}_2(\theta)$ established in Theorem 2.1.

Theorem 2.3. 1) *The space dual to X can be naturally identified with $\mathcal{T}(\theta)$. Namely, continuous linear functionals over X are of the form*

$$\Phi_A(f) = \sum_k (Ax_k, y_k), \quad f = \sum_k x_k \bar{y}_k \in X, \tag{7}$$

with $A \in \mathcal{T}(\theta)$, and the correspondence between the continuous functionals over X and the space $\mathcal{T}(\theta)$ is one-to-one and isometric.

2) *With respect to the duality* (7)*, the space X is dual to the class of all compact truncated Toeplitz operators.*

For the functionals we will also use the notation $\langle A, f\rangle = \Phi_A(f)$.

The next theorem establishes a connection between the factorization problem, Carleson-type embeddings, and the existence of a bounded symbol for every bounded truncated Toeplitz operator on K_θ.

Theorem 2.4. *The following are equivalent:*

1) *any bounded truncated Toeplitz operator on K_θ admits a bounded symbol*;

2) $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2)$;

3) *for any $f \in H^1 \cap \bar{z}\theta^2 \underline{H^1_-}$ there exist $x_k, y_k \in K_\theta$ with $\sum_k \|x_k\|_2 \cdot \|y_k\|_2 < \infty$ such that $f = \sum_k x_k y_k$.*

In the proof it will be shown that condition 2) can be replaced by the stronger condition

2′) $\mathcal{D}_1(\theta^2) = \mathcal{D}_2(\theta^2)$.

Condition 3) also admits formally stronger but in fact equivalent reformulations. If condition 3) is fulfilled, then, by the Closed Graph Theorem, one can always find x_k, y_k such that $\sum_k \|x_k\|_2 \cdot \|y_k\|_2 \leqslant C\|f\|_1$ for some constant C independent of f. Thus, 3) means that $X = H^1 \cap \bar{z}\theta^2 \underline{H^1_-}$ and the norm in X is equivalent to the L^1-norm. Moreover, it follows from Proposition 4.1 that one can require that the sum contain at most four summands.

If θ is a one-component inner function, then all classes $\mathcal{C}_p(\theta)$ coincide, see [6, Theorem 1.4]. If θ is one-component, then θ^2 is, too, hence $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2)$. As an immediate consequence of Theorem 2.4 we obtain the following result conjectured in [9]:

Corollary 2.5. *If θ is a one-component inner function, then the equivalent conditions of Theorem* 2.4 *are fulfilled.*

We do not know if the converse is true, that is, whether the equality $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2)$ implies that θ is one-component. If this is true, it would give us a nice geometrical description of inner functions θ satisfying the equivalent conditions of Theorem 2.4.

Conjecture. *The equivalent conditions of Theorem* 2.4 *are fulfilled if and only if θ is one-component.*

Theorem 2.4 also allows us to considerably extend the class of counterexamples to the conjecture about the existence of a bounded symbol. Let us recall the definition of the Clark measures σ_α [17]. For each $\alpha \in \mathbb{T}$ there exists a finite (singular) positive measure σ_α on $\mathbb{T}$ such that

$$\operatorname{Re}\frac{\alpha+\theta(z)}{\alpha-\theta(z)} = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{\tau}z|^2}\, d\sigma_\alpha(\tau), \quad z \in \mathbb{D}. \tag{8}$$

By Clark's results, the embedding $K_\theta \hookrightarrow L^2(\sigma_\alpha)$ is a unitary operator.

It is shown in [4, Theorem 8] that the condition $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2)$ implies that all measures σ_α are discrete.

Corollary 2.6. *If, for some $\alpha \in \mathbb{T}$, the Clark measure σ_α of an inner function θ is not discrete, then the equivalent conditions of Theorem* 2.4 *fail, and, in particular, there exist operators from $\mathcal{T}(\theta)$ that do not admit a bounded symbol.*

3. Embeddings $K_\theta \hookrightarrow L^2(\mu)$: The radial L^2-convergence

In this section we present a more elementary approach to embedding theorems which is different from that of Theorem 1.1. Sometimes it may be more convenient to work in the L^2-convergence setting than with continuous functions from K_θ. Here we impose an extra assumption $\theta(0) = 0$, or, equivalently, $1 \in K_\theta$, to which the general case can easily be reduced (via transform (12) defined below), but we omit the details of the reduction.

We show that the condition $\mu \in \mathcal{C}_2^+(\theta)$ is equivalent to the existence of an operator $J : K_\theta \to L^2(\mu)$ such that

(i) if $f, zf \in K_\theta$ then $Jzf = zJf$,

(ii) $J1 = 1$.

Moreover, these properties uniquely determine the operator, which turns out to coincide with the embedding operator $K_\theta \hookrightarrow L^2(\mu)$ defined in Theorem 1.1. The proofs are based on the following proposition, which is essentially due to Poltoratski (see [29, Theorem 1.1] and also [23]).

For $g \in K_\theta$, g_r denotes the function $g_r(z) = g(rz)$.

Proposition 3.1. *Let $\theta(0) = 0$. If a bounded operator $J : K_\theta \to L^2(\mu)$ satisfies properties* (i), (ii), *then for any $g \in K_\theta$ we have $\|g_r\|_{L^2(\mu)} \leqslant 2 \cdot \|J\| \cdot \|g\|_2$ and $g_r \to Jg$ in $L^2(\mu)$ as $r \nearrow 1$.*

Proof. Consider the Taylor expansion of $g \in K_\theta$,

$$g(z) = \sum_{k=0}^{\infty} a_k z^k,$$

and introduce the functions $h_n \in K_\theta$,

$$h_n(z) = \sum_{k=0}^{\infty} a_{k+n} z^k.$$

By induction from the relation $Jh_n = a_n + zJh_{n+1}$ we obtain the formula

$$\sum_{k=0}^{n} a_k z^k = Jg - z^{n+1} Jh_{n+1}. \tag{9}$$

We have $\|h_n\|_2 \leqslant \|g\|_2$ and $\|h_n\|_2 \to 0$ as $n \to +\infty$. Since J is a bounded operator from K_θ to $L^2(\mu)$, we have $Jh_{n+1} \to 0$ in the norm of $L^2(\mu)$, and hence $\sum_{k=0}^{n} a_k z^k \to Jg$ in $L^2(\mu)$.

The Abel means of the sequence $(\sum_{k=0}^{n} a_k z^k)_{n\geqslant 0}$ have the form

$$(1-r)\sum_{n=0}^{\infty} r^n \left(\sum_{k=0}^{n} a_k z^k\right) = \sum_{k=0}^{\infty} a_k r^k z^k = g_r(z),$$

and thus $g_r \to Jg$. For the norms by (9) we obtain

$$\|g_r\|_{L^2(\mu)} \leqslant (1-r)\sum_{n=0}^{\infty} r^n \left\| Jg - z^{n+1} Jh_{n+1} \right\|_{L^2(\mu)} \leqslant 2 \cdot \|J\| \cdot \|g\|_2,$$

as required. □

Since for a continuous function $g \in K_\theta$, Jg coincides with g μ-almost everywhere, J is the same operator as the embedding from Theorem 1.1.

The function $\frac{\theta(z)}{z}$ (or $\frac{\theta(z)-\theta(0)}{z}$ in the general case, if $\theta(0) \neq 0$) belongs to K_θ and thus by Proposition 3.1 it has the boundary function in $L^2(\mu)$. This allows us to define the boundary values of θ μ-almost everywhere as the limit in $L^2(\mu)$ of θ_r, $\theta_r(z) = \theta(rz)$.

Proposition 3.2. *If $\mu \in \mathcal{C}_2(\theta)$, then $|\theta| = 1$ $|\mu|$-almost everywhere.*

This fact is mentioned in [4] and its proof there seems to use the techniques of convergence μ-almost everywhere. A more elementary proof is given below for the reader's convenience.

Proof of Proposition 3.2. We may assume that $\mu \in \mathcal{C}_2^+(\theta)$. It is easy to check the relation

$$M_z J - JA_z = (\cdot, \bar{z}\theta)\theta,$$

where M_z is the operator of multiplication by z on $L^2(\mu)$, J is the embedding $K_\theta \hookrightarrow L^2(\mu)$, A_z is the truncated Toeplitz operator with symbol z, i.e., the model contraction S_θ. Indeed, on vectors orthogonal to $\bar{z}\theta$ both sides vanish, and for $\bar{z}\theta$ the formula can be verified by a simple straightforward calculation. Similarly, $M_{\bar{z}} J - JA_{\bar{z}} = (\cdot, 1)\bar{z}$, hence

$$J^* M_z - A_z J^* = (\cdot, \bar{z})1.$$

We obtain

$$\begin{aligned} JJ^*M_z - M_z JJ^* &= J\big(J^*M_z - A_z J^*\big) - (M_z J - JA_z)J^* \\ &= (\cdot, \bar{z})J1 - (\cdot, J\bar{z}\theta)\theta = (\cdot, \bar{z})1 - (\cdot, \bar{z}\theta)\theta. \end{aligned}$$

Theorem 6.1 of [24] says that if $K = \sum(\cdot, \bar{u}_k)v_k$ is a finite rank (or even trace class) operator on $L^2(\mu)$, where μ is a singular measure on $\mathbb{T}$, and if $K = XM_z - M_z X$ for some bounded linear operator X on $L^2(\mu)$, then $\sum u_k v_k = 0$ μ-almost everywhere. By this theorem $z - z|\theta|^2 = 0$, hence $|\theta| = 1$ μ-almost everywhere, as required. $\square$

4. The space X

As above, the space X is defined by formula (6),

$$X = \Big\{\sum x_k \bar{y}_k \colon\ x_k, y_k \in K_\theta,\ \sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty\Big\}.$$

We also consider the analytic analog X_a of the space X,

$$X_a = \Big\{\sum x_k y_k \colon\ x_k, y_k \in K_\theta,\ \sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty\Big\}. \tag{10}$$

Similarly to the norm in the space X, the norm in X_a is also defined as the infimum of $\sum \|x_k\|_2 \cdot \|y_k\|_2$ over all possible representations, thus X, X_a are Banach spaces.

By (4),

$$X \subset \bar{\theta} z H^1 \cap \theta \overline{zH^1}$$

and

$$X_a = \{\bar{z}\theta f \colon\ f \in X\} \subset H^1 \cap \bar{z}\theta^2 H^1_- \subset K^1_{\theta^2}.$$

Proposition 4.1. 1) *Any nonnegative element of X can be written as $|g|^2$, $g \in K_\theta$.*

2) *Any element of X can be represented as a linear combination of four nonnegative elements of X.*

3) *Every element of X, X_a admits a representation as a sum containing only four summands in the definition of these spaces, and the norm of each summand in X or X_a does not exceed the norm of the initial element of the space.*

Proof. 1) Let $f = \sum x_k \bar{y}_k \in X$, $f \geqslant 0$. Since $\bar{z}\theta\bar{y}_k \in K_\theta$, we have $\bar{z}\theta f \in H^1$. Then, by Dyakonov's result [20], $f = |g|^2$ for some $g \in K_\theta$ (proof: take the outer function with modulus $f^{1/2}$ on $\mathbb{T}$ as g; then $\bar{z}\theta|g|^2 = \bar{z}\theta f \in H^1$, hence $\bar{z}\theta\bar{g} \in H^2$ and $g \in K_\theta$).

2) Since X is symmetric with respect to complex conjugation, it suffices to show that a real function from X may be represented as the difference of two nonnegative functions from X. The real part of a function from X of the form $\sum x_k \bar{y}_k$ with $x_k, y_k \in K_\theta$, $\sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty$, can be written as

$$\frac{1}{2}\sum(x_k\bar{y}_k + \bar{x}_k y_k) = \sum\left|\frac{x_k + y_k}{2}\right|^2 - \sum\left|\frac{x_k - y_k}{2}\right|^2,$$

which is the desired representation. We may suppose that $\|x_k\|_2 = \|y_k\|_2$ for every k, then each of the norms $\|\sum|\frac{x_k+y_k}{2}|^2\|_X$, $\|\sum|\frac{x_k-y_k}{2}|^2\|_X$ obviously does not exceed $\sum\|x_k\|_2\cdot\|y_k\|_2$.

3) For the space X this directly follows from 1) and 2), for X_a it remains to use the relation $X_a = \bar{z}\theta X$, which is a consequence of (4). □

Given a function f in the unit disk, define functions f_r, $0<r<1$, by $f_r(z) = f(rz)$. We may think of functions $f \in X_a$ as analytic functions in $\mathbb{D}$. For $f \in X_a$ write $f = \sum x_k y_k$ with $x_k, y_k \in K_\theta$ and $\sum\|x_k\|_2\cdot\|y_k\|_2 < \infty$. We have $f_r = \sum(x_k)_r(y_k)_r$. Now it follows from Proposition 3.1 that for any $\mu \in \mathcal{C}_2^+(\theta)$, the functions f_r have a limit in $L^1(\mu)$ as $r \nearrow 1$. Therefore, the embedding of the space X_a into $L^1(\mu)$ is a well-defined bounded map realized by the limit of f_r.

For the proof of Theorem 2.3 we will need the following important lemma.

Lemma 4.2. *Let $\mu \in \mathcal{C}_2(\theta)$ and let $x_k, y_k \in K_\theta$, $\sum\|x_k\|_2\cdot\|y_k\|_2 < \infty$. If $\sum x_k\bar{y}_k = 0$ in the space X, then also $\sum x_k\bar{y}_k = 0$ $|\mu|$-almost everywhere.*

In other words, the embedding $X \hookrightarrow L^1(|\mu|)$ is well defined.

Proof. There is no loss of generality if we assume that $\mu \in \mathcal{C}_2^+(\theta)$. As above, let J stand for the embedding $K_\theta \hookrightarrow L^2(\mu)$. If $g \in K_\theta$, then $\tilde{g} \in K_\theta$, where $\tilde{g} = \bar{z}\theta\bar{g}$. By Proposition 3.1 the functions $\tilde{g}_r$ have a limit as $r \nearrow 1$, and we want to show that

$$\lim_{r\nearrow 1}\tilde{g}_r = \bar{z}\theta\bar{g} \quad \text{in } L^2(\mu). \tag{11}$$

It suffices to verify this relation on the dense set of all linear combinations of the reproducing kernels $k_\lambda = \frac{1-\overline{\theta(\lambda)}\theta}{1-\bar{\lambda}z}$, $\lambda \in \mathbb{D}$. It is easily seen that for these functions this property is equivalent to the fact that $|\theta_r|^2 \to 1$ μ-almost everywhere, which was proved in Proposition 3.2.

Take $x_k, y_k \in K_\theta$ such that $\sum\|x_k\|_2\cdot\|y_k\|_2 < \infty$ and $\sum x_k\bar{y}_k = 0$. Consider the functions $\tilde{y}_k \in K_\theta$, $\tilde{y}_k = \bar{z}\theta\bar{y}_k$. By (11) we have $(\tilde{y}_k)_r \to \bar{z}\theta\bar{y}_k$ in $L^2(\mu)$. The formula $\sum x_k\tilde{y}_k$ determines the zero element of X_a, hence $\sum(x_k)_r(\tilde{y}_k)_r = (\sum x_k\tilde{y}_k)_r = 0$. We obtain

$$\sum x_k\cdot\bar{z}\theta\bar{y}_k = \lim_{r\nearrow 1}\sum(x_k)_r(\tilde{y}_k)_r = 0$$

in norm in the space $L^1(\mu)$. Since $\theta \neq 0$ μ-almost everywhere (e.g., by Proposition 3.2), we conclude that $\sum x_k\bar{y}_k = 0$ μ-almost everywhere. □

5. Proofs of Theorems 2.1–2.3

Proof of Theorem 2.1. 1) Let A be a nonnegative bounded truncated Toeplitz operator with symbol φ. Define the continuous functional l on the set of all continuous functions from X by $l: f \mapsto \int\varphi f\,dm$. If $f \in X$ is a continuous function, $f \geqslant 0$, then $lf \geqslant 0$. Indeed, by item 1) of Proposition 4.1, there exists a function $g \in K_\theta$ such that $|g|^2 = f$ (g turns out to be bounded), and hence

$$lf = \int\varphi f\,dm = \int\varphi g\bar{g}\,dm = (Ag, g) \geqslant 0.$$

Assume first that $\theta(0) = 0$; then $1 \in X$. Extend the functional l to the space of all continuous functions on $\mathbb{T}$ by the Hahn–Banach theorem so that the norm of the extended functional equals the norm of l. Since $1 \in X$, it will be nonnegative automatically, hence $lf = \int f\,d\mu$ for some nonnegative Borel measure μ on $\mathbb{T}$. The map $K_\theta \to L^2(\mu)$, which takes continuous functions to their traces on the support of μ, is bounded. Indeed, if $g \in K_\theta$ is a continuous function, then $|g|^2$ is continuous as well, and

$$\int |g|^2\,d\mu = l|g|^2 = (Ag, g) \leqslant \|A\| \cdot \|g\|_2^2.$$

This proves that $\mu \in \mathcal{C}_2^+(\theta)$. By linearity and continuity the relation $\int |g|^2 d\mu = (Ag, g)$, $g \in K_\theta$, implies $\int x\bar{y}\,d\mu = (Ax, y)$ for all $x, y \in K_\theta$, hence $A = A_\mu$.

If $w = \theta(0) \neq 0$, consider the so-called Crofoot transform

$$U : f \mapsto \sqrt{1 - |w|^2}\,\frac{f}{1 - \overline{w}\theta}, \tag{12}$$

which unitarily maps K_θ onto K_Θ, where $\Theta = \frac{\theta - w}{1 - \overline{w}\theta}$ is the Frostman shift of θ. Take a bounded truncated Toeplitz operator $A \geqslant 0$ on K_θ. By [31, Theorem 13.2] the operator $UAU^* \geqslant 0$ is a bounded truncated Toeplitz operator on K_Θ. Note that $\Theta(0) = 0$. Find a measure $\nu \in \mathcal{C}_2^+(\Theta)$ such that UAU^* coincides with the truncated Toelpitz operator A_ν on K_Θ and define $\mu = \frac{1-|w|^2}{|1-\overline{w}\theta|^2}\nu$. Then A coincides with the operator A_μ (on K_θ). Indeed, $\mu \in \mathcal{C}_2^+(\theta)$, and from (12) it follows that for any $f, g \in K_\theta$ we have

$$(Af, g) = \bigl(UAU^*\,Uf, Ug\bigr) = \int Uf \cdot \overline{Ug}\,d\nu = \int f\bar{g}\,d\mu,$$

as required.

2) Let A be a bounded truncated Toeplitz operator. It may be represented in the form $A = A_1 - A_2 + iA_3 - iA_4$, where all A_i, $i = 1, 2, 3, 4$, are nonnegative truncated Toeplitz operators. Indeed, A^* is a truncated Toeplitz operators as well, which allows us to consider only selfadjoint operators. The identity operator I is trivially a truncated Toeplitz operator (with symbol 1), and A is the difference of two nonnegative operators $\|A\| \cdot I$ and $\|A\| \cdot I - A$. For each A_i construct μ_i as above. It remains to take $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$. $\square$

Proof of Theorem 2.2. If A has a bounded symbol φ, then $A = A_\mu$ with $d\mu = \varphi\,dm$, and $\mu \in \mathcal{C}_1(\theta^2)$.

Now let $\mu \in \mathcal{C}_1(\theta^2)$. We need to prove that A_μ coincides with a truncated Toeplitz operator having a bounded symbol. Define the functional $l : f \mapsto \int f\,d\mu$ on functions from X that are finite sums of functions of the form $x_k\bar{y}_k$ with $x_k, y_k \in K_\theta$. Since $\theta\bar{z}f \in K^1_{\theta^2}$ and $\mu \in \mathcal{C}_1(\theta^2)$, we get

$$\left|\int f\,d\mu\right| \leqslant \int |f|\,d|\mu| = \int |\theta\bar{z}f|\,d|\mu| \leqslant C \cdot \|\theta\bar{z}f\|_1 = C \cdot \|f\|_1.$$

Therefore, the functional l can be continuously extended to L^1 and there exists a function $\varphi \in L^\infty$ such that $lf = \int \varphi f\, dm$, $f \in X$. Hence for any $x, y \in K_\theta^2$ we have

$$\int x\bar{y}\, d\mu = l(x\bar{y}) = \int \varphi x\bar{y}\, dm = (A_\varphi x, y),$$

that is, $A_\mu = A_\varphi$. $\square$

Proof of Theorem 2.3. 1) First, we verify that the functional (7) is well defined for any operator $A \in \mathcal{T}(\theta)$. This fact looks natural, but it does not seem to be obvious and its proof is essentially based on Theorem 2.1 and Lemma 4.2.

We need to prove that $\sum (Ax_k, y_k) = 0$ if $x_k, y_k \in K_\theta$, $\sum \|x_k\|_2 \cdot \|y_k\|_2 < \infty$, and $\sum x_k \bar{y}_k = 0$ almost everywhere with respect to the Lebesgue measure. To this end, apply Theorem 2.1 and find a measure $\mu \in \mathcal{C}_2(\theta)$ such that $(Ax, y) = \int x\bar{y}\, d\mu$ for all $x, y \in K_\theta$. Then, by Lemma 4.2, $\sum x_k \bar{y}_k = 0$ μ-almost everywhere. By the definition of A_μ we have

$$\sum (Ax_k, y_k) = \int \Bigl(\sum x_k \bar{y}_k\Bigr)\, d\mu.$$

Thus $\sum (Ax_k, y_k) = 0$, and the functional is defined correctly.

Now prove that $\|\Phi_A\| = \|A\|$. Indeed, for any element $\sum x_k \bar{y}_k \in X$ we have

$$\Bigl|\Phi_A\Bigl(\sum x_k \bar{y}_k\Bigr)\Bigr| = \Bigl|\sum (Ax_k, y_k)\Bigr| \leqslant \|A\| \cdot \sum \|x_k\|_2 \cdot \|y_k\|_2.$$

Hence $\|\Phi_A\| \leqslant \|A\|$. Conversely, for any $x, y \in K_\theta$ with $\|x\|_2 = \|y\|_2 = 1$ we have $\|x\bar{y}\|_X \leqslant 1$ and

$$\|A\| = \sup_{\|x\|_2, \|y\|_2 \leqslant 1} \bigl|(Ax, y)\bigr| = \sup_{\|x\|_2, \|y\|_2 \leqslant 1} \bigl|\Phi_A(x\bar{y})\bigr| \leqslant \|\Phi_A\|.$$

It remains to show that any linear continuous functional Φ on X may be represented in the form $\Phi = \Phi_A$ for some (unique) truncated Toeplitz operator A. Define the operator $A = A_\Phi$ by the bilinear form: $(A_\Phi x, y) \overset{\text{def}}{=} \Phi(x\bar{y})$. If $f, zf \in K_\theta$, we have

$$(A_\Phi f, f) = \Phi\bigl(|f|^2\bigr) = \Phi\bigl(|zf|^2\bigr) = (A_\Phi zf, zf).$$

Now, applying Theorem 1.2, we obtain $A_\Phi \in \mathcal{T}(\theta)$. The uniqueness of A is a consequence of the relation $\|A_\Phi\| = \|\Phi\|$.

2) We need to prove that every continuous functional over the space of all compact truncated Toeplitz operators is realized by an element of X. Take a functional Φ and extend it by the Hahn–Banach theorem to the space of all compact operators on K_θ. The trace class is the dual space to the class of all compact operators, hence the functional may be written in the form $\Phi(A) = \sum (Ax_k, y_k)$ for some $x_k, y_k \in K_\theta$ with $\sum \|x_k\|_2 \cdot \|y_k\|_2 = \|\Phi\|$. This means that $f = \sum x_k \bar{y}_k \in X$ and $\Phi(A) = \langle A, f\rangle$.

Repeating the arguments from 1) we conclude that $\|\Phi\| = \|f\|_X$. $\square$

The results obtained allow us to get additional information on the structure of the space of truncated Toeplitz operators.

For $\lambda \in \mathbb{D}$, denote by k_λ, $\tilde{k}_\lambda$ the functions from K_θ,

$$k_\lambda(z) = \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\bar{\lambda}z}, \qquad \tilde{k}_\lambda(z) = \frac{\theta(z)-\theta(\lambda)}{z-\lambda}.$$

If $x, y \in K_\theta$, then $(x, k_\lambda) = x(\lambda)$, $(\tilde{k}_\lambda, y) = \tilde{y}(\lambda)$. It is shown in [31] that the operators

$$T_\lambda = (\cdot, k_\lambda)\tilde{k}_\lambda$$

are rank-one truncated Toeplitz operators. One of the possible choices for a bounded symbol of T_λ is $\frac{1}{1-|\lambda|^2}\bar{b}_\lambda\theta$, where $b_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}$. Take $f \in X$ as an element of the dual space to the class of all compact truncated Toeplitz operators and pre-dual to the space of all truncated Toeplitz operators, let $f = \sum x_k\bar{y}_k$ with $x_k, y_k \in K_\theta$. The following formula illustrates the duality on rank-one truncated Toeplitz operators:

$$\begin{aligned}\langle T_\lambda, f\rangle &= \Big\langle T_\lambda, \sum x_k\bar{y}_k\Big\rangle = \sum (T_\lambda x_k, y_k)\\ &= \sum (x_k, k_\lambda)\cdot(\tilde{k}_\lambda, y_k) = \sum x_k(\lambda)\cdot\tilde{y}_k(\lambda) = g(\lambda),\end{aligned} \tag{13}$$

where $g = \sum x_k\tilde{y}_k = \bar{z}\theta f \in X_a$.

Corollary 5.1. 1. *The closure in the $*$-weak topology of the set of all finite-rank truncated Toeplitz operators coincides with the set of all truncated Toeplitz operators.*

2. *The closure in the norm of the set of all finite-rank truncated Toeplitz operators coincides with the set of all compact truncated Toeplitz operators.*

Moreover, in place of the set of all finite-rank operators it suffices to take only the linear span of the operators T_λ, where λ runs over a sequence $\Lambda = (\lambda_k)$ in the unit disk with $\sum(1-|\lambda_k|) = \infty$.

Proof. Take $f \in X$ such that $\langle T_\lambda, f\rangle = 0$ for every $\lambda \in \Lambda$. By formula (13), then $g(\lambda) = 0$ for all $\lambda \in \Lambda$. Hence $g \equiv 0$ and f is the zero element of X. Now the claim easily follows. □

The space of all bounded linear operators on a Hilbert space is dual to the space of trace class operators. This duality generates the $*$-weak topology on the former space. Formally, the $*$-weak topology is stronger than the weak operator topology, but on the subspace of all truncated Toeplitz operators they coincide.

Corollary 5.2. *The weak operator topology on $\mathcal{T}$ coincides with the $*$-weak topology.*

Proof. Any $*$-weakly continuous functional Φ on $\mathcal{T}$ is generated by some trace class operator $\sum_k(\cdot, y_k)x_k$, where $x_k, y_k \in K_\theta$, $\sum_k \|x_k\|_2 \cdot \|y_k\|_2 < \infty$, and is of the form

$$\Phi(A) = \sum_k (Ax_k, y_k).$$

The function $h = \sum_k x_k \bar{y}_k$ belongs to the space X. It follows from Proposition 4.1 that there exist $f_1, g_1 \dots f_4, g_4 \in K_\theta$ such that $h = \sum_{k=1}^4 f_k \bar{g}_k$. Therefore, by the duality from Theorem 2.3,

$$\Phi(A) = \langle A, h \rangle = \left\langle A, \sum_{k=1}^{4} f_k \bar{g}_k \right\rangle = (Af_1, g_1) + \dots + (Af_4, g_4).$$

Now the statement is obvious. □

6. Proof of Theorem 2.4

Throughout this section we will assume, for simplicity, that $\theta(0) = 0$. The general case follows immediately by means of the transform (12) (note that in this case $\mathcal{C}_p(\Theta) = \mathcal{C}_p(\theta)$ for any p, see, e.g., [5, Theorem 1.1]).

For the proof of Theorem 2.4 we need the following obvious lemma (see [5]) based on the relations $K_{\theta^2} = K_\theta \oplus \theta K_\theta$ and $K_\theta \cdot K_\theta \subset K^1_{\theta^2}$.

Lemma 6.1. *For any inner function θ we have $\mathcal{C}_2(\theta) = \mathcal{C}_2(\theta^2)$ and $\mathcal{C}_1(\theta^2) \subset \mathcal{C}_2(\theta^2)$. If $\theta(0) = 0$, we also have $\mathcal{C}_p(\theta^2) = \mathcal{C}_p(\theta^2/z)$ for any p. The same equalities or inclusions hold for the classes $\mathcal{D}_p(\theta)$.*

Proof of Theorem 2.4. 3) $\Rightarrow$ 2). We will establish condition 2′), which is formally stronger than 2). By Lemma 6.1 it suffices to prove the inclusion $\mathcal{D}_2(\theta) \subset \mathcal{D}_1(\theta^2/z)$. Take a complex measure $\mu \in \mathcal{D}_2(\theta)$. We must check that the embedding $K^1_{\theta^2/z} \hookrightarrow L^1(|\mu|)$ is a bounded operator. By the Closed Graph Theorem, condition 3) yields the existence of a positive constant c_1 such that any function $h \in K^1_{\theta^2/z}$ can be represented in the form $h = \sum_{k=1}^\infty f_k g_k$, where $f_k, g_k \in K_\theta$ and $\sum_{k=1}^\infty \|f_k\|_2 \cdot \|g_k\|_2 \leqslant c_1 \|h\|_1$. Since $\mu \in \mathcal{D}_2(\theta)$, we have $\sum_{k=1}^\infty \|f_k\|_{L^2(|\mu|)} \cdot \|g_k\|_{L^2(|\mu|)} < \infty$, so the series converges in $L^1(|\mu|)$ and $h \in L^1(|\mu|)$. Moreover,

$$\int |h|\, d|\mu| = \int \left| \sum_{k=1}^\infty f_k g_k \right| d|\mu| \leqslant \sum_{k=1}^\infty \|f_k\|_{L^2(|\mu|)} \cdot \|g_k\|_{L^2(|\mu|)}$$
$$\leqslant c_2 \sum_{k=1}^\infty \|f_k\|_2 \cdot \|g_k\|_2 \leqslant c_1 c_2 \|h\|_1,$$

and thus $\mu \in \mathcal{D}_1(\theta^2/z)$. Therefore, $\mathcal{D}_2(\theta) \subset \mathcal{D}_1(\theta^2/z)$, which implies $\mathcal{D}_2(\theta^2) = \mathcal{D}_1(\theta^2)$ (and, in particular, $\mathcal{C}_2(\theta^2) = \mathcal{C}_1(\theta^2)$).

The implication 2) $\Rightarrow$ 1) directly follows from Theorems 2.1 and 2.2.

1) $\Rightarrow$ 3). Condition 3) can be written in the form $X_a = K^1_{\theta^2/z}$ or, equivalently, as $X = z\bar{\theta} K^1_{\theta^2/z}$, see Section 4; in the general case X is a dense subset of $z\bar{\theta} K^1_{\theta^2/z}$. By the Closed Graph Theorem, $X = z\bar{\theta} K^1_{\theta^2/z}$ if and only if the norms in the spaces X and $K^1_{\theta^2/z}$ are equivalent. Take an arbitrary function $h = \sum x_k \bar{y}_k \in X$. Clearly, $\|h\|_1 \leqslant \|h\|_X$. We need to show that $\|h\|_X \leqslant c\|h\|_1$ for some constant $c > 0$.

From Theorem 2.3 we obtain

$$\|h\|_X = \sup\left\{\left|\sum (Ax_k, y_k)\right|:\ A \in \mathcal{T}(\theta),\ \|A\| \leqslant 1\right\}. \tag{14}$$

The Closed Graph Theorem and condition 1) guarantee that any operator $A \in \mathcal{T}(\theta)$ admits a bounded symbol $\varphi \in L^\infty$ with $\|\varphi\|_\infty \leqslant c\|A\|$. Therefore, the supremum in (14) does not exceed

$$\begin{aligned}\sup\left\{\left|\sum (\varphi x_k, y_k)\right|:\ \varphi \in L^\infty,\ \|\varphi\|_\infty \leqslant c\right\} &= c \sup_{\|\varphi\|_\infty \leqslant 1} \left|\int \varphi \cdot \sum x_k \bar{y}_k\, dm\right| \\ &= c \sup_{\|\varphi\|_\infty \leqslant 1} \left|\int \varphi h\, dm\right| = c\|h\|_1,\end{aligned}$$

as required.

The theorem is proved. □

7. One-component inner functions

As we noted in Section 2, one-component inner functions satisfy condition 2) of Theorem 2.4: $\mathcal{C}_2(\theta) = \mathcal{C}_1(\theta^2)$ (recall that $\mathcal{C}_2(\theta) = \mathcal{C}_2(\theta^2)$). It is also possible to show directly that one-component inner functions satisfy the factorization condition 3) of Theorem 2.4. We will start with the particular case of the Paley–Wiener spaces $\mathcal{PW}_a^p$ (see Section 1.3).

Example 7.1. Let $\Theta_a(z) = \exp(iaz)$, $a > 0$, be an inner function in the upper half-plane. Then for the corresponding model subspace we have $K_{\Theta_a}^p = \mathcal{PW}_a^p \cap H^p$. Note that the model subspaces on the half-plane are defined as $K_\Theta^p = H^p \cap \Theta\overline{H^p}$, the involution is given by $f \mapsto \Theta\bar{f}$, and hence $fg \in K_{\Theta^2}^1$ for any $f, g \in K_\Theta^2$. Thus, in view of Proposition 4.1, the factorization for the corresponding space X is equivalent to the following property: for any $f \in \mathcal{PW}_{2a}^1$ that takes real values on the real line $\mathbb{R}$ there exists a function $g \in \mathcal{PW}_{2a}^1$ such that $|f| \leqslant g$ on $\mathbb{R}$. This can easily be achieved. Let $a = \pi/2$. Put

$$g(z) = \sum_{n\in\mathbb{Z}} c_n \frac{\sin^2 \frac{\pi}{2}(t-n)}{(t-n)^2},$$

where $c_n = \max_{[n,n+1]} |f|$. By the Plancherel–Pólya inequality (see, e.g., [25, Lecture 20]), $\sum_n c_n \leqslant C\|f\|_1$, hence $g \in \mathcal{PW}_\pi^1$. Also, if $t \in [n, n+1]$, then

$$\left|f(t)\right| \leqslant c_n \leqslant c_n \frac{\sin^2 \frac{\pi}{2}(t-n)}{(t-n)^2} \leqslant g(t). \qquad \square$$

An analogous argument works for general one-component inner functions. Let θ be an inner function in the unit disk. As usual, $k_\lambda = \frac{1-\overline{\theta(\lambda)}\theta}{1-\bar{\lambda}z}$, $\lambda \in \mathbb{D}$, the reproducing kernel of the space K_θ, that is, $(f, k_\lambda) = f(\lambda)$ for any $f \in K_\theta$. In some cases this formula may be extended to $\lambda = t \in \mathbb{T}$.

In view of Proposition 4.1, property 3) of Theorem 2.4 will be established as soon as we prove the following theorem.

Theorem 7.2. *Let θ be a one-component inner function. For any real-valued element $f \in X$ there exist $t_n \in \mathbb{T}$ and $c_n > 0$ such that*

$$g = \sum c_n |k_{t_n}|^2 \in X,$$

and for some constants $C, M > 0$ we have $\|g\|_1 \leqslant C \cdot \|f\|_1$, and $|f| \leqslant M \cdot g$ on $\mathbb{T}$.

First, we collect some known properties of one-component inner functions.

(i) Let $\rho(\theta)$ be the so-called *spectrum* of the inner function θ, that is, the set of all $\zeta \in \overline{\mathbb{D}}$ such that $\liminf_{z\to\zeta, z\in\mathbb{D}} |\theta(z)| = 0$. Then θ, as well as any function from K^p_θ, has an analytic extension across any subarc of the set $\mathbb{T} \setminus \rho(\theta)$.

Let σ_α be the Clark measures defined by (8). Recall that the embedding $K_\theta \hookrightarrow L^2(\sigma_\alpha)$ is a unitary operator [17]. Moreover, if σ_α is discrete, i.e., if $\sigma_\alpha = \sum_n a_n \delta_{t_n}$, then the system $\{k_{t_n}\}$ is an orthogonal basis in K_θ; in particular, $k_{t_n} \in K_\theta$ and $\|k_{t_n}\|_2^2 = |\theta'(t_n)|$ (cf. [1]). It is shown in [6] that for a one-component inner function $\sigma_\alpha(\rho(\theta)) = 0$ for any $\alpha \in \mathbb{T}$. Thus, all Clark measures are purely atomic and supported on the set $\mathbb{T} \setminus \rho(\theta)$ (cf. Corollary 2.6).

(ii) On each arc of the set $\mathbb{T} \setminus \rho(\theta)$, there exists a smooth increasing branch of the argument of θ (denote it by ψ) and the change of the argument between two neighboring points from the support of each Clark measure is exactly 2π.

(iii) By $[t_n, t_{n+1}]$ we denote the closed arc with endpoints t_n, t_{n+1}, which contains no other points from the Clark measure support. There exists a constant $A = A(\theta)$ such that for any two points t_n, t_{n+1} satisfying $|\psi(t_{n+1}) - \psi(t_n)| = 2\pi$ and for any s, t from the arc $[t_n, t_{n+1}]$,

$$A^{-1} \leqslant \frac{|\theta'(s)|}{|\theta'(t)|} \leqslant A, \tag{15}$$

that is, $|\theta'|$ is almost constant, when the change of the argument is small. This follows from the results of [6], a detailed proof may be found in [10, Lemma 5.1].

(iv) If θ is one-component, then $\mathcal{C}_1(\theta) = \mathcal{C}_2(\theta)$. The same holds for the function θ^2 which is also one-component. By Lemma 6.1, $\mathcal{C}_1(\theta^2) = \mathcal{C}_2(\theta^2) = \mathcal{C}_2(\theta)$, and there is a constant B such that for any measure $\mu \in \mathcal{C}_2^+(\theta)$ we have

$$\sup_{f\in K^1_{\theta^2}} \frac{\|f\|_{L^1(\mu)}}{\|f\|_1} \leqslant B \cdot \sup_{f\in K^2_\theta} \frac{\|f\|_{L^2(\mu)}}{\|f\|_2}. \tag{16}$$

(v) Let $\{t_n\}$ be the support of some Clark measure for θ and let $s_n \in [t_n, t_{n+1}]$. There exists a constant $C = C(\theta)$ which does not depend on $\{t_n\}$ and $\{s_n\}$ such that for any $f \in K^2_\theta$ we have

$$\sum_n \frac{|f(s_n)|^2}{|\theta'(s_n)|} \leqslant C\|f\|_2^2.$$

This follows from the stability result [19, Theorem 3] due to Cohn.

So (v) means that for the measures of the form $\sum_n |\theta'(s_n)|^{-1}\delta_{s_n}$ the supremum in the right-hand side of (16) is uniformly bounded. From this, (iii) and (iv) we have the following Plancherel–Pólya type inequality.

Corollary 7.3. *Let $\{t_n\}$ be the support of some Clark measure for θ and let $s_n, u_n \in [t_n, t_{n+1}]$. There exists a constant $C = C(\theta)$, which does not depend on $\{t_n\}$, $\{s_n\}$, $\{u_n\}$, such that for any $f \in X \subset K^1_{\theta^2}$ we have*

$$\sum_n \frac{|f(s_n)|}{|\theta'(u_n)|} \leqslant C\|f\|_1. \tag{17}$$

Proof of Theorem 7.2. Let $f \in X$. Take two Clark bases corresponding to $\alpha = 1, -1$, and let $\{t_n\}$ be the union of their supports. If t_n, t_{n+1} are two neighbor points of this set, then

$$\int\limits_{[t_n, t_{n+1}]} \left|\theta'(t)\right| dm(t) = \pi.$$

If we write $t_n = e^{ix_n}$ and take the branch of the argument ψ so that $\theta(e^{ix}) = e^{2i\psi(x)}$, then $|\psi(x_{n+1}) - \psi(x_n)| = \pi/2$.

Let $c_n = \sup_{t\in[t_n,t_{n+1}]} |f(t)|$ and put

$$g(z) = \sum c_n \frac{|k_{t_n}(z)|^2}{|\theta'(t_n)|^2}.$$

Then the series converges in L^1-norm. Indeed, $c_n = |f(s_n)|$ for some $s_n \in [t_n, t_{n+1}]$ and

$$\sum |c_n| \frac{\|k^2_{t_n}\|_1}{|\theta'(t_n)|^2} = \sum \frac{|f(s_n)|}{|\theta'(t_n)|} \leqslant C\|f\|_1$$

by Corollary 7.3. Hence $g \in L^1$; moreover, $g \in X$ and $g \geqslant 0$.

It remains to show that $|f| \leqslant M \cdot g$ for some $M > 0$. Let $t = e^{ix} \in [t_n, t_{n+1}]$. We have

$$\left|k_{t_n}(t)\right| = \left|\frac{\theta(t) - \theta(t_n)}{t - t_n}\right| = \left|2\frac{\sin(\psi(x) - \psi(x_n))}{e^{ix} - e^{ix_n}}\right|. \tag{18}$$

Since $|\psi(x) - \psi(x_n)| \leqslant \pi/2$, we obtain $|\sin(\psi(x) - \psi(x_n))| \geqslant 2|\psi(x) - \psi(x_n)|/\pi$. Since $|e^{ix} - e^{ix_n}| \leqslant |x - x_n|$, the last quantity in (18) is

$$\geqslant \frac{4}{\pi} \cdot \left|\frac{\psi(x) - \psi(x_n)}{x - x_n}\right| = \frac{4\psi'(y_n)}{\pi}$$

for some $y_n \in [x_n, x]$. If we put $u_n = e^{iy_n}$, we get $\psi'(y_n) = |\theta'(u_n)|/2$. Thus, we have shown that $|k_{t_n}(t)| \geqslant 2|\theta'(u_n)|/\pi$ for some $u_n \in [t_n, t_{n+1}]$. Hence, if we take $M > \pi^2 A^2/4$, then

$$M \cdot g(t) > M \cdot c_n \frac{|k_{t_n}(t)|^2}{|\theta'(t_n)|^2} \geqslant M \cdot \frac{4|\theta'(u_n)|^2}{\pi^2|\theta'(t_n)|^2} c_n \geqslant \frac{4M}{\pi^2 A^2} c_n \geqslant c_n \geqslant \left|f(t)\right|.$$

The theorem is proved. □

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