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Positive mass theorem in extended supergravities

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Abstract

Following the Witten–Nester formalism, we present a useful prescription using Weyl spinors towards the positivity of mass. As a generalization of arXiv:1310.1663, we show that some "positivity conditions" must be imposed upon the gauge connections appearing in the supercovariant derivative acting on spinors. A complete classification of the connection fulfilling the positivity conditions is given. It turns out that these positivity conditions are indeed satisfied for a number of extended supergravity theories. It is shown that the positivity property holds for the Einstein-complex scalar system, provided that the target space is Hodge–Kähler and the potential is expressed in terms of the superpotential. In the Einstein–Maxwell-dilaton theory with a dilaton potential, the dilaton coupling function and the superpotential are fixed by the positive mass property. We also explore the $N = 8$ gauged supergravity and demonstrate that the positivity of the mass holds independently of the gaugings and the deformation parameters.

1. Introduction

The positive mass theorem [1,2] is one of the major achievements in mathematical theory of general relativity. If the positivity property of the mass fails to be valid, the vacuum Minkowski spacetime which obviously has a vanishing mass possibly decays into configurations with lower energy, and a dynamical "chasing instability" is unavoidable due to the weak

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equivalence principle [3]. The positive mass theorem therefore forbids these undesirable phenomena and accounts for the stability of the lowest energy states.

Since the first proof given by Schoen and Yau [1,2], various attempts have been done towards the generalization. This subject is stimulated not only by a purely mathematical interest. First of all, the proof of Schoen and Yau cannot be applied to $D \geq 9$ dimensions, since the smoothness of the deformation of the n-dimensional minimal surface S_n is guaranteed for $n \leq 6$. The proof based on the inverse mean curvature flow [4,5] provides a physically clear interpretation. However, it has been successful only in $D = 4$ since the Gauss–Bonnet theorem over the two-surface was explicitly used therein. Furthermore, both of these methods work only in the spacetimes that tend asymptotically to the Minkowski spacetime. Compared to these proofs, a remarkably simple and elegant proof was given by Witten [6], later refined by Nester [7]. A distinguished feature of their proof is the use of a spinor field. The bilinear vector built out of the spinor field used in their proof plays the role of the infinitesimal generator of the asymptotic symmetry. Although the use of spinor imposes a mild restriction upon the spacetime topology,[1] this proof is sometimes more powerful since it is able to give a strictly positive bound on the mass, rather than a simple positivity thereof. Moreover, the Witten–Nester approach has additional advantages that it works in arbitrary dimensions, it is applicable also for asymptotically anti-de Sitter (AdS) spacetimes and it does not necessarily require the dominant energy condition. Another utility of using spinors is that it possess an intimate relationship to supergravity theories [11–14].

Recently, a number of widespread revival interests in extended supergravities have been growing from the viewpoint of string theory and AdS/CFT correspondence. Among others, the supersymmetric solutions in supergravity have played a central role in their theoretical development. Since supersymmetric solutions belong to the short multiplets, they are essentially nonperturbative objects, hence they usually evade instabilities. They are characterized by the existence of Killing spinors obeying the first-order differential equations [15]. Similar to the Bogomol'nyi–Prasad–Sommerfield (BPS) states in solitons, they are often identified as states saturating a certain kind of inequality between conserved quantities implied by the positive mass theorem. Note that this is not obvious since the quantities in the superalgebra are associated with the invariance of the background spacetime, i.e., they do not correspond to the conserved quantities in the general curved spacetime which approaches asymptotically to that background.

Thus far, various supergravity theories have been shown to admit the BPS bound [16–24], in terms of globally conserved quantities. It should be worth commenting that the converse statement is not always true, namely, the theory admitting the BPS-type inequality does not necessarily have a supergravity origin. For example, the Einstein–Maxwell-dilaton theory admits the BPS-type inequality [10]. It was realized, however, that the first-order BPS equation for the saturation of inequality is incompatible with the equations of motion except for the particular values of the coupling constant [25]. This implies that it is not always possible to embed the theory admitting BPS-type inequality into supergravity.

At the current moment, it is also less obvious which theories admit the BPS-type inequality, when the supergravity embedding is unknown. In our previous paper [26], we tackled this problem pursuant to the Witten–Nester argument, and found that a certain condition should be

[1] The condition that the spacetime admits the spin structure amounts to requiring that the second Stiefel–Whitney class should vanish. Some five-dimensional asymptotically flat soliton solutions found in Refs. [8,9] violate the mass bound proven by the spinorial method in [10], since they fail to possess the spin structure. It is an interesting but a challenging task to derive the lower bound of five-dimensional Arnowitt–Deser–Misner (ADM) mass in Einstein–Maxwell–Chern–Simons gravity without assuming the spin structure.

imposed toward the positivity bound upon the connection in the supercovariant derivative acting on a Dirac spinor. By virtue of this condition, we were able to construct the first instance of noncanonical scalar-field system admitting the BPS-type inequality [26] (see also [27]). In the current article, we generalize the argument in [26] and reformulate the "positivity conditions" in terms of Weyl spinors. We also provide a proof for the classification of connections satisfying the positivity conditions. This would make it clear the relationship to the four-dimensional extended supergravity theories. In $N = 1$ supergravity, it has been widely known that the theory admits the positive mass [11]. The $N > 1$ case is less clear since extended supergravities do not always have an $N = 1$ description except for the consistent truncation. The purpose of the present paper is to examine the positivity property of various theories inspired by extended supergravities. Using the positivity conditions, we resolve some issues about the BPS-type inequality in extended supergravities, and demonstrate that the positivity property is indeed true for wider theories than formerly considered.

The plan of the present paper is as follows. In the next section, we formulate the Witten–Nester method in terms of Weyl spinors and address the positivity of the Witten–Nester energy. We find that the gauge connections appearing in the supercovariant derivatives should satisfy the "positivity conditions." This is a generalization of our previous work [26]. A classification of the connections satisfying the positivity conditions is given in Appendix A, where it is shown that the possible connections take the same form as those appearing in extended supergravity, provided we impose an additional condition that the bilinear vector is a Killing field for the BPS geometry. In Section 3, we apply this formalism to various theories inspired by supergravity. We resolve some problems in the literature and find that the positivity of energy holds in much broader class of theories than previously studied. In particular the maximal gauged supergravity turns out to admit the mass positivity, independent of the gaugings and symplectic frames. The final conclusion with some future prospective works is described in Section 4.

Our conventions for the metric is taken to be mostly plus sign. $\mu, \nu, \dots$ refer to the spacetime indices, whereas $a, b, \dots$ to the frame indices. We adopt the units $c = 8\pi G = 1$ throughout the paper.

2. Positive mass theorem à la Witten–Nester

In our previous paper [26], we derived a minimal condition toward the positive mass for the gauge connection in the supercovariant derivative acting on a Dirac spinor. This condition provides a universally simple formula and is able to easily recover all of the previous positive mass results. In the present paper we are interested in theories inspired by extended supergravities. Hence it turns out to be more advantageous to generalize the analysis [26] in terms of Weyl spinors. We shall restrict exclusively to four dimensions for simplicity, although the higher (even) dimensional extension is straightforward. We will work in mostly plus metric signature and the Clifford algebra reads $\{\gamma_a, \gamma_b\} = 2\eta_{ab} = 2\,\mathrm{diag}(-1, 1, 1, 1)_{ab}$. Taking the orientation as $\epsilon_{0123} = 1$, the chiral matrix is defined by $\gamma_5 = -(i/4!)\epsilon_{abcd}\gamma^{abcd} = i\gamma_{0123}$ with $\gamma_5^2 = 1$. The imaginary (anti-)self dual part $H^\pm$ of the 2-form $H_{\mu\nu}$ is $H^\pm = \frac{1}{2}(H \mp i \star H)$, satisfying $\star H^\pm = \pm i H^\pm$.

We denote the set of Weyl spinors in four dimensions by ϵ_i $(i = 1, \dots, N)$. We take these spinors to have a negative chirality $\gamma_5 \epsilon_i = -\epsilon_i$. If we define the Dirac conjugate of ϵ_i by $\bar{\epsilon}^i \equiv i(\epsilon_i)^\dagger \gamma^0$, the charge conjugation of ϵ_i is denoted as

$$\epsilon^i \equiv (\epsilon_i)^c = C(\bar{\epsilon}^i)^T = -i\gamma^0 C(\epsilon_i)^*, \tag{1}$$

where C is the charge conjugation matrix satisfying $C^{-1}\gamma^\mu C = -\gamma_\mu^T$. In this paper we adopt the representation such that the charge conjugation matrix is given by $C = -i\gamma^0$. This enables us to raise and lower the indices $i, j, \dots$ simply by the complex conjugation and the gamma matrices are all real, hence $\gamma_\mu^T = \gamma^0\gamma_\mu\gamma^0$. It then follows that the spinors ϵ^i with upper index have a positive chirality $\gamma_5\epsilon^i = \epsilon^i$ and the Dirac conjugate of ϵ^i is given by $\bar{\epsilon}_i = -i(\epsilon^i)^\dagger\gamma^0$. Accordingly, the bilinears constructed out of the spinor satisfy

$$\bar{\epsilon}_i\epsilon_j = -\bar{\epsilon}_j\epsilon_i = \left(\bar{\epsilon}^i\epsilon^j\right)^*, \qquad i\bar{\epsilon}^i\gamma^\mu\epsilon_j = -i\bar{\epsilon}_j\gamma^\mu\epsilon^i, \qquad i\bar{\epsilon}_i\gamma_{\mu\nu}\epsilon_j = i\bar{\epsilon}_j\gamma_{\mu\nu}\epsilon_i. \tag{2}$$

Following the argument given in [26], we define the supercovariant derivative operator as follows

$$\hat{\nabla}_\mu\epsilon_i \equiv \nabla_\mu\epsilon_i + \mathscr{A}_{\mu i}{}^j\epsilon_j + \mathscr{B}_{\mu ij}\epsilon^j, \tag{3}$$

where ∇_μ is an ordinary Lorentz-covariant derivative acting on a spinor. The $N \times N$ numbers of 4×4 matrix-valued vector fields $\mathscr{A}_{\mu i}{}^j$ and $\mathscr{B}_{\mu ij}$ represent the deviation from the Levi-Cività connection. These connections obey different commutation relations with the chirality matrix

$$\left[\mathscr{A}_{\mu i}{}^j, \gamma_5\right] = 0, \qquad \{\mathscr{B}_{\mu ij}, \gamma_5\} = 0. \tag{4}$$

The Dirac conjugate of the supercovariant derivative is given by

$$\overline{\hat{\nabla}_\mu\epsilon^i} = \overline{\nabla_\mu\epsilon^i} - \bar{\epsilon}^j\gamma^0\left(\mathscr{A}_\mu{}^i{}_j\right)^T\gamma^0 + \bar{\epsilon}_j\gamma^0\left(\mathscr{B}_\mu{}^{ij}\right)^T\gamma^0. \tag{5}$$

Here the transpose operator T is understood as acting on the space spanned by 4×4 gamma matrices, whereas the raising and lowering the indices $i, j, \dots$ are done by complex conjugation. We wish to put some constraints on the connections $\mathscr{A}_{\mu i}{}^j$ and $\mathscr{B}_{\mu ij}$ by requiring the positivity of energy.

Using the supercovariant derivative defined above, let us introduce the anti-symmetric Nester tensor [7]

$$N^{\mu\nu} = -i\left(\bar{\epsilon}^i\gamma^{\mu\nu\rho}\hat{\nabla}_\rho\epsilon_i - \overline{\hat{\nabla}_\rho\epsilon^i}\gamma^{\mu\nu\rho}\epsilon_i\right), \tag{6}$$

which reduces to the one in [7,26] for $N = 2$. The strategy employed by Witten and Nester for the mass positivity is two-folds. Let us suppose that the asymptotically flat/AdS spacetime is foliated by some spacelike slice Σ. If Σ is an orientable 3-surface, it turns out that the spacetime admits a spin structure. This allows us to specify the appropriate fall-off rate of the metric, fluxes and spinors on the spacelike surface Σ in such a way that the following energy function is finite and conserved

$$E_{\rm WN} = \frac{1}{2}\int_{\partial\Sigma} N_{\mu\nu}{\rm d}S^{\mu\nu}, \tag{7}$$

where $\partial\Sigma$ is the two-dimensional boundary of Σ at infinity. In the asymptotically flat case, the Witten–Nester energy is related to the ADM momentum P^μ [28] as $E_{\rm WN} = -V_\infty^\mu P_\mu$, where $V_\infty^\mu = i\bar{\epsilon}_\infty^i\gamma^\mu\epsilon_{\infty i}$ corresponds to the generator of the asymptotic translational symmetry and $\epsilon_{\infty i}$ are the asymptotic value of the spinors. The next step is to convert the surface integral at infinity–using the Stokes theorem–to the volume integral over Σ,

$$E_{\rm WN} = \int_{\partial\Sigma} \nabla_\nu N^{\mu\nu}{\rm d}\Sigma_\mu, \tag{8}$$

where $\mathrm{d}\Sigma_\mu$ is a past-directed volume element of Σ. If we can show $\nabla_a N^{0a} \geq 0$, where $0, a$ means the frame component, the Witten–Nester energy turns out to be positive semi-definite $E_{\mathrm{WN}} \geq 0$. This leads to an inequality involving globally conserved quantities such as mass, angular momentum, electromagnetic charges and so on.

Since the explicit form of E_{WN} is sensitive both to the asymptotic spacetime structures and to the field contents of the theory, we tentatively suppose that we can prescribe the boundary condition so that the Witten–Nester energy converges. Hence our primary concern at the moment is the positivity of $\nabla_a N^{0a}$, or a lack thereof. After some computations, the divergence of the Nester tensor can be brought into the following form,

$$\begin{aligned}
\nabla_\nu N^{\mu\nu} = {}& 2i\overline{\hat\nabla_\rho \epsilon^i}\gamma^{\mu\nu\rho}\hat\nabla_\nu \epsilon_i - G^\mu{}_\nu\left(i\bar\epsilon^i\gamma^\nu\epsilon_i\right) \\
& - \frac{i}{2}\bar\epsilon^i\left[\gamma^{\mu\nu\rho}\mathscr{F}_{\nu\rho i}{}^j + \gamma^0\left(\mathscr{F}_{\nu\rho}{}^j{}_i\right)^T\gamma^0\gamma^{\mu\nu\rho}\right]\epsilon_j \\
& - \frac{i}{2}\left[\bar\epsilon^i\gamma^{\mu\nu\rho}\mathscr{H}_{\nu\rho ij}\epsilon^j - \bar\epsilon_i\gamma^0\left(\mathscr{H}_{\nu\rho}{}^{ji}\right)^T\gamma^0\gamma^{\mu\nu\rho}\epsilon_j\right] \\
& - i\bar\epsilon^i\left[\gamma^{\mu\nu\rho}\mathscr{B}_{\nu ik}\mathscr{B}_\rho{}^{kj} + \gamma^0\left(\mathscr{B}_\nu{}^{jk}\mathscr{B}_{\rho ki}\right)^T\gamma^0\gamma^{\mu\nu\rho}\right]\epsilon_j \\
& + i\bar\epsilon^i\left[\gamma^{\mu\nu\rho}\mathscr{A}_{\nu i}{}^j - \gamma^0\left(\mathscr{A}_\nu{}^j{}_i\right)^T\gamma^0\gamma^{\mu\nu\rho}\right]\hat\nabla_\rho\epsilon_j \\
& - i\overline{\hat\nabla_\rho\epsilon^i}\left[\gamma^{\mu\nu\rho}\mathscr{A}_{\nu i}{}^j - \gamma^0\left(\mathscr{A}_\nu{}^j{}_i\right)^T\gamma^0\gamma^{\mu\nu\rho}\right]\epsilon_j \\
& - i\bar\epsilon_i\left[\gamma^{\mu\nu\rho}\mathscr{B}_\nu{}^{ij} - \gamma^0\left(\mathscr{B}_\nu{}^{ji}\right)^T\gamma^0\gamma^{\mu\nu\rho}\right]\hat\nabla_\rho\epsilon_j \\
& - i\overline{\hat\nabla_\rho\epsilon^i}\left[\gamma^{\mu\nu\rho}\mathscr{B}_{\nu ij} - \gamma^0(\mathscr{B}_{\nu ji})^T\gamma^0\gamma^{\mu\nu\rho}\right]\epsilon^j,
\end{aligned} \tag{9}$$

where we have defined the two kinds of curvatures

$$\mathscr{F}_{\mu\nu i}{}^j = 2\left(\nabla_{[\mu}\mathscr{A}_{\nu]i}{}^j + \mathscr{A}_{[\mu i}{}^k\mathscr{A}_{\nu]k}{}^j\right), \tag{10a}$$

$$\mathscr{H}_{\mu\nu ij} = 2\left(\nabla_{[\mu}\mathscr{B}_{\nu]ij} + \mathscr{A}_{[\mu i}{}^k\mathscr{B}_{\nu]kj} + \mathscr{B}_{[\mu ik}\mathscr{A}_{\nu]}{}^k{}_j\right). \tag{10b}$$

Readers should observe the following relation in deriving Eq. (9),

$$i\bar\epsilon^i\gamma^{\mu\nu\rho}\mathscr{B}_{\nu ij}\hat\nabla_\rho\epsilon^j = \left(-i\bar\epsilon_i\gamma^{\mu\nu\rho}\mathscr{B}_\nu{}^{ij}\hat\nabla_\rho\epsilon_j\right)^\dagger = i\overline{\hat\nabla_\rho\epsilon^i}\gamma^0(\mathscr{B}_{\nu ji})^T\gamma^0\gamma^{\mu\nu\rho}\epsilon^j. \tag{11}$$

We assume that the spinors ϵ_i satisfy the Dirac–Witten condition on Σ [6],

$$\gamma^I\hat\nabla_I\epsilon_i = 0, \quad I = 1, 2, 3. \tag{12}$$

If there exist spinors satisfying this differential equation and giving a finite Witten–Nester energy, the first term of the right side of (9) gives the nonnegative contribution to the volume integral due to $\overline{\hat\nabla_\rho\epsilon^i}\gamma^{0\nu\rho}\hat\nabla_\nu\epsilon_i = g^{IJ}(\hat\nabla_I\epsilon_i)^\dagger(\hat\nabla_J\epsilon_i) \geq 0$. According to our convention, the vector field $V^\mu \equiv i\bar\epsilon^i\gamma^\mu\epsilon_i$ is future-directed and nonspacelike because of $V^0 = \epsilon_i^\dagger\epsilon_i > 0$. It follows that the term $-G^\mu{}_\nu V^\nu$ turns out to have a positive contribution to the Witten–Nester energy, provided Einstein's equations hold and matter fields satisfy the suitable energy conditions. On the other hand, the last four terms proportional to $\hat\nabla_\rho\epsilon_i$ in Eq. (9) do not to have a definite sign. Hence we demand as a minimal requirement for the positivity of mass that the gauge connections should be subjected to the subsequent conditions,

$$\gamma^0\left(\mathscr{A}_\rho{}^j{}_i\right)^T\gamma^0\gamma^{\mu\nu\rho} = \gamma^{\mu\nu\rho}\mathscr{A}_{\rho i}{}^j, \tag{13a}$$

$$\gamma^0(\mathscr{B}_{\rho ji})^T\gamma^0\gamma^{\mu\nu\rho} = \gamma^{\mu\nu\rho}\mathscr{B}_{\rho ij}. \tag{13b}$$

We shall refer to these conditions as "positivity conditions." Although we have not shown that these conditions are necessary, this requirement seems persuasive since all the theories which have been shown to admit the mass positivity in Refs. [10,16–24] indeed satisfy this property. Under the positivity conditions, the divergence of the Nester tensor takes a remarkably simple form

$$\nabla_\nu N^{\mu\nu} = 2i\hat{\nabla}_\rho \epsilon^i \gamma^{\mu\nu\rho}\hat{\nabla}_\nu \epsilon_i - G^\mu{}_\nu V^\nu + S^\mu, \tag{14}$$

where the current $S^\mu = S^\mu_{(1)} + S^\mu_{(2)} + S^\mu_{(3)}$ is built out of three different contributions,

$$S^\mu_{(1)} \equiv -i\bar{\epsilon}^i \gamma^{\mu\nu\rho}\mathscr{F}_{\nu\rho i}{}^j \epsilon_j, \tag{15a}$$

$$S^\mu_{(2)} \equiv -\frac{i}{2}\big(\bar{\epsilon}^i \gamma^{\mu\nu\rho}\mathscr{H}_{\nu\rho ij}\epsilon^j - \bar{\epsilon}_i \gamma^{\mu\nu\rho}\mathscr{H}_{\nu\rho}{}^{ij}\epsilon_j\big), \tag{15b}$$

$$S^\mu_{(3)} \equiv -2i\bar{\epsilon}^i \gamma^{\mu\nu\rho}\mathscr{B}_{\nu ik}\mathscr{B}_\rho{}^{kj}\epsilon_j. \tag{15c}$$

Hence if we can show that the zero-th component of the current

$$J^\mu \equiv -G^\mu{}_\nu V^\nu + S^\mu \tag{16}$$

is nonnegative $J^0 \geq 0$ modulo the field equations, we can conclude the positivity of the Witten–Nester energy. Due to the simplicity of the formula (14), our approach can circumvent complications encountered in the model-dependent analysis.

A possible way to find the gauge connections satisfying (13) is to expand them in terms of the Clifford basis. We give the classification of the connections in Appendix A. It turns out that the possible connections take the same form as those in extended supergravity if we impose an additional condition that $V^\mu = i\bar{\epsilon}^i\gamma^\mu\epsilon_i$ is a Killing field when $\hat{\nabla}_\mu\epsilon_i = 0$ is satisfied. Note that this does not immediately imply that the Witten–Nester energy is positive and finite, since these conditions are not sufficient to prove $J^0 \geq 0$, and the finiteness of the surface integral is sensitive to the boundary conditions for the metric, gauge fields and scalars.

3. Explicit examples

Exploiting the formulation developed in the previous section, we shall now demonstrate the positivity of the Witten–Nester energy for various theories. The models we shall discuss are all motivated by extended supergravities. The following analysis illustrates that $\mathscr{A}_{\mu i}{}^j$ and $\mathscr{B}_{\mu ij}$ correspond respectively to the connection of the spinor bundle and to the contribution coming from the flux torsion. It turns out that the positivity conditions (13) are indeed true for all models inspired by extended supergravities.

3.1. $N = 2$ minimal gauged supergravity

Let us begin with the positivity of Witten–Nester energy in $N = 2$ minimal gauged supergravity, i.e., the Einstein–Maxwell theory with a negative cosmological constant

$$L = R - F_{\mu\nu}F^{\mu\nu} - 2\Lambda, \tag{17}$$

where $F = \mathrm{d}A$ and $\Lambda = -3\ell^{-2} < 0$. The field equations of this system are given by[2]

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T^{(\mathrm{em})}_{\mu\nu} \equiv 2\left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right), \qquad \nabla_\nu\left(F^{\mu\nu} + i \star F^{\mu\nu}\right) = 0. \tag{18}$$

This subject was first discussed in [22] by using a single Dirac spinor. We demonstrate below that the argument in [22] concerning the surface integral should be refined.

The connections in the supercovariant derivative are given by

$$\mathscr{A}_{\mu i}{}^j = \frac{i}{\ell}\left(\sigma^3\right)_i{}^j A_\mu, \qquad \mathscr{B}_{\mu ij} = \frac{1}{4}\epsilon_{ij} F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu + \frac{i}{2\ell}\left(\sigma^3\right)_{ij}\gamma_\mu, \tag{19}$$

where $(\sigma^I)_i{}^j$ is a standard Pauli matrix, whose index is lowered by the alternate tensor ϵ_{ij} with $\epsilon_{12} = -\epsilon_{21} = 1$ as $(\sigma^I)_{ij} \equiv \epsilon_{ki}(\sigma^I)_j{}^k$, viz $(\sigma^3)_{ij} = (\sigma^1)_i{}^j$. Note that our convention leads to $(\sigma^I)^{ij} = [(\sigma^I)_{ij}]^*$ which differs from the one in [29]. It is a simple exercise to verify that the connections (19) obey the positivity conditions (13) and (77). Hence we get

$$\begin{aligned}
S^\mu_{(1)} &= -\frac{2}{\ell} \star F^{\mu\nu}\left(\sigma^3\right)_i{}^j\left(i\bar\epsilon^i\gamma_\nu\epsilon_j\right), \\
S^\mu_{(2)} &= \left(T^{(\mathrm{em})\mu}{}_\nu - \Lambda\delta^\mu{}_\nu\right)V^\nu - S^\mu_{(1)}, \\
S^\mu_{(3)} &= \epsilon_{ij}\left(\nabla_\nu F^{\mu\nu} + i\nabla_\nu \star F^{\mu\nu}\right) i\bar\epsilon^i\epsilon^j + \text{c.c.},
\end{aligned} \tag{20}$$

thereby the current $J^\mu = -G^\mu{}_\nu V^\nu + S^\mu$ vanishes when the equations of motion (18) are satisfied. This probes that the Witten–Nester energy is indeed positive semi-definite. It is worth commenting that the negativity of the cosmological constant is essential. An attempt to give a positive cosmological constant does not work, since the positivity conditions (13) fail to hold.[3] This would convince us that the positivity conditions (13) are indeed related to the mass positivity.

The surface integral can be expressed in terms of globally conserved quantities as follows. It is convenient here to exploit the Dirac spinor $\eta = \epsilon^1 - i\epsilon_2$ to evaluate the surface integral. Let us assume that the spacetime asymptotes to the AdS at infinity following the notion of Refs. [31–34]. We require that the Dirac spinor η tends to the Killing spinor ζ of AdS at infinity and obeys

$$\hat\nabla_\mu\eta = O\left(1/r^2\right), \quad \text{as } r \to \infty. \tag{21}$$

The expression of Witten–Nester energy was derived in [22] and reads

$$E_{\mathrm{WN}} = \bar\zeta J_{AB}\sigma^{AB}\zeta - \bar\zeta\left(Q_e - i\gamma_5 Q_m\right)\zeta, \tag{22}$$

where σ^{AB} is the generator of SO(3, 2) in the spinor representation and J_{AB} is the SO(3, 2) momentum $(A, B, = 0, \ldots, 4)$. Q_e and Q_m denote the electric and magnetic charges defined by

$$Q_e = \int_{\partial\Sigma} \star F, \qquad Q_m = \int_{\partial\Sigma} F. \tag{23}$$

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[2] We do not consider here the extra source terms $T^{(\mathrm{mat})}_{\mu\nu}$ and $J^\mu + i\tilde J^\mu$ to the right side of Einstein's and Maxwell's equations, respectively. The positive mass property continues to be valid provided that $T^{(\mathrm{mat})}_{\mu\nu}$ satisfies the dominant energy condition, and that J^μ and $\tilde J^\mu$ are future-pointing timelike vectors.

[3] Unlike in the Dirac spinor formulation in [26], the "fake" Killing spinor equations for $\Lambda = 3H^2 > 0$ are not obtained by the simple Wick-rotation $\ell \to iH^{-1}$ of (19), since it is incompatible with raising and lowering the SU(2) indices via complex conjugation. In the $\Lambda > 0$ case, we have to choose $\mathscr{A}_{\mu i}{}^j = HA_\mu\delta_i{}^j$ and $\mathscr{B}_{\mu ij} = \epsilon_{ij}(\frac{1}{4}F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu + \frac{1}{2}H\gamma_\mu)$ in order to produce the correct equations of motion [30]. The latter connection does not satisfy the positivity conditions, as expected. We thank D. Klemm for useful comments about this.

Kostelecky and Perry [22] then concluded that the BPS bound should be given by $M \geq \ell^{-1}|J| + \sqrt{Q_e^2 + Q_m^2}$, where $M = J_{04}$ and $J = \ell J_{12}$ represent the mass and angular momentum [31–34]. However, it has been pointed out in Refs. [35,36] that the magnetically charged Reissner–Norsdtröm-AdS solution cannot be supersymmetric.

This apparent contradiction can be resolved in the following manner. The Killing spinor ζ in AdS satisfies $[\nabla_\mu + (1/2\ell)\gamma_\mu]\zeta = 0$ and is given by [32],

$$\zeta = \left(\cosh\frac{\rho}{2} + \sinh\frac{\rho}{2}\gamma^1\right)\left(\cos\frac{t}{2\ell} + \sin\frac{t}{2\ell}\gamma^0\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\gamma^{12}\right)$$
$$\times\left(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\gamma^{23}\right)\zeta_0, \tag{24}$$

where ζ_0 is a constant Dirac spinor. Here we have employed the global coordinates,

$$\mathrm{d}s^2 = -\cosh\rho^2\mathrm{d}t^2 + \ell^2\left[\mathrm{d}\rho^2 + \sinh^2\rho\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right]. \tag{25}$$

The standard radial coordinate is given by $r = \ell\sinh\rho$. Given the explicit form of the Killing spinor (24), one can compute the spinor bilinears appearing in the electric and magnetic charges of (22) as

$$\bar\zeta\zeta = c_0, \tag{26a}$$
$$i\bar\zeta\gamma_5\zeta = \left[c_1\cos(t/\ell) + c_2\sin(t/\ell)\right]\cosh\rho$$
$$+\left[c_3\cos\theta + \sin\theta(c_4\cos\phi + c_5\sin\phi)\right]\sinh\rho, \tag{26b}$$

where c_{0-5} are real constants built out of ζ_0. For the choice $c_{1-5} = 0$, one can verify that $U_\mu \equiv i\bar\zeta\gamma_\mu\gamma_5\zeta = -\ell\nabla_\mu(i\bar\zeta\gamma_5\zeta)$ vanishes, which in turn implies that $V_\mu = i\bar\zeta\gamma_\mu\zeta$ is null due to the Fierz identity. This boundary condition is not the case that we are interested in, so at least one of c_{1-5} is nonvanishing. It then follows that the bilinear $i\bar\zeta\gamma_5\zeta$ appearing in the magnetic charge of (22) diverges at infinity ($\rho \to \infty$), rendering the Witten–Nester energy (22) ill-defined. Moreover, the presence of magnetic charge implies that the gauge potential A_μ cannot be globally defined, otherwise the integral of Q_m in (23) vanishes. It is typically singular on the axis, leading to the source of delta-function. Since the gauge potential A_μ appears explicitly in the supercovariant derivative (19), the Dirac–Witten operator $\gamma^i\hat\nabla_i$ therefore cannot be straightforwardly invertible using Green's function. If one attempts to remove this distributional singularity, the topological structure of spacetime must change, resulting in a different BPS bound [37,38]. In the case of asymptotically globally AdS case in the Einstein–Maxwell-Λ system, we therefore arrive at the inequality[4]

$$M \geq \frac{1}{\ell}|J| + Q_e. \tag{27}$$

From the standpoint of Osp(4|2) superalgebra, the introduction of magnetic charge as central extension is forbidden since it fails to satisfy the Jacobi identity due to the breakdown of the SO(3, 2) covariance [40]. Our explanation seems more convincing in the present context since the positive mass theorem does not assume the underlying supergravity theories in advance.

[4] The appearance of angular momentum into the Witten–Nester energy can also be understood from the fact that in the framework of $N = 2$ gauged supergravity, the bilinear vector field $V^\mu = i\bar\zeta\gamma^\mu\zeta$ in AdS is rotating by the constant angular velocity ℓ^{-1} with respect to the static observer at infinity [39].

3.2. Kähler target space

Let us next discuss the case in which the set of the complex scalar fields parameterizing the Kähler manifold is the source of Einstein's equations. Namely we shall concentrate on the theory

$$L = R - 2G_{\alpha\bar{\beta}} g^{\mu\nu} \partial_\mu z^\alpha \partial_\nu \bar{z}^{\bar{\beta}} - V(z, \bar{z}), \qquad G_{\alpha\bar{\beta}} = \frac{\partial^2 K}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}}, \tag{28}$$

where $K(z, \bar{z})$ is a real Kähler potential and $V(z, \bar{z})$ is the potential to be determined by requiring the positivity of the Witten–Nester energy. The indices $\alpha, \bar{\beta}$ run over any positive integers corresponding to the number of complex scalars. The Einstein equations following from this Lagrangian read

$$G_{\mu\nu} = T_{\mu\nu}, \qquad T_{\mu\nu} = 2G_{\alpha\bar{\beta}} \left(\nabla_{(\mu} z^\alpha \nabla_{\nu)} \bar{z}^{\bar{\beta}} - \frac{1}{2} g_{\mu\nu} \nabla_\rho z^\alpha \nabla^\rho \bar{z}^{\bar{\beta}} \right) - \frac{1}{2} g_{\mu\nu} V. \tag{29}$$

We assume that $G_{\alpha\bar{\beta}}$ is a positive matrix, or equivalently the null energy condition, in such a way that there appear no ghosts.

We take $\mathscr{A}_{\mu i}{}^j$ and $\mathscr{B}_{\mu ij}$ satisfying the positivity condition (13) and (77) as

$$(\mathscr{A}_\mu)_i{}^j = \frac{1}{4} \left(K_\alpha \partial_\mu z^\alpha - K_{\bar{\alpha}} \partial_\mu \bar{z}^{\bar{\alpha}} \right) \delta_i{}^j, \qquad \mathscr{B}_{\mu ij} = \frac{1}{2} e^{K/2} W \gamma_\mu \delta_{ij}, \tag{30}$$

where $K_\alpha \equiv \partial K / \partial z^\alpha$. Here $W = W(z)$ is a holomorphic function of z^α and is referred to as a superpotential. The superpotential is assumed to transform as $W \to W e^{-f}$ under the Kähler gauge transformation $K \to K + f + \bar{f}$. The connection $\mathscr{A}_{\mu i}{}^j$ represents the $\mathrm{U}(1)^N$ connection.

We define the "variation of dilatini" as

$$\delta\lambda^\alpha{}_i = \delta_{ij} \gamma^\mu \partial_\mu z^\alpha \epsilon^j - e^{K/2} G^{\alpha\bar{\beta}} D_{\bar{\beta}} \bar{W} \epsilon_i, \tag{31}$$

which has negative chirality $\gamma_5 \delta\lambda^\alpha{}_i = -\delta\lambda^\alpha{}_i$. D_α denotes the Kähler U(1) covariant derivative, which acts on the superpotential as

$$D_\alpha W = \partial_\alpha W + K_\alpha W. \tag{32}$$

In this case, the on-shell current J^μ takes the manifestly nonnegative form

$$J^\mu = -\left(G^\mu{}_\nu - T^\mu{}_\nu \right) V^\nu + G_{\alpha\bar{\beta}} i \overline{\delta\lambda^{\beta i}} \gamma^\mu \delta\lambda^\alpha{}_i, \tag{33}$$

provided the potential is given by

$$V = 2e^K \left(G^{\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} - 3|W|^2 \right), \tag{34}$$

where $G^{\alpha\bar{\beta}}$ is the inverse of $G_{\alpha\bar{\beta}}$ and $\overline{\delta\lambda^{\beta i}} = i(\delta\lambda^\beta{}_i)^\dagger \gamma^0$. The existence of the superpotential obeying the desired transformation under the Kähler gauge transformation implies that the first Chern class of the line bundle coincides with the Kähler class, i.e., the manifold must be Hodge. The above discussion means that the volume integral is positive semi-definite as far as the Hodge–Kähler target space is concerned, irrespective of the choice of superpotential.

The surface integral is finite if we impose Eq. (21) for the spinors and that the scalar fields fall off faster than $r^{-3/2}$. This boundary condition for the scalar implies that the mass eigenvalues should be above the Breitenlohner–Freedman bound $m_{\mathrm{BF}}^2 = -9/(4\ell^2)$ [41], where ℓ is the curvature radius of the AdS vacua.[5] The finiteness of the Witten–Nester energy is not guaranteed for the boundary condition employed in [18], in which case the negative mass initial data can be constructed along the line of [42].

Despite the fact that the potential constructed from the Kähler potential and superpotential is in general unbounded from below and above, the AdS vacua above the Breitenlohner–Freedman bound are stabilized to allow the positive mass. For example, the positivity of mass for the bosonic sector of the gravity multiplet in $N = 4$ SO(4) gauged supergravity was discussed in Ref. [18]. In this case, the Kähler metric is given by the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset and the superpotential takes a constant value [18],

$$K = -\ln\bigl(1 - |\tau|^2\bigr), \qquad W = \sqrt{2}g, \tag{35}$$

where τ is an axidilaton and g is an SO(4) gauge coupling constant. We can see that the potential is indeed unbounded from below and the origin is the unique vacuum with the mass spectrum $m^2 = -2\ell^{-2}(\times 2)$. The analysis given in this section implies that the positivity of the mass holds in more general settings than the model considered in [18].

It is worthwhile to comment that the stress energy tensor for the complex scalar field does not respect the dominant energy condition in general. The essential requirement that has played a crucial role here is the null energy condition, viz., $\mathrm{eig}(G_{\alpha\bar{\beta}}) \geq 0$ [26].

It should be also noticed that the number N of the Weyl spinors can be arbitrary. One may be suspicious that this cannot be done since $N > 8$ extended supersymmetric theory implies the necessity of introducing higher spin $s > 2$ fields, which is a main obstacle to construct a local theory. However, the spinors only play a subsidiary role in the Witten–Nester formulation as the bilinear vector of the asymptotic symmetry. It therefore follows that the above argument actually has nothing to do with the full supergravity theories incorporating the fermion interactions even if it implies the underlying bosonic sector of supergravity theories. Hence our analysis continues to be valid also in the case involving $N > 8$ spinors, and also in the (even) $D > 11$ case, although its relevance to the physically interesting theories is less obvious.

3.3. Einstein–Maxwell-dilaton theory

In this subsection, we consider the Einstein–Maxwell theory coupled to the dilaton field with a potential,

$$L = R - 2(\nabla\phi)^2 - h(\phi)F_{\mu\nu}F^{\mu\nu} - 2V(\phi), \tag{36}$$

where $h(\phi)$ is the dilaton coupling function and $V(\phi)$ is the potential of the dilation, both of which are to be determined by requiring the positive mass. The Einstein equations are given by

$$\begin{aligned} &G_{\mu\nu} = T_{\mu\nu}, \\ &T_{\mu\nu} = 2\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right) - Vg_{\mu\nu} + 2h\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right). \end{aligned} \tag{37}$$

[5] It is important to note that if the mass of the scalar field is in the range $m^2_{\mathrm{BF}} \leq m^2 \leq m^2_{\mathrm{BF}} + \ell^{-2}$, the slowly decaying solution is also normalizable, admitting any boundary conditions. In this case, it is unclear if the Witten–Nester energy coincides with other definitions of charges, e.g., the one introduced in [41]. See e.g., Refs. [42,43] for the recent work addressing this problem. We content ourselves here by imposing the Dirichlet boundary conditions.

The dilaton and Maxwell equations read

$$\nabla^2\phi - \frac{1}{2}V'(\phi) - \frac{1}{4}h'(\phi)F_{\mu\nu}F^{\mu\nu} = 0, \qquad \nabla_\nu\big[h(\phi)F^{\mu\nu}\big] = 0, \qquad \mathrm{d}F = 0. \tag{38}$$

Here the prime denotes the differentiation with respect to ϕ.

Setting $N = 2$, we choose the gauge connections satisfying (13) and (77) as follows

$$\mathscr{A}_{\mu i}{}^{j} = -ig\big(\sigma^3\big)_i{}^{j}A_\mu, \qquad \mathscr{B}_{\mu ij} = \epsilon_{ij}K_1(\phi)F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu + iW(\phi)\big(\sigma^3\big)_{ij}\gamma_\mu, \tag{39}$$

where $F = \mathrm{d}A$, g is the gauge coupling constant and $K_1(\phi)$ and $W(\phi)$ are some real functions of ϕ. We further define the variation of the spin 1/2 fields as

$$\delta\lambda_i = i\gamma^\mu\nabla_\mu\phi\big(\sigma^3\big)_{ij}\epsilon^j + K_2(\phi)\epsilon_i + \big(\sigma^2\big)_i{}^{j}K_3(\phi)F_{\mu\nu}\gamma^{\mu\nu}\epsilon_j, \tag{40}$$

where $K_{2,3}(\phi)$ are again real functions. A straightforward computation shows that the on-shell current J^μ takes the nonnegative form,

$$J^\mu = -\big(G^\mu{}_\nu - T^\mu{}_\nu\big)V^\nu + i\overline{\delta\lambda^i}\gamma^\mu\delta\lambda_i, \tag{41}$$

provided the Maxwell equations and the Bianchi identity $\mathrm{d}F = 0$ hold, and if the following relations are satisfied

$$V = K_2^2 - 12W^2, \qquad K_2 = -2W', \tag{42a}$$

$$h = 4\big(K_3^2 + 4K_1^2\big), \qquad K_1^2 \propto h, \qquad K_3 = 2K_1', \tag{42b}$$

$$0 = g + 8WK_1 + 2K_2K_3. \tag{42c}$$

Eq. (42a) implies that the potential is expressed by the (real) superpotential as

$$V(\phi) = 4\big[W'(\phi)^2 - 3W(\phi)^2\big]. \tag{43}$$

The differential equation (42b) can be integrated to give

$$h = e^{-2\alpha\phi}, \qquad K_1 = \frac{1}{4\sqrt{1+\alpha^2}}e^{-\alpha\phi}, \qquad K_3 = -\frac{\alpha}{2\sqrt{1+\alpha^2}}e^{-\alpha\phi}, \tag{44}$$

where $\alpha \in \mathbb{R}$ is the coupling constant of the dilaton. Finally, Eq. (42c) is solved as

$$W(\phi) = W_0e^{-\phi/\alpha} - \frac{g}{2\sqrt{1+\alpha^2}}e^{\alpha\phi}, \tag{45}$$

where W_0 is the integration constant. Thus, in terms of the Dirac spinor $\eta = \epsilon^1 - i\epsilon_2$, we have

$$\hat{\nabla}_\mu\eta = \left[\nabla_\mu + \frac{i}{4\sqrt{1+\alpha^2}}e^{-\alpha\phi}F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu + W(\phi)\gamma_\mu + igA_\mu\right]\eta, \tag{46}$$

$$\delta\lambda = \left[\gamma^\mu\nabla_\mu\phi - 2W'(\phi) - \frac{i\alpha}{2\sqrt{1+\alpha^2}}e^{-\alpha\phi}F_{\mu\nu}\gamma^{\mu\nu}\right]\eta. \tag{47}$$

When the potential vanishes, this recovers the result in [10]. It is interesting that the superpotential and the dilaton coupling function $h(\phi)$ are completely determined by requiring the positivity within the class (39) and (40). If W_0 is tuned suitably, the above system with $\alpha = \pm\sqrt{3}$ is obtained by the $\mathrm{U}(1)^4$ truncation of SO(8) maximal gauged supergravity [44].

The Einstein–Maxwell-dilaton theory admitting the positive mass allows a free parameter α, which characterizes how the four-dimensional theory is derived from the higher-dimensional theory. One may hope from the positive mass theorem above that this theory can be embedded into supergravity for an arbitrary value of α. In order to see this, let us consider the BPS system described by [25]

$$\hat{\nabla}_\mu \eta = 0, \qquad \delta\lambda = 0. \tag{48}$$

The first relation is first-order differential equation, while the second is purely algebraic. Then it is not obvious for this BPS system to have a solution, thereby we have to check the integrability. Acting $\gamma^\nu \nabla_\nu$ to $\delta\lambda = 0$ and using $\hat{\nabla}_\mu \eta = 0$, we obtain

$$\begin{aligned} 0 = \Bigg[& \nabla^2\phi + 4W'\big(3W - W''\big) + \frac{\alpha}{2} e^{-2\alpha\phi} F_{\mu\nu} F^{\mu\nu} \\ & + \frac{i e^{-\alpha\phi}}{\sqrt{1+\alpha^2}} \big\{ \alpha\big(W - W''\big) + \big(\alpha^2 - 1\big) W' \big\} \\ & - \frac{i\alpha}{\sqrt{1+\alpha^2}} \big\{ e^{-\alpha\phi} \big(\nabla_\mu \star F^{\mu\nu}\big)\gamma_\nu\gamma_5 + e^{\alpha\phi} \nabla_\mu \big(e^{-2\alpha\phi} F^{\mu\nu}\big)\gamma_\nu \big\} \\ & + \frac{i\gamma_5 \alpha(\alpha^2 - 3)}{2(1+\alpha^2)} e^{-2\alpha\phi} F_{\mu\nu} \star F^{\mu\nu} \Bigg] \eta. \end{aligned} \tag{49}$$

Assuming the dilaton field equation, the Maxwell equation and the Bianchi identity, the first three lines of (49) drop out [note that the second line vanishes due to (45)]. However, the term at the last line remains nonvanishing unless $\alpha(\alpha^2 - 3) F \wedge F = 0$. Hence it follows that the first-order system (48) does not allow solutions in general except for $\alpha = 0$ or $\alpha = \pm\sqrt{3}$. The latter is obtained by the Kaluza–Klein reduction of five-dimensional gravity if the potential is absent [44]. This means that the general coupling case cannot be embedded into supergravity. Since the $F \wedge F \neq 0$ case corresponds to the dyonic metric, the purely electric/magnetic solution may admit Killing spinors as in the massless case [25].

3.4. $N = 8$ *supergravity*

Finally, let us see the case of $N = 8$ gauged supergravity. In Ref. [18], the positivity of the Witten–Nester energy was explored for the electric SO(8) gauging models constructed by de Wit and Nicolai [45]. In recent years we have witnessed a lot of progress in $N = 8$ gauged supergravity. Of particular interest is the discovery of the one-parameter family of the deformation of SO(8) gaugings [46] (see [47] for the deformation of the SL(8)-type gaugings). The deformed theory displays considerably rich physics compared to the undeformed one, since it admits new kinds of vacua [48,49] and new supersymmetry breaking patterns [48,50]. The deformation parameter might give rise to a new interpretation to M-theory embeddings and their field theory duals. Although the higher-dimensional origin of the noncompact gaugings is not identified yet, it has been extensively studied recently from the viewpoint of generalized geometry (see e.g., [51] and references therein). Hence it is intriguing to see whether the positivity of Witten–Nester energy depends on the deformation and the underlying gauging group. In particular, the noncompact gaugings might be associated to the ghost contribution, hence the positive mass property is quite nontrivial.

The recent development of $N = 8$ gauged supergravity is based on the embedding tensor formalism [52], by which we can discuss in a duality covariant manner how to gauge a group by introducing additional 28 magnetic vector fields. The embedding tensor $\Theta_M{}^\alpha$ specifies how to choose the gauge group G inside $E_{7(7)}$, and defined by the relation $X_M = \Theta_M{}^\alpha t_\alpha$, where

t_α and X_M are the generators of $E_{7(7)}$ and G, respectively. Here $\alpha, \beta, \ldots = 1, \ldots, 133$ and $M, N, \ldots = 1, \ldots, 56$ are the adjoint and fundamental of $E_{7(7)}$. The consistent gaugings amount to requiring that the embedding tensor obeys linear and quadratic constraints [52]. The linear constraint implies that $\Theta_M{}^\alpha$ is sitting in the **912** representation of $E_{7(7)}$, whereas the quadratic constraint corresponds to the closure condition $\Omega^{MN}\Theta_M{}^\alpha\Theta_N{}^\beta = 0$, where $\Omega^{MN} = i\sigma_2 \otimes \mathbb{I}_{28}$ is the $\mathrm{Sp}(56, \mathbb{R})$ invariant metric. Once the symplectic frame is chosen, the gauging can be done by the replacement $\partial_\mu \to D_\mu = \partial_\mu - gA_\mu{}^M\Theta_M{}^\alpha t_\alpha$, where g is the gauge coupling constant and $A_\mu{}^M$ consists of electric and magnetic vectors $A_\mu{}^M = (A_\mu{}^\Lambda, A_{\mu\Lambda})$.

Einstein's equations read [53]

$$G_{\mu\nu} = T_{\mu\nu},$$
$$T_{\mu\nu} \equiv \frac{1}{6}\mathcal{P}_{(\mu}{}^{ijkl}\mathcal{P}_{\nu)ijkl} - \left(\frac{1}{12}|\mathcal{P}_\rho|^2 + V\right)g_{\mu\nu} + \mathcal{H}^+{}_{(\mu}{}^\rho{}_{ij}\mathcal{H}^-{}_{\nu)\rho}{}^{ij}, \tag{50}$$

where $\mathcal{P}_{\mu ijkl}$ is the self-dual vector field which corresponds to the kinetic term for scalars parameterizing the $E_{7(7)}/\mathrm{SU}(8)$ coset space, and given in terms of the mixed coset representative $\mathcal{V}_M{}^{\underline{N}}$ as $\mathcal{P}_{\mu ijkl} = i\Omega^{MN}\mathcal{V}_{Mij}\mathcal{D}_\mu\mathcal{V}_{Nkl}$, where $\mathcal{D}_\mu$ is the SU(8) covariant derivative [52]. The potential V arises from the $O(g^2)$ corrections for the supersymmetry transformation and is constructed out of the T-tensor as [52]

$$V = g^2\left(\frac{1}{24}\left|A_{2i}{}^{jkl}\right|^2 - \frac{3}{4}\left|A_1{}^{ij}\right|^2\right). \tag{51}$$

Here A_1 and A_2 denote the **36** and **420** irrep of the SU(8).

The embedding tensor keeps the U-duality covariance at the price of introducing additional 28 magnetic vector fields $A_{\mu\Lambda}$, in addition to the usual electric vector fields $A_\mu{}^\Lambda$. This renders the usual field strength $\mathcal{F}_{\mu\nu}{}^M = 2\partial_{[\mu}A_{\nu]}{}^M + gX_{[NP]}{}^M A_\mu^N A_\nu{}^P$ defined by the Ricci identity $[D_\mu, D_\nu] = -g\mathcal{F}_{\mu\nu}{}^M X_M$ no longer covariant. A proposed prescription to overcome this is to introduce a tensorial auxiliary field $B_{\mu\nu\alpha}$ [54], which can be used to construct a covariant field strength $\mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + gZ^{M,\alpha}B_{\mu\nu\alpha}$, where $Z^{M,\alpha} \equiv \frac{1}{2}\Omega^{MN}\Theta_N{}^\alpha$. Using the electric part of this field strength, it turns out that the following vector field strength transforms as a symplectic vector,

$$\mathcal{G}^{+\ M}_{\mu\nu} = \begin{pmatrix} \mathcal{H}^+{}_{\mu\nu}{}^\Lambda \\ \mathcal{N}_{\Lambda\Sigma}\mathcal{H}^+{}_{\mu\nu}{}^\Sigma + 2i\mathcal{O}^+{}_{\mu\nu\Lambda} \end{pmatrix}, \tag{52}$$

where $\mathcal{N}_{\Lambda\Sigma} = \mathcal{N}_{(\Lambda\Sigma)}$ is the kinetic term for the vector fields defined by $\mathcal{V}^{\Sigma ij}\mathcal{N}_{\Lambda\Sigma} = -\mathcal{V}_\Lambda{}^{ij}$, and $\mathcal{O}^+{}_{\mu\nu\Lambda}$ describes the fermion contribution [52] which is taken to vanish in our computation, and "+" stands for the self-dual part, i.e., $\mathcal{G}^{+\ M}_{\mu\nu} = \frac{1}{2}(\mathcal{G}_{\mu\nu}{}^M - i\star\mathcal{G}_{\mu\nu}{}^M)$. The quantity $\mathcal{H}^+{}_{\mu\nu ij}$ appearing in Einstein's equation is dressed by a mixed coset representative as $\mathcal{H}^+{}_{\mu\nu ij} \equiv \mathcal{V}_{Mij}\mathcal{G}^{+\ M}_{\mu\nu}$. In terms of these ingredients, the equation for the vector fields is given by [53]

$$E^\mu{}_{ij} \equiv \mathcal{D}_\nu\mathcal{H}^{+\mu\nu}{}_{ij} + \mathcal{P}_{\nu ijkl}\mathcal{H}^{-\mu\nu kl} + \frac{g}{3}A_{2[i}{}^{nkl}\mathcal{P}^\mu{}_{j]nkl} = 0. \tag{53}$$

We now turn to the discussion for the positive mass. As a connection $\mathscr{A}_{\mu i}{}^j$, we choose (half of) the SU(8) connection $\mathscr{A}_{\mu i}{}^j = \frac{1}{2}\mathcal{Q}_{\mu i}{}^j$ with $\mathcal{Q}_\mu{}^i{}_j = -\mathcal{Q}_{\mu j}{}^i$ and $\mathcal{Q}_{\mu i}{}^i = 0$. This connection obviously satisfies the positivity condition. Hence the current $S^\mu_{(1)}$ can be written in terms of the SU(8) curvature $F_{\mu\nu}(\mathcal{Q})_i{}^j = 2\partial_{[\mu}\mathcal{Q}_{\nu]}{}^i{}_j + \mathcal{Q}_{[\mu i}{}^k\mathcal{Q}_{\nu]k}{}^j$. Note that a physical degree of freedom is not encoded in this field, since SU(8) is the maximal compact subgroup of $E_{7(7)}$. Using the

Maurer–Cartan equations [see Eq. (3.5) of [53]], the SU(8) curvature is expressed by other fields and the current $S^\mu_{(1)}$ is given by

$$S^\mu_{(1)} = -\frac{2}{3}\mathcal{P}_\nu{}^{jklm}\mathcal{P}_{\rho iklm}\big(i\bar\epsilon^i\gamma^{\mu\nu\rho}\epsilon_j\big) + g\big(\star\mathcal{F}^{\mu\nu M}\big)\mathcal{Q}_{Mi}{}^j\bar\epsilon^i\gamma_\nu\epsilon_j, \tag{54}$$

where $\mathcal{Q}_{Mi}{}^k = \frac{2}{3}i\Omega^{NP}\mathcal{V}_{Nij}X_{MP}{}^Q\mathcal{V}_Q{}^{kj}$.

Let us take the connection $\mathscr{B}_{\mu ij}$ satisfying the positivity condition as

$$\mathscr{B}_{\mu ij} = \frac{\sqrt{2}}{8}\mathcal{H}^{+\rho\sigma}{}_{ij}\gamma_{\rho\sigma}\gamma_\mu + \frac{g}{\sqrt{2}}A_{1ij}\gamma_\mu. \tag{55}$$

The T-tensor variational identity [Eq. (D2) of [53]] leads to $\mathcal{D}_\mu A_1{}^{ij} = -\frac{1}{3}\mathcal{P}_\mu{}^{klm(i}A_2{}^{j)}{}_{klm}$, which yields

$$\begin{aligned} S^\mu_{(2)} &= \sqrt{2}\big[\mathcal{D}_\nu\mathcal{H}^{+\mu\nu}{}_{ij}\big(i\bar\epsilon^i\epsilon^j\big) - \mathcal{D}_\nu\mathcal{H}^{-\mu\nu ij}(i\bar\epsilon_i\epsilon_j)\big] \\ &\quad + \frac{\sqrt{2}g}{3}\big(\mathcal{P}_{\nu klm(i}A_{2j)}{}^{klm}i\bar\epsilon^i\gamma^{\mu\nu}\epsilon^j - \mathcal{P}_\nu{}^{klm(i}A_2{}^{j)}{}_{klm}i\bar\epsilon_i\gamma^{\mu\nu}\epsilon_j\big). \end{aligned} \tag{56}$$

A simple computation shows that

$$\begin{aligned} S_{(3)\mu} &= -2\mathcal{H}^+{}_{(\mu}{}^\rho{}_{ik}\mathcal{H}^-{}_{\nu)\rho}{}^{kj}\big(i\bar\epsilon^i\gamma^\nu\epsilon_j\big) + 6g^2A_{1ik}A_1{}^{kj}\big(i\bar\epsilon^i\gamma_\mu\epsilon_j\big) \\ &\quad + 2g\big[A_{1ik}\mathcal{H}^-{}_{\mu\nu}{}^{kj}\big(i\bar\epsilon^i\gamma^\nu\epsilon_j\big) - \mathcal{H}^+{}_{\mu\nu ik}A_1{}^{kj}\big(i\bar\epsilon^i\gamma^\nu\epsilon_j\big)\big]. \end{aligned} \tag{57}$$

Finally we define the variation of dilatini as

$$\delta\chi_{ijk} = -2\sqrt{2}\mathcal{P}_{\mu ijkl}\gamma^\mu\epsilon^l + \frac{3}{2}\gamma_{\mu\nu}\mathcal{H}^{+\mu\nu}{}_{[ij}\epsilon_{k]} - 2gA_2{}^l{}_{ijk}\epsilon_l. \tag{58}$$

The self-dual property of $\mathcal{P}_{\mu ijkl}$ implies $\mathcal{P}_{(\nu}{}^{ijkl}\mathcal{P}_{\rho)ijkm} = \frac{1}{8}\mathcal{P}_{\nu ijkn}\mathcal{P}_\rho{}^{ijkn}\delta_m{}^l$, hence after some calculations we find

$$\begin{aligned} i\overline{\delta\chi^{ijk}}\gamma^\mu\delta\chi_{ijk} &= -18\big(\mathcal{H}^{+\mu\rho}{}_{ij}\mathcal{H}^-{}_{\nu\rho}{}^{[ij} + \mathcal{H}^+{}_{\nu\rho ij}\mathcal{H}^{-\mu\rho[ij}\big)i\bar\epsilon^{k]}\gamma^\nu\epsilon_k \\ &\quad - 12\sqrt{2}\big[\mathcal{H}^{+\mu\nu}{}_{ij}\mathcal{P}_\nu{}^{ijkm}(i\bar\epsilon_m\epsilon_k) + \mathcal{H}^{-\mu\nu ij}\mathcal{P}_{\nu ijkm}\big(i\bar\epsilon^k\epsilon^m\big)\big] \\ &\quad - 12g\big[\mathcal{H}^{+\mu}{}_{\nu ij}A_{2m}{}^{ijk}\big(i\bar\epsilon^m\gamma^\nu\epsilon_k\big) + \mathcal{H}^{-\mu}{}_\nu{}^{ij}A^m{}_{ijk}\big(i\bar\epsilon^k\gamma^\nu\epsilon_m\big)\big] \\ &\quad + 8i\bar\epsilon^l\gamma^{\mu\nu\rho}\epsilon_m\mathcal{P}_{\nu ijkl}\mathcal{P}_\rho{}^{ijkm} + 4g^2A_{2l}{}^{ijk}A_2{}^m{}_{ijk}i\bar\epsilon^l\gamma^\mu\epsilon_m \\ &\quad - \big(\mathcal{P}^{\mu ijkl}\mathcal{P}_{\nu ijkl} + \mathcal{P}^\mu{}_{ijkl}\mathcal{P}_\nu{}^{ijkl} - \delta^\mu{}_\nu|\mathcal{P}|^2\big)i\bar\epsilon^m\gamma^\nu\epsilon_m \\ &\quad + 4\sqrt{2}g\big(i\bar\epsilon^l\epsilon^m\mathcal{P}^\mu{}_{ijkm}A_{2l}{}^{ijk} + i\bar\epsilon_l\epsilon_m\mathcal{P}^{\mu ijkl}A_2{}^m{}_{ijk}\big) \\ &\quad + 4\sqrt{2}g\big(i\bar\epsilon^l\gamma^{\mu\nu}\epsilon^m\mathcal{P}_{\nu ijkm}A_{2l}{}^{ijk} - i\bar\epsilon_l\gamma^{\mu\nu}\epsilon_m\mathcal{P}_\nu{}^{ijkl}A_2{}^m{}_{ijk}\big). \end{aligned} \tag{59}$$

Focusing on terms proportional to $i\bar\epsilon^i\gamma^\nu\epsilon_j$ in (57) and (59), the following relation holds

$$\begin{aligned} &\mathcal{H}^+{}_{\mu\nu kl}\big(A_{2i}{}^{jkl} + 2A_1{}^{j[k}\delta^{l]}{}_i\big) + \mathcal{H}^-{}_{\mu\nu}{}^{kl}\big(A_2{}^j{}_{ikl} + 2A_{1i[k}\delta_{l]}{}^j\big) \\ &\quad = -\frac{4}{3}\big(\mathcal{H}^+{}_{\mu\nu kl}T_i{}^{jkl} + \mathcal{H}^-{}_{\mu\nu}{}^{kl}T^j{}_{ikl}\big) \\ &\quad = -i\Omega^{MN}\mathcal{Q}_{Mi}{}^j\big(\mathcal{G}^+{}_{\mu\nu}{}^P\mathcal{V}_{Pkl}\mathcal{V}_N{}^{kl} + \mathcal{G}^-{}_{\mu\nu}{}^P\mathcal{V}_P{}^{kl}\mathcal{V}_{Nkl}\big) \\ &\quad = -\Omega^{MN}\Omega_{PN}\mathcal{Q}_{Mi}{}^j\big(\mathcal{G}^+{}_{\mu\nu}{}^P - \mathcal{G}^-{}_{\mu\nu}{}^P\big) = i\mathcal{Q}_{Mi}{}^j\star\mathcal{G}_{\mu\nu}{}^M, \end{aligned} \tag{60}$$

where we have used $\mathcal{V}_M{}^{ij}\mathcal{V}_{Nij} - \mathcal{V}_{Mij}\mathcal{V}_N{}^{ij} = i\Omega_{MN}$, $\mathcal{G}^+{}_{\mu\nu}{}^P\mathcal{V}_P{}^{kl} = -\frac{1}{2}\mathcal{O}^+{}_{\mu\nu}{}^{kl}$ and $\Omega^{MN}\Omega_{PN} = \delta_P{}^M$. Due to the property $Z^{M,\alpha}X_M = 0$, we have $\star(\mathcal{F}^M - \mathcal{G}^M)_{\mu\nu}\mathcal{Q}_{Mi}{}^j = \star(\mathcal{H}^M - \mathcal{G}^M)_{\mu\nu} \times \mathcal{Q}_{Mi}{}^j = 0$, where the second equality follows from the equations of motion of $B_{\mu\nu\alpha}$. Combined with the fact that the T-tensor identity [Eq. (3.30) of [52]] implies

$$V\delta_l{}^m = g^2\left(\frac{1}{3}A_{2l}{}^{ijk}A_2{}^m{}_{ijk} - 6A_{1li}A_1{}^{mi}\right), \tag{61}$$

it follows that the terms involving $i\bar{\epsilon}^i\gamma^\nu\epsilon_j$ are canceled out except for the stress energy tensor. We therefore arrive at

$$S^\mu = T^\mu{}_\nu\left(i\bar{\epsilon}^i\gamma^\nu\epsilon_i\right) + \frac{1}{12}i\overline{\delta\chi^{ijk}}\gamma^\mu\delta\chi_{ijk} + \sqrt{2}\left[\left(i\bar{\epsilon}^i\epsilon^j\right)E_{ij}{}^\mu - (i\bar{\epsilon}_i\epsilon_j)E^{ij\mu}\right]. \tag{62}$$

This is the desired one which gives rise to the positive contribution to the Witten–Nester energy when the bosonic equations of motion (53) are satisfied, that is, the on-shell current J^μ becomes

$$J^\mu = \frac{1}{12}i\overline{\delta\chi^{ijk}}\gamma^\mu\delta\chi_{ijk}. \tag{63}$$

In deriving Eq. (62), we have used the Maurer–Cartan equations and the T-tensor identities. The Maurer–Cartan equation is derived based upon the closure relation $[X_M, X_N] = -X_{MN}{}^PX_P$, whereas the T-tensor identities are coming from the branching of **912** of $E_{7(7)}$ into irrep of SU(8). This means that the positivity of Witten–Nester energy holds as long as the linear and quadratic constraints on the embedding tensor are satisfied (but the explicit solutions for these constraints are unnecessary). Namely, the positivity property continues to be valid for the consistent gaugings for any symplectic frames. This is a generalization of the result in [18], where the positivity has been shown for the SO(8) electric gaugings of de Wit and Nicolai [45].

4. Summary

Inspired by the recent sparkling development of our understanding the extended supergravities, this article studied the positivity of mass in these theories. We presented a formulation for the positivity of Witten–Nester energy in terms of Weyl spinors. We found that the positivity conditions (13) should be satisfied as a minimal requirement for the positivity. These conditions are the direct generalization of the one proposed in our previous paper [26].

We derived the universal formula (14) under the positivity conditions. Of particular use of this formula is its simplicity, allowing one to evaluate the mass positivity without lengthy computations as have been done in the literature. Although we have explored the "positivity conditions" for particular theories inspired by supergravity, we have verified that this is indeed true for all ungauged models in [55]. We gave a detailed proof for the classification of the connection in Appendix A. If we required that the bilinear vector field is a Killing vector for BPS states, it turned out that the possible connections take the same form as those appearing in extended supergravities (except for the unusual type of "trombone gaugings"). There should presumably be a profound reason for this. We leave the deeper investigation for future study.

We revealed various new aspects that have been overlooked in the past studies and provided a generalization of the mass positivity proof considered in the literature. We first revisited the minimal $N = 2$ gauged supergravity, for which the contribution of the magnetic charge to the BPS-type inequality was reconsidered. We argued the absence of magnetic charge without resorting the supersymmetry algebra. We expect that the similar argument can be carried out for the matter-coupled $N = 2$ supergravity [29]. As a generalization of the result in [18], it was shown that the positivity holds as far as the target space of the complex scalar is the Hodge–Kähler. This is a gratifying result from the viewpoint of scalar-multiplet in supergravity. In Einstein–Maxwell-dilaton theory, we showed that the dilaton coupling function and the superpotential are severely constrained due to the positive mass property. The supergravity embedding was explored by investigating the integrability condition for the dilation variation, allowing us to find that this is the case for the particular values of the coupling constant. We also extended the result of Ref. [18] concerning the $N = 8$ gauged supergravity by making use of the modern formulation based upon the embedding tensor. Recent development of the maximal gauged supergravity revealed that the deformed theories display interesting physics quite different from those predicted in undeformed theory. Despite that the positive mass property is obscure for deformed theories and for noncompact gaugings, we nevertheless demonstrated that the mass positivity is insensitive to the gauging and deformation parameter, as far as the linear and quadratic constraints on the embedding tensor are satisfied.

Recently, several gravitational theories have been considered motivated by dark energy. Most of these theories are phenomenological and suffer from various stability problems. These theories may be constrained by requiring the positive mass, as discussed in [26,27]. For these purposes, the results of Section 2 and Appendix A would be of great help, since the possible connections are highly restricted. Looking for modified gravitational theories admitting the positive mass and the (bosonic sector of) supergravity with noncanonical scalar fields are interesting future work. We hope to report the results in a separate paper.

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Appendix A. Classifying the positivity conditions

In the body of text, we imposed the conditions (13) on the connections in the supercovariant derivatives for the positivity of the Witten–Nester energy. Here we give a classification of the connection satisfying the positivity conditions (13).

Our strategy here is to expand the connections in terms of the Clifford basis $\{\mathbb{I}, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \gamma_{\mu\nu}\}$. Taking into account the commutation relation (4), the connections $\mathscr{A}_{\mu i}{}^j$ and $\mathscr{B}_{\mu ij}$ can be expanded as[6]

[6] Expressions (64) and (65) are actually redundant, since $\mathscr{A}_{\mu i}{}^j$ ($\mathscr{B}_{\mu ij}$) acts on the spinors with negative (positive) chirality. Hence $a_{(2)}$ and $b_{(2)}$ can be absorbed respectively into $a_{(1)}$ and $b_{(1)}$, and $a_{(3)}$ can be chosen to satisfy $\frac{1}{2}\epsilon_{\nu\rho}{}^{\sigma\tau}a_{(3)\mu\sigma\tau} = ia_{(3)\mu\nu\rho}$. If this is done, however, the positivity conditions (13a) and (13b) must be projected by $1-\gamma_5$ and $1+\gamma_5$, respectively. In this case, one must take great care of the dual of the form fields. In order to circumvent this, we leave the chiral matrix in (64) and (65), for which the basis $\{\mathbb{I}, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \gamma_{\mu\nu}\}$ is independent. The redundancy can be removed by taking $a_{(2)} \to -a_{(1)}$, $b_{(2)} \to b_{(1)}$, $\frac{1}{2}\epsilon_{\nu\rho}{}^{\sigma\tau}\tilde{a}_{(3)\mu\sigma\tau} = i\tilde{a}_{(3)\mu\nu\rho}$ and $a_{(3)}{}^\rho{}_{\rho\mu} = -\frac{i}{2}\epsilon_{\nu\rho\sigma\mu}a_{(3)}{}^{[\nu\rho\sigma]}$

$$\mathscr{A}_{\mu i}{}^{j} = a_{(1)\mu i}{}^{j}\mathbb{I} + a_{(2)\mu i}{}^{j}\gamma_5 + a_{(3)\mu\nu\rho i}{}^{j}\gamma^{\nu\rho}, \tag{64}$$

$$\mathscr{B}_{\mu ij} = b_{(1)\mu\nu ij}\gamma^{\nu} + b_{(2)\mu\nu ij}\gamma^{\nu}\gamma_5, \tag{65}$$

where $a_{(1-3)}$ and $b_{(1-2)}$ are $N \times N$ matrix-valued tensorial fields with $a_{(3)\mu\nu\rho} = a_{(3)\mu[\nu\rho]}$.

Let us begin with the case of $\mathscr{B}_{\mu ij}$. Substituting (65) into (13), expanding again by the Clifford basis and comparing the coefficients of the both sides of equation, we can get two set of relations

$$b_{(1)[\mu\nu]ij} = -b_{(1)[\mu\nu]ji}, \qquad b_{(2)[\mu\nu]ij} = -b_{(2)[\mu\nu]ji}, \tag{66}$$

$$b_{(1)\rho}{}^{[\mu}{}_{ij}\delta^{\nu}{}_{[\tau}\delta^{\rho]}{}_{\lambda]} + \frac{i}{2}b_{(2)\rho}{}^{[\mu}{}_{ij}\epsilon^{\nu\rho]}{}_{\tau\lambda} = b_{(1)\rho}{}^{[\mu}{}_{ji}\delta^{\nu}{}_{[\tau}\delta^{\rho]}{}_{\lambda]} + \frac{i}{2}b_{(2)\rho}{}^{[\mu}{}_{ji}\epsilon^{\nu\rho]}{}_{\tau\lambda}. \tag{67}$$

Contracting indices of (67) and using (66), we obtain

$$b_{(1)(\mu\nu)ij} = b_{(1)(\mu\nu)ji}, \quad b_{(2)(\mu\nu)ij} = b_{(2)(\mu\nu)ji}, \quad b_{(1)[\mu\nu]ij} = -\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}b_{(2)}{}^{\rho\sigma}{}_{ij}. \tag{68}$$

$b_{(1,2)(\mu\nu)}$ can be further decomposed into trace and trace-free parts as

$$b_{(I)(\mu\nu)ij} = \frac{1}{4}g_{\mu\nu}b_{(I)\rho}{}^{\rho}{}_{ij} + \hat{b}_{(I)(\mu\nu)ij}, \qquad \hat{b}_{(I)\rho}{}^{\rho}{}_{ij} = 0, \quad I = 1, 2. \tag{69}$$

One can similarly obtain the relation for the coefficients $a_{(1-3)}$ as above. Suppressing the indices $i, j, \dots$, the 3-tensor $a_{(3)\mu\nu\rho}$ has 24 components. Hence it is decomposable into the irreducible parts $\mathbf{24} \to \mathbf{4} + \mathbf{16} + \mathbf{4}$ as

$$a_{(3)\mu\nu\rho} = a_{(3)[\mu\nu\rho]} + \tilde{a}_{(3)\mu\nu\rho} - \frac{2}{3}a_{(3)}{}^{\sigma}{}_{\sigma[\nu}g_{\rho]\mu}, \tag{70}$$

where $\tilde{a}_{(3)\mu\nu\rho} \equiv \frac{2}{3}(a_{(3)\mu\nu\rho} - a_{(3)[\nu\rho]\mu} + a_{(3)\sigma}{}^{\sigma}{}_{[\nu}g_{\rho]\mu})$ satisfies

$$\tilde{a}_{(3)\mu\nu\rho} = \tilde{a}_{(3)\mu[\nu\rho]}, \qquad \tilde{a}_{(3)}{}^{\sigma}{}_{\sigma\mu} = \tilde{a}_{(3)}{}^{\sigma}{}_{\mu\sigma} = 0, \qquad \tilde{a}_{(3)[\mu\nu\rho]} = 0. \tag{71}$$

Noting that γ_5 is pure-imaginary and anti-symmetric in our convention, insertion of (64) into (13) yields

$$a_{(3)[\mu\nu\rho]i}{}^{j} = -a_{(3)[\mu\nu\rho]}{}^{j}{}_{i}, \qquad \tilde{a}_{(3)\mu\nu\rho i}{}^{j} = 0, \qquad a_{(3)}{}^{\rho}{}_{\rho\mu i}{}^{j} = a_{(3)}{}^{\rho}{}_{\rho\mu}{}^{j}{}_{i},$$
$$a_{(1)\mu i}{}^{j} + a_{(1)\mu}{}^{j}{}_{i} = -\frac{4}{3}a_{(3)}{}^{\rho}{}_{\rho\mu i}{}^{j}, \qquad a_{(2)\mu i}{}^{j} + a_{(2)\mu}{}^{j}{}_{i} = -\frac{2}{3}i\epsilon_{\nu\rho\sigma\mu}a_{(3)}{}^{[\nu\rho\sigma]}{}_{i}{}^{j}. \tag{72}$$

Eqs. (66), (68) and (72) are exhaustive constraints arising from the positivity conditions (13) (see also the comments in footnote 6). One can easily verify that all connections considered in the body of text satisfy these relations. Comparing with the model of Einstein–Maxwell-dilaton theory in Section 3.3, one sees that $b_{(1)[\mu\nu][ij]}$ term correspond to the Maxwell field, $a_{(1)\mu i}{}^{j}$ is the gauge connection and $b_{(1)\rho}{}^{\rho}{}_{(ij)}$ denotes the superpotential contributions.

Although the positivity conditions (13) put some restrictions to the possible form of the connections, some unfamiliar terms ($a_{(3)}{}^{\rho}{}_{\rho\mu}$ and $\hat{b}_{(1)\mu\nu}$) remain. Eq. (72) implies that $a_{(1)}$ fails to describe the connection contained in the subgroup of U(N) if $a_{(3)}{}^{\rho}{}_{\rho\mu}$ is nonvanishing. Also, there exist no supergravity models which contain $\hat{b}_{(1)(\mu\nu)ij} = \hat{b}_{(1)(\mu\nu)(ij)}$ (see e.g., [55] for

[see (70) for definition] at the final expression. Because of this, the imaginary self-dual property of $b_{(1)[\mu\nu]}$ follows from (68).

ungauged models). Hence the positivity conditions leave some more freedom than extended supergravity models, although it is not clear yet such terms in fact produce the *positive and finite* Witten–Nester energy.

Nevertheless, we can fix these remaining terms as follows. Let us consider the case in which $\hat{\nabla}_\mu \epsilon_i = 0$ is satisfied, for which the spacetime is in "BPS." If the supergravity embedding is indeed possible, the bilinear vector $V^\mu = i\bar{\epsilon}^i \gamma^\mu \epsilon_i$ vector turns out to be a Killing field for the BPS metric [15].[7] Hence it might be reasonable to require that $V^\mu = i\bar{\epsilon}^i \gamma^\mu \epsilon_i$ satisfies the Killing equation when $\hat{\nabla}_\mu \epsilon_i = 0$ is satisfied. This gives

$$\begin{aligned} 0 &= \nabla_{(\mu} V_{\nu)} \\ &= i\bar{\epsilon}^i \big[\gamma^0 \big(\mathscr{A}_{(\mu}{}^j{}_i\big)^T \gamma^0 \gamma_{\nu)} - \gamma_{(\nu} \mathscr{A}_{\mu)i}{}^j\big]\epsilon_j + i\bar{\epsilon}_i \gamma_{(\mu} \mathscr{B}_{\nu)}{}^{ij} \epsilon_j - i\bar{\epsilon}^i \gamma_{(\mu} \mathscr{B}_{\nu)ij} \epsilon^j, \end{aligned} \tag{73}$$

where we have used $i\bar{\epsilon}_i \gamma^0 (\mathscr{B}_{(\mu}{}^{ji})^T \gamma^0 \gamma_{\nu)} \epsilon_j = -i\bar{\epsilon}_i \gamma_{(\nu} \mathscr{B}_{\mu)}{}^{ij} \epsilon_j$. It follows that $\mathscr{A}_{\mu i}{}^j$ obeys

$$\gamma^0 \big(\mathscr{A}_{(\mu}{}^j{}_i\big)^T \gamma^0 \gamma_{\nu)} = \gamma_{(\nu} \mathscr{A}_{\mu)i}{}^j. \tag{74}$$

Substituting (64) into the above equation and using (72), we have further constraints

$$a_{(3)\rho}{}^\rho{}_{\mu i}{}^j = 0, \qquad a_{(3)[\mu\nu\rho]i}{}^j = 0. \tag{75}$$

This implies that $a_{(1)i}{}^j = -a_{(1)}{}^j{}_i$, viz., $a_{(1)}$ is anti-hermitian and therefore describes the connection contained in $\mathrm{U}(N)$.

For the connection $\mathscr{B}_{\mu ij}$, Eq. (73) does not imply $\gamma_{(\mu} \mathscr{B}_{\nu)ij} = 0$ since the property (2) must be taken into account. With this remark in mind, the condition $\bar{\epsilon}^i \gamma_{(\mu} \mathscr{B}_{\nu)ij} \epsilon^j = 0$ yields

$$\hat{b}_{(I)(\mu\nu)ij} = 0, \quad I = 1, 2. \tag{76}$$

After the replacement $a_{(2)} \to -a_{(1)}, b_{(2)} \to b_{(1)}$ with chiral projections, we finally arrive at

$$\mathscr{A}_{\mu i}{}^j = A_{\mu i}{}^j \mathbb{I}, \qquad \mathscr{B}_{\mu ij} = W_{(ij)} \gamma_\mu + F_{\mu\nu[ij]} \gamma^\nu, \tag{77}$$

where A_μ is anti-hermitian, $W_{(ij)}$ is an $N \times N$ symmetric matrix and $F_{\mu\nu} = F_{[\mu\nu]}$ is imaginary self-dual $\star F_{\mu\nu} = iF_{\mu\nu}$. This is exactly the same form as those appearing in extended supergravity models considered thus far.[8] It therefore turns out that the conditions (13) and (73) are closely related to the construction of extended supergravity. Note however that the condition (77) is not sufficient to probe that the Witten–Nester energy is positive nor the supergravity embedding is possible. For example, the connection $\mathscr{A}_{\mu i}{}^j$ in the maximal gauged supergravity is not U(8) but SU(8) [i.e., $\mathrm{Tr}(A_\mu) = 0$], which corresponds to the R-symmetry.

Since $V^\mu = i\bar{\epsilon}^i \gamma^\mu \epsilon_i$ generates an asymptotic time translation at infinity, the condition (73) requires that this asymptotic symmetry is enhanced to the exact symmetry for the configuration in which $\hat{\nabla}_\mu \epsilon_i = 0$ is satisfied. This is in accordance with the intuition that the BPS states are in mechanical equilibrium for which gravitational attractions and moduli fields are compensated by the electromagnetic repulsive forces, implying the existence of the Killing field.

[7] The Einstein–Maxwell-dilaton theory does not have a supergravity origin for the general coupling as shown in Section 3.3, yet this property continues to hold and the positivity condition is also met. In the Einstein-$\Lambda(> 0)$ system for which the positivity condition is not satisfied, the bilinear vector field also fails to be a Killing vector.

[8] More precisely, this is true except for the gauged supergravity in which the "trombone symmetry" is gauged. In this case, $b_{(1)\rho}{}^\rho{}_{[ij]} = 0$ is not satisfied [53]. This accords with the intuition since this kind of gaugings contributes positively to the cosmological constant and even more this theory does not have a covariant action (even in the electric gaugings). It would be interesting to understand better this fact and its relation to the positivity conditions.

Though we did not discuss the new types of connections in the body of text, the results of this appendix will be instrumental for constructing (bosonic sector of) supergravity incorporating noncanonical scalar fields and constraining modified theories of gravity.

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