Periodic Solutions of Functional Differential Equations of Neutral Type

M. Ait Babram and K. Ezzinbi

*Département de Mathématiques, Faculté des Sciences Semlalia, B.P.S.15 Marrakech, Morocco*

and

R. Benkhalti

*Department of Mathematics, Pacific Lutheran University, Tacoma, Washington 98447*

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In this paper we give necessary and sufficient conditions for the existence of periodic solutions for convex functional differential equations of neutral type with finite and infinite delay.

1. INTRODUCTION

Periodic solutions of dynamical systems are of great interest both in applications and theory. In different set-ups and by many different authors, various methods were developed in order to ensure the existence of periodic solutions, see for instance the work of Arino, Burton, and Haddock [1], Burton and Hatvani [3], Chow [4], Hale and Mawhin [8], Hino, Murakani, and Naito [9], Makay [10], and Massera [11]. It has been shown in [11] that if there is a bounded solution of a periodic ordinary differential equation and the solutions can be continued for all future times then the O.D.E. has a periodic solution. In [4], a similar result was extended to functional differential equations with finite delay. Recently, in [10] Makay generalized Chow's result to functional differential equations with infinite delay and to integral equations.
In this paper we are concerned with convex neutral functional differential equations of the form

$$\frac{d}{dt}\left[ x(t) - G(t, x) \right] = F(t, x),$$

(1)

where $F$ and $G$ are $T$ periodic in $t$. We will investigate the existence of periodic solutions of Eq. (1), when the delay is finite or infinite. By using the Massera approach, we will derive sufficient conditions on the parameter of Eq. (1) that ensure the existence of periodic solutions. We will present two different methods, one for the finite delay and the other for the infinite delay. In the latter, we introduce a new phase space satisfying the axiomatic framework required in [7, 8].

The paper will be organized in 4 sections. The second section deals with the finite delay problem. The third section considers the problem with infinite delay. The last section is devoted to an example.

2. FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE WITH FINITE DELAY

Consider the following functional differential equation of neutral type,

$$\begin{cases} \frac{d}{dt}\left[ x(t) - G(t, x) \right] = F(t, x), & \text{for } t \geq 0, \\ x_0 = \varphi \in C = C([-h, 0], \mathbb{R}^n), \end{cases}$$

(2)

where $C$ is the space of all continuous functions from $[-h, 0]$ into $\mathbb{R}^n$ endowed with the uniform norm topology. The functions $F, G : \mathbb{R} \times C \rightarrow \mathbb{R}^n$ are continuous and for every $t \geq 0$, the function $x_0 \in C$ is defined by $x_0(t) = x(t + \theta)$, for every $\theta \in [-h, 0]$. The initial value problem associated with Eq. (2) is that given $\varphi \in C$, find a continuous function $x : [-h, A] \rightarrow \mathbb{R}^n$, such that $x_0 = \varphi$ and $t \rightarrow x(t) - G(t, x)$ is continuously differentiable and satisfies Eq. (2) in $[0, A]$ for some $A > 0$.

(H₁) $G : \mathbb{R} \times C \rightarrow \mathbb{R}^n$ is a continuously locally Lipschitz $T$-periodic function such that for all $r > 0$, there exists a real function $W$ defined on $\mathbb{R}^+$ with $\lim_{b \rightarrow 0} W(b) = 0$ and for $t, s \in \mathbb{R}$ we have $|G(t, x) - G(s, x)| \leq W(|t - s|)$, for all continuous functions $x$ defined on $[-h, \infty)$ with values in $\mathbb{R}^n$ and satisfying $\sup_{t \in [-h, \infty)} |x(t)| \leq r$ (otherwise, we may call $G(t, x)$ equicontinuous in $x$ uniformly for $t$ defined on $[-h, \infty)$ with values in $\mathbb{R}^n$ such that $\sup_{t \in [-h, \infty)} |x(t)| \leq r$).

(H₂) $F : \mathbb{R} \times C \rightarrow \mathbb{R}^n$ is a continuous locally Lipschitz $T$-periodic function in $t$. 
We say that Eq. (2) is convex, if for every \( x, y \) solutions of Eq. (2) and \( \alpha \in [0,1] \), then \( \alpha x + (1 - \alpha)y \) is also a solution of (2).

**Theorem 1.** Assume that \( (H_1) \) and \( (H_2) \) are satisfied and Eq. (2) is convex. If there is a bounded solution of (2) defined on the interval \([0, +\infty)\), then Eq. (2) has a \( T \)-periodic solution.

**Proof.** Without loss of generality, we can assume that \( T > h \). Otherwise, we consider a multiple of \( T \).

Let \( S = C([T, 0], \mathbb{R}^n) \), and consider the set

\[
U = \{ \psi \in S : \begin{array}{l}
|\psi(s)| \leq r, \text{ for } t \in [-T, 0] \\
|x(t, 0, \psi)| \leq r, \text{ for } t \geq 0
\end{array} \quad (a)
\]

where \( r \) is the bound for the existing bounded solution \( x \). To show that \( U \) is non-empty, let \( x \) be the bounded solution. Then, since \( F \) and \( G \) are \( T \)-periodic, one can suppose that \( x \) is defined on the interval \([-T, 0] \), i.e., \( x_0 = x|_{[-T, 0]} \), which allows us to see that \( U \) is non-empty. Next, we can see that \( U \) is convex by using the uniqueness property and the fact that Eq. (2) is convex. For \( \psi_1, \psi_2 \in U \), since

\[
x(t, 0, \alpha \psi_1 + (1 - \alpha) \psi_2) = \alpha x(t, 0, \psi_1) + (1 - \alpha) x(t, 0, \psi_2),
\]

one has

\[
|\alpha \psi_1(s) + (1 - \alpha) \psi_2(s)| \leq \alpha |\psi_1(s)| + (1 - \alpha) |\psi_2(s)| \leq r.
\]

In addition, for \( s \in [-T, 0] \) we have

\[
|\alpha \psi_1(s) + (1 - \alpha) \psi_2(s)| \leq \alpha |\psi_1(s)| + (1 - \alpha) |\psi_2(s)| \leq r.
\]

That is, \( U \) is a convex set. To prove that \( U \) is a closed set, consider a sequence \( (\psi_n)_{n \in \mathbb{N}} \subseteq U \), such that \( \psi_n \to \psi \) with the supremum norm. Note that \( \psi \) satisfies condition (a), and if \( |x(t, 0, \psi)| > r \), for some \( t > 0 \), then by the continuous dependence on the initial data, there exists an \( n_2 \in \mathbb{N} \) such that \( |x(t, 0, \psi_n)| > r \), for \( n \geq n_2 \). This is a contradiction to the fact that \( \psi_n \in U \). Thus, \( U \) is a closed bounded set.

Now, let \( P : U \to U \) be the operator defined by \( P(\psi)(s) = x(s + T, 0, \psi) \), for \( s \in [-T, 0] \). Then for every \( s_1, s_2 \in [-T, 0] \) and \( \psi \in U \), we have

\[
|x(s_1 + T, 0, \psi) - x(s_2 + T, 0, \psi)|
\]

\[
\leq |c(s_1 + T, x_{s_1 + T}) - c(s_2 + T, x_{s_2 + T})| + \int_{s_1 + T}^{s_2 + T} |F(s, x_s)| \, ds
\]

\[
\leq W(|s_1 - s_2|) + k|x_1 - x_2|,
\]
where $k = \sup \{ |F(s, \varphi)| : s \in [0, T] \text{ and } \|\varphi\|_{\infty} \leq r \}$. Then, by the Ascoli–Arzela theorem, we conclude that $P(U)$ is a compact set. And the operator $P$ is continuous. Hence by the Schauder fixed point theorem, $P$ has a fixed point, that is, Eq. (2) has a $T$-periodic solution.

3. FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE WITH INFINITE DELAY

In the infinite delay interval, the state $x$, at time $t$, always contains the initial data. As a consequence, the introduction of new phase spaces in particular applications requires a new and separate development of the theory. In our case, we consider as a phase space $C_g = C([0, \infty), \mathbb{R}^n)$ of continuous bounded functions on $[0, \infty)$, endowed with the norm $\|\cdot\|_g$ defined by

$$
\|\varphi\|_g = \sup_{s \in [0, \infty)} \left[ \frac{|\varphi(s)|}{g(s)} \right],
$$

where $g : (0, \infty) \to [1, +\infty)$ is a continuous decreasing function such that $\lim_{t \to \infty} g(t) = +\infty$.

Recently, some attention have been focused on the space $C_g$ as a phase space for functional differential equations with infinite delay (see [7, 9]). In fact, Arino, Burton, and Haddock in [1] were the first who introduced the space $C_g$ and established the existence of periodic solutions of some integro-differential equations using Horn's fixed point theorem. For more details on these topics, we refer the reader to Burton's book [2].

Consider the functional differential equation of neutral type with infinite delay,

$$
\frac{d}{dt} \left[ x(t) - G(t, x_t) \right] = F(t, x_t), \quad \text{for } t > 0
$$

$$
x_0 = \varphi \in C_g,
$$

where $F, G : \mathbb{R} \times C_g \to \mathbb{R}^n$ are continuous $T$-periodic functions and for every $t \geq 0$ the function $x_t \in C_g$ is defined by $x_t(\theta) = x(t + \theta)$, for $\theta \in (-\infty, 0]$.

Remark. For every $\varphi \in C_g$, $\|\varphi\|_g \leq \sup_{\theta \in (-\infty, 0]} |\varphi(\theta)|$.

The importance of introducing the norm $\|\cdot\|_g$ is shown in the following proposition.
**Proposition 2.** Let \( \Psi = (\psi_n)_{n \in \mathbb{K}} \subset C_g \) be a bounded and equicontinuous sequence. Then, there is a subsequence \((\psi_k)_{n \in \mathbb{K}}\) of \( \Psi \) and \( \psi \in C_g \) such that \( \psi_n \to_{n \to \infty} \psi \) in \( \| \cdot \|_{\infty} \)-norm.

**Proof.** Applying the Ascoli–Arzela theorem to the sequence \((\psi_n)_{n \in \mathbb{K}}\) in the interval \([-1, 0]\), one can see that there exists a \( \psi^{(1)} \in C((-\infty, 0], \mathbb{R}^n) \) and a subsequence \((\psi_n^{(1)})_{n \in \mathbb{K}}\), such that \( \psi_n^{(1)} \to_{n \to \infty} \psi^{(1)} \) with the supremum norm in the interval \([-1, 0]\). Similarly, one can find \( \psi^{(2)} \in C_s \), and a subsequence \((\psi_n^{(2)})_{n \in \mathbb{K}}\) of \((\psi_n^{(1)})_{n \in \mathbb{K}}\), such that \( \psi_n^{(2)} \to_{n \to \infty} \psi^{(2)} \) with the supremum norm in the interval \([-2, 0]\).

Following the same procedure, one can find \( \psi \in C_g \) and a subsequence \((\psi_k) = (\psi^{(n)})_{n \in \mathbb{K}}\), such that \( \psi_{[\infty, 0]} = \psi^{(n)} \), for \( n \in \mathbb{K} \), and

\[
\psi_{k_n} \to_{n \to \infty} \psi \text{ compactly in } (-\infty, 0].
\]

Moreover, for every \( K > 0 \),

\[
\| \psi_{k_n} - \psi \|_{\infty} \leq \max \left\{ \sup_{s < -K} \frac{|\psi_{k_n}(s)|}{g(s)}; \sup_{-K \leq s \leq 0} \frac{|\psi_{k_n}(s) - \psi(s)|}{g(s)} \right\}
\]

\[
\leq \max \left\{ \sup_{s < -K} \frac{2L}{g(s)}; \sup_{-K \leq s \leq 0} \frac{|\psi_{k_n}(s) - \psi(s)|}{g(s)} \right\},
\]

where \( L = \sup_{n \in \mathbb{K}} \sup_{\theta \in [\infty, 0]}|\psi_k(\theta)| \). Thus, for every \( \varepsilon > 0 \), if we choose \( K \) sufficiently large such that \( 2L/g(s) < \varepsilon \), for \( s \in (-\infty, K] \), then by (4), there exists \( N(\varepsilon) \in \mathbb{K} \) such that, for each \( n \geq N(\varepsilon) \),

\[
\sup_{0 \leq s \geq -K} \frac{|\psi_{k_n}(s) - \psi(s)|}{g(s)} < \varepsilon.
\]

Hence, for \( \varepsilon > 0 \) there exists \( N(\varepsilon) \in \mathbb{K} \) such that \( \| \psi_{k_n} - \psi \|_{\infty} < \varepsilon \), for \( n \geq N(\varepsilon) \).</p>

Now, we assume that

(\( H_4 \)) \( G : \mathbb{R} \times C_s \to \mathbb{R}^n \) is a continuous \( T \)-periodic function such that for all \( r > 0 \), there exists a real function \( W \) defined on \( \mathbb{R}^+ \) with \( \lim_{t \to \infty} W(t) = 0 \) and \( |G(t, x_t) - G(s, x_s)| \leq W(t - s) \), for all continuous functions \( x \) which are defined from \( \mathbb{R} \) into \( \mathbb{R}^n \) and satisfy \( \sup_{t \in \mathbb{R}} |x(t)| \leq r \) (otherwise, we may call \( G(t, x_t) \) equicontinuous in \( t \) uniformly in \( x \) where \( x \) is a continuous function from \( \mathbb{R} \) into \( \mathbb{R}^n \) that satisfies \( \sup_{t \in \mathbb{R}} |x(t)| \leq r \).)

(\( H_5 \)) \( F : \mathbb{R} \times C_s \to \mathbb{R}^n \) is a continuous \( T \)-periodic in \( t \) function.

(\( H_6 \)) Equation (3) has a bounded solution which is defined on \( \mathbb{R} \).
Let $\Lambda = \{x : x$ is a solution of (3) on $\mathbb{R}$ such that $\sup_{t \in \mathbb{R}} |x(t)| \leq r\}$, where $r$ is the bound of the existing bounded solution. Note that by assumption (H3), $\Lambda$ is non-empty.

For every $x \in \Lambda$ and $t \in \mathbb{R}$, define

$$T(t, x) = \int_{-\infty}^{+\infty} \Theta(s)|x(t + s)|^2 \, ds,$$

where $\Theta : \mathbb{R} \to \mathbb{R}^+$ is an integrable function and let $l(x) = \sup_{t \in \mathbb{R}} T(t, x)$ and $L(F, G) = \inf_{x \in \Lambda} l(x)$.

**Theorem 3.** Suppose that Eq. (3) is convex. Then, under assumptions (H3), (H4), and (H5), Eq. (3) has a $T$-periodic solution.

To show Theorem 3 the following two lemmas are needed:

**Lemma 4.** Under assumptions (H3), (H4), and (H5), there exists at least one $x \in \Lambda$, such that $l(x) = L(F, G)$.

**Proof.** For every $n \in \mathbb{N}$, there exists a certain $x^n \in \Lambda$ satisfying $l(x^n) \leq L(F, G) + 1/n$. By assumptions (H3) and (H4), $(x^n)_{n \in \mathbb{N}}$ is equicontinuous. Moreover, it is bounded. Hence, by the Ascoli–Arzelà theorem, there is an $x \in C^0(\mathbb{R} ; \mathbb{R}^n)$ (the space of bounded continuous functions) and a subsequence $(x^{n_k})_{k \in \mathbb{N}}$ of $(x^n)_{n \in \mathbb{N}}$ so that $x^{n_k} \to x$ compactly in $\mathbb{R}$. Then, for every $t \in \mathbb{R}$, $x^{n_k} \to x$ compactly in $(-\infty, 0]$. Hence, by Proposition 2, $x^{n_k} \to x$ in $\|\cdot\|_\infty$-norm. Moreover, for $t \in [0, \infty)$ we have

$$x^k(t) = x^k(0) - G(0, x^k_0) + G(t, x^k_t) + \int_0^t F(s, x^k_s) \, ds.$$

Hence if we let $k$ go to infinity, then by the continuity of $G(t, \cdot)$ and $F(t, \cdot)$ with respect to the norm $\|\cdot\|_\infty$, we conclude that

$$x(t) = x(0) - G(0, x_0) + G(t, x_t) + \int_0^t F(s, x_s) \, ds,$$

for $t \in [0, \infty)$. Therefore $x \in \Lambda$. In addition, by the dominated convergence theorem, we have $T(t, x^n) \to_T T(t, x)$, for every $t \in \mathbb{R}$. Thus $l(x) \leq L(F, G)$. And it follows that $l(x) = L(F, G)$. \( \blacksquare \)

**Lemma 5.** Under assumptions (H3), (H4), and (H5), if there is $u, v \in \Lambda$, such that $l(u) = l(v) = L(F, G)$, then there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, such that, for $s \in \mathbb{R}$,

$$\lim_{n \to \infty} [u(t_n + s) - v(t_n + s)] = 0.$$


Proof. For \( t \in \mathbb{R} \), define \( y(t) = (u(t) + v(t))/2 \) and \( z(t) = (u(t) - v(t))/2 \). Then \( y \in \Lambda \) and
\[
|T(t, y)| + |T(t, z)| = \frac{|T(t, u)| + |T(t, v)|}{2}.
\]
Consequently,
\[
|T(t, y)| \leq L(F, G) - \inf_{t \in \mathbb{R}} |T(t, z)|.
\]
Thus \( \inf_{t \in \mathbb{R}} |T(t, z)| = 0 \). This implies the existence of a sequence \((t_n)_{n \in \mathbb{R}} \subset \mathbb{R} \) such that
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \Theta(s)|u(t_n + s) - v(t_n + s)|^2 \, ds = 0. \tag{5}
\]
On the other hand, \( u(t_n + \cdot) - v(t_n + \cdot) \) is equicontinuous and bounded. Then, by the Ascoli–Arzela theorem, there is a subsequence \((t_{n_k})_{k \in \mathbb{N}} \subset \mathbb{R} \) and \( p \in C^2(\mathbb{R}, \mathbb{R}^2) \) such that \( u(t_{n_k} + \cdot) - v(t_{n_k} + \cdot) \to p \) compactly in \( \mathbb{R} \). Hence, by the Dominated Convergence theorem and \( (5) \), it follows that
\[
\int_{-\infty}^{\infty} \Theta(s)|p(s)|^2 \, ds = 0.
\]
Finally, using the fact that \( \Theta > 0 \), we deduce that \( p = 0 \). Consequently,
\[
\lim_{n \to \infty} |u(t_n + s) - v(t_n + s)| = 0 \quad \text{for} \quad s \in \mathbb{R}. \tag{6}
\]

Proof of Theorem 3. Let \( u \in \Lambda \), such that \( l(u) = L(F, G) \). Then, \( u(\cdot + T) \) is also a solution of \( (3) \) and \( l(u) = l(u(\cdot + T)) \). Thus, by Lemma 5, there is a sequence \((t_{n_k})_{k \in \mathbb{N}} \subset \mathbb{R} \) such that \( \lim_{n \to +\infty} |u(t + t_n) - u(t + t_n + T)| = 0 \), for \( t \in \mathbb{R} \). Now, by assumptions (H_3) and (H_4), it follows that the sequence \((u(\cdot + t_n))_{n \in \mathbb{N}} \) is equicontinuous. Moreover, it is bounded. Then, by the Ascoli–Arzela theorem, there exists a continuous function \( q \in C^2(\mathbb{R}, \mathbb{R}^2) \) and a subsequence \((t_{n_k})_{k \geq 0} \subset \mathbb{R} \) such that
\[
\lim_{n \to +\infty} u(\cdot + t_{n_k}) = q(\cdot) \quad \text{compactly in} \ \mathbb{R}. \tag{6}
\]
In particular, for \( t \in \mathbb{R} \), we have
\[
q(t + T) = \lim_{n \to +\infty} u(t + T + t_n)
= \lim_{n \to +\infty} u(t + t_n)
= q(t). \tag{7}
\]
Thus $q$ is $T$-periodic. To see this, let $t_n = k_n T + \sigma_n$, where $k_n \in \mathbb{N}$ and $\sigma_n \in [0, T)$. Then there exists $\sigma \in \mathbb{R}$ and a subsequence $(\sigma_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to +\infty} \sigma_n = \sigma$. Moreover, for every $t \in \mathbb{R}^+$, we have

$$\left| u(t + t_n - \sigma_n) - q(t - \sigma) \right| \leq \left| u(t + t_n - \sigma_n) - q(t - \sigma_n) \right|$$

$$+ \left| q(t - \sigma_n) - q(t - \sigma) \right|$$

$$\leq \sup_{s \in [t-T, t]} \left| u(t_n + s) - q(s) \right|$$

$$+ \left| q(t - \sigma_n) - q(t - \sigma) \right|.$$  

So, by (6) and the continuity of $q$, it follows that $\lim_{n \to +\infty} u(+k_nT) = q(-\sigma)$, compactly in $\mathbb{R}$. Consequently, for each $t \geq 0$ and by Proposition 2 we have $\lim_{n \to +\infty} u_{t+k_nT} = q_{t-\sigma}$ in $\|\cdot\|_x$-norm. Moreover,

$$u(t + k_nT) = u(k_nT) - G(0, u_{k_nT}) + G(t, u_{t+k_nT}) + \int_0^t F(s, u_{t+k_nT}) \, ds.$$  

Hence, if we let $n$ go to infinity, then by the continuity of $G(t, \cdot)$ and $F(t, \cdot)$ with respect to the norm $\|\cdot\|_x$, we conclude that

$$q(t - \sigma) = q(-\sigma) - G(0, q_{-\sigma}) + G(t, q_{t-\sigma}) + \int_0^t F(s, q_{t-\sigma}) \, ds. \quad (8)$$

Therefore, by (7) and (8), it follows that $q(-\sigma)$ is a $T$-periodic solution of (3). \[\square\]

**Remark.** The method used in the infinite delay case to obtain $T$-periodic solutions can be extended successfully to problems with finite delay, without conditions on the delay. One can obtain the following result:

**Theorem 6.** Assume that $(H_1)$ and $(H_2)$ are satisfied and Eq. (2) is convex. If there is a bounded solution of (2) defined on an interval of the form $[a, +\infty)$, then Eq. (2) has a $T$-periodic solution.

4. **Example**

Let $\Delta = \{(t, s) \in \mathbb{R}^2 : s \leq t\}$ and consider the scalar equation

$$\begin{cases}
\frac{d}{dt} \left[ x(t) - \int_{-\infty}^t k(t, s)x(s) \, ds \right] = b(t)x(t) + \sum_{i=1}^{N} a_i(t)x(t-h_i(t)) + f(t), & \text{for } t \geq 0 \\
x_0 = \varphi \in C((-\infty, 0]; \mathbb{R}).
\end{cases} \quad (9)$$
We assume that

(H₁) The function \( k : \Delta \to \mathbb{R} \) and its first derivative with respect to \( t \) are continuous and \( T \)-periodic, such that

\[
\int_{-\infty}^{\infty} |k(t, s)| \, ds < +\infty, \quad \int_{-\infty}^{\infty} \left| \frac{\partial k(t, s)}{\partial t} \right| \, ds < +\infty, \quad \text{and}
\]

\[
|k(t, s)| \leq M.
\]

(By periodicity, we mean \( k(t + T, s + T) = k(t, s) \).)

(H₂) \((h_i)_{i=1}^N, (a_i)_{i=1}^N, b, f \) are continuous and \( T \)-periodic functions and \( h_i \), for \( i = 1, \ldots, N \).

For every \((t, \varphi) \in \mathbb{R} \times C((-\infty, 0], \mathbb{R})\), define

\[
G(t, \varphi) = \int_{-\infty}^{0} k(t, t + s) \varphi(s) \, ds,
\]

then we have the following assertions.

**Proposition 7.** Assume that (H₁) and (H₂) are satisfied, then for all \( r > 0 \) there exists a real function \( W : \mathbb{R}^+ \to \mathbb{R}^+ \), with \( \lim_{b \to 0^+} W(b) = 0 \) such that for every continuous function \( x : \mathbb{R} \to \mathbb{R} \) satisfying \( \sup_{t \in \mathbb{R}} |x(t)| \leq r \), one has

\[
\left| G(t_1, x_{t_1}) - G(t_2, x_{t_2}) \right| \leq W(|t_1 - t_2|), \quad \text{for each } t_1, t_2 \in \mathbb{R}.
\]

**Proof.** Let \( G(t, x) = \int_{-\infty}^{t} k(t, s)x(s) \, ds \) and \( x : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( \sup_{t \in \mathbb{R}} |x(t)| \leq r \). Then for every \( t_1, t_2 \in \mathbb{R} \) with \( t_1 < t_2 \), we have

\[
\left| G(t_1, x_{t_1}) - G(t_2, x_{t_2}) \right| \leq r \int_{-\infty}^{t_1} |k(t_1, s) - k(t_2, s)| \, ds
\]

\[
+ r \int_{t_1}^{t_2} |k(t_2, s)| \, ds.
\]

(10)

Now, it remains to show that the second term of (10) tends to zero uniformly as \( |t_1 - t_2| \to 0 \). But, by assumption (H₁) we have \( \int_{-\infty}^{t_2} |k(t_2, s)| \, ds \leq M|t_1 - t_2| \). Moreover, \( \int_{-\infty}^{t} |k(t, s)| \, ds \) is continuous and periodic, and it follows that for every \( \varepsilon > 0 \), there exists \( P > 0 \), independent of \( t \), such that

\[
\int_{-\infty}^{t-P} |k(t, s)| \, ds < \frac{\varepsilon}{4}, \quad \text{for } t \in \mathbb{R}.
\]

(11)
In addition, since \( k \) is continuous, there exists \( \delta > 0 \), such that

\[
|k(t_1, s) - k(t_2, s)| < \frac{\varepsilon}{2F},
\]

(12)

for \( |t_1 - t_2| < \delta \) and \( s \in [t_1 - P, t_1] \). Consequently by (11) and (12), we get

\[
\int_{t_1 - P}^{t_1} |k(t_1, s) - k(t_2, s)| ds \leq \int_{t_1 - P}^{t_1} |k(t_1, s) - k(t_2, s)| ds
\]

\[
+ \int_{t_1 - P}^{t_1} |k(t_1, s) - k(t_2, s)| ds \leq \varepsilon,
\]

for \( |t_1 - t_2| < \delta \).

**Proposition 8.** Under assumptions (H_8) and (H_7), there exists a continuous decreasing function \( g : (-\infty, 0) \to [1, +\infty) \), such that \( |G(t, \varphi)| \leq c\|\varphi\|_g \) for each \( \varphi \in C((-\infty, 0], \mathbb{R}) \), where \( c \) is a constant independent of \( (t, \varphi) \).

**Proof.** Define \( g \) by \( g(s) = 1/M\gamma(s) \), for \( s \in (-\infty, 0] \), where \( \gamma : (-\infty, 0] \to (0, 1/M] \) is a continuous non-decreasing function, such that \( \lim_{s \to -\infty} \gamma(s) = 0 \) and \( \int_{-\infty}^{0} (1/\gamma(s)) ds < +\infty \). Thus, for every \( \varphi \in C(\mathbb{R}) \),

\[
|G(t, \varphi)| \leq \int_{-\infty}^{0} |k(t, t + s)| |\varphi(s)| ds
\]

\[
\leq \left( \int_{-\infty}^{0} \frac{|k(t, t + s)|}{M} \frac{1}{\gamma(s)} ds \right) \|\varphi\|_g
\]

\[
\leq \left( \int_{-\infty}^{0} \frac{1}{\gamma(s)} ds \right) \|\varphi\|_g.
\]

Consequently, if we let \( c = \int_{-\infty}^{0} (1/\gamma(s)) ds \), it follows that \( |G(t, \varphi)| \leq c\|\varphi\|_g \).

Finally, since the assumptions (H_3) and (H_4) are satisfied by Propositions 7 and 8, all that remains to be shown is the existence of a bounded solution of (9), in order to apply Theorem 3.

**Proposition 9.** Under assumptions (H_8) and (H_7), suppose that there exists \( \alpha > 0 \) such that, for \( t \in \mathbb{R} \),

\[
k(t, t) + b(t) - \sum_{i=1}^{N} |a_i(t)| + \int_{-\infty}^{t} \left| \frac{\partial k(t, s)}{\partial t} \right| ds + |f(t)| > \alpha.
\]

Then, there exists a T-periodic solution of (9).
Proof. Consider the equation
\[
\frac{d}{dt}x(t) = (k(t, t) + b(t))x(t) + \int_{-\infty}^{t} \left| \frac{\partial k(t, s)}{\partial t} \right| ds + \sum_{i=1}^{N} a_i(t) x(t - h_i(t)) + f(t)
\]
where \( g \) is given by Proposition 8. If \( x(\cdot, \sigma, \varphi) \) is a solution of (14) then \( x(\cdot, \sigma, \varphi) \) is a solution of (4). In [7, p. 401; 9] it was established that for every \( \varphi \in C_g \), Eq. (14) has a solution \( x(\cdot, \sigma, \varphi) \) defined on \( \mathbb{R} \). Let \( \varphi \in C_g \) such that \( \sup_{\sigma \in (-\infty, 0]} |\varphi(0)| = \varphi(0) \) and \( \varphi(0) > K \), where \( K = 1 + B/\alpha \) and \( B = \sup_{t \in \mathbb{R}} \| f(t) \| \). Then, for each \( \sigma \in \mathbb{R}^+ \) we have
\[
x'(\sigma, \sigma, \varphi) = (k(\sigma, \sigma) + b(\sigma)) \varphi(0) + \sum_{i=1}^{N} a_i(\sigma) \varphi(-h_i(\sigma)) + \int_{-\infty}^{\sigma} \frac{\partial k(\sigma, s)}{\partial t} \varphi(s - \sigma) ds + f(\sigma)
\]
\[
\geq \left( k(\sigma, \sigma) + b(\sigma) - \sum_{i=1}^{N} |a_i(\sigma)| \right) \varphi(0) - \int_{-\infty}^{\sigma} \left| \frac{\partial k(\sigma, s)}{\partial t} \right| ds \varphi(0)
\]
\[
= -f(\sigma) > K\alpha - B = \alpha.
\]
Similarly, for every \( \varphi \in C_g \) such that \( \|\varphi\|_\infty = |\varphi(0)| \) and \( \varphi(0) < -K \), it follows that for each \( \sigma \in \mathbb{R} \) one can find
\[
x'(\sigma, \sigma, \varphi) < -\alpha.
\]
Consider the set
\[
S = \{ \varphi \in C((-\infty, 0], \mathbb{R}) : \theta \text{ is a constant function} \}.
\]
Denote by
\[
S^+ = \{ \varphi \in S : \text{There is } t > 0 \text{ such that } x(t, 0, \varphi) > K \text{ and } x(s, 0, \varphi) > -K \text{ for } s \in [0, t] \},
\]
and
\[
S^- = \{ \varphi \in S : \text{There is } t > 0 \text{ such that } x(t, 0, \varphi) < -K \text{ and } x(s, 0, \varphi) < K \text{ for } s \in [0, t] \}.
\]
By the continuous dependence of the solution on the initial condition one can show that $S^+$ and $S^-$ are open sets in $S$. Moreover by the estimates in (15) and (16), one has that $\{ \varphi \in S : \varphi > K \} \subset S^+$ and $\{ \varphi \in S : \varphi < -K \} \subset S^-$. This implies that $S^+ \cup S^-$ does not cover all of $S$ and therefore there exists a nontrivial $\varphi \in S$, such that

$$|x(t,0,\varphi)| < K, \quad \text{for } t \in \mathbb{R}.$$  

Finally, by Theorem 3 we deduce the existence of a $T$-periodic solution of (9). 

**REFERENCES**