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# A non-hyponormal operator generating Stieltjes moment sequences ☆

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**Abstract**

A linear operator $S$ in a complex Hilbert space $\mathcal{H}$ for which the set $\mathcal{D}^\infty(S)$ of its $C^\infty$-vectors is dense in $\mathcal{H}$ and $\{\|S^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $f \in \mathcal{D}^\infty(S)$ is said to generate Stieltjes moment sequences. It is shown that there exists a closed non-hyponormal operator $S$ which generates Stieltjes moment sequences. What is more, $\mathcal{D}^\infty(S)$ is a core of any power $S^n$ of $S$. This is established with the help of a weighted shift on a directed tree with one branching vertex. The main tool in the construction comes from the theory of indeterminate Stieltjes moment sequences. As a consequence, it is shown that there exists a non-hyponormal composition operator in an $L^2$-space (over a $\sigma$-finite measure space) which is injective, paranormal and which generates Stieltjes moment sequences. The independence assertion of Barry Simon's theorem which parameterizes von Neumann extensions of a closed real symmetric operator with deficiency indices $(1, 1)$ is shown to be false.



*Keywords:* Indeterminate moment problem; N-extremal measure; Krein and Friedrichs measures; Directed tree; Weighted shift on a directed tree; Hyponormal operator; Operator generating Stieltjes moment sequences; Composition operator in an $L^2$-space

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## 1. Preliminaries

### 1.1. Introduction

A linear operator $S$ in a complex Hilbert space $\mathcal{H}$ is said to *generate* Stieltjes moment sequences if the set $\mathcal{D}^\infty(S)$ of all its $C^\infty$-vectors is dense in $\mathcal{H}$ and $\{\|S^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $f \in \mathcal{D}^\infty(S)$. The celebrated Lambert characterization of subnormality [24] states that a (closed) bounded linear operator is subnormal if and only if it generates Stieltjes moment sequences. As shown in [7,8,39], this result remains true for some classes of unbounded operators (see [16,38–41] for the foundations of the theory of bounded and unbounded subnormal operators). To the best of our knowledge, the only known examples of non-subnormal operators generating Stieltjes moment sequences are those coming from formally normal ones[1] (see [7, Section 3.2] for a more detailed discussion of this question). Unfortunately, the operators so constructed, though closable, are not closed. In the present paper we provide an example of a non-hyponormal (and thus a non-subnormal) closed paranormal operator $S$ which generates Stieltjes moment sequences[2] and which has the property that $\mathcal{D}^\infty(S)$ is a core of any power $S^n$ of $S$ (see Example 4.2.1). This is a carefully constructed weighted shift on an enumerable leafless directed tree (we refer the reader to [20] for the foundations of the theory of weighted shifts on directed trees). As a byproduct, we obtain an example of a paranormal operator which is not hyponormal (see [17,10,20] for other examples of this kind).

Using N-extremal measures (including the Friedrichs one) of an indeterminate moment sequence as well as some facts from moment theory which relate the determinacy of sequences $\{a_n\}_{n=0}^\infty$ and $\{a_{n+1}\}_{n=0}^\infty$, we construct a non-hyponormal weighted shift on a directed tree $\mathscr{T}_{\infty,\kappa}$ which generates Stieltjes moment sequences (cf. Example 4.2.1). The $\mathscr{T}_{\infty,\kappa}$ is an enumerable leafless directed tree which has only one branching vertex denoted by 0. If $\kappa < \infty$, then $\mathscr{T}_{\infty,\kappa}$ has a root and 0 belongs to the $\kappa$th generation of the root; otherwise $\mathscr{T}_{\infty,\kappa}$ is rootless. The weighted shift so constructed does not satisfy the consistency condition (3.1.6) at $u = 0$ and it has no consistent system of measures (in the sense of [7]). The case of $\kappa = \infty$ is especially interesting because it leads to an example of a non-hyponormal composition operator in an $L^2$-space over a $\sigma$-finite measure space which generates Stieltjes moment sequences (cf. Theorem 4.3.3). In view of [9], this example is the first showing that Lambert's characterization of subnormality of composition operators (cf. [25]) is no longer true in the unbounded case. As proved in [9], each formally normal composition operator in an $L^2$-space is normal. This means that an example of a non-subnormal formally normal operator $N$ with dense set of $C^\infty$-vectors $f$ having the property that $\{\|N^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence, could not be realized as a composition operator in an $L^2$-space.

Since our main example (Example 4.2.1) depends heavily on some subtle properties of indeterminate Stieltjes moment sequences, we provide necessary facts concerning N-extremal measures including Krein and Friedrichs ones (see Sections 2.1 and 2.2). In Section 2.3 we supply examples of exotic Stieltjes moment sequences that are used in Example 4.2.1. The necessary facts concerning weighted shifts $S_{\boldsymbol{\lambda}}$ on directed trees are given in Section 3.1. Powers of such operators are described in Section 3.2. As a consequence, it is shown that if $\mathcal{D}^\infty(S_{\boldsymbol{\lambda}})$ is

[1] Formally normal operators are always hyponormal but not necessarily subnormal (see [14,31,36]).

[2] Note that if $S$ is a Hilbert space operator which generates Stieltjes moment sequences, then the operator $S|_{\mathcal{D}^\infty(S)}$ is paranormal; see (4.1.21).

dense in the underlying Hilbert space, then $\mathcal{D}^\infty(S_{\boldsymbol\lambda})$ is a core of any power $S_{\boldsymbol\lambda}^n$ of $S_{\boldsymbol\lambda}$. A sufficient condition for $S_{\boldsymbol\lambda}$ to generate Stieltjes moment sequences, written in terms of basic vectors, is given in Theorem 3.2.4. Section 4.1 offers a general scheme for constructing weighted shifts on the directed tree $\mathscr{T}_{\eta,\kappa}$ with assorted properties (cf. Theorem 4.1.1). Section 4.2 contains the main example of the paper. Appendix A shows that the independence assertion of Barry Simon's theorem which parameterizes von Neumann extensions of a closed real symmetric operator with deficiency indices (1, 1) is false (cf. Proposition A.4.1). This theorem was used by Simon to describe N-extremal measures of indeterminate moment sequences in [34]. Fortunately, this fault does not spoil[3] the main idea of his paper which is based on the formula (4.20) in [34].

### *1.2. Notation and terminology*

In what follows, $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$ stand for the sets of complex numbers, real numbers and integer numbers, respectively. Set

$$\mathbb{N} = \{n \in \mathbb{Z}\colon n \geqslant 1\}, \qquad \mathbb{Z}_+ = \mathbb{N} \cup \{0\}, \qquad \mathbb{R}_+ = \{x \in \mathbb{R}\colon x \geqslant 0\}.$$

For a Borel set $\Omega$ in $\mathbb{R}_+$, we denote by $\mathfrak{B}(\Omega)$ the $\sigma$-algebra of all Borel sets in $\Omega$. Given $a \in \mathbb{R}_+$, we write $\delta_a$ for the Borel probability measure on $\mathbb{R}_+$ concentrated on $\{a\}$. The closed support of a finite positive Borel measure $\mu$ on $\mathbb{R}$ will be denoted by $\operatorname{supp}(\mu)$. We write $\operatorname{card}(X)$ for the cardinal number of a set $X$.

Let $A$ be a (linear) operator in a complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\ker(A)$, $\bar{A}$ and $A^*$ the domain, the range, the kernel, the closure and the adjoint of $A$ (in case they exist). Set $\mathcal{D}^\infty(A) = \bigcap_{n=0}^\infty \mathcal{D}(A^n)$; members of $\mathcal{D}^\infty(A)$ are called $C^\infty$*-vectors*. A linear subspace $\mathcal{E}$ of $\mathcal{D}(A)$ is said to be a *core* of $A$ if the graph of $A$ is contained in the closure of the graph of the restriction $A|_{\mathcal{E}}$ of $A$ to $\mathcal{E}$. We say that $A$ is *symmetric* if $A$ is densely defined, $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $Af = A^*f$ for all $f \in \mathcal{D}(A)$. If $A$ is densely defined and $A = A^*$, then $A$ is called *selfadjoint*. The operator $A$ is said to be *essentially* selfadjoint if $A$ is closable and the closure of $A$ is selfadjoint. The orthogonal dimensions of $\ker(A^* \mp \mathrm{i}I)$, which are denoted by $d_\pm = d_\pm(A)$, are called the *deficiency indices* of a symmetric operator $A$ ($I$ is the identity operator on $\mathcal{H}$). It is well known that if $A$ is symmetric, then $A$ is essentially selfadjoint if and only if its deficiency indices are both equal to 0. If $A$ is symmetric, then $A$ has equal deficiency indices if and only if it has a selfadjoint extension in $\mathcal{H}$; such an extension will be called a *von Neumann* extension of $A$. Note that a symmetric operator may have no von Neumann extension, though it always has a selfadjoint one in a larger complex Hilbert space (cf. [1, Theorem 1 in Appendix I.2]). This means that each symmetric operator is subnormal. We say that $A$ is *nonnegative* if $\langle Ah, h\rangle \geqslant 0$ for all $h \in \mathcal{D}(A)$. Given two nonnegative selfadjoint operators $C$ and $D$ in $\mathcal{H}$, we write $C \preccurlyeq D$ if $\mathcal{D}(D^{1/2}) \subseteq \mathcal{D}(C^{1/2})$ and $\|C^{1/2}h\| \leqslant \|D^{1/2}h\|$ for all $h \in \mathcal{D}(D^{1/2})$; note that $C \preccurlyeq D$ if and only if $(D + xI)^{-1} \leqslant (C + xI)^{-1}$ for all real $x > 0$ or equivalently for some real $x > 0$ (cf. [23, Theorem VI.2.21]). If $A$ is densely defined and nonnegative, then there exist nonnegative selfadjoint operators $B_{\mathrm{K}}$ and $B_{\mathrm{F}}$ in $\mathcal{H}$ that extend $A$ and such that $B_{\mathrm{K}} \preccurlyeq B \preccurlyeq B_{\mathrm{F}}$ for every nonnegative selfadjoint extension $B$ of $A$ in $\mathcal{H}$. The operators $B_{\mathrm{K}}$ and $B_{\mathrm{F}}$ are called the *Krein* and the *Friedrichs* extensions of $A$. We refer the reader to [6,44,15,32,28,29] for more information on these subjects.

[3] May be with an exception of Remark 2 on page 104 in [34].

An operator $A$ in $\mathcal{H}$ is called *paranormal* if $\|Af\|^2 \leqslant \|f\|\|A^2f\|$ for all $f \in \mathcal{D}(A^2)$. We say that an operator $A$ in $\mathcal{H}$ is *hyponormal* if $A$ is densely defined, $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^*f\| \leqslant \|Af\|$ for all $f \in \mathcal{D}(A)$. A densely defined operator $N$ in $\mathcal{H}$ is said to be *normal* if $N$ is closed and $N^*N = NN^*$ (or equivalently if and only if $N$ is closed and both operators $N$ and $N^*$ are hyponormal, cf. [44, Section 5.6]). A densely defined operator $S$ in $\mathcal{H}$ is called *subnormal* if there exist a complex Hilbert space $\mathcal{K}$ and a normal operator $N$ in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Sh = Nh$ for all $h \in \mathcal{D}(S)$. It is well known that normality implies subnormality, subnormality implies hyponormality and hyponormality implies paranormality, but none of these implications can be reversed in general, i.e.,

$$\{\text{normals}\} \subsetneq \{\text{subnormals}\} \subsetneq \{\text{hyponormals}\} \subsetneq \{\text{paranormals}\}.$$

For details on this we refer the reader to [16,19,18,20] (see also [44,6,40,22,27,42] for the unbounded case).

## 2. The classical moment problem revisited

### 2.1. Indeterminate moment problems

A sequence $\{\gamma_n\}_{n=0}^\infty$ of real numbers is said to be a *Stieltjes moment sequence* if there exists a positive Borel measure $\mu$ on $\mathbb{R}_+$ such that (from now on, we abbreviate $\int_{\mathbb{R}_+}$ to $\int_0^\infty$)

$$\gamma_n = \int_0^\infty x^n \,\mathrm{d}\mu(x), \quad n \in \mathbb{Z}_+.$$

Call such $\mu$ an *S-representing measure* of the Stieltjes moment sequence $\{\gamma_n\}_{n=0}^\infty$. A Stieltjes moment sequence is said to be *S-determinate* if it has only one S-representing measure; otherwise, we call it *S-indeterminate*. By the Stieltjes theorem (cf. [5, Theorem 6.2.5]), a sequence $\{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is a Stieltjes moment sequence if and only if the sequences $\{\gamma_n\}_{n=0}^\infty$ and $\{\gamma_{n+1}\}_{n=0}^\infty$ are positive definite (recall that a sequence $\{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is said to be *positive definite* if $\sum_{k,l=0}^n \gamma_{k+l}\alpha_k\overline{\alpha_l} \geqslant 0$ for all $\alpha_0, \ldots, \alpha_n \in \mathbb{C}$ and $n \in \mathbb{Z}_+$). It is clear that if $\{\gamma_n\}_{n=0}^\infty$ is a Stieltjes moment sequence, then so is $\{\gamma_{n+1}\}_{n=0}^\infty$. The converse is easily seen to be false (consider, e.g., the sequence $\{\gamma_n\}_{n=0}^\infty := \{\gamma_0, 1, 0, 0, \ldots\}$). Moreover, if a Stieltjes moment sequence $\{\gamma_n\}_{n=0}^\infty$ is S-indeterminate, then so is $\{\gamma_{n+1}\}_{n=0}^\infty$ (see [34, Proposition 5.12]; see also Lemma 2.1.1 below). The converse implication fails to hold (cf. [34, Corollary 4.21]; see also the discussion below).

The following result has been established in [7] (see also [45] and [43] for the question of backward extendibility of Hamburger moment sequences).

**Lemma 2.1.1.** *(See [7, Lemma 2.4.1].) Let $\{\gamma_n\}_{n=0}^\infty$ be a Stieltjes moment sequence and let $\gamma_{-1}$ be a positive real number. Then the following are equivalent*[4]:

(i) *$\{\gamma_{n-1}\}_{n=0}^\infty$ is a Stieltjes moment sequence,*

[4] We adhere to the convention that $\frac{1}{0} := \infty$. Hence, $\int_0^\infty \frac{1}{x}\,\mathrm{d}\mu(x) < \infty$ implies $\mu(\{0\}) = 0$.

(ii) *there exists an S-representing measure $\mu$ of $\{\gamma_n\}_{n=0}^\infty$ such that*

$$\int_0^\infty \frac{1}{x}\,\mathrm{d}\mu(x) \leqslant \gamma_{-1}. \tag{2.1.1}$$

*Moreover, if* (i) *holds, then the mapping $\mathscr{M}_0(\gamma_{-1}) \ni \mu \to \nu_\mu \in \mathscr{M}_{-1}(\gamma_{-1})$ defined by*

$$\nu_\mu(\sigma) = \int_\sigma \frac{1}{x}\,\mathrm{d}\mu(x) + \left(\gamma_{-1} - \int_0^\infty \frac{1}{x}\,\mathrm{d}\mu(x)\right)\delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{2.1.2}$$

*is a bijection with the inverse $\mathscr{M}_{-1}(\gamma_{-1}) \ni \nu \to \mu_\nu \in \mathscr{M}_0(\gamma_{-1})$ given by*

$$\mu_\nu(\sigma) = \int_\sigma x\,\mathrm{d}\nu(x), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

*where $\mathscr{M}_0(\gamma_{-1})$ is the set of all S-representing measures $\mu$ of $\{\gamma_n\}_{n=0}^\infty$ such that $\int_0^\infty \frac{1}{x}\,\mathrm{d}\mu(x) \leqslant \gamma_{-1}$, and $\mathscr{M}_{-1}(\gamma_{-1})$ is the set of all S-representing measures $\nu$ of $\{\gamma_{n-1}\}_{n=0}^\infty$. In particular, $\nu_\mu(\{0\}) = 0$ if and only if $\int_0^\infty \frac{1}{x}\,\mathrm{d}\mu(x) = \gamma_{-1}$.*

*If* (i) *holds and the sequence $\{\gamma_n\}_{n=0}^\infty$ is S-determinate, then $\{\gamma_{n-1}\}_{n=0}^\infty$ is S-determinate, the unique S-representing measure $\mu$ of $\{\gamma_n\}_{n=0}^\infty$ satisfies the inequality $\int_0^\infty \frac{1}{x}\,\mathrm{d}\mu(x) \leqslant \gamma_{-1}$, and $\nu_\mu$ is the unique S-representing measure of $\{\gamma_{n-1}\}_{n=0}^\infty$.*

A sequence $\{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is said to be a *Hamburger moment sequence* if there exists a positive Borel measure $\mu$ on $\mathbb{R}$ such that

$$\gamma_n = \int_{-\infty}^\infty x^n\,\mathrm{d}\mu(x), \quad n \in \mathbb{Z}_+.$$

Call such $\mu$ an *H-representing measure* of the Hamburger moment sequence $\{\gamma_n\}_{n=0}^\infty$. A Hamburger moment sequence is said to be *H-determinate* if it has only one H-representing measure; otherwise, we call it *H-indeterminate*. By the Hamburger theorem (cf. [5, Theorem 6.2.2]), a sequence $\{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is a Hamburger moment sequence if and only if it is positive definite. It is clear that if a Stieltjes moment sequence is S-indeterminate, then it is H-indeterminate. The reverse implication is not true in general (cf. [34, p. 96]).

Let $\{\gamma_n\}_{n=0}^\infty$ be an H-indeterminate Hamburger moment sequence. By an *N-extremal* measure of $\{\gamma_n\}_{n=0}^\infty$ we mean an H-representing measure $\mu$ of $\{\gamma_n\}_{n=0}^\infty$ for which the complex polynomials in one variable are dense in $L^2(\mu)$. It is well known that there is a bijection $t \mapsto \mu_t$ between the set $\mathbb{R} \cup \{\infty\}$ and the set of all N-extremal measures of $\{\gamma_n\}_{n=0}^\infty$ such that (cf. [34, Remark, p. 96])

$$\int_{-\infty}^\infty \frac{\mathrm{d}\mu_t(x)}{x} = t, \quad t \in \mathbb{R} \cup \{\infty\}. \tag{2.1.3}$$

The parametrization $t \mapsto \mu_t$ can be done as follows (cf. [34]). Denote by $\mathcal{P}$ the ring of all polynomials in one formal variable $X$ with complex coefficients. Since $\{\gamma_n\}_{n=0}^\infty$ is indeterminate, there exists a unique inner product $\langle\cdot,\text{-}\rangle$ on $\mathcal{P}$ such that

$$\big\langle X^m, X^n\big\rangle = \gamma_{m+n}, \quad m,n \in \mathbb{Z}_+. \tag{2.1.4}$$

Let $\mathcal{H}$ be the complex Hilbert space completion of $(\mathcal{P}, \langle\cdot,\text{-}\rangle)$. Since $\langle Xp, q\rangle = \langle p, Xq\rangle$ for all $p, q \in \mathcal{P}$, we deduce that there exists a unique symmetric operator $A$ in $\mathcal{H}$ such that $\mathcal{D}(A) = \mathcal{P}$ and $A(p) = X \cdot p$ for all $p \in \mathcal{P}$. Then clearly $\mathcal{D}(A)$ is equal to the linear span of $\{A^n e\colon n \in \mathbb{Z}_+\}$ and, by (2.1.4),

$$\gamma_n = \big\langle A^n e, e\big\rangle, \quad n \in \mathbb{Z}_+ \ \big(e := X^0\big). \tag{2.1.5}$$

Hence, if $B$ is a von Neumann extension of $A$, then $\mu_B(\cdot) := \langle E_B(\cdot)e, e\rangle$ is an H-representing measure of $\{\gamma_n\}_{n=0}^\infty$, where $E_B$ is the spectral measure of $B$. By the H-indeterminacy of $\{\gamma_n\}_{n=0}^\infty$, the symmetric operator $A$ is not essentially selfadjoint and its deficiency indices are both equal to 1, and thus there exists a bijection $t \mapsto B_t$ between the set $\mathbb{R} \cup \{\infty\}$ and the set of all von Neumann extensions of $A$ such that for every $t \in \mathbb{R}$, the spectrum of $B_t$ does not contain 0 and $t = \langle B_t^{-1} e, e\rangle$, and 0 is an eigenvalue of $B_\infty$ (see [34, formulas (4.20)] and[5] [34, Theorem 2.6]). This immediately implies (2.1.3) with $\mu_t(\cdot) := \langle E_{B_t}(\cdot)e, e\rangle$ for $t \in \mathbb{R} \cup \{\infty\}$. It turns out that for every $t \in \mathbb{R} \cup \{\infty\}$, $\mu_t$ is an N-extremal measure of $\{\gamma_n\}_{n=0}^\infty$ (and that there are no other N-extremal measures), the closed support of $\mu_t$ (which coincides with the spectrum of $B_t$) has no accumulation point in $\mathbb{R}$, and consequently it is infinite and countable. Moreover, $\operatorname{supp}(\mu_s) \cap \operatorname{supp}(\mu_t) = \emptyset$ for all $s, t \in \mathbb{R} \cup \{\infty\}$ such that $s \neq t$, and $\mathbb{R} = \bigcup_{t \in \mathbb{R}\cup\{\infty\}} \operatorname{supp}(\mu_t)$, which means that the family $\{\operatorname{supp}(\mu_t)\}_{t\in\mathbb{R}\cup\{\infty\}}$ forms a partition of $\mathbb{R}$.

Now suppose that $\{\gamma_n\}_{n=0}^\infty$ is an S-indeterminate Stieltjes moment sequence. Then $\{\gamma_n\}_{n=0}^\infty$ is H-indeterminate. Let $(\mathcal{H}, e, A)$ be as above. Then $A$ is nonnegative (in fact $A - \alpha I$ is nonnegative for some real $\alpha > 0$) and it has many nonnegative selfadjoint extensions in $\mathcal{H}$. As a consequence, the Krein extension $B_{\mathrm{K}}$ of $A$ is different from the Friedrichs extension $B_{\mathrm{F}}$ of $A$. It follows from [34, Theorem 4.18] that $B_{\mathrm{K}} = B_\infty$ and $B_{\mathrm{F}} = B_{t_0}$, where $t_0 = \langle B_{\mathrm{F}}^{-1} e, e\rangle \in (0,\infty)$, and[6]

$$\forall t \in \mathbb{R}\cup\{\infty\}\colon \quad \operatorname{supp}(\mu_t) \subseteq [0,\infty) \quad \Longleftrightarrow \quad t \in [t_0,\infty) \cup \{\infty\}. \tag{2.1.6}$$

In other words, $\{\mu_t\}_{t\in[t_0,\infty)\cup\{\infty\}}$ are the only N-extremal measures of $\{\gamma_n\}_{n=0}^\infty$ which are simultaneously S-representing measures of $\{\gamma_n\}_{n=0}^\infty$. Call the N-extremal measures $\mu_{\mathrm{K}}(\cdot) := \langle E_{B_\infty}(\cdot)e, e\rangle$ and $\mu_{\mathrm{F}}(\cdot) := \langle E_{B_{t_0}}(\cdot)e, e\rangle$ the *Krein* and the *Friedrichs* measures of $\{\gamma_n\}_{n=0}^\infty$, respectively. Note that $\mu_{\mathrm{K}} = \mu_\infty$ and $\mu_{\mathrm{F}} = \mu_{t_0}$. Arguing as in the proof of [34, Proposition 3.1], we deduce that $\min(\operatorname{supp}(\mu_t)) < \min(\operatorname{supp}(\mu_{\mathrm{F}}))$ for all $t \in (t_0,\infty) \cup \{\infty\}$. Hence, by the preceding paragraph and (2.1.6), we have

$$0 \in \operatorname{supp}(\mu_{\mathrm{K}}) \text{ and } 0 < \min\big(\operatorname{supp}(\mu_t)\big) < \min\big(\operatorname{supp}(\mu_{\mathrm{F}})\big) \quad \text{for all } t \in (t_0,\infty).$$

[5] Unfortunately, the independence assertion of [34, Theorem 2.6], saying that the family $\{B_t\}_{t\in\mathbb{R}\cup\{\infty\}}$ is independent of the choice of $\psi$, is not true (see Appendix A). Fortunately, the choice of $\psi$ made in [34, (4.20)] suits both the Hamburger and Stieltjes moment problems.

[6] See also [2, Theorem 5.2] for a Nevanlinna type parametrization of solutions of an S-indeterminate Stieltjes moment sequence. Both parameterizations are equivalent.

This in turn implies that

$$0<\int_0^\infty \frac{1}{x^n}\,\mathrm{d}\mu_t(x)<\infty \quad \text{for all } n\in\mathbb{N} \text{ and } t\in[t_0,\infty). \tag{2.1.7}$$

In particular $0<\int_0^\infty \frac{1}{x^n}\,\mathrm{d}\mu_{\mathrm{F}}(x)<\infty$ for all $n\in\mathbb{N}$.

### 2.2. Krein and Friedrichs measures

Now we state some crucial inequalities for the Krein and Friedrichs measures.

**Theorem 2.2.1.** *(See [34, Theorem 4.19 and Corollary 4.20].) Let $\{\gamma_n\}_{n=0}^\infty$ be an S-indeterminate Stieltjes moment sequence and let $\mu_{\mathrm{K}}$, $\mu_{\mathrm{F}}$ be the corresponding Krein and Friedrichs measures. If $\rho$ is an S-representing measure of $\{\gamma_n\}_{n=0}^\infty$ such that $\rho\neq\mu_{\mathrm{F}}$, then*

$$\int_0^\infty \frac{\mathrm{d}\mu_{\mathrm{F}}(x)}{x+y}<\int_0^\infty \frac{\mathrm{d}\rho(x)}{x+y}\leqslant\int_0^\infty \frac{\mathrm{d}\mu_{\mathrm{K}}(x)}{x+y},\quad y\in[0,\infty). \tag{2.2.1}$$

**Corollary 2.2.2.** *Let $\boldsymbol{\gamma}=\{\gamma_n\}_{n=0}^\infty$ be an S-indeterminate Stieltjes moment sequence and let $\mathcal{M}^{\mathrm{S}}(\boldsymbol{\gamma})$ be the set of all its S-representing measures. Then the Friedrichs measure $\mu_{\mathrm{F}}$ of $\boldsymbol{\gamma}$ is a unique measure $\rho\in\mathcal{M}^{\mathrm{S}}(\boldsymbol{\gamma})$ such that*

$$\int_0^\infty \frac{1}{x}\,\mathrm{d}\rho(x)=\min\left\{\int_0^\infty \frac{1}{x}\,\mathrm{d}\sigma(x)\colon\ \sigma\in\mathcal{M}^{\mathrm{S}}(\boldsymbol{\gamma})\right\}.$$

We will show that the right-hand inequality in (2.2.1) is in fact strict for all real $y>0$ (but not for $y=0$ as explained just after the proof of Proposition 2.2.3). This is an answer to a question raised by C. Berg [4].

**Proposition 2.2.3.** *Let $\{\gamma_n\}_{n=0}^\infty$ be an S-indeterminate Stieltjes moment sequence and let $\mu_{\mathrm{K}}$ be its Krein measure. If $\rho$ is an S-representing measure of $\{\gamma_n\}_{n=0}^\infty$ such that $\rho\neq\mu_{\mathrm{K}}$, then*

$$\int_0^\infty \frac{\mathrm{d}\rho(x)}{x+y}<\int_0^\infty \frac{\mathrm{d}\mu_{\mathrm{K}}(x)}{x+y},\quad y\in(0,\infty). \tag{2.2.2}$$

**Proof.** It follows from [34, Theorem 4.18] that there are entire functions $A$, $B$, $C$, $D$ (determined by the sequence $\{\gamma_n\}_{n=0}^\infty$) such that for all $t\in[t_0,\infty)\cup\{\infty\}$,

$$F(-y)(t):=-\frac{C(-y)t+A(-y)}{D(-y)t+B(-y)}=\int_0^\infty \frac{\mathrm{d}\mu_t(x)}{x+y},\quad y\in(0,\infty), \tag{2.2.3}$$

where the middle term in (2.2.3) is understood as $-\frac{C(-y)}{D(-y)}$ for $t=\infty$. Since $A, B, C, D$ take real values on the real line and $AD-BC\equiv 1$ (cf. [34, Theorem 4.8(iii)]), we deduce that the derivative of $F(-y)(\cdot)$ is positive on $[t_0,\infty)$, and thus the map $F(-y)(\cdot)$ is strictly increasing on $[t_0,\infty)$. Then for all $t\in[t_0,\infty)$,

$$F(-y)(t)=-\frac{C(-y)t+A(-y)}{D(-y)t+B(-y)} \underset{(t\to\infty)}{\nearrow} -\frac{C(-y)}{D(-y)}=F(-y)(\infty). \tag{2.2.4}$$

If the measure $\rho$ is N-extremal, then by our assumption and [34, Theorem 4.18] there is $t\in[t_0,\infty)$ such that $\rho=\mu_t$. Then, by (2.2.4), we have

$$\int_0^\infty \frac{\mathrm{d}\mu_t(x)}{x+y} \overset{(2.2.3)}{=} F(-y)(t)<F(-y)(\infty) \overset{(2.2.3)}{=} \int_0^\infty \frac{\mathrm{d}\mu_{\mathrm{K}}(x)}{x+y}, \quad y\in(0,\infty).$$

If $\rho$ is not N-extremal, then, again by [34, Theorem 4.18], there is a non-constant Pick function $\Phi:\mathbb{C}\setminus[0,\infty)\to\mathbb{C}$ such that $\Phi(-y)\in[t_0,\infty)$ for all $y\in(0,\infty)$, and

$$\int_0^\infty \frac{\mathrm{d}\rho(x)}{x-z}=-\frac{C(z)\Phi(z)+A(z)}{D(z)\Phi(z)+B(z)}, \quad z\in\mathbb{C}\setminus[0,\infty). \tag{2.2.5}$$

Hence, substituting $z=-y$ into (2.2.5), we get

$$\int_0^\infty \frac{\mathrm{d}\rho(x)}{x+y}=F(-y)\big(\Phi(-y)\big) \overset{(2.2.4)}{<} F(-y)(\infty) \overset{(2.2.3)}{=} \int_0^\infty \frac{\mathrm{d}\mu_{\mathrm{K}}(x)}{x+y}, \quad y\in(0,\infty).$$

This completes the proof. □

**Remark 2.2.4.** Let $\{\gamma_n\}_{n=0}^\infty$ be any S-indeterminate Stieltjes moment sequence. Fix $\alpha\in(0,1)$ and set $\rho_\alpha=\alpha\mu_{\mathrm{K}}+(1-\alpha)\mu_{\mathrm{F}}$, where $\mu_{\mathrm{K}}$ and $\mu_{\mathrm{F}}$ are the Krein and the Friedrichs measures of $\{\gamma_n\}_{n=0}^\infty$. Then $\rho_\alpha$ is an S-representing measure of $\{\gamma_n\}_{n=0}^\infty$ such that $\rho_\alpha\neq\mu_{\mathrm{K}}$, $\rho_\alpha$ is not N-extremal and, because 0 is an atom of $\mu_{\mathrm{K}}$,

$$\int_0^\infty \frac{\mathrm{d}\rho_\alpha(x)}{x}=\int_0^\infty \frac{\mathrm{d}\mu_{\mathrm{K}}(x)}{x}=\infty.$$

In other words, the strict inequality in (2.2.2) may turn into equality when $y=0$. This is never the case for an N-extremal measure $\rho$ (apply (2.1.3)).

Before stating the next result, we prove a lemma which is of some independent interest.

**Lemma 2.2.5.** *If $\{\gamma_n\}_{n=0}^\infty$ is an S-determinate Stieltjes moment sequence whose S-representing measure $\tau$ has the property that $\tau(\{0\})=0$, then $\{\gamma_n\}_{n=0}^\infty$ is H-determinate.*

**Proof.** Suppose that, contrary to our claim, $\{\gamma_n\}_{n=0}^\infty$ is H-indeterminate. Then the operator $A$ attached to $\{\gamma_n\}_{n=0}^\infty$ via (2.1.5) is not essentially selfadjoint. Since $\{\gamma_n\}_{n=0}^\infty$ is S-determinate, we deduce from [34, Theorem 2] (see also [13, Theorem 5]) that $A$ has a unique nonnegative selfadjoint extension in $\mathcal{H}$ which is evidently the Friedrichs extension $B_{\mathrm{F}}$ of $A$. Hence, by [34, Proposition 3.1], 0 is an eigenvalue of $B_{\mathrm{F}}$. Denote by $E$ the spectral measure of $B_{\mathrm{F}}$. Then clearly $\mu(\cdot) := \langle E(\cdot)e, e\rangle$ is an N-extremal measure of $\{\gamma_n\}_{n=0}^\infty$. Since the closed support of any N-extremal measure has no accumulation point in $\mathbb{R}$ and $\operatorname{supp}(\mu)$ coincides with the spectrum of $B_{\mathrm{F}}$ (see [13, Theorem 5] and also [40, Theorem 5]), we deduce that $\mu$ is an S-representing measure of $\{\gamma_n\}_{n=0}^\infty$ and 0 is an atom of $\mu$. By the S-determinacy of $\{\gamma_n\}_{n=0}^\infty$, we have $\tau = \mu$, which implies that $\tau(\{0\}) \neq 0$, a contradiction. This completes the proof. □

Note that if $\{\gamma_n\}_{n=0}^\infty$ is an H-determinate Stieltjes moment sequence, then its unique H-representing measure may have an atom at 0 (any compactly supported finite positive Borel measure on $[0,\infty)$ with an atom at 0 is an H-representing measure of such a sequence). This means that the converse of the implication in Lemma 2.2.5 does not hold in general.

The following characterization of the H-determinacy of a borderline backward extension of an S-indeterminate Stieltjes moment sequence will be used in the proof of Theorem 2.2.7. Let us mention that the implication (ii) ⇒ (i) and the “moreover” part of Theorem 2.2.6 below has appeared in [34, Corollary 4.21]. We include their proofs to keep the exposition self-contained.

**Theorem 2.2.6.** *Let $\{\gamma_n\}_{n=0}^\infty$ be an S-indeterminate Stieltjes moment sequence, $\mu_{\mathrm{F}}$ be its Friedrichs measure and $\gamma_{-1}$ be a nonnegative real number. Then the following two conditions are equivalent*:

(i) *$\{\gamma_{n-1}\}_{n=0}^\infty$ is an S-determinate Stieltjes moment sequence,*
(ii) $\gamma_{-1} = \int_0^\infty \frac{\mathrm{d}\mu_{\mathrm{F}}(x)}{x}$.

*Moreover, if any of the above equivalent conditions holds, then $\{\gamma_{n-1}\}_{n=0}^\infty$ is H-determinate.*

**Proof.** Let $\{\mu_t\}_{t\in\mathbb{R}\cup\{\infty\}}$ be the parametrization of N-extremal measures of $\{\gamma_n\}_{n=0}^\infty$ given by (2.1.3).

(i) ⇒ (ii) Note that $\gamma_{-1} > 0$ (otherwise $\gamma_n = 0$ for all $n \in \mathbb{Z}_+$). By the S-determinacy of $\{\gamma_{n-1}\}_{n=0}^\infty$ and Lemma 2.1.1, there is a unique S-representing measure $\rho$ of $\{\gamma_n\}_{n=0}^\infty$ such that $\int_0^\infty \frac{1}{x}\,\mathrm{d}\rho(x) \leqslant \gamma_{-1}$. In view of Theorem 2.2.1, we have

$$t_0 \overset{(2.1.3)}{=} \int_0^\infty \frac{1}{x}\,\mathrm{d}\mu_{\mathrm{F}}(x) \leqslant \int_0^\infty \frac{1}{x}\,\mathrm{d}\rho(x) \leqslant \gamma_{-1} \overset{(2.1.3)}{=} \int_0^\infty \frac{1}{x}\,\mathrm{d}\mu_{\gamma_{-1}}(x), \tag{2.2.6}$$

which, by (2.1.6), implies that $\mu_{\gamma_{-1}}$ is an S-representing measure of $\{\gamma_n\}_{n=0}^\infty$. Since, by (2.2.6), $\mu_{\mathrm{F}}$ and $\mu_{\gamma_{-1}}$ satisfy inequality (2.1.1), we conclude that $\mu_{\mathrm{F}} = \rho = \mu_{\gamma_{-1}}$. This gives (ii).

(ii) $\Rightarrow$ (i) If $\{\gamma_{n-1}\}_{n=0}^{\infty}$ were not S-determinate, then by Lemma 2.1.1, there would exist an S-representing measure $\rho$ of $\{\gamma_n\}_{n=0}^{\infty}$ such that $\rho \neq \mu_{\mathrm{F}}$ and

$$\int_0^\infty \frac{1}{x}\,\mathrm{d}\rho(x) \leqslant \int_0^\infty \frac{1}{x}\,\mathrm{d}\mu_{\mathrm{F}}(x),$$

which would contradict (2.2.1).

If $\{\gamma_{n-1}\}_{n=0}^{\infty}$ is S-determinate, then by (2.1.2) with $\mu = \mu_{\mathrm{F}}$ we see that $\mathrm{d}\tau(x) := \frac{1}{x}\,\mathrm{d}\mu_{\mathrm{F}}(x)$ is an S-representing measure of $\{\gamma_{n-1}\}_{n=0}^{\infty}$ such that $0 \notin \operatorname{supp}(\tau)$ (because $0 \notin \operatorname{supp}(\mu_{\mathrm{F}})$). Hence the "moreover" part follows from Lemma 2.2.5. $\square$

We are now ready to state a result which is the main tool for constructing an operator with properties mentioned in the title of the paper.

**Theorem 2.2.7.** *Suppose that $\{\gamma_n\}_{n=0}^{\infty}$ is an S-indeterminate Stieltjes moment sequence, $\mu_{\mathrm{F}}$ is its Friedrichs measure and $\gamma_{-1}$ is a nonnegative real number. Then the following assertions hold.*

(i) *If $\gamma_{-1} < \int_0^\infty \frac{1}{x}\,\mathrm{d}\mu_{\mathrm{F}}(x)$, then $\{\gamma_{n-1}\}_{n=0}^{\infty}$ is not a Stieltjes moment sequence.*
(ii) *If $\gamma_{-1} = \int_0^\infty \frac{1}{x}\,\mathrm{d}\mu_{\mathrm{F}}(x)$, then $\{\gamma_{n-1}\}_{n=0}^{\infty}$ is an H-determinate Stieltjes moment sequence.*
(iii) *If $\gamma_{-1} > \int_0^\infty \frac{1}{x}\,\mathrm{d}\mu_{\mathrm{F}}(x)$, then $\{\gamma_{n-1}\}_{n=0}^{\infty}$ is an S-indeterminate Stieltjes moment sequence.*

**Proof.** Assertions (i) and (ii) follow from Lemma 2.1.1 and Theorems 2.2.1 and 2.2.6.

(iii) By Lemma 2.1.1, the sequence $\{\gamma_{n-1}\}_{n=0}^{\infty}$ is a Stieltjes moment sequence. In view of (2.1.3) and (2.1.6), the measures $\mu_{\mathrm{F}}$ and $\mu_{\gamma_{-1}}$ are two distinct S-representing measures of $\{\gamma_n\}_{n=0}^{\infty}$ which satisfy (2.1.1). Hence, by Lemma 2.1.1, $\{\gamma_{n-1}\}_{n=0}^{\infty}$ is an S-indeterminate Stieltjes moment sequence. $\square$

**Corollary 2.2.8.** *Let $\{\gamma_n\}_{n=0}^{\infty}$ be an S-indeterminate Stieltjes moment sequence and let $\mu_{\mathrm{F}}$ be the Friedrichs measure of $\{\gamma_n\}_{n=0}^{\infty}$. Then*

$$\int_0^\infty \frac{1}{x}\,\mathrm{d}\mu_{\mathrm{F}}(x) = \min\bigl\{\gamma_{-1} \in (0,\infty)\colon\ \forall n \geqslant 0\ \det[\gamma_{i+j-1}]_{i,j=0}^{n} > 0\bigr\}. \tag{2.2.7}$$

**Proof.** Set $t_0 = \int_0^\infty \frac{1}{x}\,\mathrm{d}\mu_{\mathrm{F}}(x)$. It follows from Theorem 2.2.7 that

$$t_0 = \min\bigl\{\gamma_{-1} \in (0,\infty)\colon\ \{\gamma_{n-1}\}_{n=0}^{\infty} \text{ is a Stieltjes moment sequence}\bigr\}.$$

Applying the Stieltjes and Hamburger theorems (cf. [5, Theorems 6.2.5 and 6.2.2]) and using the fact that $\{\gamma_n\}_{n=0}^{\infty}$ is a Hamburger moment sequence, we deduce that

$$t_0 = \min\bigl\{\gamma_{-1} \in (0,\infty)\colon\ \{\gamma_{n-1}\}_{n=0}^{\infty} \text{ is a Hamburger moment sequence}\bigr\}.$$

This equality, when combined with [33, Theorem 1.2] and the fact that $\{\gamma_{n-1}\}_{n=0}^{\infty}$ can never have a finitely supported H-representing measure completes the proof. $\square$

### 2.3. Peculiar Stieltjes moment sequences

Our main objective here is to construct S-indeterminate Stieltjes moment sequences with specific properties that will be used later to build non-hyponormal operators generating Stieltjes moment sequences.

**Example 2.3.1.** Fix $\kappa \in \mathbb{Z}_+ \sqcup \{\infty\}$. We will indicate a system $\{\gamma_n\}_{n=-\kappa}^{\infty}$ of positive real numbers which has the following properties:

(i) $\gamma_0 = 1$,
(ii) there exists a positive Borel measure $\nu$ on $(0, \infty)$ such that

$$\gamma_n = \int_0^\infty x^n \,\mathrm{d}\nu(x), \quad n \in \mathbb{Z},\ n \geqslant -\kappa,$$

(iii) $\{\gamma_{n+1}\}_{n=0}^{\infty}$ is an S-indeterminate Stieltjes moment sequence,
(iv) there exists an S-representing measure $\rho$ of $\{\gamma_{n+1}\}_{n=0}^{\infty}$ such that

$$\operatorname{supp}(\rho) \text{ has no accumulation point in } (0, \infty), \tag{2.3.1}$$

$$0 < \int_0^\infty \frac{1}{x^n} \,\mathrm{d}\rho(x) < \infty, \quad n = 1, \dots, \kappa + 1, \tag{2.3.2}$$

and

$$\int_0^\infty \frac{1}{x} \,\mathrm{d}\rho(x) > 1. \tag{2.3.3}$$

What is more, we can always construct a system $\{\gamma_n\}_{n=-\kappa}^{\infty}$ of positive real numbers which satisfies the conditions (i) to (iv) and which has the property that the sequence $\{\gamma_n\}_{n=0}^{\infty}$ is either H-determinate or S-indeterminate according to our needs.

For this purpose, we fix $q \in (0, 1)$ and define

$$\zeta_n = q^{-\frac{1}{2}n^2}, \quad n \in \mathbb{Z}.$$

It is easily seen that for every $\theta \in [-1, 1]$,

$$\zeta_n = \int_0^\infty x^n \omega_\theta(x) \,\mathrm{d}x, \quad n \in \mathbb{Z},$$

where the density function $\omega_\theta$ is given by

$$\omega_\theta(x) = \frac{1}{\sqrt{2\pi}\,\sigma} x^{-1} \exp\biggl(-\frac{(\log x)^2}{2\sigma^2}\biggr)\biggl(1 + \theta \sin\biggl(\frac{2\pi}{\sigma^2} \log x\biggr)\biggr), \quad x \in (0, \infty),$$

with $\sigma = \sqrt{-\log q}$. This means that for every $l \in \mathbb{Z}$, the sequence $\{\zeta_{n+l}\}_{n=0}^{\infty}$ is an S-indeterminate Stieltjes moment sequence. This is a famous example due to Stieltjes (cf. [35]). It was noticed much later by Chihara [12] and Leipnik [26] (see also [3]) that for every $a \in (0, \infty)$, the Borel probability measure $\lambda_a$ defined by

$$\lambda_a = \frac{1}{L(a)} \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2} \delta_{aq^k}, \quad L(a) = \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2}, \tag{2.3.4}$$

solves the moment problem

$$\zeta_n = \int_0^{\infty} x^n \, \mathrm{d}\lambda_a(x), \quad n \in \mathbb{Z}. \tag{2.3.5}$$

Therefore, for every fixed $l \in \mathbb{Z}$, the absolutely continuous measures $x^l \omega_\theta(x)\,\mathrm{d}x$, $\theta \in [-1, 1]$, and the pure point measures $x^l \,\mathrm{d}\lambda_a(x)$, $a \in (0, \infty)$, are S-representing measures of $\{\zeta_{n+l}\}_{n=0}^{\infty}$. Since 0 is an accumulation point of the closed support of each of these measures, we conclude that neither of them is N-extremal.

Let $\{\mu_t\}_{t \in \mathbb{R} \cup \{\infty\}}$ be the set of all N-extremal measures of $\{\zeta_n\}_{n=0}^{\infty}$ (cf. (2.1.3)) and let $\mu_{\mathrm{F}}$ be the Friedrichs measure of $\{\zeta_n\}_{n=0}^{\infty}$. Set $t_0 = \int_0^{\infty} \frac{1}{x} \,\mathrm{d}\mu_{\mathrm{F}}(x)$. Take $t \in [t_0, \infty)$ and define the system $\{\gamma_n(t)\}_{n=-\kappa}^{\infty}$ by

$$\gamma_n(t) = \begin{cases} t^{-1} \int_0^{\infty} x^{n-1} \,\mathrm{d}\mu_t(x) & \text{if } -\kappa \leqslant n \leqslant 0, \\ t^{-1} \zeta_{n-1} & \text{if } n \geqslant 1. \end{cases}$$

By (2.1.7), the above definition is correct. It is clear that the system $\{\gamma_n(t)\}_{n=-\kappa}^{\infty}$ satisfies the conditions (i) and (ii) with a measure $\nu$ given by $\mathrm{d}\nu(x) = t^{-1} \frac{1}{x} \,\mathrm{d}\mu_t(x)$. Since $\gamma_{n+1}(t) = t^{-1}\zeta_n$ for all $n \in \mathbb{Z}_+$, we see that the system $\{\gamma_n(t)\}_{n=-\kappa}^{\infty}$ satisfies the condition (iii) and that for every $s \in (t, \infty)$, $\rho_s := t^{-1}\mu_s$ is an S-representing measure of $\{\gamma_{n+1}(t)\}_{n=0}^{\infty}$ which satisfies (2.3.1) and (2.3.2) (see (2.1.7)). Moreover, we have

$$\int_0^{\infty} \frac{1}{x} \,\mathrm{d}\rho_s(x) = t^{-1} \int_0^{\infty} \frac{1}{x} \,\mathrm{d}\mu_s(x) \overset{(2.1.3)}{=} t^{-1} s > 1, \quad s \in (t, \infty),$$

which means that $\rho_s$ satisfies (2.3.3) for every $s \in (t, \infty)$. It follows from (2.1.3) and Theorem 2.2.7 that the Stieltjes moment sequence $\{\gamma_n(t)\}_{n=0}^{\infty}$ is H-determinate for $t = t_0$ and S-indeterminate for $t \in (t_0, \infty)$.

Since the closed supports of the measures $\rho_s$, $s \in (t, \infty)$, are not explicitly known, we will provide other examples of measures satisfying the conditions (2.3.1), (2.3.2) and (2.3.3), the closed supports of which are precisely given. According to Theorem 2.2.1 and the fact that $\lambda_a$ is not N-extremal, we have

$$t_0 = \int_0^{\infty} \frac{1}{x} \,\mathrm{d}\mu_{\mathrm{F}}(x) < \int_0^{\infty} \frac{1}{x} \,\mathrm{d}\lambda_a(x) \overset{(2.3.5)}{=} \zeta_{-1}, \quad a \in (0, \infty),$$

which means that $[t_0, \zeta_{-1}) \neq \emptyset$. Take $t \in [t_0, \zeta_{-1})$ and set $\tilde{\rho}_a = \frac{1}{t}\lambda_a$ for $a \in (0, \infty)$. Using (2.3.5), we can easily verify that for every $a \in (0, \infty)$, $\tilde{\rho}_a$ is an S-representing measure of $\{\gamma_{n+1}(t)\}_{n=0}^{\infty}$ which satisfies (2.3.1), (2.3.2) and (2.3.3). By (2.3.4), $\operatorname{supp}(\tilde{\rho}_a) = \{aq^k\colon k \in \mathbb{Z}\} \cup \{0\}$ for every $a \in (0, \infty)$.

Note that the constant $t_0$ which plays an essential role in Example 2.3.1 can be estimated by using (2.2.7).

## 3. Relating moments to directed trees

### 3.1. Weighted shifts on directed trees

Let $\mathscr{T} = (V, E)$ be a directed tree ($V$ and $E$ stand for the sets of vertices and edges of $\mathscr{T}$, respectively). If $\mathscr{T}$ has a root, which will always be denoted by $\mathsf{root}$, then we write $V^\circ := V \setminus \{\mathsf{root}\}$; otherwise, we put $V^\circ = V$. Set

$$\mathsf{Chi}(u) = \big\{v \in V\colon (u, v) \in E\big\}, \quad u \in V.$$

A member of $\mathsf{Chi}(u)$ is called a *child* (or *successor*) of $u$. For every vertex $u \in V^\circ$ there exists a unique vertex, denoted by $\mathsf{par}(u)$, such that $(\mathsf{par}(u), u) \in E$. The correspondence $u \mapsto \mathsf{par}(u)$ is a partial function from $V$ to $V$. For an integer $n \geqslant 1$, the $n$-fold composition of the partial function $\mathsf{par}$ with itself will be denoted by $\mathsf{par}^n$. Let $\mathsf{par}^0$ stand for the identity map on $V$. We call $\mathscr{T}$ *leafless* if $V = V'$, where $V' := \{u \in V\colon \mathsf{Chi}(u) \neq \emptyset\}$. It is clear that every leafless directed tree is infinite. A vertex $u \in V$ is said to be a *branching vertex* of $\mathscr{T}$ if $\mathsf{Chi}(u)$ consists of at least two vertices. If $W \subseteq V$, we put $\mathsf{Chi}(W) = \bigcup_{v \in W} \mathsf{Chi}(v)$ and $\mathsf{Des}(W) = \bigcup_{n=0}^{\infty} \mathsf{Chi}^{\langle n \rangle}(W)$, where $\mathsf{Chi}^{\langle 0 \rangle}(W) = W$ and $\mathsf{Chi}^{\langle n+1 \rangle}(W) = \mathsf{Chi}(\mathsf{Chi}^{\langle n \rangle}(W))$ for all integers $n \geqslant 0$. For $u \in V$, we set $\mathsf{Chi}^{\langle n \rangle}(u) = \mathsf{Chi}^{\langle n \rangle}(\{u\})$ and $\mathsf{Des}(u) = \mathsf{Des}(\{u\})$. It follows from [20, Proposition 2.1.2] and [7, Proposition 2.2.1] that

$$V^\circ = \bigsqcup_{u \in V} \mathsf{Chi}(u), \tag{3.1.1}$$

$$\mathsf{Chi}^{\langle n+1 \rangle}(u) = \bigsqcup_{v \in \mathsf{Chi}(u)} \mathsf{Chi}^{\langle n \rangle}(v), \quad n \in \mathbb{Z}_+,\ u \in V, \tag{3.1.2}$$

where the symbol $\bigsqcup$ is reserved to denote pairwise disjoint union of sets.

Given a directed tree $\mathscr{T}$, we tacitly assume that $V$ and $E$ stand for the sets of vertices and edges of $\mathscr{T}$, respectively. Denote by $\ell^2(V)$ the complex Hilbert space of all square summable complex functions on $V$ with the standard inner product. For $u \in V$, we define $e_u$ to be the characteristic function of the one-point set $\{u\}$. The family $\{e_u\}_{u \in V}$ is an orthonormal basis of $\ell^2(V)$. We write $\mathscr{E}_V$ for the linear span of the set $\{e_u\colon u \in V\}$.

Given $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$, we define the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ by

$$\mathcal{D}(S_{\boldsymbol{\lambda}}) = \big\{f \in \ell^2(V)\colon \Lambda_{\mathscr{T}} f \in \ell^2(V)\big\},$$

$$S_{\boldsymbol{\lambda}} f = \Lambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\boldsymbol{\lambda}}),$$

where $\Lambda_{\mathscr{T}}$ is the map defined on functions $f : V \to \mathbb{C}$ via

$$(\Lambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\mathsf{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}. \end{cases} \tag{3.1.3}$$

The operator $S_{\boldsymbol{\lambda}}$ is called a *weighted shift* on the directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$. Combining Propositions 3.1.2, 3.1.3(iii) and 3.1.7 of [20], we get the ensuing properties of $S_{\boldsymbol{\lambda}}$ (from now on, we adopt the convention that $\sum_{v \in \emptyset} x_v = 0$).

**Proposition 3.1.1.** *Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$. Then the following assertions hold.*

(i) *$S_{\boldsymbol{\lambda}}$ is closed.*
(ii) *$e_u$ is in $\mathcal{D}(S_{\boldsymbol{\lambda}})$ if and only if $\sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 < \infty$; if $e_u \in \mathcal{D}(S_{\boldsymbol{\lambda}})$, then*

$$S_{\boldsymbol{\lambda}} e_u = \sum_{v \in \mathsf{Chi}(u)} \lambda_v e_v \quad \text{and} \quad \|S_{\boldsymbol{\lambda}} e_u\|^2 = \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2.$$

(iii) *$S_{\boldsymbol{\lambda}}$ is injective if and only if $\mathscr{T}$ is leafless and $\sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 > 0$ for every $u \in V$.*

Let us now recall a characterization of hyponormality of weighted shifts on leafless directed trees with nonzero weights.

**Theorem 3.1.2.** *(See [20, Theorem 5.1.2 and Remark 5.1.5].) Let $S_{\boldsymbol{\lambda}}$ be a densely defined weighted shift on a leafless directed tree $\mathscr{T}$ with nonzero weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$. Then $S_{\boldsymbol{\lambda}}$ is hyponormal if and only if*

$$\sum_{v \in \mathsf{Chi}(u)} \frac{|\lambda_v|^2}{\|S_{\boldsymbol{\lambda}} e_v\|^2} \leqslant 1, \quad u \in V. \tag{3.1.4}$$

The following lemma relates representing measures of Stieltjes moment sequences induced by basic vectors coming from the parent and its children. Inequality (3.1.6) below will be referred to as the *consistency condition* at $u$.

**Lemma 3.1.3.** *(See [7, Lemma 4.1.3].) Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$ such that $\mathscr{E}_V \subseteq \mathcal{D}^\infty(S_{\boldsymbol{\lambda}})$. Let $u \in V'$. Suppose that for every $v \in \mathsf{Chi}(u)$ the sequence $\{\|S_{\boldsymbol{\lambda}}^n e_v\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence with a representing measure $\mu_v$. Consider the following two conditions*[7]:

$$\big\{\big\|S_{\boldsymbol{\lambda}}^n e_u\big\|^2\big\}_{n=0}^\infty \text{ is a Stieltjes moment sequence}, \tag{3.1.5}$$

$$\sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{x} \, \mathrm{d}\mu_v(x) \leqslant 1. \tag{3.1.6}$$

[7] We adhere to the standard convention that $0 \cdot \infty = 0$; see also footnote 4.

*Then the following assertions are valid.*

(i) *If* (3.1.6) *holds, then so does* (3.1.5) *and the positive Borel measure $\mu_u$ on $\mathbb{R}_+$ defined by*

$$\mu_u(\sigma) = \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \int_\sigma \frac{1}{x}\,\mathrm{d}\mu_v(x) + \varepsilon_u \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{3.1.7}$$

*with*

$$\varepsilon_u = 1 - \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{x}\,\mathrm{d}\mu_v(x), \tag{3.1.8}$$

*is a representing measure of* $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$.

(ii) *If* (3.1.5) *holds and* $\{\|S_{\boldsymbol\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$ *is determinate, then* (3.1.6) *holds, the Stieltjes moment sequence* $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ *is determinate and its unique representing measure $\mu_u$ is given by* (3.1.7) *and* (3.1.8).

### *3.2. Generating Stieltjes moments on directed trees*

We begin by recalling the action of powers of $S_{\boldsymbol\lambda}$ on basic vectors $e_u$, $u \in V$.

**Lemma 3.2.1.** *(See [7, Lemma 2.3.1].) Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$. Then the following assertions hold for all $u \in V$ and $n \in \mathbb{Z}_+$.*

(i) $e_u \in \mathcal{D}(S_{\boldsymbol\lambda}^n)$ *if and only if* $\sum_{v \in \mathsf{Chi}^{\langle m \rangle}(u)} |\lambda_{u|v}|^2 < \infty$ *for all integers $m$ such that $1 \leqslant m \leqslant n$.*

(ii) *If* $e_u \in \mathcal{D}(S_{\boldsymbol\lambda}^n)$, *then*

$$S_{\boldsymbol\lambda}^n e_u = \sum_{v \in \mathsf{Chi}^{\langle n \rangle}(u)} \lambda_{u|v} e_v, \tag{3.2.1}$$

$$\|S_{\boldsymbol\lambda}^n e_u\|^2 = \sum_{v \in \mathsf{Chi}^{\langle n \rangle}(u)} |\lambda_{u|v}|^2, \tag{3.2.2}$$

*where*

$$\lambda_{u|v} = \begin{cases} 1 & \text{if } v = u, \\ \prod_{j=0}^{n-1} \lambda_{\mathsf{par}^j(v)} & \text{if } v \in \mathsf{Chi}^{\langle n \rangle}(u),\ n \geqslant 1. \end{cases} \tag{3.2.3}$$

One can deduce from (3.2.3) that

$$\lambda_{\mathsf{par}(v)|w} = \lambda_v \lambda_{v|w}, \quad v \in V^\circ,\ w \in \mathsf{Des}(v). \tag{3.2.4}$$

The above lemma enables us to describe the powers of $S_{\boldsymbol\lambda}$. Below we write $\sum^{\oplus}$ for the sum of a series whose terms are mutually orthogonal.

**Theorem 3.2.2.** *Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then the following assertions hold for any $n \in \mathbb{Z}_+$.*

(i) *A function $f : V \to \mathbb{C}$ belongs to $\mathcal{D}(S_{\boldsymbol\lambda}^n)$ if and only if*

$$\sum_{u\in V} \big|f(u)\big|^2 \Bigg(\sum_{j=0}^{n} \sum_{v\in \mathsf{Chi}^{\langle j\rangle}(u)} |\lambda_{u|v}|^2\Bigg) < \infty, \tag{3.2.5}$$

*with the usual convention that $0\cdot\infty = 0$.*

(ii) *If $f \in \mathcal{D}(S_{\boldsymbol\lambda}^n)$, then $e_u \in \mathcal{D}(S_{\boldsymbol\lambda}^n)$ for every $u \in V$ such that $f(u) \neq 0$, and*

$$S_{\boldsymbol\lambda}^n f = \sum_{u\in V:\ f(u)\neq 0}^{\oplus} f(u) S_{\boldsymbol\lambda}^n e_u, \quad f \in \mathcal{D}\big(S_{\boldsymbol\lambda}^n\big), \tag{3.2.6}$$

$$\big\|S_{\boldsymbol\lambda}^n f\big\|^2 = \sum_{u\in V:\ f(u)\neq 0} \big|f(u)\big|^2 \big\|S_{\boldsymbol\lambda}^n e_u\big\|^2, \quad f \in \mathcal{D}\big(S_{\boldsymbol\lambda}^n\big). \tag{3.2.7}$$

(iii) *If $\mathscr{E}_V \subseteq \mathcal{D}(S_{\boldsymbol\lambda}^n)$, then $\mathscr{E}_V$ is a core of $S_{\boldsymbol\lambda}^n$.*

(iv) *$S_{\boldsymbol\lambda}^n$ is densely defined if and only if $\mathscr{E}_V \subseteq \mathcal{D}(S_{\boldsymbol\lambda}^n)$.*

**Proof.** (ii) We proceed by induction on $n$. The case of $n = 0$ is obvious. Assume that assertion (ii) holds for a fixed $n \in \mathbb{Z}_+$. Take $f$ in $\mathcal{D}(S_{\boldsymbol\lambda}^{n+1})$. It follows from (3.1.1) that

$$\big\{v \in V^\circ:\ f\big(\mathsf{par}(v)\big) \neq 0,\ \lambda_v \neq 0\big\} = \bigsqcup_{u\in V:\ f(u)\neq 0} \big\{v \in \mathsf{Chi}(u):\ \lambda_v \neq 0\big\}. \tag{3.2.8}$$

Applying the induction hypothesis to the function $S_{\boldsymbol\lambda} f$ which clearly belongs to $\mathcal{D}(S_{\boldsymbol\lambda}^n)$, we obtain

$$\begin{aligned}
S_{\boldsymbol\lambda}^{n+1} f = S_{\boldsymbol\lambda}^n (S_{\boldsymbol\lambda} f) &\overset{(3.2.6)}{=} \sum_{v\in V:\ (S_{\boldsymbol\lambda} f)(v)\neq 0}^{\oplus} (S_{\boldsymbol\lambda} f)(v) S_{\boldsymbol\lambda}^n e_v \\
&\overset{(3.1.3)}{=} \sum_{v\in V^\circ:\ f(\mathsf{par}(v))\neq 0,\, \lambda_v\neq 0}^{\oplus} \lambda_v f\big(\mathsf{par}(v)\big) S_{\boldsymbol\lambda}^n e_v \\
&\overset{(3.2.8)}{=} \sum_{u\in V:\ f(u)\neq 0}^{\oplus} f(u) \Bigg(\sum_{v\in \mathsf{Chi}(u):\ \lambda_v\neq 0}^{\oplus} \lambda_v S_{\boldsymbol\lambda}^n e_v\Bigg) \\
&\overset{(3.2.1)}{=} \sum_{u\in V:\ f(u)\neq 0}^{\oplus} f(u) \Bigg(\sum_{v\in \mathsf{Chi}(u):\ \lambda_v\neq 0}^{\oplus} \sum_{w\in \mathsf{Chi}^{\langle n\rangle}(v)}^{\oplus} \lambda_v \lambda_{v|w} e_w\Bigg) \\
&\overset{(3.2.4)}{=} \sum_{u\in V:\ f(u)\neq 0}^{\oplus} f(u) \Bigg(\sum_{v\in \mathsf{Chi}(u):\ \lambda_v\neq 0}^{\oplus} \sum_{w\in \mathsf{Chi}^{\langle n\rangle}(v)}^{\oplus} \lambda_{u|w} e_w\Bigg)
\end{aligned}$$

$$\stackrel{(3.1.2)}{=} \sum_{u\in V:\ f(u)\neq 0}^{\oplus} f(u)\biggl(\sum_{w\in \mathsf{Chi}^{\langle n+1\rangle}(u):\ \lambda_{\mathsf{par}^n(w)}\neq 0}^{\oplus} \lambda_{u|w} e_w\biggr)$$

$$\stackrel{(3.2.3)}{=} \sum_{u\in V:\ f(u)\neq 0}^{\oplus} f(u)\biggl(\sum_{w\in \mathsf{Chi}^{\langle n+1\rangle}(u)}^{\oplus} \lambda_{u|w} e_w\biggr), \tag{3.2.9}$$

where the penultimate inequality is valid because the vectors $\{e_w\}_{w\in V}$ are pairwise orthogonal. Since the series in (3.2.9) are orthogonal, we deduce that

$$\sum_{u\in V} \bigl|f(u)\bigr|^2 \sum_{v\in \mathsf{Chi}^{\langle n+1\rangle}(u)} |\lambda_{u|v}|^2 = \bigl\|S_{\boldsymbol\lambda}^{n+1} f\bigr\|^2, \tag{3.2.10}$$

and hence that

$$\sum_{v\in \mathsf{Chi}^{\langle n+1\rangle}(u)} |\lambda_{u|v}|^2 < \infty \quad \text{for every } u\in V \text{ such that } f(u)\neq 0. \tag{3.2.11}$$

As $f$ belongs to $\mathcal{D}(S_{\boldsymbol\lambda}^n)$, we infer from Lemma 3.2.1(i) and the induction hypothesis applied to $f$ that $\sum_{v\in \mathsf{Chi}^{\langle m\rangle}(u)} |\lambda_{u|v}|^2 < \infty$ for $m = 0,\dots,n$ and for every $u\in V$ such that $f(u)\neq 0$. But this, together with (3.2.11) and Lemma 3.2.1(i), implies that $e_u \in \mathcal{D}(S_{\boldsymbol\lambda}^{n+1})$ for all $u\in V$ such that $f(u)\neq 0$. Combining Lemma 3.2.1(ii) with (3.2.9) and (3.2.10), we obtain (3.2.6) and (3.2.7) with $n+1$ in place of $n$, which completes the induction argument. Therefore, (ii) holds.

(i) It follows from (ii) and (3.2.2) that the "only if" part of assertion (i) holds for all $n\in\mathbb{Z}_+$. To prove the reverse implication in (i), we proceed by induction on $n$. The case of $n=0$ is obvious. Assume that for a fixed $n\in\mathbb{Z}_+$, the "if" part of assertion (i) holds. Let $f: V\to\mathbb{C}$ be a function satisfying (3.2.5) with $n+1$ in place of $n$. Since $n+1\geqslant 1$, this implies that

$$\sum_{u\in V} \bigl|f(u)\bigr|^2 \biggl(1 + \sum_{v\in \mathsf{Chi}(u)} |\lambda_v|^2\biggr) < \infty,$$

which in view of [20, Proposition 3.1.3(i)] yields $f\in\mathcal{D}(S_{\boldsymbol\lambda})$. Note that

$$\sum_{u\in V} \bigl|(S_{\boldsymbol\lambda} f)(u)\bigr|^2 \biggl(\sum_{j=0}^{n} \sum_{v\in \mathsf{Chi}^{\langle j\rangle}(u)} |\lambda_{u|v}|^2\biggr)$$

$$\stackrel{(3.1.3)}{=} \sum_{u\in V^\circ} \bigl|\lambda_u f\bigl(\mathsf{par}(u)\bigr)\bigr|^2 \biggl(\sum_{j=0}^{n} \sum_{v\in \mathsf{Chi}^{\langle j\rangle}(u)} |\lambda_{u|v}|^2\biggr)$$

$$\stackrel{(3.1.1)}{=} \sum_{x\in V} \sum_{u\in \mathsf{Chi}(x)} |\lambda_u|^2 \bigl|f(x)\bigr|^2 \biggl(\sum_{j=0}^{n} \sum_{v\in \mathsf{Chi}^{\langle j\rangle}(u)} |\lambda_{u|v}|^2\biggr)$$

$$= \sum_{x\in V} \bigl|f(x)\bigr|^2 \sum_{j=0}^{n} \sum_{u\in \mathsf{Chi}(x)} \sum_{v\in \mathsf{Chi}^{\langle j\rangle}(u)} |\lambda_u \lambda_{u|v}|^2$$

$$\overset{(3.2.4)}{=} \sum_{x\in V} \big|f(x)\big|^2 \sum_{j=0}^{n} \sum_{u\in\mathsf{Chi}(x)} \sum_{v\in\mathsf{Chi}^{\langle j\rangle}(u)} |\lambda_{x|v}|^2$$

$$\overset{(3.1.2)}{=} \sum_{x\in V} \big|f(x)\big|^2 \sum_{j=0}^{n} \sum_{v\in\mathsf{Chi}^{\langle j+1\rangle}(x)} |\lambda_{x|v}|^2 < \infty.$$

Hence, by the induction hypothesis, we see that $S_{\boldsymbol\lambda} f$ is in $\mathcal{D}(S_{\boldsymbol\lambda}^n)$. This completes the proof of (i).

(iii) Suppose that $\mathscr{E}_V \subseteq \mathcal{D}(S_{\boldsymbol\lambda}^n)$. Thus, by (i), (ii) and (3.2.2), the domain and the graph norm of $S_{\boldsymbol\lambda}^n$ are given by the following formulas:

$$\mathcal{D}\big(S_{\boldsymbol\lambda}^n\big) = \left\{ f \in \mathbb{C}^V \colon \sum_{u\in V} \big|f(u)\big|^2 \left( \sum_{j=0}^{n} \big\| S_{\boldsymbol\lambda}^j e_u \big\|^2 \right) < \infty \right\},$$

$$\|f\|^2 + \big\|S_{\boldsymbol\lambda}^n f\big\|^2 = \sum_{u\in V} \big|f(u)\big|^2 \big(1 + \big\|S_{\boldsymbol\lambda}^n e_u\big\|^2\big), \quad f \in \mathcal{D}\big(S_{\boldsymbol\lambda}^n\big).$$

Since $\mathscr{E}_V$, being the set of all complex functions on $V$ which vanish off finite sets, is dense in the weighted $\ell^2$-space on $V$ with weights $\{1 + \|S_{\boldsymbol\lambda}^n e_u\|^2\}_{u\in V}$ and $\mathcal{D}(S_{\boldsymbol\lambda}^n)$ is between these two spaces, we see that $\mathscr{E}_V$ is a core of $S_{\boldsymbol\lambda}^n$.

(iv) Since $\mathscr{E}_V$ is dense in $\ell^2(V)$, we see that the "if" part of assertion (iv) is valid. Suppose that the reverse implication in (iv) does not hold. Then $S_{\boldsymbol\lambda}^n$ is densely defined and $e_u \notin \mathcal{D}(S_{\boldsymbol\lambda}^n)$ for some $u \in V$. Hence, by (ii), $f(u) = 0$ for every $f \in \mathcal{D}(S_{\boldsymbol\lambda}^n)$. This and the density of $\mathcal{D}(S_{\boldsymbol\lambda}^n)$ in $\ell^2(V)$ imply that $e_u \perp \ell^2(V)$, which is a contradiction. This completes the proof of Theorem 3.2.2. □

Regarding Theorem 3.2.2, we note that classical unilateral and bilateral weighted shifts are always closed, but their higher powers may not be closed.

**Corollary 3.2.3.** *Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then the following conditions are equivalent*:

(i) $\mathscr{E}_V \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$,
(ii) $\mathcal{D}^\infty(S_{\boldsymbol\lambda})$ *is dense in* $\ell^2(V)$,
(iii) $S_{\boldsymbol\lambda}^n$ *is densely defined for every* $n \in \mathbb{Z}_+$.

*Moreover, if any of the above equivalent conditions holds, then $\mathcal{D}^\infty(S_{\boldsymbol\lambda})$ is a core of $S_{\boldsymbol\lambda}^n$ for every $n \in \mathbb{Z}_+$.*

It is worth pointing out that the equivalence (ii) ⇔ (iii) which appears in Corollary 3.2.3 remains true in the class of composition operators in $L^2$-spaces (cf. [9]).

We conclude this section by proving that a weighted shift $S_{\boldsymbol\lambda}$ on a directed tree generates Stieltjes moment sequences if and only if each basic vector $e_u$, $u \in V$, induces a Stieltjes moment sequence.

**Theorem 3.2.4.** *Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Suppose that $\mathscr{E}_V \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$ and $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $u \in V$. Then $\{\|S_{\boldsymbol\lambda}^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $f \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$.*

**Proof.** Since, by the Stieltjes theorem (cf. [5, Theorem 6.2.5]), the class of Stieltjes moment sequences is closed under both, the operation of taking linear combinations with nonnegative coefficients and the operation of taking pointwise limits, we can infer Theorem 3.2.4 from (3.2.7). □

## 4. Examples of exotic non-hyponormal operators

### 4.1. General scheme

In this section we introduce a class of weighted shifts on an enumerable leafless directed tree with one branching vertex. Such a directed tree (which is, roughly speaking, one step more complicated than the directed trees involved in the definitions of classical weighted shifts) can be modelled as follows (cf. [20, (6.2.10)]). Given $\eta, \kappa \in \mathbb{Z}_+ \sqcup \{\infty\}$ with $\eta \geqslant 2$, we define the directed tree $\mathscr{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ by

$$\begin{aligned} V_{\eta,\kappa} &= \{-k\colon k \in J_\kappa\} \sqcup \{0\} \sqcup \big\{(i,j)\colon i \in J_\eta,\ j \in \mathbb{N}\big\}, \\ E_{\eta,\kappa} &= E_\kappa \sqcup \big\{\big(0,(i,1)\big)\colon i \in J_\eta\big\} \sqcup \big\{\big((i,j),(i,j+1)\big)\colon i \in J_\eta,\ j \in \mathbb{N}\big\}, \\ E_\kappa &= \big\{(-k,-k+1)\colon k \in J_\kappa\big\}, \end{aligned}$$

where

$$J_\iota = \{k \in \mathbb{N}\colon k \leqslant \iota\}, \quad \iota \in \mathbb{Z}_+ \sqcup \{\infty\}.$$

Note that 0 is the only branching vertex of $\mathscr{T}_{\eta,\kappa}$ and $V^\circ_{\eta,\kappa} = V_{\eta,\kappa} \setminus \{-\kappa\}$.

Let $\{\gamma_n\}_{n=-\kappa}^\infty$ be a system of positive real numbers such that

$$\gamma_0 = 1, \tag{4.1.1}$$

$$\gamma_n = \int_0^\infty x^n \,\mathrm{d}\nu(x), \quad n \in \mathbb{Z},\ n \geqslant -\kappa, \tag{4.1.2}$$

for some positive Borel measure $\nu$ on $\mathbb{R}_+$ (note that if $\kappa > 0$, then (4.1.2) implies that $\nu(\{0\}) = 0$). It follows from (4.1.2) that

$$\{\gamma_{n-k}\}_{n=0}^\infty \text{ is a Stieltjes moment sequence for every integer } k \leqslant \kappa. \tag{4.1.3}$$

Suppose that there exists an S-representing measure $\rho$ of $\{\gamma_{n+1}\}_{n=0}^\infty$ such that

$$0 < \int_0^\infty \frac{1}{x^n} \,\mathrm{d}\rho(x) < \infty, \quad n \in J_{\kappa+1}, \tag{4.1.4}$$

$$\operatorname{card}\big(\operatorname{supp}(\rho)\big) \geqslant \begin{cases} \eta & \text{if } \eta < \infty, \\ \aleph_0 & \text{if } \eta = \infty. \end{cases} \tag{4.1.5}$$

Let $\{\Omega_i\}_{i=1}^\eta$ be a sequence of pairwise disjoint Borel subsets of $(0,\infty)$ such that

$$\rho(\Omega_i) > 0, \quad i \in J_\eta, \tag{4.1.6}$$

$$\bigsqcup_{i \in J_\eta} \Omega_i = (0, \infty). \tag{4.1.7}$$

Since, by (4.1.4), 0 is not an atom of $\rho$, one can deduce from (4.1.5) that such $\{\Omega_i\}_{i=1}^{\eta}$ always exists (see also Proposition 4.1.2 for the case of $\operatorname{card}(\operatorname{supp}(\rho)) = \aleph_0$). In view of (4.1.6), we can define the sequence $\{\mu_{i,1}\}_{i \in J_\eta}$ of Borel probability measures on $\mathbb{R}_+$ by

$$\mu_{i,1}(\sigma) = \frac{1}{\rho(\Omega_i)} \rho(\Omega_i \cap \sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ i \in J_\eta, \tag{4.1.8}$$

and the family $\{\lambda_{i,j}\colon i \in J_\eta, j \in \mathbb{N}\}$ of positive real numbers by

$$\lambda_{i,j} = \begin{cases} \sqrt{\rho(\Omega_i)} & \text{for } j = 1, \\ \sqrt{\frac{\int_0^\infty x^{j-1}\, \mathrm{d}\mu_{i,1}(x)}{\int_0^\infty x^{j-2}\, \mathrm{d}\mu_{i,1}(x)}} & \text{for } j \geqslant 2, \end{cases} \quad i \in J_\eta. \tag{4.1.9}$$

If $\kappa > 0$, then we define the sequence $\{\lambda_{-k}\}_{k=0}^{\kappa-1}$ of positive real numbers by

$$\lambda_{-k} = \sqrt{\frac{\gamma_{-k}}{\gamma_{-(k+1)}}}, \quad k \in \mathbb{Z}_+,\ 0 \leqslant k < \kappa. \tag{4.1.10}$$

Let $S_{\boldsymbol\lambda}$ be a weighted shift on the directed tree $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ defined by (4.1.9) and (4.1.10) (we adhere to notation $\lambda_{i,j}$ instead of a more formal expression $\lambda_{(i,j)}$). The reader should be aware of the fact that the operator $S_{\boldsymbol\lambda}$ just constructed depends not only on $\{\gamma_n\}_{n=-\kappa}^\infty$ and $\rho$, but also on the partition $\{\Omega_i\}_{i=1}^\eta$ of $(0, \infty)$. Now we can prove some crucial properties of $S_{\boldsymbol\lambda}$.

**Theorem 4.1.1.** *Let $\{\gamma_n\}_{n=-\kappa}^\infty$, $\rho$, $\{\Omega_i\}_{i \in J_\eta}$, $\{\mu_{i,1}\}_{i \in J_\eta}$, $\boldsymbol\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ and $S_{\boldsymbol\lambda}$ be as above. Then the following assertions hold.*

(i) $\mathcal{E}_{V_{\eta,\kappa}} \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$.
(ii) $\{\|S_{\boldsymbol\lambda}^n f\|^2\}_{n=0}^\infty$ *is a Stieltjes moment sequence for every* $f \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$.
(iii) $S_{\boldsymbol\lambda}$ *is paranormal.*
(iv) *The consistency condition* (3.1.6) *holds at* $u = 0$ *if and only if*

$$\int_0^\infty \frac{1}{x}\, \mathrm{d}\rho(x) \leqslant 1. \tag{4.1.11}$$

(v) $S_{\boldsymbol\lambda}$ *is hyponormal if and only if*

$$\sum_{i \in J_\eta} \frac{\lambda_{i,1}^2}{\|S_{\boldsymbol\lambda} e_{i,1}\|^2} \leqslant 1. \tag{4.1.12}$$

(vi) *The following inequality holds*

$$\sum_{i\in J_\eta} \frac{\lambda_{i,1}^2}{\|S_{\boldsymbol\lambda} e_{i,1}\|^2} \leqslant \int_0^\infty \frac{1}{x}\,\mathrm{d}\rho(x). \tag{4.1.13}$$

(vii) *The inequality in* (4.1.13) *turns into equality if and only if for every $i \in J_\eta$, there exists $q_i \in \Omega_i$ such that*

$$\rho(\sigma\cap\Omega_i) = \rho(\Omega_i)\cdot\delta_{q_i}(\sigma), \quad \sigma\in\mathfrak{B}(\mathbb{R}_+),\ i\in J_\eta. \tag{4.1.14}$$

(viii) *If $\{\gamma_{n+1}\}_{n=0}^\infty$ is S-determinate, then $S_{\boldsymbol\lambda}$ is subnormal and* (4.1.11) *holds.*

**Proof.** We prove (i) and (ii) simultaneously. It follows from Lemma 3.2.1(i) that $e_{i,j}\in\mathcal{D}^\infty(S_{\boldsymbol\lambda})$ for all $(i,j)\in\mathsf{Des}(0)\setminus\{0\}$ (we abbreviate $e_{(i,j)}$ to $e_{i,j}$), and

$$\left\|S_{\boldsymbol\lambda}^n e_{i,j}\right\|^2 \overset{(4.1.9)}{=} \int_0^\infty x^n\,\mathrm{d}\mu_{i,j}(x), \quad n\in\mathbb{Z}_+,\ (i,j)\in\mathsf{Des}(0)\setminus\{0\}, \tag{4.1.15}$$

where

$$\mu_{i,j}(\sigma) = \frac{1}{\int_0^\infty x^{j-1}\,\mathrm{d}\mu_{i,1}(x)}\int_\sigma x^{j-1}\,\mathrm{d}\mu_{i,1}(x), \quad \sigma\in\mathfrak{B}(\mathbb{R}_+),\ (i,j)\in\mathsf{Des}(0)\setminus\{0\}.$$

Noting that

$$\begin{aligned}\sum_{v\in\mathsf{Chi}^{\langle n\rangle}(0)}\lambda_{0|v}^2 &= \sum_{i\in J_\eta}\lambda_{0|(i,n)}^2 \overset{(3.2.3)}{=} \sum_{i\in J_\eta}\prod_{j=1}^n\lambda_{i,j}^2\\ &\overset{(4.1.8)\&(4.1.9)}{=} \sum_{i\in J_\eta}\int_{\Omega_i} x^{n-1}\,\mathrm{d}\rho(x) \overset{(4.1.7)}{=} \int_0^\infty x^{n-1}\,\mathrm{d}\rho(x)<\infty, \quad n\geqslant 1,\end{aligned}$$

and applying Lemma 3.2.1, we deduce that $e_0\in\mathcal{D}^\infty(S_{\boldsymbol\lambda})$ and

$$\left\|S_{\boldsymbol\lambda}^{n+1} e_0\right\|^2 = \int_0^\infty x^n\,\mathrm{d}\rho(x), \quad n\in\mathbb{Z}_+. \tag{4.1.16}$$

As $\rho$ is an S-representing measure of $\{\gamma_{n+1}\}_{n=0}^\infty$, we infer from (4.1.1) and (4.1.16) that

$$\left\|S_{\boldsymbol\lambda}^n e_0\right\|^2 = \gamma_n, \quad n\in\mathbb{Z}_+. \tag{4.1.17}$$

Now combining (4.1.15) with (4.1.17), we conclude that $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $u\in\mathsf{Des}(0)$.

Consider now the case of $\kappa > 0$. By using Lemma 3.2.1 and the fact that $e_0 \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$, we deduce that $e_{-k} \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$ for every $k \in J_\kappa$, which means that (i) holds. Now we show that

$$\left\| S_{\boldsymbol\lambda}^n e_{-k} \right\|^2 = \frac{\gamma_{n-k}}{\gamma_{-k}}, \quad n \in \mathbb{Z}_+,\ k \in J_\kappa. \tag{4.1.18}$$

Indeed, if $k \in J_\kappa$, then

$$\left\| S_{\boldsymbol\lambda}^n e_{-k} \right\|^2 \overset{(3.2.2)}{=} \prod_{j=k-n}^{k-1} \lambda_{-j}^2 \overset{(4.1.10)}{=} \prod_{j=k-n}^{k-1} \frac{\gamma_{-j}}{\gamma_{-(j+1)}} = \frac{\gamma_{n-k}}{\gamma_{-k}}, \quad n \in J_k, \tag{4.1.19}$$

which, in view of (4.1.17) and (4.1.1), yields

$$\left\| S_{\boldsymbol\lambda}^n e_{-k} \right\|^2 = \prod_{j=0}^{k-1} \lambda_{-j}^2 \left\| S_{\boldsymbol\lambda}^{n-k} e_0 \right\|^2 = \frac{\gamma_0}{\gamma_{-k}} \gamma_{n-k} = \frac{\gamma_{n-k}}{\gamma_{-k}}, \quad n \in \mathbb{Z},\ n > k. \tag{4.1.20}$$

Combining (4.1.19) with (4.1.20), we obtain (4.1.18). It follows from (4.1.3) and (4.1.18) that the sequence $\{\|S_{\boldsymbol\lambda}^n e_{-k}\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $k \in J_\kappa$. Together with (4.1.15) and (4.1.17), this implies that $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $u \in V_{\eta,\kappa}$. Thus, by Theorem 3.2.4, assertion (ii) is proved.

(iii) Fix $h \in \mathcal{D}^\infty(S_{\boldsymbol\lambda})$. Then, by (ii), there exists a positive Borel measure $\mu_h$ on $\mathbb{R}_+$ such that $\|S_{\boldsymbol\lambda}^n h\|^2 = \int_0^\infty x^n \,\mathrm{d}\mu_h(x)$ for all $n \in \mathbb{Z}_+$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left\| S_{\boldsymbol\lambda} h \right\|^2 &= \int_0^\infty x^0 x^1 \,\mathrm{d}\mu_h(x) \\ &\leqslant \left( \int_0^\infty x^0 \,\mathrm{d}\mu_h(x) \right)^{\frac{1}{2}} \left( \int_0^\infty x^2 \,\mathrm{d}\mu_h(x) \right)^{\frac{1}{2}} = \|h\| \left\| S_{\boldsymbol\lambda}^2 h \right\|. \end{aligned} \tag{4.1.21}$$

Take $f \in \mathcal{D}(S_{\boldsymbol\lambda}^2)$. It follows from (i) and Corollary 3.2.3 that there exists a sequence $\{h_n\}_{n=1}^\infty \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$ such that $h_n \to f$ and $S_{\boldsymbol\lambda}^2 h_n \to S_{\boldsymbol\lambda}^2 f$ as $n \to \infty$. This and (4.1.21) yield

$$\|S_{\boldsymbol\lambda} h_m - S_{\boldsymbol\lambda} h_n\|^2 \leqslant \|h_m - h_n\| \left\| S_{\boldsymbol\lambda}^2 h_m - S_{\boldsymbol\lambda}^2 h_n \right\|, \quad m, n \in \mathbb{N},$$

which, by the completeness of $\mathcal{H}$, implies that the sequence $\{S_{\boldsymbol\lambda} h_n\}_{n=1}^\infty$ is convergent in $\mathcal{H}$. Since $S_{\boldsymbol\lambda}$ is closed (cf. Proposition 3.1.1(i)), we deduce that $S_{\boldsymbol\lambda} h_n \to S_{\boldsymbol\lambda} f$ as $n \to \infty$. Hence, by passage to the limit in the inequality $\|S_{\boldsymbol\lambda} h_n\|^2 \leqslant \|h_n\| \|S_{\boldsymbol\lambda}^2 h_n\|$ (see (4.1.21)), we obtain $\|S_{\boldsymbol\lambda} f\|^2 \leqslant \|f\| \|S_{\boldsymbol\lambda}^2 f\|$. This shows that $S_{\boldsymbol\lambda}$ is paranormal.

(iv) Since $\rho(\{0\})=0$, we obtain

$$\sum_{v\in\mathsf{Chi}(0)} \lambda_v^2 \int_0^\infty \frac{1}{x}\,\mathrm{d}\mu_v(x) = \sum_{i\in J_\eta} \lambda_{i,1}^2 \int_0^\infty \frac{1}{x}\,\mathrm{d}\mu_{i,1}(x) \overset{(4.1.8)\&(4.1.9)}{=} \sum_{i\in J_\eta} \int_{\Omega_i} \frac{1}{x}\,\mathrm{d}\rho(x) \overset{(4.1.7)}{=} \int_0^\infty \frac{1}{x}\,\mathrm{d}\rho(x),$$

which yields (iv).

(v) Inequality (3.1.4), written for $u=(i,j)\in\mathsf{Des}(0)\setminus\{0\}$, takes the form $\lambda_{i,j+1}\leqslant\lambda_{i,j+2}$, which in view of (4.1.9) is equivalent to

$$\left(\int_0^\infty x^j\,\mathrm{d}\mu_{i,1}(x)\right)^2 = \left(\int_0^\infty \sqrt{x^{j-1}}\sqrt{x^{j+1}}\,\mathrm{d}\mu_{i,1}(x)\right)^2 \leqslant \int_0^\infty x^{j-1}\,\mathrm{d}\mu_{i,1}(x)\int_0^\infty x^{j+1}\,\mathrm{d}\mu_{i,1}(x). \tag{4.1.22}$$

Since the latter is always true due to the Cauchy–Schwarz inequality, we see that (3.1.4) is valid for all $u\in\mathsf{Des}(0)\setminus\{0\}$. Clearly, inequality (3.1.4) is valid for $u=0$ if and only if (4.1.12) holds. Finally, if $\kappa>0$ and $k\in J_\kappa$, then using (4.1.1) and (4.1.2) and arguing as in (4.1.22), we verify that $\gamma_{-(k-1)}^2\leqslant\gamma_{-k}\gamma_{-(k-2)}$ for any integer $k$ such that $2\leqslant k\leqslant\kappa$, and that $\gamma_0=\gamma_0^2\leqslant\gamma_{-1}\gamma_1$. Hence, by (4.1.10) and (4.1.17) applied to $n=1$, we conclude that inequality (3.1.4) is valid for $u=-k$ whenever $k\in J_\kappa$. Applying Theorem 3.1.2 yields (v).

(vi) It follows from (4.1.7) and the Cauchy–Schwarz inequality that

$$\rho(\Omega_i)^2 = \left(\int_{\Omega_i} \frac{1}{\sqrt{x}}\sqrt{x}\,\mathrm{d}\rho(x)\right)^2 \leqslant \int_{\Omega_i}\frac{1}{x}\,\mathrm{d}\rho(x)\cdot\int_{\Omega_i} x\,\mathrm{d}\rho(x), \quad i\in J_\eta, \tag{4.1.23}$$

which together with (4.1.6) implies that

$$\frac{\rho(\Omega_i)^2}{\int_{\Omega_i} x\,\mathrm{d}\rho(x)} \leqslant \int_{\Omega_i}\frac{1}{x}\,\mathrm{d}\rho(x), \quad i\in J_\eta. \tag{4.1.24}$$

Therefore, by (4.1.8), (4.1.9) and (4.1.15), we have (recall that $\rho(\{0\})=0$)

$$\sum_{i\in J_\eta}\frac{\lambda_{i,1}^2}{\|S_{\boldsymbol\lambda} e_{i,1}\|^2} = \sum_{i\in J_\eta}\frac{\rho(\Omega_i)^2}{\int_{\Omega_i} x\,\mathrm{d}\rho(x)} \overset{(4.1.24)}{\leqslant} \sum_{i\in J_\eta}\int_{\Omega_i}\frac{1}{x}\,\mathrm{d}\rho(x) \overset{(4.1.7)}{=} \int_0^\infty\frac{1}{x}\,\mathrm{d}\rho(x), \tag{4.1.25}$$

which gives (vi).

(vii) If we have equality in (4.1.13), then one can deduce from (4.1.24) and (4.1.25) that the inequality in (4.1.24) turns into equality for every $i \in J_\eta$. The latter is equivalent to the fact that the Cauchy–Schwarz inequality (4.1.23) becomes an equality for every $i \in J_\eta$. Since this is possible if and only if the functions $\frac{1}{\sqrt{x}}$ and $\sqrt{x}$ are linearly dependent as vectors in $L^2(\Omega_i, \mathfrak{B}(\Omega_i), \rho)$ for every $i \in J_\eta$, we conclude that (4.1.14) holds for some sequence $\{q_i\}_{i\in J_\eta}$ such that $q_i \in \Omega_i$ for all $i \in J_\eta$. The reverse implication is obvious.

(viii) Since the Stieltjes moment sequence $\{\gamma_{n+1}\}_{n=0}^\infty$ is S-determinate, we infer from (4.1.17) that $\{\|S_{\boldsymbol\lambda}^{n+1} e_0\|^2\}_{n=0}^\infty$ is an S-determinate Stieltjes moment sequence. This fact together with (i) and (ii) implies that the weighted shift $S_{\boldsymbol\lambda}$ (which has nonzero weights) satisfies all the assumptions of [8, Corollary 4.5]. Hence, by this corollary, $S_{\boldsymbol\lambda}$ is subnormal and it satisfies the consistency condition (3.1.6) at $u = 0$. Applying (iv) completes the proof. □

Note that, in virtue of Theorem 4.1.1, the validity of the consistency condition (3.1.6) at $u = 0$ implies the hyponormality of $S_{\boldsymbol\lambda}$.

Regarding Theorem 4.1.1(vii) it is worth mentioning that if $\operatorname{card}(\operatorname{supp}(\rho)) = \aleph_0$, then we can always find a Borel partition $\{\Omega_i\}_{i=1}^\infty$ of $(0,\infty)$ satisfying (4.1.6) and (4.1.14) with $\eta = \infty$.

**Proposition 4.1.2.** *Let $\rho$ be a finite positive Borel measure on $\mathbb{R}_+$ such that* $\operatorname{card}(\operatorname{supp}(\rho)) = \aleph_0$. *Then there exist a Borel partition $\{\Omega_i\}_{i=1}^\infty$ of $(0,\infty)$ and a sequence $\{q_i\}_{i=1}^\infty \subseteq (0,\infty)$ such that $\rho(\Omega_i) > 0$, $q_i \in \Omega_i$ and $\rho(\sigma \cap \Omega_i) = \rho(\Omega_i)\cdot\delta_{q_i}(\sigma)$ for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ and $i \in \mathbb{N}$.*

**Proof.** Clearly $\operatorname{supp}(\rho) = A \sqcup B$, where $A := \{x \in \operatorname{supp}(\rho)\colon \rho(\{x\}) > 0\}$ and $B := \operatorname{supp}(\rho) \setminus A$. Since $\operatorname{card}(\operatorname{supp}(\rho)) = \aleph_0$, we deduce that $B \subseteq A'$, where $A'$ is the set of all accumulation points of $A$ in $\mathbb{R}_+$. This and the equality $\operatorname{card}(\operatorname{supp}(\rho)) = \aleph_0$ imply that $\operatorname{card}(A) = \aleph_0$. Hence there exists a sequence $\{q_i\}_{i=1}^\infty$ of distinct positive real numbers such that $A \setminus \{0\} = \{q_1, q_2, q_3, \ldots\}$. Set

$$\Omega_i = \begin{cases} ((0,\infty)\setminus\operatorname{supp}(\rho)) \sqcup \{q_1\} & \text{if } i = 1,\\ (B\setminus\{0\}) \sqcup \{q_2\} & \text{if } i = 2,\\ \{q_i\} & \text{if } i \geqslant 3.\end{cases}$$

It is a simple matter to verify that $\{\Omega_i\}_{i=1}^\infty$ is the required Borel partition of $(0,\infty)$, which completes the proof. □

### *4.2. The main example*

The following example was announced in the title of this paper.

**Example 4.2.1.** Fix $\kappa \in \mathbb{Z}_+ \sqcup \{\infty\}$. Let $\{\gamma_n\}_{n=-\kappa}^\infty$, $\nu$ and $\rho$ be as in Example 2.3.1, i.e., $\{\gamma_n\}_{n=-\kappa}^\infty$ is a system of positive real numbers and $\nu, \rho$ are positive Borel measures on $\mathbb{R}_+$ satisfying the conditions (i) to (iv) of this example. From (iii) and (iv) we infer that $\operatorname{card}(\operatorname{supp}(\rho)) = \aleph_0$. Hence the triplet $(\{\gamma_n\}_{n=-\kappa}^\infty, \nu, \rho)$ satisfies the conditions (4.1.1), (4.1.2), (4.1.4) and (4.1.5) with $\eta = \infty$. It follows from Proposition 4.1.2 that there exist a Borel partition $\{\Omega_i\}_{i=1}^\infty$ of $(0,\infty)$ and a sequence $\{q_i\}_{i=1}^\infty \subseteq (0,\infty)$ which satisfy (4.1.6) and (4.1.14) with $\eta = \infty$ (note that if $\rho = \tilde\rho_a$ for some $a \in (0,\infty)$, where $\tilde\rho_a$ is as in Example 2.3.1, then we may simply consider the sequence $\{q_i\}_{i=1}^\infty := \{a, aq, aq^{-1}, aq^2, aq^{-2}, \ldots\}$). Let $S_{\boldsymbol\lambda}$ be a weighted shift on the directed tree $\mathscr{T}_{\infty,\kappa}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\infty,\kappa}^\circ}$ defined by (4.1.8), (4.1.9) and (4.1.10) with $\eta = \infty$.

By (2.3.3) and assertions (v), (vi) and (vii) of Theorem 4.1.1, the operator $S_{\boldsymbol\lambda}$ is not hyponormal. In turn, assertions (i), (ii) and (iii) of Theorem 4.1.1 imply that $S_{\boldsymbol\lambda}$ is a paranormal operator which generates Stieltjes moment sequences; moreover, by Corollary 3.2.3, $\mathcal{D}^\infty(S_{\boldsymbol\lambda})$ is a core of $S_{\boldsymbol\lambda}^n$ for every $n \in \mathbb{Z}_+$. In view of (2.3.3) and assertion (iv) of Theorem 4.1.1, the weighted shift $S_{\boldsymbol\lambda}$ does not satisfy the consistency condition (3.1.6) at $u = 0$. Since $\mathcal{E}_{V_{\infty,\kappa}} \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$ and $S_{\boldsymbol\lambda}$ is not subnormal, we deduce from [7, Theorem 5.1.1] that the weighted shift $S_{\boldsymbol\lambda}$ has no consistent system of measures (in the sense of [7]). Finally, by making an appropriate choice of the triplet $(\{\gamma_n\}_{n=-\kappa}^\infty, \nu, \rho)$, we can guarantee that $\{\|S_{\boldsymbol\lambda}^{n+1} e_0\|^2\}_{n=0}^\infty$ is S-indeterminate, while $\{\|S_{\boldsymbol\lambda}^n e_0\|^2\}_{n=0}^\infty$ is either H-determinate or S-indeterminate according to our needs (cf. Example 2.3.1).

The directed tree $\mathscr{T}_{\eta,\kappa}$ can also be used to construct examples of unbounded subnormal weighted shifts $S_{\boldsymbol\lambda}$ for which $\{\|S_{\boldsymbol\lambda}^n e_0\|^2\}_{n=0}^\infty$ and $\{\|S_{\boldsymbol\lambda}^{n+1} e_0\|^2\}_{n=0}^\infty$ are H-determinate Stieltjes moment sequences.

**Example 4.2.2.** The following example is an adaptation of [37, Example 7.1] to our needs. Set $c = \sum_{j=2}^\infty j2^{-j} + \sum_{j=2}^\infty j^{-1}e^{-j^2}$. It is easily seen that the two-sided sequence $\{\gamma_n\}_{n=-\infty}^\infty$ given by

$$\gamma_n = c^{-1}\left(\sum_{j=2}^\infty \frac{1}{2^j j^{n-1}} + \sum_{j=2}^\infty \frac{j^{n-1}}{e^{j^2}}\right), \quad n \in \mathbb{Z},$$

is well defined, $\gamma_0 = 1$ and

$$\gamma_n = \int_0^\infty x^n \, \mathrm{d}\nu(x), \quad n \in \mathbb{Z}, \tag{4.2.1}$$

where $\nu := c^{-1}(\sum_{j=2}^\infty j2^{-j}\delta_{\frac{1}{j}} + \sum_{j=2}^\infty j^{-1}e^{-j^2}\delta_j)$. Note also that

$$\operatorname{supp}(\nu) = \{0\} \cup \left\{\ldots, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\} \cup \{2, 3, 4, \ldots\}. \tag{4.2.2}$$

It was proved in [37, Example 7.1] that $\gamma_{2n+1} \leqslant 4c^{-1}n^n$ for all integers $n \geqslant 4$. This implies that $\gamma_{2n} \leqslant 5c^{-1}n^n$ for all integers $n \geqslant 4$. Hence, by Carleman's criterion (see e.g., [34, Corollary 4.5]), the Stieltjes moment sequences $\{\gamma_n\}_{n=0}^\infty$ and $\{\gamma_{n+1}\}_{n=0}^\infty$ are H-determinate. In view of (4.2.1), the positive Borel measure

$$\rho := c^{-1}\left(\sum_{j=2}^\infty 2^{-j}\delta_{\frac{1}{j}} + \sum_{j=2}^\infty e^{-j^2}\delta_j\right)$$

is a unique representing measure of $\{\gamma_{n+1}\}_{n=0}^\infty$. Putting all these together, we conclude that for every $\eta \in \{2, 3, \ldots\} \sqcup \{\infty\}$ and for every $\kappa \in \mathbb{Z}_+ \cup \{\infty\}$, the system $\{\gamma_n\}_{n=-\kappa}^\infty$ and the measures $\nu$ and $\rho$ satisfy (4.1.1), (4.1.2), (4.1.4) and (4.1.5). Take any Borel partition $\{\Omega_i\}_{i=1}^\eta$ of $(0, \infty)$ which satisfies (4.1.6). Let $S_{\boldsymbol\lambda}$ be the weighted shift on the directed tree $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ defined by (4.1.8), (4.1.9) and (4.1.10). Then, by assertions (iv) and (viii) of

Theorem 4.1.1, the operator $S_{\boldsymbol{\lambda}}$ is subnormal and it satisfies the consistency condition (3.1.6) at $u=0$ (in fact, $1=\gamma_0=\int_0^\infty \frac{1}{x}\,\mathrm{d}\rho(x)$). Moreover, by (4.1.17), the Stieltjes moment sequences $\{\|S_{\boldsymbol{\lambda}}^n e_0\|^2\}_{n=0}^\infty$ and $\{\|S_{\boldsymbol{\lambda}}^{n+1} e_0\|^2\}_{n=0}^\infty$ are H-determinate. Note that the operator $S_{\boldsymbol{\lambda}}$ is unbounded. Indeed, otherwise by [20, Notation 6.1.9 and Theorem 6.1.3], a unique H-representing measure of $\{\|S_{\boldsymbol{\lambda}}^n e_0\|^2\}_{n=0}^\infty$ is compactly supported. This fact, together with (4.1.17) and (4.2.1), contradicts (4.2.2).

### 4.3. The case of composition operators

It turns out that Example 4.2.1 can be realized as a composition operator in an $L^2$-space. Before proving this, we show that a great deal of weighted shifts on directed trees can be identified with composition operators in $L^2$-spaces.

**Lemma 4.3.1.** *Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on a rootless directed tree $\mathscr{T}=(V,E)$ with positive weights $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V^\circ}$. Suppose* $\operatorname{card}(V)=\aleph_0$. *Then $S_{\boldsymbol{\lambda}}$ is unitarily equivalent to a composition operator $C$ in an $L^2$-space over a $\sigma$-finite measure space. Moreover, if the directed tree $\mathscr{T}$ is leafless, then $C$ can be made injective.*

**Proof.** We begin by proving that for any $(w,\beta)\in V\times(0,\infty)$ there exists a function $\alpha:\mathsf{Des}(w)\to(0,\infty)$ such that

$$\alpha(w)=\beta \quad \text{and} \quad \alpha(v)=\lambda_v^2\alpha(u) \quad \text{for all } v\in\mathsf{Chi}(u) \text{ and } u\in\mathsf{Des}(w). \tag{4.3.1}$$

Indeed, since $\mathsf{Des}(w)=\bigsqcup_{n=0}^\infty \mathsf{Chi}^{\langle n\rangle}(w)$ (cf. [20, (2.1.10)]), we can proceed by induction. For the base step of the induction, set $\alpha(v)=\lambda_v^2\beta$ for $v\in\mathsf{Chi}(w)$. Fix $n\geqslant 1$, and assume that we already have a function $\alpha:\bigsqcup_{j=0}^n \mathsf{Chi}^{\langle j\rangle}(w)\to(0,\infty)$ such that $\alpha(w)=\beta$ and $\alpha(v)=\lambda_v^2\alpha(u)$ for all $v\in\mathsf{Chi}(u)$ and $u\in\bigsqcup_{j=0}^{n-1}\mathsf{Chi}^{\langle j\rangle}(w)$. Since $\mathsf{Chi}^{\langle n+1\rangle}(w)=\bigsqcup_{u\in\mathsf{Chi}^{\langle n\rangle}(w)}\mathsf{Chi}(u)$ (cf. [20, (6.1.3)]), we can extend the function $\alpha$ to $\bigsqcup_{j=0}^{n+1}\mathsf{Chi}^{\langle j\rangle}(w)$ by setting $\alpha(v)=\lambda_v^2\alpha(u)$ for all $v\in\mathsf{Chi}(u)$ and $u\in\mathsf{Chi}^{\langle n\rangle}(w)$. Therefore the induction step is valid, and so our claim is proved.

Fix $z\in V$. Let $\alpha_0:\mathsf{Des}(z)\to(0,\infty)$ be a function satisfying (4.3.1) with $\alpha=\alpha_0$, $w=z$ and $\beta=1$. By [21, (3.4)], we have

$$\mathsf{Des}\big(\mathsf{par}(z)\big)\setminus\mathsf{Des}(z)=\big\{\mathsf{par}(z)\big\}\sqcup\bigsqcup_{w\in\mathsf{Chi}(\mathsf{par}(z))\setminus\{z\}}\mathsf{Des}(w). \tag{4.3.2}$$

Using (4.3.2), we will extend the function $\alpha_0$ to a function $\alpha_1:\mathsf{Des}(\mathsf{par}(z))\to(0,\infty)$ which satisfies (4.3.1) with $\alpha=\alpha_1$, $w=\mathsf{par}(z)$ and $\beta=1/\lambda_z^2$. By the preceding paragraph, for every $w\in\mathsf{Chi}(\mathsf{par}(z))\setminus\{z\}$ there exists a function $\alpha_{1,w}:\mathsf{Des}(w)\to(0,\infty)$ satisfying (4.3.1) with $\alpha=\alpha_{1,w}$ and $\beta=\lambda_w^2/\lambda_z^2$. Set $\alpha_1(\mathsf{par}(z))=1/\lambda_z^2$ and $\alpha_1(v)=\alpha_{1,w}(v)$ for $v\in\mathsf{Des}(w)$ and $w\in\mathsf{Chi}(\mathsf{par}(z))\setminus\{z\}$. Then, by (4.3.2), the function $\alpha_1$ is well defined and it satisfies our requirements. Using the decomposition $V=\bigcup_{k=0}^\infty\mathsf{Des}(\mathsf{par}^k(z))$ (cf. [20, Proposition 2.1.6]) and induction with $\alpha(\mathsf{par}^k(z))=\prod_{j=0}^{k-1}\lambda_{\mathsf{par}^j(z)}^{-2}$ for $k\geqslant 1$, we get a function $\alpha:V\to(0,\infty)$ such that

$$\alpha(v)=\lambda_v^2\alpha(u), \quad v\in\mathsf{Chi}(u),\ u\in V. \tag{4.3.3}$$

Define a measure space $(V, \Sigma, \mu)$ by $\Sigma = 2^V$ and $\mu(\{u\}) = \alpha(u)$ for every $u \in V$. Since $\operatorname{card}(V) = \aleph_0$, the measure $\mu$ is $\sigma$-finite. Let $\phi : V \to V$ be a transformation given by $\phi(u) = \mathsf{par}(u)$ for all $u \in V$ ($\phi$ is well defined because $\mathscr{T}$ is rootless) and let $C$ be a composition operator in $L^2(\mu)$ defined by

$$\mathcal{D}(C) = \{f \in L^2(\mu)\colon f \circ \phi \in L^2(\mu)\} \quad \text{and} \quad Cf = f \circ \phi \quad \text{for } f \in \mathcal{D}(C).$$

If the directed tree $\mathscr{T}$ is leafless, then the transformation $\phi$ is surjective, and thus the operator $C$ is injective. It is clear that the operator $C$ is closed.[8] Now we define the mapping $U : \ell^2(V) \to L^2(\mu)$ by

$$(Uf)(u) = \frac{f(u)}{\sqrt{\alpha(u)}}, \quad u \in V,\ f \in \ell^2(V). \tag{4.3.4}$$

It is easily seen that $U$ is a well-defined unitary isomorphism such that

$$\begin{aligned} \big((Uf) \circ \phi\big)(v) &\overset{(4.3.4)}{=} \frac{f(\phi(v))}{\sqrt{\alpha(\phi(v))}} \\ &\overset{(4.3.3)}{=} \lambda_v \frac{f(\mathsf{par}(v))}{\sqrt{\alpha(v)}} \overset{(3.1.3)}{=} \frac{(\Lambda_{\mathscr{T}} f)(v)}{\sqrt{\alpha(v)}}, \quad v \in V,\ f \in \ell^2(V). \end{aligned} \tag{4.3.5}$$

Hence if $f \in \mathcal{D}(S_{\boldsymbol\lambda})$, then $((Uf) \circ \phi)(v) = (US_{\boldsymbol\lambda} f)(v)$ for every $v \in V$, which implies that $Uf \in \mathcal{D}(C)$ and $CUf = US_{\boldsymbol\lambda} f$. This shows that $US_{\boldsymbol\lambda} \subseteq CU$. In turn, if $f \in \ell^2(V)$ and $Uf \in \mathcal{D}(C)$, then by (4.3.5) the function $g : V \to \mathbb{C}$ given by

$$g(v) = \frac{(\Lambda_{\mathscr{T}} f)(v)}{\sqrt{\alpha(v)}}, \quad v \in V,$$

belongs to $L^2(\mu)$. It follows from (4.3.4) that $(U^{-1}g)(v) = (\Lambda_{\mathscr{T}} f)(v)$ for every $v \in V$, which means that $f \in \mathcal{D}(S_{\boldsymbol\lambda})$. Putting all these together, we conclude that $US_{\boldsymbol\lambda} = CU$, or equivalently that $S_{\boldsymbol\lambda} = U^*CU$. This completes the proof. □

**Remark 4.3.2.** A close inspection of the proof of Lemma 4.3.1 reveals that if functions $\alpha, \alpha' : V \to (0, \infty)$ satisfy (4.3.3), then there exists $t \in (0, \infty)$ such that $\alpha'(v) = t\alpha(v)$ for all $v \in V$. If we drop the assumption "$\operatorname{card}(V) = \aleph_0$" in Lemma 4.3.1, then the composition operator $C$ constructed in its proof acts in an $L^2$-space over a measure space which is not necessarily $\sigma$-finite. It follows from [20, Proposition 3.1.10] that if there exists a densely defined weighted shift on a directed tree $\mathscr{T}$ with nonzero weights, then $\operatorname{card}(V) \leqslant \aleph_0$.

The following surprising fact follows directly from Example 4.2.1 with $\kappa = \infty$ and Lemma 4.3.1.

[8] In fact, composition operators in $L^2$-spaces are always closed (see [9], see also [11, Lemma 6.2] for the case of densely defined composition operators).

**Theorem 4.3.3.** *There exists a non-hyponormal composition operator $C$ in an $L^2$-space over a $\sigma$-finite measure space which is injective, paranormal and which generates Stieltjes moment sequences. Moreover, $C$ has the property that $\mathcal{D}^\infty(C)$ is a core of $C^n$ for every $n \in \mathbb{Z}_+$.*

It is worth pointing out that every composition operator $C$ in an $L^2$-space over a $\sigma$-finite measure space which generates Stieltjes moment sequences has the property that $\mathcal{D}^\infty(C)$ is a core of $C^n$ for every $n \in \mathbb{Z}_+$ (cf. [9]).

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## Appendix A

*A.1.* As announced in the Introduction, the independence assertion of Barry Simon's theorem which parameterizes von Neumann extensions of a closed real symmetric operator with deficiency indices $(1,1)$ is false (see Propositions A.4.1, A.4.2 and A.4.4). For the reader's convenience, we state the Simon theorem without typos that appeared in the original version. We have also added a missing assumption that $\varphi \neq 0$.

**Caution.** The reader should be aware of the fact that the inner products considered in Simon's paper [34] are linear in the second factor and anti-linear in the first. From now on we follow his convention.

**Theorem A.1.1.** *(See [34, Theorem 2.6].) Suppose that $A$ is a closed symmetric operator so that there exists a complex conjugation under which $A$ is real. Suppose that $d_+ = 1$ and that $\ker(A) = \{0\}$, $\dim\ker(A^*) = 1$. Pick $\varphi \in \ker(A^*) \setminus \{0\}$, $C\varphi = \varphi$, and $\eta \in \mathcal{D}(A^*)$, not in $\mathcal{D}(A) + \ker(A^*)$. Then $\langle \varphi, A^*\eta\rangle \neq 0$ and $\psi = \{\eta - [\langle \eta, A^*\eta\rangle / \langle \varphi, A^*\eta\rangle]\varphi\}/\langle \varphi, A^*\eta\rangle$ are such that in $\varphi$, $\psi$ basis, $\langle \cdot, A^*\cdot\rangle$ has the form*

$$\langle \cdot, A^*\cdot\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.7}$$

*The self-adjoint extensions, $B_t$, can be labelled by a real number or $\infty$ where*

$$\begin{aligned} \mathcal{D}(B_t) &= \mathcal{D}(A) + \{\alpha(t\varphi + \psi) \mid \alpha \in \mathbb{C}\} \quad && t \in \mathbb{R} \\ &= \mathcal{D}(A) + \{\alpha\varphi \mid \alpha \in \mathbb{C}\} && t = \infty. \end{aligned}$$

*The operators $B_t$ are independent of which real $\psi$ in $\mathcal{D}(A^*) \setminus \mathcal{D}(A)$ is chosen so that* (2.7) *holds.*

*A.2.* Let $C$ be a complex conjugation on a complex Hilbert space $\mathcal{H}$ (i.e., $C$ is an anti-linear map from $\mathcal{H}$ to $\mathcal{H}$ such that $C(Cf) = f$ and $\langle Cf, Cg\rangle = \langle g, f\rangle$ for all $f, g \in \mathcal{H}$). We say that a vector $f$ in $\mathcal{H}$ is $C$-*real* (or briefly *real*) if $Cf = f$. Set

$$\Re_C f = \frac{1}{2}(f + Cf) \quad \text{and} \quad \Im_C f = \frac{1}{2\mathrm{i}}(f - Cf), \quad f \in \mathcal{H}.$$

Then clearly for every $f \in \mathcal{H}$,

$$\Re_C f \text{ and } \Im_C f \text{ are } C\text{-real, and } f = \Re_C f + \mathrm{i} \cdot \Im_C f. \tag{A.2.1}$$

Hence $\langle \Re_C f, \Im_C f \rangle = \langle C(\Re_C f), C(\Im_C f) \rangle = \langle \Im_C f, \Re_C f \rangle$ for all $f \in \mathcal{H}$, and thus $\langle \Re_C f, \Im_C f \rangle \in \mathbb{R}$ for all $f \in \mathcal{H}$, which gives

$$\|f\|^2 = \|\Re_C f\|^2 + \|\Im_C f\|^2, \quad f \in \mathcal{H}.$$

Recall that if $A$ is a symmetric operator in $\mathcal{H}$ such that $CA \subseteq AC$ (i.e., $C(\mathcal{D}(A)) \subseteq \mathcal{D}(A)$ and $CAf = ACf$ for all $f \in \mathcal{D}(A)$), then $CA^* \subseteq A^*C$, i.e.,

$$C\big(\mathcal{D}\big(A^*\big)\big) \subseteq \mathcal{D}\big(A^*\big) \quad \text{and} \quad CA^*f = A^*Cf \quad \text{for } f \in \mathcal{D}\big(A^*\big). \tag{A.2.2}$$

For much of the rest of the paper we will be considering the following situation.

> Let $A$ be a closed symmetric operator in a complex Hilbert space $\mathcal{H}$ such that $\ker(A) = \{0\}$. Suppose that there exists a complex conjugation $C$ on $\mathcal{H}$ such that $A$ is *$C$-real* (or briefly *real*), i.e., $CA \subseteq AC$. (A.2.3)

The next two lemmata are of technical importance.

**Lemma A.2.1.** *Suppose that* (A.2.3) *holds and* $\mathcal{H} \neq \{0\}$. *Then there exists* $f \in \mathcal{D}(A)$ *such that either* $\langle f, Af \rangle > 0$ *or* $\langle f, Af \rangle < 0$. *In the former case, there exists* $h \in \mathcal{D}(A)$ *such that* $Ch = h$ *and* $\langle h, Ah \rangle > 0$. *In the latter case, there exists* $h \in \mathcal{D}(A)$ *such that* $Ch = h$ *and* $\langle h, Ah \rangle < 0$.

**Proof.** Since the possibility that $\langle f, Af \rangle = 0$ for all $f \in \mathcal{D}(A)$ is excluded by the fact that $\mathcal{H} \neq \{0\}$ and $\ker(A) = \{0\}$, and $\langle f, Af \rangle \in \mathbb{R}$ for all $f \in \mathcal{D}(A)$, it remains to prove the last two statements of the conclusion. By symmetry, it suffices to consider the case when $\langle f, Af \rangle > 0$ for some $f \in \mathcal{D}(A)$. Since $A$ is $C$-real, we deduce that $u := \Re_C f \in \mathcal{D}(A)$, $v := \Im_C f \in \mathcal{D}(A)$, $Cu = u$, $Cv = v$ and

$$\langle u, Av \rangle = \langle CAv, Cu \rangle = \langle ACv, u \rangle = \langle Av, u \rangle = \overline{\langle u, Av \rangle},$$

which together with $A \subseteq A^*$ implies that

$$0 < \big\langle u + \mathrm{i}v, A(u + \mathrm{i}v)\big\rangle = \langle u, Au \rangle + 2\,\mathrm{Re}\big(\mathrm{i}\langle u, Av \rangle\big) + \langle v, Av \rangle = \langle u, Au \rangle + \langle v, Av \rangle.$$

Therefore either $\langle u, Au \rangle > 0$ or $\langle v, Av \rangle > 0$, which completes the proof. □

**Lemma A.2.2.** *If $T$ is an operator in $\mathcal{H}$ and $C$ is a complex conjugation on $\mathcal{H}$ such that $CT \subseteq TC$ and* $\ker(T) \neq \{0\}$, *then there exists* $f \in \ker(T) \setminus \{0\}$ *such that* $Cf = f$.

**Proof.** Take $f \in \ker(T) \setminus \{0\}$. Since $CT \subseteq TC$ implies $C(\ker(T)) = \ker(T)$, we get $\Re_C f$, $\Im_C f \in \ker T$, which together with (A.2.1) gives that either $\Re_C f \neq 0$ or $\Im_C f \neq 0$. This completes the proof. □

*A.3.* Now, we concentrate on the class of vectors satisfying the assumptions of the independence assertion of Theorem A.1.1. Given a symmetric operator $A$ in $\mathcal{H}$, a complex conjugation $C$ on $\mathcal{H}$ and a vector $\varphi$ in $\ker(A^*)$, we write

$$\mathscr{S}_{A,C}^{\varphi}=\{\psi\in\mathcal{D}(A^*)\setminus\mathcal{D}(A)\colon\ C\psi=\psi,\ \langle\psi,A^*\psi\rangle=0,\ \langle\varphi,A^*\psi\rangle=1\}.$$

Clearly, the equality $\langle\varphi,A^*\psi\rangle=1$ implies that

$$\mathscr{S}_{A,C}^{\varphi}=\{\psi\in\mathcal{D}(A^*)\colon\ C\psi=\psi,\ \langle\psi,A^*\psi\rangle=0,\ \langle\varphi,A^*\psi\rangle=1\}. \tag{A.3.1}$$

**Remark A.3.1.** Note that if $\varphi\in\ker(A^*)$ and $\psi\in\mathcal{D}(A^*)$, then

$$\langle\alpha\varphi+\beta\psi,A^*(\gamma\varphi+\delta\psi)\rangle=\left\langle\begin{bmatrix}\alpha\\ \beta\end{bmatrix},\begin{bmatrix}0 & \langle\varphi,A^*\psi\rangle\\ 0 & \langle\psi,A^*\psi\rangle\end{bmatrix}\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\right\rangle,\quad \alpha,\beta,\gamma,\delta\in\mathbb{C}.$$

Hence, if additionally $\varphi\neq 0$, then $\langle\varphi,A^*\psi\rangle=1$ and $\langle\psi,A^*\psi\rangle=0$ if and only if the vectors $\varphi,\psi$ are linearly independent and $\langle\cdot,A^*\cdot\rangle$ has the matrix representation (2.7) in the basis $(\varphi,\psi)$.

The following lemma is a modified version of what can be found in [34, Theorem 2.6]. For the reader's convenience we include its proof.

**Lemma A.3.2.** *Suppose that* (A.2.3) *holds,* $d_+(A)=1$ *and* $\dim\ker(A^*)=1$. *Let* $\varphi$ *be a* $C$*-real vector in* $\ker(A^*)\setminus\{0\}$ (*cf. Lemma* A.2.2). *Then* $\mathscr{S}_{A,C}^{\varphi}\neq\emptyset$.

**Proof.** Since $A$ is $C$-real, we infer from the von Neumann theorem that $d_-(A)=d_+(A)=1$. This and the equality

$$\mathcal{D}(A^*)=\mathcal{D}(A)\dotplus\ker(A^*+\mathrm{i}I)\dotplus\ker(A^*-\mathrm{i}I)\quad\text{(direct sum)},$$

which is true for arbitrary closed symmetric operators (cf. [30, Lemma, p. 138]), imply that $\dim(\mathcal{D}(A^*)/\mathcal{D}(A))=2$. Since $A\subseteq A^*$ and $\ker(A)=\{0\}$, we get $\mathcal{D}(A)\cap\ker A^*=\{0\}$. Hence $\dim[(\mathcal{D}(A)\dotplus\ker(A^*))/\mathcal{D}(A)]=1$ (because $\dim\ker(A^*)=1$), and thus there exists $\eta\in\mathcal{D}(A^*)\setminus(\mathcal{D}(A)\dotplus\ker(A^*))$. Since, by (A.2.2), the vectors $\mathfrak{R}_C\eta$ and $\mathfrak{I}_C\eta$ are in $\mathcal{D}(A^*)$, we deduce from (A.2.1) that either $\mathfrak{R}_C\eta\notin\mathcal{D}(A)\dotplus\ker(A^*)$ or $\mathfrak{I}_C\eta\notin\mathcal{D}(A)\dotplus\ker(A^*)$. Therefore, we can assume without loss of generality that $C\eta=\eta$. Putting all these together, we get

$$\mathcal{D}(A^*)=\mathcal{D}(A)\dotplus\ker(A^*)\dotplus\mathbb{C}\cdot\eta. \tag{A.3.2}$$

Now we show that $\langle\varphi,A^*\eta\rangle\neq 0$. Suppose that contrary to our claim $\langle\varphi,A^*\eta\rangle=0$. Define a sesquilinear form $Q$ on $\mathcal{D}(A^*)$ by $Q(f,g)=\langle f,A^*g\rangle-\langle A^*f,g\rangle$ for $f,g\in\mathcal{D}(A^*)$. Since $A$ is symmetric, we have $Q(f,g)=0$ for $f,g\in\mathcal{D}(A)$. Note also that

$$\langle A^*\eta,\eta\rangle=\langle C\eta,CA^*\eta\rangle\overset{\text{(A.2.2)}}{=}\langle\eta,A^*C\eta\rangle=\langle\eta,A^*\eta\rangle. \tag{A.3.3}$$

Thus $Q(\eta,\eta)=0$. Using (A.3.2) and $\langle\varphi,A^*\eta\rangle=0$, it is now easily seen that $Q\equiv 0$, which means that $A^*$ is symmetric. This and $A=\bar{A}$ imply that $A$ is selfadjoint, which contradicts $d_+(A)=1$, and finally shows that $\langle\varphi,A^*\eta\rangle\neq 0$. Since $\varphi$ and $\eta$ are $C$-real, we infer from (A.2.2) that $\langle\varphi, A^*\eta\rangle \in \mathbb{R}$. Therefore, we can assume without loss of generality that $\langle\varphi, A^*\eta\rangle = 1$. Now, by setting $\psi = \eta - \langle\eta, A^*\eta\rangle\varphi$, we infer from (A.3.1) and (A.3.3) that $\psi \in \mathscr{S}^{\varphi}_{A,C}$ (our particular choice of $\psi$ guarantees that $\psi \notin \mathcal{D}(A) \dotplus \ker(A^*)$). $\square$

The next lemma is a main ingredient of the proof of Proposition A.4.1.

**Lemma A.3.3.** *Suppose that* (A.2.3) *holds,* $d_+(A) = 1$ *and* $\dim\ker(A^*) = 1$*. Let* $\varphi$ *and* $h$ *be* $C$*-real vectors such that* $\varphi \in \ker(A^*) \setminus \{0\}$*,* $h \in \mathcal{D}(A)$ *and* $\langle h, Ah\rangle \neq 0$ *(cf. Lemmata* A.2.1 *and* A.2.2)*, and let* $\psi \in \mathscr{S}^{\varphi}_{A,C}$ *(cf. Lemma* A.3.2)*. Set*

$$\hat{\psi}(x) = \hat{\eta}(x) - \big\langle\hat{\eta}(x), A^*\hat{\eta}(x)\big\rangle\varphi \quad \text{with } \hat{\eta}(x) = xh + \varphi + \psi \text{ for } x \in \mathbb{R}. \tag{A.3.4}$$

*Then* $\{\hat{\psi}(x)\colon x \in \mathbb{R}\} \subseteq \mathscr{S}^{\varphi}_{A,C}$ *and*

$$\hat{\psi}(x) - \hat{\psi}(y) = (x-y)h - \big(\Delta(x) - \Delta(y)\big)\varphi, \quad x, y \in \mathbb{R}, \tag{A.3.5}$$

*where* $\Delta(x) = \Delta_{h,\psi}(x) := x^2\langle h, Ah\rangle + 2x\operatorname{Re}\langle\psi, Ah\rangle$ *for* $x \in \mathbb{R}$*. Moreover, for every* $\vartheta \in \mathbb{R}\setminus\{0\}$ *there exist* $x, y \in \mathbb{R}$ *such that*

$$\hat{\psi}(x) - \hat{\psi}(y) = (x-y)h + \vartheta\varphi \in \big(\mathcal{D}(A) \dotplus \ker\big(A^*\big)\big) \setminus \mathcal{D}(A). \tag{A.3.6}$$

**Proof.** Since $h, \varphi, \psi$ are $C$-real, so are $\hat{\eta}(x)$, $x \in \mathbb{R}$. It follows from $\varphi \perp \mathcal{R}(A)$, $\langle\varphi, A^*\psi\rangle = 1$ and $\langle\psi, A^*\psi\rangle = 0$ that

$$\big\langle\hat{\eta}(x), A^*\hat{\eta}(x)\big\rangle = \big\langle xh + \varphi + \psi, xAh + A^*\psi\big\rangle = \Delta(x) + 1, \quad x \in \mathbb{R}. \tag{A.3.7}$$

These two facts imply that $C\hat{\psi}(\cdot) = \hat{\psi}(\cdot)$. As $\varphi \perp \mathcal{R}(A)$, we have for all $x \in \mathbb{R}$

$$\big\langle\varphi, A^*\hat{\psi}(x)\big\rangle = \big\langle\varphi, A^*\hat{\eta}(x)\big\rangle = x\langle\varphi, Ah\rangle + \big\langle\varphi, A^*\psi\big\rangle = 1. \tag{A.3.8}$$

Using the fact that $\hat{\eta}(x)$ is $C$-real for all $x \in \mathbb{R}$ and arguing as in (A.3.3), we see that $\langle\hat{\eta}(x), A^*\hat{\eta}(x)\rangle \in \mathbb{R}$ for all $x \in \mathbb{R}$. This yields

$$\big\langle\hat{\psi}(x), A^*\hat{\psi}(x)\big\rangle = \big\langle\hat{\eta}(x), A^*\hat{\eta}(x)\big\rangle - \overline{\big\langle\hat{\eta}(x), A^*\hat{\eta}(x)\big\rangle}\big\langle\varphi, A^*\hat{\eta}(x)\big\rangle \overset{(A.3.8)}{=} 0, \quad x \in \mathbb{R}.$$

Hence $\{\hat{\psi}(x)\colon x \in \mathbb{R}\} \subseteq \mathscr{S}^{\varphi}_{A,C}$. Noting that for every $x \in \mathbb{R}$,

$$\hat{\psi}(x) \overset{(A.3.4)}{=} xh + \big(1 - \big\langle\hat{\eta}(x), A^*\hat{\eta}(x)\big\rangle\big)\varphi + \psi \overset{(A.3.7)}{=} xh - \Delta(x)\varphi + \psi, \tag{A.3.9}$$

we obtain (A.3.5). The latter together with $\langle h, Ah\rangle \neq 0$ and the equality

$$\Delta(x) - \Delta(y) = \big(x^2 - y^2\big)\langle h, Ah\rangle + 2(x-y)\operatorname{Re}\langle\psi, Ah\rangle, \quad x, y \in \mathbb{R},$$

imply the "moreover" part of the conclusion. $\square$

*A.4.* Clearly, the description of the operator $B_\infty$ given in Theorem A.1.1 does not depend on the choice of $\psi \in \mathscr{S}^{\varphi}_{A,C}$. However, as shown in Proposition A.4.1 below, the description of the operators $\{B_t\colon t \in \mathbb{R}\}$ is extremely dependent on the choice of the vector $\psi \in \mathscr{S}^{\varphi}_{A,C}$.

**Proposition A.4.1.** *Suppose that* (A.2.3) *holds,* $d_+(A) = 1$ *and* $\dim\ker(A^*) = 1$*. Let* $\varphi$ *be a* $C$*-real vector in* $\ker(A^*) \setminus \{0\}$ *(cf. Lemma* A.2.2*). Then for every* $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$ *such that* $t_1 \neq t_2$*, there exist* $(\psi_1, \psi_2) \in \mathscr{S}^{\varphi}_{A,C} \times \mathscr{S}^{\varphi}_{A,C}$ *such that*

$$\mathcal{D}^{\varphi}_{t_1,\psi_1} = \mathcal{D}^{\varphi}_{t_2,\psi_2},$$

*where* $\mathcal{D}^{\varphi}_{t,\psi} = \mathcal{D}(A) + \{\alpha(t\varphi + \psi)\colon \alpha \in \mathbb{C}\}$ *for* $t \in \mathbb{R}$ *and* $\psi \in \mathscr{S}^{\varphi}_{A,C}$*.*

**Proof.** Take $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$ such that $t_1 \neq t_2$, and fix $\psi \in \mathscr{S}^{\varphi}_{A,C}$ (cf. Lemma A.3.2). Let $h$ be a $C$-real vector in $\mathcal{D}(A)$ such that $\langle h, Ah\rangle \neq 0$ (cf. Lemma A.2.1), and let $\hat{\psi}(\cdot)$ be as in (A.3.4). Set $\vartheta = t_2 - t_1$. Then by Lemma A.3.3, there exist $x, y \in \mathbb{R}$ such that $\psi_1 := \hat{\psi}(x) \in \mathscr{S}^{\varphi}_{A,C}$, $\psi_2 := \hat{\psi}(y) \in \mathscr{S}^{\varphi}_{A,C}$ and (A.3.6) holds. Since $h \in \mathcal{D}(A)$, we have

$$\begin{aligned} \mathcal{D}^{\varphi}_{t_1,\psi_1} &\overset{(A.3.6)}{=} \mathcal{D}(A) + \mathbb{C} \cdot \bigl(t_1\varphi + \psi_2 + (x-y)h + \vartheta\varphi\bigr) \\ &= \mathcal{D}(A) + \mathbb{C} \cdot \bigl((t_1 + \vartheta)\varphi + \psi_2\bigr) = \mathcal{D}^{\varphi}_{t_2,\psi_2}, \end{aligned}$$

which completes the proof. □

Calculating the vectors $\hat{\psi}(x) - \psi$ with the help of (A.3.9) and considering them instead of $\hat{\psi}(x) - \hat{\psi}(y)$ in the proof of Proposition A.4.1, we obtain the following result which itself implies Proposition A.4.1 (note that by Lemma A.2.1 there is no loss of generality in assuming that the vector $h$ in Proposition A.4.2 below is $C$-real).

**Proposition A.4.2.** *Suppose that the assumptions of Proposition* A.4.1 *are satisfied and* $(\psi, t) \in \mathscr{S}^{\varphi}_{A,C} \times \mathbb{R}$*. If there exists* $h \in \mathcal{D}(A)$ *such that* $\langle h, Ah\rangle > 0$ *(respectively,* $\langle h, Ah\rangle < 0$*), then for every real* $t' > t$ *(respectively,* $t' < t$*), there exists* $\psi' \in \mathscr{S}^{\varphi}_{A,C}$ *such that* $\mathcal{D}^{\varphi}_{t,\psi} = \mathcal{D}^{\varphi}_{t',\psi'}$*.*

The following proposition together with Lemma A.3.3 shows that the term $\ker(A^*)$ which appears in the formula (A.4.1) below could not be removed without spoiling the conclusion of Proposition A.4.3 (in contrast to what is written in the proof of the independence assertion of [34, Theorem 2.6]).

**Proposition A.4.3.** *Suppose that the assumptions of Proposition* A.4.1 *are satisfied. If* $\psi$ *and* $\psi'$ *are any two vectors in* $\mathscr{S}^{\varphi}_{A,C}$*, then*

$$\psi' - \psi \in \mathcal{D}(A) \dotplus \ker\bigl(A^*\bigr) = \mathcal{D}(A) \dotplus \mathbb{C} \cdot \varphi. \tag{A.4.1}$$

**Proof.** First we note that $\psi \notin \mathcal{D}(A) \dotplus \mathbb{C} \cdot \varphi$. Indeed, otherwise $\psi = f + \gamma \cdot \varphi$ for some $f \in \mathcal{D}(A)$ and $\gamma \in \mathbb{C}$, which implies

$$1 \overset{(A.3.1)}{=} \bigl\langle \varphi, A^*\psi\bigr\rangle = \langle \varphi, Af\rangle = \bigl\langle A^*\varphi, f\bigr\rangle = 0,$$

a contradiction. Since $\dim(\mathcal{D}(A^*)/\mathcal{D}(A)) = 2$ (see the proof of Lemma A.3.2), we deduce that $\mathcal{D}(A^*) = \mathcal{D}(A) \dotplus \mathbb{C} \cdot \varphi \dotplus \mathbb{C} \cdot \psi$. Hence, there exist $h \in \mathcal{D}(A)$ and $\alpha, \beta \in \mathbb{C}$ such that $\psi' - \psi = h + \alpha \cdot \varphi + \beta \cdot \psi$, which yields

$$0 \overset{(A.3.1)}{=} \langle A^*\psi', \varphi\rangle - \langle A^*\psi, \varphi\rangle = \langle A^*(\psi' - \psi), \varphi\rangle = \langle Ah, \varphi\rangle + \bar{\beta}\langle A^*\psi, \varphi\rangle \overset{(A.3.1)}{=} \bar{\beta}.$$

Thus $\psi' - \psi = h + \alpha \cdot \varphi \in \mathcal{D}(A) \dotplus \mathbb{C} \cdot \varphi$, which together with the equality $\ker(A^*) = \mathbb{C} \cdot \varphi$ completes the proof. □

The question of when two vectors $\psi, \psi' \in \mathscr{S}^{\varphi}_{A,C}$ represent the same operators $\{B_t\colon t \in \mathbb{R}\}$ in the sense that $\mathcal{D}^{\varphi}_{t,\psi} = \mathcal{D}^{\varphi}_{t,\psi'}$ for all $t \in \mathbb{R}$ has a simple answer.

**Proposition A.4.4.** *Suppose that the assumptions of Proposition* A.4.1 *are satisfied. If* $\psi, \psi' \in \mathscr{S}^{\varphi}_{A,C}$, *then the following conditions are equivalent*:

(i) $\mathcal{D}^{\varphi}_{t,\psi} = \mathcal{D}^{\varphi}_{t,\psi'}$ *for all* $t \in \mathbb{R}$,
(ii) *there exist* $\alpha, \beta \in \mathbb{C}$ *such that* $|\alpha|^2 + |\beta|^2 > 0$ *and* $\alpha\psi + \beta\psi' \in \mathcal{D}(A)$,
(iii) $\psi' - \psi \in \mathcal{D}(A)$.

**Proof.** (i) ⇒ (ii) Since $\psi' \in \mathcal{D}^{\varphi}_{0,\psi} = \mathcal{D}^{\varphi}_{0,\psi'}$, we see that $\psi' = h + \alpha\psi$ for some $h \in \mathcal{D}(A)$ and $\alpha \in \mathbb{C}$.

(ii) ⇒ (iii) Since $h := \alpha\psi + \beta\psi' \in \mathcal{D}(A)$, $\varphi \in \ker(A^*)$ and $\psi, \psi' \in \mathscr{S}^{\varphi}_{A,C}$, we get

$$0 = \langle\varphi, A^*(\alpha\psi + \beta\psi')\rangle = \alpha\langle\varphi, A^*\psi\rangle + \beta\langle\varphi, A^*\psi'\rangle \overset{(A.3.1)}{=} \alpha + \beta.$$

Hence, by the inequality $|\alpha|^2 + |\beta|^2 > 0$, we have $\beta \neq 0$ and $\psi' - \psi = \beta^{-1}h \in \mathcal{D}(A)$.

(iii) ⇒ (i) Obvious. □

**Remark A.4.5.** In view of the above discussion it is natural to ask whether the following implication is valid (still under the assumptions of Proposition A.4.1):

$$\psi \in \mathscr{S}^{\varphi}_{A,C}, \quad \psi' \in \mathcal{D}(A^*), \quad C\psi' = \psi', \quad \psi' - \psi \in \mathcal{D}(A) \quad \Longrightarrow \quad \psi' \in \mathscr{S}^{\varphi}_{A,C}. \tag{A.4.2}$$

We show that the answer is in the negative (note, however, that $\langle\varphi, A^*\psi'\rangle = 1$). Suppose that, contrary to our claim, the implication (A.4.2) is valid. Take $\psi \in \mathscr{S}^{\varphi}_{A,C}$ and a $C$-real vector $h \in \mathcal{D}(A)$. Then, by (A.4.2) applied to $\psi' = th + \psi$, we obtain

$$0 = \langle th + \psi, A^*(th + \psi)\rangle = t^2\langle h, Ah\rangle + 2t\,\mathrm{Re}\langle h, A^*\psi\rangle, \quad t \in \mathbb{R}.$$

Hence $\langle h, Ah\rangle = 0$ for all $C$-real vectors $h \in \mathcal{D}(A)$. This contradicts Lemma A.2.1.

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