Journal of Computational and Applied Mathematics 233 (2009) 165–172

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

New method based on the HPM and RKHSM for solving forced Duffing equations with integral boundary conditions

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ARTICLE INFO

Article history:
Received 17 March 2009
Received in revised form 1 July 2009

Keywords:
Duffing equation
Integral boundary conditions
Homotopy perturbation method
Reproducing kernel Hilbert space method

ABSTRACT

This paper investigates the forced Duffing equation with integral boundary conditions. Its approximate solution is developed by combining the homotopy perturbation method (HPM) and the reproducing kernel Hilbert space method (RKHSM). HPM is based on the use of the traditional perturbation method and the homotopy technique. The HPM can reduce nonlinear problems to some linear problems and generate a rapid convergent series solution in most cases. RKHSM is also an analytical technique, which can solve powerfully linear boundary value problems. Therefore, the forced Duffing equation with integral boundary conditions can be solved using advantages of these two methods. Two numerical examples are presented to illustrate the strength of the method.

1. Introduction

In this paper, we consider the following forced Duffing equation with integral boundary conditions:

$$\begin{cases} u''(t) + \sigma u'(t) + f(t, u) = 0, \quad 0 < t < 1, \sigma \in R - \{0\}, \\ u(0) - \mu_1 u'(0) = \int_0^1 h_1(s) u(s) \mathrm{d}s, \quad u(1) + \mu_2 u'(1) = \int_0^1 h_2(s) u(s) \mathrm{d}s, \end{cases} \tag{1.1}$$

where $f : [0, 1] \times R \to R$ and μ_i are nonnegative constants.

Various problems arising in heat conduction [1–3], chemical engineering [4], thermo-elasticity [5], and plasma physics [6] can be reduced to the nonlocal problems. Boundary value problems with integral conditions constitute a very interesting and important class of problems. The integral boundary problems have been investigated by many authors in recent years [7,8]. The Duffing equation is a well known nonlinear equation of applied science which is used as a powerful tool to discuss some important practical phenomena such as orbit extraction, nonuniformity caused by an infinite domain, nonlinear mechanical oscillators, etc. Another important application of the Duffing equation is in the field of the prediction of diseases. The numerical solutions of the forced Duffing equations with two-point boundary conditions have been widely investigated. However, there are few references on the forced Duffing equation with integral boundary conditions. The existence and uniqueness of the solution of the forced Duffing equation with integral boundary conditions are presented by means of a constructive method [9]. Dehghan presented some effective methods for solving problems with nonlocal conditions [10–14].

In this work, we will give the analytic approximation of the solution to the forced Duffing equation with integral boundary conditions (1.1) by combining HPM and RKHSM.

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doi:10.1016/j.cam.2009.07.007

The HPM is based on the use of the traditional perturbation method and the homotopy technique. Using this method, a rapid convergent series solution can be obtained in most cases. Usually, a few number of terms of the series solution can be used for numerical purposes with a high degree of accuracy. Furthermore, the HPM does not require the discretization of the problem. Thus it is suitable for finding the approximation of the solution without discretization of the problem. The method was successfully applied to boundary value problems, partial differential equations and other fields [15–29].

Reproducing kernel theory has important application in numerical analysis, differential equation, probability and statistics and so on [30–40]. Recently, using the RKHSM, Xie, Yao, Cui, Geng and Chen discussed many linear and nonlinear differential equations [32–40].

The rest of the paper is organized as follows. In the next section, the HPM is introduced. The RKHSM is introduced in Section 3. The HPM and RKHSM are applied to (1.1) in Section 4. The numerical examples are presented in Section 5. Section 6 ends this paper with a brief conclusion.

2. Analysis of HPM

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.1}$$

with the boundary conditions of

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

Generally speaking, the operator A can be divided into parts which are L and N, where L is linear, but N is nonlinear. (2.1) can therefore be rewritten as

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \tag{2.3}$$

By the homotopy technique, we construct a homotopy $V(r, p) : \Omega \times [0, 1] \to R$ which satisfies:

$$H(V, p) = (1 - p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, \quad p \in [0, 1],\ r \in \Omega, \tag{2.4}$$

or

$$H(V, p) = L(V) - L(u_0) + pL(u_0) + p[N(V) - f(r)] = 0, \quad p \in [0, 1],\ r \in \Omega, \tag{2.5}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of (2.1), which satisfies the boundary conditions. Obviously, from (2.4) or (2.5), one obtains

$$H(V, 0) = L(V) - L(u_0) = 0, \tag{2.6}$$

$$H(V, 1) = A(V) - f(r) = 0, \tag{2.7}$$

the changing process of p from zero to unity is just that of $V(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(V) - L(u_0)$ and $A(V) - f(r)$ are called homotopies.

According to the HMP, we can first use the embedding parameter p as a "small parameter", and assume that the solution of (2.4) or (2.5) can be written as a power series in p:

$$V = V_0 + pV_1 + p^2V_2 + \cdots. \tag{2.8}$$

Setting $p = 1$ results in the approximate solution of Eq. (2.1):

$$u = \lim_{p\to 1} V = V_0 + V_1 + V_2 + \cdots. \tag{2.9}$$

The combination of perturbation method and homotopy method is called the HPM, which has eliminated the limitations of traditional perturbation methods. On the other hand, this technique is of full advantage of traditional perturbation techniques. The series (2.9) is convergent in most cases. However, the convergent rate depends on the nonlinear operator $A(V)$ (the following opinions are suggested in [18])

(1) The second derivative of $N(V)$ with respect to V must be small because the parameter may be relatively large, i.e., $p \to 1$.
(2) The norm of $L^{-1}(\partial N/\partial V)$ must be smaller than one so that the series converges.

3. Analysis of RKHSM

In this section, we illustrate how to solve the following linear second order ordinary differential equations with integral boundary conditions using RKHSM:

$$\begin{cases} Lu(x) = f(x), \quad 0 < x < 1, \\ u(0) - \mu_1 u'(0) = \displaystyle\int_0^1 h_1(s)u(s)\mathrm{d}s, \qquad u(1) + \mu_2 u'(1) = \displaystyle\int_0^1 h_2(s)u(s)\mathrm{d}s, \end{cases} \tag{3.1}$$

where $Lu = u''(x) + b(x)u'(x) + c(x)u(x)$, $b(x)$, $c(x)$ are continuous and $u(x) \in W_2^3[0, 1]$, $f(x) \in W_2^1[0, 1]$.

In order to solve (3.1) using RKHSM, we first construct a reproducing kernel Hilbert space $W_2^3[0, 1]$ in which every function satisfies the integral boundary conditions of (3.1).

Definition 3.1 (*Reproducing Kernel*). Let E be a nonempty abstract set. A function $K : E \times E \to \mathrm{C}$ is a reproducing kernel of the Hilbert space H if and only if

(a) $\forall\, t \in E, \quad K(\cdot,\ t) \in \mathrm{H}$
(b) $\forall\, t \in E, \forall\, \varphi \in \mathrm{H}, \quad (\varphi(\cdot),\ K(\cdot,\ t)) = \varphi(t).$

The last condition is called "the reproducing property": the value of the function φ at the point t is reproduced by the inner product of $\varphi(\cdot)$ with $K(\cdot,\ t)$.

A Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space (RKHS).

3.1. The reproducing kernel Hilbert space $W_2^3[0, 1]$

The inner product space $W_2^3[0, 1]$ is defined as $W_2^3[0, 1] = \{u(x) \mid u, u', u''$ are absolutely continuous real-valued functions, $u^{(3)} \in L^2[0, 1]$, $u(0) - \mu_1 u'(0) = \int_0^1 h_1(s)u(s)\mathrm{d}s$, $u(1) + \mu_2 u'(1) = \int_0^1 h_2(s)u(s)\mathrm{d}s\}$. The inner product in $W_2^3[0, 1]$ is given by

$$(u(y), v(y))_{W_2^3} = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + \int_0^1 u^{(3)}v^{(3)}\mathrm{d}y, \tag{3.2}$$

and the norm $\|u\|_{W_2^3}$ is denoted by $\|u\|_{W_2^3} = \sqrt{(u, u)_{W_2^3}}$, where $u, v \in W_2^3[0, 1]$.

Remark. In the above space $W_2^3[0, 1]$, the reproducing kernel satisfying integral boundary conditions cannot be obtained by using the method in [34–36]. The following theorem gives a new method for obtaining such a reproducing kernel.

Theorem 3.1. *The space $W_2^3[0, 1]$ is a reproducing kernel Hilbert space. That is, there exists $R_x(y) \in W_2^3[0, 1]$, for any $u(y) \in W_2^3[0, 1]$ and each fixed $x \in [0, 1]$, $y \in [0, 1]$, such that $(u(y), R_x(y))_{W_2^3} = u(x)$. The reproducing kernel $R_x(y)$ can be denoted by*

$$R_x(y) = \begin{cases} \displaystyle\sum_{i=0}^{5} a_i y^i - c_1 H_1(y) - c_2 H_2(y), & y \le x, \\ \displaystyle\sum_{i=0}^{5} b_i y^i - c_1 H_1(y) - c_2 H_2(y), & y > x, \end{cases} \tag{3.3}$$

where $H_i(y) = \int_0^y \int_0^y \int_0^y \int_0^y \int_0^y \int_0^y h_i(y)\mathrm{d}y\mathrm{d}y\mathrm{d}y\mathrm{d}y\mathrm{d}y\mathrm{d}y$, $i = 1, 2$.

Proof. Note that

$$\begin{aligned}(u(y), R_x(y))_{W_2^3} &= u(0)R_x(0) + u'(0)R_x'(0) + u(1)R_x(1) + \int_0^1 u^{(3)}(y)R_x^{(3)}(y)\mathrm{d}y \\ &\quad + c_1\left[u(0) - \mu_1 u'(0) - \int_0^1 h_1(s)u(s)\mathrm{d}s\right] + c_2\left[u(1) + \mu_2 u'(1) - \int_0^1 h_2(s)u(s)\mathrm{d}s\right].\end{aligned} \tag{3.4}$$

Through several integration by parts for (3.4), it becomes

$$\begin{aligned}(u(y), R_x(y))_{W_2^3} &= u(0)[R_x(0) + c_1 - R_x^{(5)}(0)] + u(1)[R_x(1) + c_2 + R_x^{(5)}(1)] \\ &\quad + u'(0)[R_x'(0) - c_1\mu_1 + R_x^{(4)}(0)] + u'(1)[c_2\mu_2 - R_x^{(4)}] \\ &\quad - u''(0)R_x^{(3)}(0) + u''(1)R_x^{(3)}(1) - \int_0^1 u(y)[R_x^{(6)}(y) + c_1 h_1(y) + c_2 h_2(y)]\mathrm{d}y.\end{aligned} \tag{3.5}$$

Since $R_x(y) \in W_2^3[0, 1]$, it follows that

$$R_x(0) - \mu_1 R_x'(0) = \int_0^1 h_1(s)R_x(s), R_x(1) + \mu_2 R_x'(1) = \int_0^1 h_2(s)R_x(s)\mathrm{d}s. \tag{3.6}$$

If

$$\begin{aligned}&R_x(0)+c_1-R_x^{(5)}=0, \qquad R_x(1)+c_2+R_x^{(5)}(1)=0, \qquad R_x'(0)-c_1\mu_1+R_x^{(4)}(0)=0,\\ &c_2\mu_2-R_x^{(4)}=0, \qquad R_x^{(3)}(0)=0, \qquad R_x^{(3)}(1)=0,\end{aligned} \tag{3.7}$$

then

$$(u(y),R_x(y))_{W_2^3}=-\int_0^1 u(y)[R_x^{(6)}(y)+c_1h_1(y)+c_2h_2(y)]\mathrm{d}y.$$

For $\forall x\in[0,1]$, if $R_x(y)$ also satisfies

$$-[R_x^{(6)}(y)+c_1h_1(y)+c_2h_2(y)]=\delta(y-x), \tag{3.8}$$

then

$$(u(y),R_x(y))_{W_2^3}=u(x).$$

The characteristic equation of (3.8) is given by

$$\lambda^6=0,$$

then we can obtain characteristic values $\lambda=0$ whose multiplicity is 6. So, let

$$R_x(y)=\begin{cases}\displaystyle\sum_{i=0}^{5}a_iy^i-c_1H_1(y)-c_2H_2(y), & y\le x,\\ \displaystyle\sum_{i=0}^{5}b_iy^i-c_1H_1(y)-c_2H_2(y), & y>x,\end{cases}$$

where $H_i(y)=\int_0^y\int_0^y\int_0^y\int_0^y\int_0^y\int_0^y h_i(y)\mathrm{d}y\mathrm{d}y\mathrm{d}y\mathrm{d}y\mathrm{d}y\mathrm{d}y$, $i=1,2$. On the other hand, for (3.8), let $R_x(y)$ satisfy

$$R_x^{(k)}(x+0)=R_x^{(k)}(x-0),\ k=0,1,2,\ldots,4. \tag{3.9}$$

Integrating (3.8) from $x-\varepsilon$ to $x+\varepsilon$ with respect to y and letting $\varepsilon\to 0$, we have the jump degree of $R_x^{(5)}(y)$ at $y=x$

$$R_x^{(5)}(x-0)-R_x^{(5)}(x+0)=1. \tag{3.10}$$

From (3.6), (3.7), (3.9) and (3.10), the unknown coefficients of (3.3) can be obtained. □

In [39], Li and Cui defined a reproducing kernel Hilbert space $W_2^1[0,1]$ and gave its reproducing kernel

$$\overline{R}_x(y)=\frac{1}{2\sinh(1)}[\cosh(x+y-1)+\cosh(|x-y|-1)].$$

3.2. The solution of (3.1)

In (3.1), it is clear that $L:W_2^3[0,1]\to W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_i(x)=\overline{R}_{x_i}(x)$ and $\psi_i(x)=L^*\varphi_i(x)$ where L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^3[0,1]$ can be derived from the Gram–Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x)=\sum_{k=1}^{i}\beta_{ik}\psi_k(x), \quad (\beta_{ii}>0,\ i=1,2,\ldots). \tag{3.11}$$

Theorem 3.2. *For (3.1), if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$ and $\psi_i(x)=L_yR_x(y)|_{y=x_i}$. The subscript y by the operator L indicates that the operator L applies to the function of y.*

Proof. Note here that

$$\begin{aligned}\psi_i(x)&=(L^*\varphi_i)(x)=((L^*\varphi_i)(y),R_x(y))\\ &=(\varphi_i(y),L_yR_x(y))=L_yR_x(y)|_{y=x_i}.\end{aligned}$$

Clearly, $\psi_i(x)\in W_2^3[0,1]$.
For each fixed $u(x)\in W_2^3[0,1]$, let $(u(x),\psi_i(x))=0$, $(i=1,2,\ldots)$, which means that,

$$(u(x),(L^*\varphi_i)(x))=(Lu(\cdot),\varphi_i(\cdot))=(Lu)(x_i)=0. \tag{3.12}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on [0, 1], $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of Theorem 3.2 is complete. □

Theorem 3.3. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,\ 1]$ and the solution of (3.1) is unique, then the solution of (3.1) is*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\overline{\psi}_i(x). \tag{3.13}$$

Proof. Applying Theorem 3.2, it is easy to know that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^3[0, 1]$. Note that $(v(x), \varphi_i(x)) = v(x_i)$ for each $v(x) \in W_2^1[0, 1]$. Hence we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty}(u(x), \overline{\psi}_i(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(u(x), L^*\varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(Lu(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(f(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\overline{\psi}_i(x) \end{aligned} \tag{3.14}$$

and the proof of the theorem is complete. □

Now, the approximate solution $u_n(x)$ can be obtained by the n-term intercept of the exact solution $u(x)$ and

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}f(x_k)\overline{\psi}_i(x). \tag{3.15}$$

Remark. Put $Q = \overline{Span\{\{\overline{\psi}_i\}_{i=1}^{n}\}}$. Clearly, $Q \subset W_2^3[0, 1]$. In fact, $u_n(x)$ is the projection of the exact solution $u(x)$ onto space Q.

Theorem 3.4. *Assume that $u(x)$ is the solution of (3.1) and $r_n(x)$ is the error between the approximate $u_n(x)$ and the exact solution $u(x)$. Then the error $r_n(x)$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^3}$.*

Proof. From (3.14) and (3.15), it follows that

$$\begin{aligned} \|r_n\|_{W_2^3} &= \left\| \sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k)\overline{\psi}_i(x)\right\|_{W_2^3} \\ &= \sum_{i=n+1}^{\infty}\left(\sum_{k=1}^{i}\beta_{ik}f(x_k)\right)^2. \end{aligned} \tag{3.16}$$

(3.16) shows that the error r_n is monotone decreasing in the sense of $\|\cdot\|_{W_2^3}$ and the proof is complete. □

4. The application of HPM and RKHSM to (1.1)

For (1.1), according to the HPM, we construct a homotopy as follows:

$$H(u, p) = u''(t) + \sigma u'(t) + Pf(t, u) = 0 \tag{4.1}$$

where $p \in [0, 1]$ is an embedding parameter. In the case $p = 0$, (4.1) becomes a linear equation, which is easy to be solved, and when $p = 1$, (4.1) turns out to be the original one, (1.1).

In view of the HPM, we use the homotopy parameter p to expand the solution

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 \cdots \tag{4.2}$$

The approximate solution of (1.1) can be obtained by setting $p = 1$

$$u = u_0 + u_1 + u_2 + u_3 \cdots. \tag{4.3}$$

Substituting (4.2) into (4.1), and equating the coefficients of the identical powers of p yields the following equations:

$$p^0: u_0''(t) + \sigma u_0'(t) = 0, \qquad u_0(0) - \mu_1 u_0'(0) = \int_0^1 h_1(s)u_0(s)\mathrm{d}s, \qquad u_0(1) + \mu_2 u_0'(1) = \int_0^1 h_2(s)u_0(s)\mathrm{d}s,$$

$$p^1: u_1''(t) + \sigma u_1'(t) = -f(t,u)|_{p=0}, \qquad u_1(0) - \mu_1 u_1'(0) = \int_0^1 h_1(s)u_1(s)\mathrm{d}s,$$

$$u_1(1) + \mu_2 u_1'(1) = \int_0^1 h_2(s)u_1(s)\mathrm{d}s,$$

$$p^2: u_2''(t) + \sigma u_2'(t) = -\left.\frac{\mathrm{d}f(t,u)}{\mathrm{d}p}\right|_{p=0}, \qquad u_2(0) - \mu_1 u_2'(0) = \int_0^1 h_1(s)u_2(s)\mathrm{d}s,$$

$$u_2(1) + \mu_2 u_2'(1) = \int_0^1 h_2(s)u_2(s)\mathrm{d}s,$$

$$p^3: u_3''(t) + \sigma u_3'(t) = -\left.\frac{\mathrm{d}^2f(t,u)}{2!dp^2}\right|_{p=0}, \qquad u_3(0) - \mu_1 u_3'(0) = \int_0^1 h_1(s)u_3(s)\mathrm{d}s,$$

$$u_3(1) + \mu_2 u_3'(1) = \int_0^1 h_2(s)u_3(s)\mathrm{d}s,$$

$$p^4: u_4''(t) + \sigma u_4'(t) = -\left.\frac{\mathrm{d}^3f(t,u)}{3!dp^3}\right|_{p=0}, \qquad u_4(0) - \mu_1 u_4'(0) = \int_0^1 h_1(s)u_4(s)\mathrm{d}s,$$

$$u_4(1) + \mu_2 u_4'(1) = \int_0^1 h_2(s)u_4(s)\mathrm{d}s,$$

$$\cdots\cdots$$

$$p^m: u_m''(t) + \sigma u_m'(t) = -\left.\frac{\mathrm{d}^{m-1}f(t,u)}{(m-1)!dp^{m-1}}\right|_{p=0}, u_m(0) - \mu_1 u_m'(0) = \int_0^1 h_1(s)u_m(s)\mathrm{d}s,$$

$$u_m(1) + \mu_2 u_m'(1) = \int_0^1 h_2(s)u_m(s)\mathrm{d}s.$$

To solve the above equations, we use the RKHSM presented in Section 3 and obtain $u_0, u_1, u_2, u_3, \ldots$

$$\begin{aligned}
u_0(t) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f_0(t_k)\overline{\psi}_i(t),\\
u_1(t) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f_1(t_k)\overline{\psi}_i(t),\\
u_2(t) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f_2(t_k)\overline{\psi}_i(t),\\
u_3(t) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f_3(t_k)\overline{\psi}_i(t),\\
u_4(t) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f_4(t_k)\overline{\psi}_i(t),\\
&\cdots\cdots\\
u_m(t) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f_m(x_k)\overline{\psi}_i(x),
\end{aligned} \tag{4.4}$$

where $f_0(t) = 0, f_1(t) = -f(t,u)|_{p=0}(t), f_2(t) = -\frac{\mathrm{d}f(t,u)}{\mathrm{d}p}|_{p=0}(t), f_3(t) = -\frac{\mathrm{d}^2f(t,u)}{2!dp^2}|_{p=0}(t), f_4(t) = -\frac{\mathrm{d}^3f(t,u)}{3!dp^3}|_{p=0}(t)$, $f_m(t) = -\frac{\mathrm{d}^{m-1}f(t,u)}{(m-1)!dp^{m-1}}|_{p=0}(t)$.

Therefore, the approximate solution of Eq. (1.1) and the m-term approximation to this solution are obtained

$$u = \sum_{k=0}^{\infty} u_k, \qquad U_m = \sum_{k=0}^{m-1} u_k. \tag{4.5}$$

Table 1
Numerical results for Example 5.1.

x	True solution $u(x)$	Relative error ($U_{5,10}$)	Relative error ($U_{5,20}$)	Relative error ($U_{5,50}$)	Relative error ($U_{5,100}$)
0.01	0.031411	9.19E−3	2.21E−3	3.52E−4	7.49E−5
0.08	0.248690	8.13E−3	2.03E−3	3.25E−4	8.17E−5
0.16	0.481754	8.07E−3	2.01E−3	3.23E−4	8.11E−5
0.32	0.844328	8.04E−3	2.01E−3	3.22E−4	8.09E−5
0.48	0.998027	8.02E−3	2.01E−3	3.22E−4	8.08E−5
0.64	0.904827	8.00E−3	2.00E−3	3.20E−4	8.03E−5
0.80	0.587785	7.93E−3	1.98E−3	3.18E−4	7.97E−5
0.96	0.125333	7.96E−3	1.96E−3	3.14E−4	7.85E−5

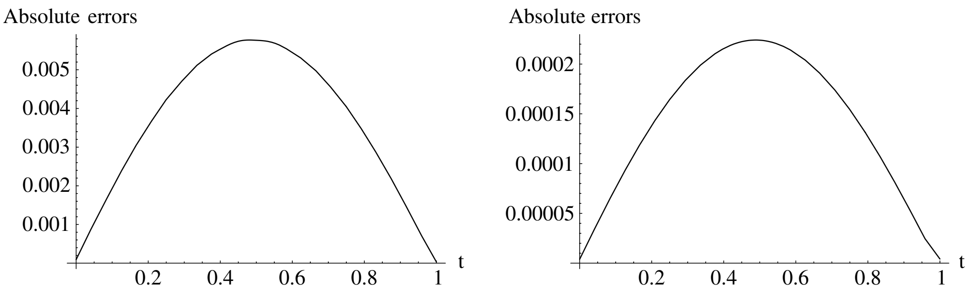

Fig. 1. Figures of absolute errors $|u(x) - U_{2,10}(x)|$, $|u(x) - U_{5,50}(x)|$ for Example 5.2.

Now, the approximate solution $U_{m,n}(t)$ can be obtained by the n-term intercept of the $u_k(t)$, $k = 0, 1, 2, \ldots,$ and

$$U_{m,n}(t) = \sum_{k=0}^{m-1} \sum_{i=1}^{n} A_{ik} \overline{\psi}_i(t), \tag{4.6}$$

where $A_{ik} = \sum_{j=1}^{i} \beta_{ij} f_k(t_j)$.

5. Numerical examples

In this section, we present and discuss the numerical results by employing the HPM and RKHSM for two examples. The results demonstrate that the present method is remarkably effective.

Example 5.1. Consider the following forced Duffing equation:

$$\begin{cases} u''(t) + u'(t) + t(1-t)u^3 = f(t), \quad 0 < t < 1, \\ u(0) - \dfrac{2}{\pi^2} u'(0) = -\displaystyle\int_0^1 u(s)\mathrm{d}s, \qquad u(1) + \dfrac{1}{\pi^2} u'(1) = -\displaystyle\int_0^1 s u(s)\mathrm{d}s, \end{cases}$$

where $f(t) = \pi \cos(\pi t) - \sin(\pi t) \left(\pi^2 + (-1+t)\ t\ \sin(\pi t)^2\right)$. It is easy to see that the exact solution is $u(t) = \sin(\pi t)$.

Solution: According to (4.4)–(4.6), one can obtain the approximation $U_{m,n}(x)$ of $u(x)$.
When we take $m = 5$, $n = 10, 20, 50, 100$, the numerical results are shown in Table 1.

Example 5.2. Consider the following forced Duffing equation:

$$\begin{cases} u''(t) - u'(t) - 2u(t) + \sin(u) = f(t), \quad 0 < t < 1, \\ u(0) - \dfrac{4}{3\pi^2} u'(0) = -\displaystyle\int_0^1 \cos\left(\frac{\pi s}{2}\right) u(s)\mathrm{d}s, \qquad u(1) + \dfrac{6}{\pi^2} u'(1) = -\displaystyle\int_0^1 (2s+2) u(s)\mathrm{d}s, \end{cases}$$

where $f(t) = -(\pi \cos(\pi t)) - \left(2 + \pi^2\right) \sin(\pi t) + \sin(\sin(\pi t))$. It is easy to see that the exact solution is $u(t) = \sin(\pi t)$.

Solution: According to (4.4)–(4.6), one can obtain the approximation $U_{m,n}(x)$ of $u(x)$.
When we take $m = 2$, $n = 10$ and $m = 5$, $n = 51$, the numerical results are shown in Fig. 1.

6. Conclusion

In this paper, the combination of HPM and RKHSM was employed successfully for solving the forced Duffing equation with integral boundary conditions. The numerical results show that the present method is an accurate and reliable analytical technique for the forced Duffing equation with integral boundary conditions.

Acknowledgments

The authors would like to express their thanks to the unknown referees for their careful reading and helpful comments. The work of the first author was supported by the Scientific Research Project of Heilongjiang Education Office (2009-11541098).

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