JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 93, 312–323 (1983)

Fuzzy Measures Assuming Their Values in the Set of Fuzzy Numbers

E. P. KLEMENT*

Institut für Mathematik, Johannes Kepler Universität, A-4040 Linz, Austria

Submitted by L. Zadeh

Fuzzy-valued fuzzy measures are defined in an axiomatic way. Extending a result of Klement (*J. Math. Anal. Appl.* **75** (1980), 330–339) it is shown that they can be characterized by a suitable family of ordinary measures and Markov kernels.

I. INTRODUCTION

It is the aim of this article to develop axiomatically a measure theory in which fuzzy sets are measured by fuzzy rather than by crisp numbers. To be precise, we consider fuzzy-valued fuzzy measures to be functions from a fuzzy σ-algebra σ (see [12]) into the set $\mathscr{H}(\overline{\mathbb{R}}_+)$ of fuzzy nonnegative numbers. The definition of $\mathscr{H}(\overline{\mathbb{R}}_+)$ and other notations and preliminaries are given in Section II. In Section III, we first give the definition of fuzzy-valued fuzzy measures, show the connection with crisp-valued fuzzy measures as studied in [13, 15], and present some natural examples of these measures.

In [13] the author has shown that, given a generated fuzzy σ-algebra, a finite crisp-valued fuzzy measure can be characterized by an ordinary measure P and a Markov kernel K. The main result of this article is an extension of the cited result to the fuzzy-valued case. Each finite fuzzy-valued fuzzy measure corresponds to a family $(P_\alpha, K_\alpha)_{\alpha\in[0,1[}$ of ordinary measures and Markov kernels, respectively, fulfilling some rather technical conditions (Section IV). Finally, using this result for some of our examples in Section III the families $(P_\alpha, K_\alpha)_{\alpha\in[0,1[}$ are constructed explicitly in these cases.

II. BASIC NOTATIONS AND PRELIMINARIES

Throughout this article, X will denote an ordinary nonempty set. A fuzzy set on X, as usual, is then a mapping $\mu: X \to [0, 1]$, the value $\mu(x)$ being

* This research was supported by a grant of the Jubiläumsfonds der Österreichischen Nationalbank (project No. 1665).

0022-247X/83 $3.00

interpreted as the *degree of membership* of the point x in the fuzzy set μ. We shall write $\mathbb{R}_+$ for $[0, \infty[$ and $\overline{\mathbb{R}}_+$ for $[0, \infty]$. Then $\mathscr{B}_+$ and $\overline{\mathscr{B}}_+$ will denote the σ-algebra of Borel subsets of $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$, respectively. If A is a nonempty subset of $\mathbb{R}$, we shall write $\mathscr{B} \cap A$ for the σ-algebra of Borel subsets of A.

In a lattice $(L, \leqslant)$ the symbols $\wedge$ and $\vee$ denote meet and join, respectively, of two elements of L. If the lattice $(L, \leqslant)$ is complete, we shall write $\bigwedge$ and $\bigvee$ for infimum and supremum, respectively. For example, $(\mathbb{R}_+, \leqslant)$ is a (noncomplete) lattice and $(\overline{\mathbb{R}}_+, \leqslant)$ and $([0, 1]^X, \leqslant)$, where in the latter case $\leqslant$ is the usual (pointwise) partial ordering of fuzzy sets on X, are complete lattices. Furthermore, $(\mathbb{R}_+, \leqslant, +)$ and $(\overline{\mathbb{R}}_+, \leqslant, +)$ form partially ordered, commutative semigroups.

The concept of fuzzy numbers we shall use for our definition of fuzzy-valued fuzzy measures follows the ideas of Höhle [9–11]. There are some significant differences to other definitions of fuzzy numbers which are summarized in Dubois and Prade [6] and which have been studied by Zadeh [21], Dubois and Prade [4, 5], Mizumoto and Tanaka [16], Dubois [3], and others. We also shall restrict ourselves to nonnegative fuzzy numbers, for an extension to a vector space of fuzzy numbers see Höhle [9].

A *nonnegative fuzzy number* is defined to be a mapping $\rho\colon \overline{\mathbb{R}}_+ \to [0, 1]$ fulfilling either

$$\rho(0) = 0, \qquad \rho(\infty) = 1, \tag{2.1}$$

$$\forall r \in \overline{\mathbb{R}}_+ : \rho(r) = \bigvee \{\rho(s) \mid s \in \mathbb{R}_+, s < r\}, \tag{2.2}$$

or being the fuzzy infinity ε_∞ defined by

$$\varepsilon_\infty = 1_{\{\infty\}}. \tag{2.3}$$

We shall denote the family of nonnegative fuzzy numbers by $\mathscr{H}(\overline{\mathbb{R}}_+)$.

The interpretation of the fuzzy number ρ is as follows: $\rho(r)$ is the degree of ρ being contained in the ordinary interval $[0, r[$. A partial ordering on $\mathscr{H}(\overline{\mathbb{R}}_+)$ is given by

$$\rho \lesssim \varphi \Leftrightarrow \forall r \in \overline{\mathbb{R}}_+ : \rho(r) \geqslant \varphi(r) \tag{2.4}$$

and $(\mathscr{H}(\overline{\mathbb{R}}_+), \lesssim)$ is a complete lattice. A natural algebraic operation $\tau_\wedge$ on $\mathscr{H}(\overline{\mathbb{R}}_+)$ is defined by

$$\tau_\wedge(\rho, \varphi)(r) = \bigvee \{\rho(s) \wedge \varphi(t) \mid s, t \in \overline{\mathbb{R}}_+, s + t = r\}. \tag{2.5}$$

Then $(\mathscr{H}(\overline{\mathbb{R}}_+), \lesssim, \tau_\wedge)$ forms a partially ordered, commutative semigroup.

There is a natural embedding of $(\bar{\mathbb{R}}_+, \leqslant, +)$ into $(\mathscr{H}(\bar{\mathbb{R}}_+), \lesssim, \tau_\wedge)$ by virtue of the complete monomorphism $x \to \varepsilon_x$, where ε_x for $x \in \mathbb{R}_+$ is defined by

$$\varepsilon_x = 1_{]x,\infty]}. \tag{2.6}$$

Next, for any $\rho \in \mathscr{H}(\bar{\mathbb{R}}_+)$ we consider its *quasi-inverse* $[\rho]^q$ to be a function from $[0, 1]$ into $\bar{\mathbb{R}}_+$ as follows:

$$[\rho]^q(\alpha) = \bigvee \{r \in \bar{\mathbb{R}}_+ \mid \rho(r) < \alpha\}. \tag{2.7}$$

This notion of the quasi-inverse was introduced by Sherwood and Taylor [18]. Since all proper prime ideals in $([0, 1], \leqslant)$ are of the form $[0, \alpha]$, $\alpha \in [0, 1[$, or $[0, \alpha[$, $\alpha \in]0, 1]$, it is also a special case of the quasi-inverse studied by Höhle [9, 11]. It is straightforward that the set $\mathscr{H}^q(\bar{\mathbb{R}}_+)$ of all quasi-inverses of elements of $\mathscr{H}(\bar{\mathbb{R}}_+)$ is precisely the set of all functions f: $[0, 1] \to \bar{\mathbb{R}}_+$ fulfilling

$$\begin{aligned} f(\alpha) &= \bigvee \{f(\beta) \mid \beta \in [0, \alpha[\}, \qquad &\text{if} \quad \alpha > 0, \\ &= 0, &\text{if} \quad \alpha = 0. \end{aligned} \tag{2.8}$$

If we endow $\mathscr{H}^q(\bar{\mathbb{R}}_+)$ with $\leqslant$, the usual (pointwise) partial ordering of real functions and $+$, the usual (pointwise) addition of real functions, then it is obvious that $(\mathscr{H}^q(\bar{\mathbb{R}}_+), \leqslant)$ is a complete lattice and $(\mathscr{H}^q(\bar{\mathbb{R}}_+), \leqslant, +)$ is a partially ordered, commutative semigroup. Moreover, $\rho \to [\rho]^q$ is a complete isomorphism from $(\mathscr{H}(\bar{\mathbb{R}}_+), \lesssim, \tau_\wedge)$ onto $(\mathscr{H}^q(\bar{\mathbb{R}}_+), \leqslant, +)$.

If we are interested in finite fuzzy numbers only, we consider $\mathscr{H}(\mathbb{R}_+)$ which consists of the restrictions of elements of $\mathscr{H}(\bar{\mathbb{R}}_+) \backslash \{\varepsilon_\infty\}$ to $\mathbb{R}_+$, i.e., of all functions fulfilling

$$\rho(0) = 0, \qquad \bigvee \{\rho(r) \mid r \in \mathbb{R}_+\} = 1, \tag{2.9}$$

$$\forall r \in \mathbb{R}_+ : \rho(r) = \bigvee \{\rho(s) \mid s \in \mathbb{R}_+, s < r\}. \tag{2.10}$$

That means, $\mathscr{H}(\mathbb{R}_+)$ is the set of all nondecreasing, left-continuous functions fulfilling the boundary condition (2.9), i.e., $\mathscr{H}(\mathbb{R}_+)$ coincides with the set $\mathscr{D}^+$ of all nonnegative probability distribution functions (see Schweizer [17]). The partial ordering $\lesssim$ and the operation $\tau_\wedge$ can be restricted to $\mathscr{H}(\mathbb{R}_+)$ in a straightforward manner. Now $(\mathscr{H}(\mathbb{R}_+), \lesssim)$ is a lattice (but not a complete one) and $(\mathscr{H}(\mathbb{R}_+), \lesssim, \tau_\wedge)$ is still a partially ordered, commutative semigroup in which, additionally, the cancellation law holds. Of course, $x \to \varepsilon_x$ is again a monomorphism from $(\mathbb{R}_+, \leqslant, +)$ into $(\mathscr{H}(\mathbb{R}_+), \leqslant, \tau_\wedge)$.

Given a finite fuzzy number $\rho \in \mathscr{H}(\mathbb{R}_+)$, its quasi-inverse $[\rho]^q$ is now a function from $[0, 1[$ into $\mathbb{R}_+$. The set $\mathscr{H}^q(\mathbb{R}_+)$ of all quasi-inverses of finite fuzzy numbers is therefore the set of all functions f: $[0, 1[\to \mathbb{R}_+$ satisfying property (2.8). Obviously, $(\mathscr{H}^q(\mathbb{R}_+), \leqslant)$ is a (noncomplete) lattice and $(\mathscr{H}^q(\mathbb{R}_+), \leqslant, +)$ is a partially ordered, commutative semigroup, where the cancellation law holds. And still $\rho \to [\rho]^q$ is an isomorphism from $(\mathscr{H}(\mathbb{R}_+), \lesssim, \tau_\wedge)$ into $(\mathscr{H}^q(\mathbb{R}_+), \leqslant, +)$.

Finally, we recall the definition of a Markov kernel which we shall use frequently in this article. If $(X, \mathscr{A})$ is an ordinary measurable space, then a function

$$K: X \times (\mathscr{B} \cap [0, 1[) \to [0, 1] \tag{2.11}$$

is called a *Markov kernel from* $(X, \mathscr{A})$ *to* $([0, 1[, \mathscr{B} \cap [0, 1[)$ if it fulfills the following properties: for each $B \in \mathscr{B} \cap [0, 1[$ the function

$$K(\cdot, B): x \to K(x, B) \tag{2.12}$$

is measurable (with respect to $\mathscr{A}$ and $\mathscr{B} \cap [0, 1]$); for each $x \in X$ the function

$$K(x, \cdot): B \to K(x, B) \tag{2.13}$$

is a probability measure on $([0, 1[, \mathscr{B} \cap [0, 1[)$. For other notations concerning lattice theory we refer to Birkhoff [2], in the fields of measure and probability theory to Halmos [7] and Bauer [1].

III. Fuzzy-Valued Fuzzy Measures

The idea of measuring fuzzy sets using fuzzy numbers rather than crisp numbers was presented for the first time by Zadeh [21] in the context of linguistic variables. An axiomatic measure theory of this type for G-fuzzy sets, i.e., mappings $\mu: X \to G$, G being a regular Boolean algebra, can be found in Höhle [8]. Since $[0, 1]$ is not a Boolean algebra, our approach is somewhat different.

First recall the definition of a fuzzy σ-algebra (cf. [12]). A *fuzzy σ-algebra* σ is a subfamily σ of $[0, 1]^X$ containing all constant fuzzy sets and being closed under complementation and countable union, i.e., fulfilling these properties:

$$\alpha \text{ constant} \Rightarrow \alpha \in \sigma, \tag{3.1}$$

$$\mu \in \sigma \Rightarrow 1 - \mu \in \sigma, \tag{3.2}$$

$$(\mu_n)_{n \in \mathbb{N}} \in \sigma^{\mathbb{N}} \Rightarrow \bigvee_{n \in \mathbb{N}} \mu_n \in \sigma. \tag{3.3}$$

The pair (X, σ) is then called a *measurable space*. Obviously, if $\mathscr{A}$ is a classical σ-algebra on X, then the family $\zeta(\mathscr{A})$ of all measurable functions $\mu: X \to [0, 1]$ forms a fuzzy σ-algebra; it is called a generated one.

A *crisp-valued fuzzy measure* m on (X, σ) is a function $m: \sigma \to \overline{\mathbb{R}}^+$ such that

$$m(0) = 0, \tag{3.4}$$

$$m(\mu \vee \nu) + m(\mu \wedge \nu) = m(\mu) + m(\nu), \tag{3.5}$$

$$\mu_1 \leqslant \mu_2 \leqslant \cdots \Rightarrow m\left(\bigvee_{n \in \mathbb{N}} \mu_n\right) = \bigvee_{n \in \mathbb{N}} m(\mu_n). \tag{3.6}$$

If $m(1) < \infty$, then the measure is said to be *finite*. These measures have been studied extensively in [13, 15]. Analogously, a *fuzzy-valued fuzzy measure* $\tilde{m}$ on (X, σ) is a function $\tilde{m}: \sigma \to \mathscr{H}(\overline{\mathbb{R}}_+)$ such that

$$\tilde{m}(0) = \varepsilon_0, \tag{3.7}$$

$$\tau_\wedge(\tilde{m}(\mu \vee \nu), \tilde{m}(\mu \wedge \nu)) = \tau_\wedge(\tilde{m}(\mu), \tilde{m}(\nu)), \tag{3.8}$$

$$\mu_1 \leqslant \mu_2 \leqslant \cdots \Rightarrow \tilde{m}\left(\bigvee_{n \in \mathbb{N}} \mu_n\right) = \bigvee_{n \in \mathbb{N}} \tilde{m}(\mu_n). \tag{3.9}$$

Again the measure is called *finite* if $\tilde{m}(1) < \varepsilon_\infty$.

Using the isomorphism $\rho \to [\rho]^q$ we can characterize fuzzy-valued fuzzy measures as follows:

PROPOSITION 3.1. *A function $\tilde{m}$: $\sigma \to \mathscr{H}(\overline{\mathbb{R}}_+)$ is a fuzzy-valued fuzzy measure if and only if we have*

$$[\tilde{m}(0)]^q = 0, \tag{3.10}$$

$$[\tilde{m}(\mu \vee \nu)]^q + [\tilde{m}(\mu \wedge \nu)]^q = [\tilde{m}(\mu)]^q + [\tilde{m}(\nu)]^q, \tag{3.11}$$

$$\mu_1 \leqslant \mu_2 \leqslant \cdots \Rightarrow \left[\tilde{m}\left(\bigvee_{n \in \mathbb{N}} \mu_n\right)\right]^q = \bigvee_{n \in \mathbb{N}} [\tilde{m}(\mu_n)]^q. \tag{3.12}$$

Now let us present some examples of fuzzy-valued measures showing that they are indeed proper generalizations of crisp-valued ones.

EXAMPLE 1. If m is a crisp-valued fuzzy measure on (X, σ), then $\tilde{\varepsilon}_m$ given by

$$\tilde{\varepsilon}_m(\mu) = \varepsilon_{m(\mu)} \tag{3.13}$$

is a (trivial) fuzzy-valued fuzzy measure. Note that we also have

$$m(\mu) = \int_{]0,1[} [\tilde{\varepsilon}_m(\mu)]^q(\alpha)\, d\alpha. \tag{3.14}$$

Generally, if $\tilde{m}$ is a fuzzy-valued fuzzy measure on (X, σ), then

$$m(\mu) = \int_{[0,1[} [\tilde{m}(\mu)]^q(\alpha)\, d\alpha \tag{3.15}$$

defines a crisp-valued fuzzy measure on (X, σ). These results have been proved in [14].

EXAMPLE 2. It was shown in [13] that, given a generated fuzzy σ-algebra $\zeta(\mathscr{A})$ on X, a function $m\colon \zeta(\mathscr{A}) \to \mathbb{R}_+$ is a finite crisp-valued fuzzy measure if and only if there is a (classical) finite measure P and a Markov kernel K from $(X, \mathscr{A})$ to $([0, 1[, \mathscr{B} \cap [0, 1[)$ such that

$$m(\mu) = \int_X K(x, [0, \mu(x)[)\, dP(x). \tag{3.16}$$

Now let m be a finite crisp-valued fuzzy measure and P and K the corresponding measure and Markov kernel, respectively. Then $\tilde{m}$ defined by

$$\tilde{m}(\mu)(r) = \bigvee \{\alpha \in [0, 1] \mid P(\{K_\mu > 1 - \alpha\}) < r\}, \tag{3.17}$$

where K_μ is the function specified by

$$K_\mu(x) = K(x, [0, \mu(x)[), \tag{3.18}$$

is a finite fuzzy-valued fuzzy measure. Again we have

$$m(\mu) = \int_{[0,1[} [\tilde{m}(\mu)]^q\, d\alpha \tag{3.19}$$

and, if $A \in \mathscr{A}$ is a crisp set, then

$$\tilde{m}(1_A) = \varepsilon_{P(A)}. \tag{3.20}$$

EXAMPLE 3. Let $\mathscr{A}$ be the power set of X. Then $\tilde{m}$ defined by

$$\tilde{m}(\mu)(r) = \bigvee \{\alpha \in [0, 1] \mid \mathrm{card}(\{\mu > 1 - \alpha\}) < r\} \tag{3.21}$$

is a fuzzy-valued fuzzy measure on $(X, [0, 1]^X)$, called the *fuzzy cardinality*. Note that again for crisp sets A we have

$$\tilde{m}(1_A) = \varepsilon_{\mathrm{card}(A)}. \tag{3.22}$$

EXAMPLE 4. Let (X, σ) and (Y, ξ) be two fuzzy measurable spaces and

$f: (X, \sigma) \to (Y, \xi)$ a fuzzy measurable function (cf. [12]). If $\tilde{m}$ is a fuzzy-valued fuzzy measure on (X, σ), then $[f\tilde{m}]$ given by

$$[f\tilde{m}](\mu) = \tilde{m}(f^{-1}(\mu)) \tag{3.23}$$

is a fuzzy-valued fuzzy measure on (Y, ξ), called the *image of* $\tilde{m}$ *under* f. Furthermore, $[f\tilde{m}]$ is finite if and only if $\tilde{m}$ is finite.

IV. Characterization of Finite Fuzzy-Valued Measures

As we mentioned in Example 2, a finite crisp-valued fuzzy measure on $(X, \zeta(\mathscr{A}))$ can be characterized completely by an ordinary finite measure P and a Markov kernel K. In this section, we show that a fuzzy-valued fuzzy measure can be identified with a suitable family $(P_\alpha, K_\alpha)_{\alpha \in [0,1[}$ of ordinary finite measures and Markov kernels, respectively. Throughout this section let $\mathscr{A}$ be an ordinary σ-algebra on X. A family $(P_\alpha, K_\alpha)_{\alpha \in [0,1[}$, where each P_α is an ordinary finite measure on $(X, \mathscr{A})$ and each K_α is a Markov kernel from $(X, \mathscr{A})$ to $([0, 1[, \mathscr{B} \cap [0, 1[)$ is called an *adapted family* provided the following properties are fulfilled:

(i) P_0 is the zero measure,

(ii) for each $A \in \mathscr{A}$ the function $\alpha \to P_\alpha(A)$ is bounded, nondecreasing and left-continuous, and

(iii) for each $\gamma \in [0, 1]$ we have

$$(\alpha_n)_{n \in \mathbb{N}} \uparrow \alpha \Rightarrow \left(K_{\alpha_n}(\cdot, [0, \gamma[) \frac{dP_{\alpha_n}}{dP_\alpha}(\cdot)\right)_{n \in \mathbb{N}} \uparrow K_\alpha(\cdot, [0, \gamma[), \qquad P_\alpha\text{-a.e.} \tag{4.1}$$

The expression dP_{α_n}/dP_α in (iii) stands for the Radon–Nikodym derivative of P_{α_n} with respect to P_α (which always exists as a consequence of (ii)).

Now we get the result,

Lemma 4.1. *Let* $(P_\alpha, K_\alpha)_{\alpha \in [0,1[}$ *be a family of ordinary measures* P_α *on* $(X, \mathscr{A})$ *and Markov kernels* K_α *from* $(X, \mathscr{A})$ *to* $([0, 1[, \mathscr{B} \cap [0, 1[)$. *Then for each* $\mu \in \zeta(\mathscr{A})$ *the function* $F_\mu: [0, 1[\to \mathbb{R}_+$ *specified by*

$$F_\mu(\alpha) = \int_X K_\alpha(x, [0, \mu(x)[)\, dP_\alpha(x) \tag{4.2}$$

is a bounded element of $\mathscr{H}^q(\mathbb{R}_+)$ *if and only if* $(P_\alpha, K_\alpha)_{\alpha \in [0,1[}$ *is an adapted family.*

Proof. Assume first that for each $\mu \in \zeta(\mathscr{A})$ F_μ is a bounded element of $\mathscr{H}^q(\mathbb{R}_+)$. Then F_μ is also nondecreasing, left-continuous, and we have $F_\mu(0) = 0$. Now choosing $A \in \mathscr{A}$ leads to

$$P_\alpha(A) = \int_X K_\alpha(x, [0, 1_A(x)[)\, dP_\alpha(x) = F_{1_A}(\alpha).$$

Therefore the validity of (i) and (ii) is checked easily using the corresponding properties of F_{1_A}. As for condition (iii), observe first that for a given $(\gamma, A) \in [0, 1] \times \mathscr{A}$ we obtain

$$F_{\gamma \times 1_A}(\alpha) = \int_A K_\alpha(x, [0, \gamma[)\, dP_\alpha.$$

Thus, if $(\alpha_n)_{n \in \mathbb{N}} \uparrow \alpha$, then

$$\begin{aligned}
&\left(\int_A K_{\alpha_n}(x, [0, \gamma[)\, dP_{\alpha_n}(x)\right)_{n \in \mathbb{N}} \\
&\quad = \left(\int_A K_{\alpha_n}(x, [0, \gamma[)\, \frac{dP_{\alpha_n}}{dP_\alpha}(x)\, dP_\alpha(x)\right)_{n \in \mathbb{N}} \\
&\quad \uparrow \int_A K_\alpha(x, [0, \gamma[)\, dP_\alpha(x).
\end{aligned}$$

Since A was choosen arbitrarily this proves that (iii) is fulfilled. Hence $(P_\alpha, K_\alpha)_{\alpha \in [0,1[}$ is an adapted family.

Conversely, if $(P_\alpha, K_\alpha)_{\alpha \in [0,1[}$ is an adapted family, it is readily seen that (i) implies $F_\mu(0) = 0$ for all $\mu \in \zeta(\mathscr{A})$. The boundedness of F_μ is an immediate consequence of the boundedness of both K and $\alpha \to P_\alpha(X)$. Next choose a measurable step function s, say,

$$s = \sum_{i=1}^{k} \gamma_i \times 1_{A_i}.$$

Then, if $(\alpha_n)_{n \in \mathbb{N}} \uparrow \alpha$, we get

$$(F_s(\alpha_n))_{n \in \mathbb{N}} = \left(\sum_{i=1}^{k} \int_{A_i} K(x, [0, \gamma_i[)\, \frac{dP_{\alpha_n}}{dP_\alpha}(x)\, dP_\alpha(x)\right)_{n \in \mathbb{N}}$$

and therefore, because of (iii) and the Lebesgue monotone convergence theorem,

$$(F_s(\alpha_n))_{n \in \mathbb{N}} \uparrow F_s(\alpha), \tag{4.3}$$

showing that F_s is nondecreasing and left-continuous.

For an arbitrary $\mu \in \zeta(\mathscr{A})$, let $(s_m)_{m\in\mathbb{N}}$ be a nondecreasing sequence of measurable step functions such that $\mu = \bigvee_{m\in\mathbb{N}} s_m$. Thus for $(\alpha_n)_{n\in\mathbb{N}} \uparrow \alpha$ the application of (iv) and of the Lebesgue monotone convergence theorem yields

$$(F_\mu(\alpha_n))_{n\in\mathbb{N}} \uparrow F_\mu(\alpha).$$

Hence F_μ is nondecreasing and left-continuous, too, and therefore an element of $\mathscr{H}^q(\mathbb{R}_+)$. ∎

Now we are ready to state the main result.

THEOREM 4.2. *A function $\tilde{m}$: $\zeta(\mathscr{A}) \to \mathscr{H}(\mathbb{R}_+)$ is a finite fuzzy-valued fuzzy measure on $(X, \zeta(\mathscr{A}))$ if and only if there is an adapted family $(P_\alpha, K_\alpha)_{\alpha\in[0,1[}$ such that for each $\mu \in \zeta(\mathscr{A})$*

$$[\tilde{m}(\mu)]^q(\alpha) = \int_X K_\alpha(x, [0, \mu(x)[)\, dP_\alpha(x). \tag{4.4}$$

Furthermore, $\tilde{m}$ is characterized by $(P_\alpha, K_\alpha)_{\alpha\in[0,1[}$ uniquely in the sense that, if $(P_\alpha, K_\alpha)_{\alpha\in[0,1[}$ and $(Q_\alpha, L_\alpha)_{\alpha\in[0,1[}$ are adapted families fulfilling (4.4), *then we have $P_\alpha = Q_\alpha$ for each $\alpha \in [0, 1[$ and $K_\alpha(\cdot, A) = L_\alpha(\cdot, A)$ P_α almost everywhere for each $(\alpha, A) \in [0, 1[\times \mathscr{A}$.*

Proof. If $\tilde{m}$ is a finite fuzzy-valued fuzzy measure, then by Proposition 3.1 $[\tilde{m}(\cdot)]^q$ fulfills (3.10)–(3.12). Then, for each $\alpha \in [0, 1[$, the function $\mu \to [\tilde{m}(\mu)]^q(\alpha)$ is a finite crisp-valued fuzzy measure. Now, from [13] we know that there is an ordinary finite measure P_α on $(X, \mathscr{A})$ and a Markov kernel K_α from $(X, \mathscr{A})$ to $([0, 1[, \mathscr{B} \cap [0, 1[)$ such that for each $\mu \in \zeta(\mathscr{A})$

$$[\tilde{m}(\mu)]^q(\alpha) = \int_X K_\alpha(x, [0, \mu(x)[)\, dP_\alpha(x).$$

Since $F_\mu = [\tilde{m}(\mu)]^q$ is bounded and an element of $\mathscr{H}^q(\mathbb{R}_+)$ Lemma 4.1 tells us that $(P_\alpha, K_\alpha)_{\alpha\in[0,1[}$ has to be an adapted family.

On the other hand, if $(P_\alpha, K_\alpha)_{\alpha\in[0,1[}$ is an adapted family, then for each $\mu \in \zeta(\mathscr{A})$ the function F_μ defined by (4.2) is bounded and belongs to $\mathscr{H}^q(\mathbb{R}_+)$. Using the properties of the Markov kernel and of the integral, especially the Lebesgue monotone convergence theorem, it is perfectly clear that the mapping $\mu \to F_\mu$ also fulfills (3.10)–(3.12). Thus, by Proposition 3.1 there is a finite fuzzy-valued fuzzy measure $\tilde{m}$ such that $[\tilde{m}(\mu)]^q = F_\mu$ for all $\mu \in \zeta(\mathscr{A})$.

Finally, suppose that both $(P_\alpha, K_\alpha)_{\alpha\in[0,1[}$ and $(Q_\alpha, L_\alpha)_{\alpha\in[0,1[}$ are adapted families fulfilling (4.4). Then putting $\mu = 1_A$ leads immediately to $P_\alpha = Q_\alpha$ for all $\alpha \in [0, 1[$. Consequently, if $\mu = \gamma \times 1_A$, we obtain

$$\int_A K_\alpha(x, [0, \gamma[)\, dP_\alpha(x) = \int_A L_\alpha(x, [0, \gamma[)\, dP_\alpha(x)$$

which implies $K_\alpha(\cdot, A) = L_\alpha(\cdot, A)$ P_α almost everywhere for all $(\alpha, A) \in [0, 1[\times \mathscr{A}$. This completes the proof. ∎

We now want to apply these results to some examples.

EXAMPLE 5. In Example 1, we mentioned that, given a fuzzy-valued measure $\tilde{m}$ on $(X, \zeta(\mathscr{A}))$ then m defined by (3.14) is a crisp-valued fuzzy measure on $(X, \zeta(\mathscr{A}))$. Obviously, if $\tilde{m}$ is finite, so is m. Then by Theorem 4.2, $\tilde{m}$ is characterized by an adapted family $(P_\alpha, K_\alpha)_{\alpha \in [0,1[}$, by the main result of [13] m is represented by an ordinary finite measure P on $(X, \mathscr{A})$ and a Markov kernel K from $(X, \mathscr{A})$ to $([0, 1[, \mathscr{B} \cap [0, 1[)$. It is now easy to construct P and K if the family $(P_\alpha, K_\alpha)_{\alpha \in [0,1[}$ is known. For each $A \in \mathscr{A}$ we have

$$P(A) = \int_{[0,1[} P_\alpha(A)\, d\alpha,$$

K is determined P almost everywhere by

$$K(x, [0, \gamma[) = \frac{dQ_\gamma}{dP}(x),$$

where dQ_γ/dP is again the Radon–Nikodym derivative of the measure Q_γ with respect to P, Q_γ being given by

$$Q_\gamma(A) = \int_{[0,1[} \int_A K_\alpha(x, [0, \gamma[)\, dP_\alpha(x)\, d\alpha$$

for each $A \in \mathscr{A}$.

EXAMPLE 6. In Example 2 we started with a finite crisp-valued fuzzy measure m on $(X, \zeta(\mathscr{A}))$ and, using the characterization of m by an ordinary finite measure P and a Markov kernel K, we constructed a finite fuzzy-valued fuzzy measure $\tilde{m}$ by means of (3.17). If we now look at the adapted family $(P_\alpha, K_\alpha)_{\alpha \in [0,1[}$ characterizing $\tilde{m}$ according to Theorem 4.2, we obtain for P_0 the zero measure and therefore we can choose an arbitrary Markov kernel K_0. If $\alpha \neq 0$, we get $P_\alpha = P$, and the Markov kernel K_α is determined P almost everywhere by

$$K_\alpha(\cdot, [0, \gamma[) = 1_{\{K(\cdot, [0,\gamma[) > 1-\alpha\}}$$

for each $\gamma \in [0, 1]$.

EXAMPLE 7. If we are given an ordinary finite measure P on $(X, \mathscr{A})$, we can also ask which finite fuzzy-valued fuzzy measures $\tilde{m}$ on $(X, \zeta(\mathscr{A}))$ fulfill Eq. (3.20) for each $A \in \mathscr{A}$. The solution is simply the set of finite fuzzy-valued fuzzy measures such that for the corresponding adapted families $(P_\alpha, K_\alpha)_{\alpha \in [0,1[}$, $P_0 = 0$, and $P_\alpha = P$ for all $\alpha > 0$ holds.

EXAMPLE 8. Suppose $(X, \mathscr{A})$ and $(Y, \mathscr{C})$ are two ordinary measurable spaces and $f: (X, \zeta(\mathscr{A})) \to (Y, \zeta(\mathscr{C}))$ is fuzzy measurable. Let $\tilde{m}$ be a finite fuzzy-valued fuzzy measure and $(P_\alpha, K_\alpha)_{\alpha \in [0,1[}$ the corresponding adapted family. Then the adapted family $(Q_\alpha, L_\alpha)_{\alpha \in [0,1[}$ associated with the finite fuzzy-valued fuzzy measure $[f\tilde{m}]$ defined by (3.23) fulfills the following properties:

$$Q_\alpha = [fP_\alpha], \qquad L_\alpha(\cdot, B) = E(K_\alpha(\cdot, B) \,|\, f^{-1}(\mathscr{C})),$$

where $[fP_\alpha]$ is the image of P_α under f and $E(K_\alpha(\cdot, B) \,|\, f^{-1}(\mathscr{C}))$ stands for the conditional expected value of $K_\alpha(\cdot, B)$ with respect to the σ-algebra $f^{-1}(\mathscr{C}) = \{f^{-1}(C) \mid C \in \mathscr{C}\}$.

ACKNOWLEDGMENTS

The author would like to thank his colleagues W. Schwyhla and R. Takacs for valuable discussions during the preparation of this article.

REFERENCES

1. H. BAUER, "Probability Theory and Elements of Measure Theory," Academic Press, 1981.
2. G. BIRKHOFF, "Lattice Theory," Amer. Math. Soc. Colloquium Publications, Vol. XXV, New York, 1973.
3. D. DUBOIS, La théorie des nombres flous: panorama des résultats actuels, *in* Actes du Table Ronde du CNRS, "Quelques Applications Concrètes Utilisant les Derniers Perfectionnements de la Théorie du Flou," Université Claude Bernard, Lyon, France, 1980.
4. D. DUBOIS AND H. PRADE, Operations on fuzzy numbers, *Internat. J. Systems Sci.* **9** (1978), 613–626.
5. D. DUBOIS AND H. PRADE, Fuzzy real algebra: some results, *Fuzzy Sets and Systems* **2** (1979), 327–348.
6. D. DUBOIS AND H. PRADE, "Fuzzy Sets and Systems: Theory and Applications," Academic Press, New York, 1980.
7. P. R. HALMOS, "Measure Theory," Van Nostrand, New York, 1968.
8. U. HÖHLE, Maße auf unscharfen Mengen, *Z. Wahrsch. Verw. Gebiete* **36** (1976), 179–188.
9. U. HÖHLE, L-fuzzy real numbers—a special order complete vector lattice, *in* Actes du Table Ronde du CNRS, "Quelques Applications Concrètes Utilisant les Derniers Perfectionnements de la Théorie du Flou," Université Claude Bernard, Lyon, France, 1980.

10. U. HÖHLE, Fuzzy measures as extensions of stochastic measures, *in* "Proceedings of the Second International Seminar on Fuzzy Set Theory" (E. P. Klement, Ed.), Johannes Kepler Universität, Linz, Austria, 1980.
11. U. HÖHLE, Representation theorems for fuzzy quantities, *Fuzzy Sets and Systems* **5** (1981), 83–107.
12. E. P. KLEMENT, Fuzzy σ-algebras and fuzzy measurable functions, *Fuzzy Sets and Systems* **4** (1980), 83–93.
13. E. P. KLEMENT, Characterization of finite fuzzy measures using Markoff-kernels, *J. Math. Anal. Appl.* **75** (1980), 330–339.
14. E. P. KLEMENT, On the relationship between different notions of fuzzy measures, *in* "Fuzzy Set and Possibility Theory. Recent Developments" (R. R. Yager, Ed.), Pergamon, Elmsford, N.Y., 1982.
15. E. P. KLEMENT, R. LOWEN, AND W. SCHWYHLA, Fuzzy probability measures, *Fuzzy Sets and Systems* **5** (1981), 21–30.
16. M. MIZUMOTO AND K. TANAKA, Some properties of fuzzy numbers, *in* "Advances in Fuzzy Set Theory and Applications" (M. M. Gupta *et al.*, Eds.), pp. 153–164, North-Holland, Amsterdam, 1978.
17. B. SCHWEIZER, Multiplication on the space of probability distribution functions, *Aequationes Math.* **12** (1975), 156–183.
18. H. SHERWOOD AND M. D. TAYLOR, Some *PM* structures on the set of distribution functions, *Rev. Roumaine Math. Pures Appl.* **19** (1974), 1251–1260.
19. L. A. ZADEH, Fuzzy Sets, *Inform. and Control* **8** (1965), 338–353.
20. L. A. ZADEH, Probability measures of fuzzy events, *J. Math. Anal. Appl.* **23** (1968), 421–427.
21. L. A. ZADEH, The concept of a linguistic variable and its application to approximate reasoning I, II, III, *Inform. Sci.* **8** (1975), 199–249, 301–357; **9** (1976), 43–80.