A new look at Jarvis’ distribution formula

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Starting from the well-known transformation law for the Klein functions, we give a proof of a fairly general multiplicative distribution formula for the Siegel functions associated to isogenous complex lattices. This formula has as an immediate consequence the remarkable distribution formula proved by Jarvis in 2000 on the occasion of Rolshausen’s thesis on the second \( K \)-group of an elliptic curve.

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In 2000, Jarvis gave two different proofs of a distribution formula (see Theorem 3.10 below) for the Siegel functions on certain divisors of elliptic curves [5,6]. Some years before, a particular instance of this formula had played a crucial role in Rolshausen’s thesis on the second \( K \)-group of an elliptic curve [10].

At the origin of this paper is the desire to understand properly that intriguing formula and to find eventually a third proof of it, as elementary and as close to other classical known distribution formulas—such as those explained for example in the book of Kubert and Lang [8]—as possible. Indeed, we prove here a fairly general distribution formula for the Siegel functions on divisors of the complex numbers that gives rise naturally to Jarvis’ formula when one descends to elliptic curves. Our proof for this general distribution formula for the Siegel functions is directly based on the transformation law for the Klein functions, a law that was probably already known to Klein himself.

As mentioned above, the formula of Jarvis was important in the method developed in the work of Rolshausen and Schappacher [11] to systematically produce elements of the second \( K \)-group of an elliptic curve \( E \) from certain divisors on \( E \), in the spirit of the elliptic analogue of Zagier’s polyloga-

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rithm conjecture [13]. Actually, the authors of this work point out, in connection with Jarvis’ formula, that the direct proof of this distribution relation is somewhat nontrivial [11, 3.5].

Jarvis’ first proof of the formula [5] generalizes a proof due to Coates and Taylor appearing in an appendix to [2], while the second one [6] consists chiefly in explicit calculations with \( q \)-expansions. In our proof, we have avoided altogether subtle considerations about endomorphisms of elliptic curves as well as the handling of complicated analytic expressions and we have proceeded on the contrary in a quite elementary manner starting from classical known results about the Klein functions. Jarvis’ formula has a remarkable beauty of its own and we thought it certainly deserved a third proof making it easily accessible to everybody.

Let us now sum up briefly the contents of the paper. In Section 1, we explain all the particulars concerning the distribution formula for the Klein functions; all this is more or less known. In Section 2, we prove a distribution formula for the Siegel functions on divisors of the complex numbers (see Theorem 2.3) using mainly the known results of the first section; this formula is rather rigid in the sense that it depends strongly on the selected bases for the lattices which appear in it. In Section 3, we explain how the Siegel functions can be adapted to produce functions on elliptic curves and give a detailed and complete proof of Jarvis’ formula (see Theorem 3.10) which is a direct consequence of the formula proved in the second section for some appropriate choice of divisors. The main point in Section 3 is the fact that the Siegel function is not periodic for the corresponding lattice, so that all the subtlety of the affair consists in the control of the value of a certain real alternating bilinear form on divisors of the complex numbers.

1. Some properties of Klein functions

As explained above, we start by recalling the more or less known facts about Klein functions which we shall need in the next paragraph to prove a distribution formula for the Siegel functions on divisors of the complex numbers.

If \( z \in \mathbb{C} \), let \( \Re(z) \) and \( \Im(z) \) denote the real and imaginary part of \( z \), respectively. Let \( L \) be a complex lattice and let \( (w_1, w_2) \) be an ordered basis of \( L \) over \( \mathbb{Z} \) such that \( \Im(w_1/w_2) > 0 \). Consider the positive real number \( a(L; w_1, w_2) \) defined by

\[
a(L; w_1, w_2) = \frac{1}{2i} \begin{vmatrix} w_1 & \bar{w}_1 \\ w_2 & \bar{w}_2 \end{vmatrix} = \frac{w_1 \bar{w}_2 - \bar{w}_1 w_2}{2i} = |w_2|^2 \Im(w_1/w_2);
\]

it is straightforward to verify that, if \( (w'_1/w'_2) \) is another ordered basis of \( L \) over \( \mathbb{Z} \) such that \( \Im(w'_1/w'_2) > 0 \), then \( a(L; w_1, w_2) = a(L; w'_1, w'_2) \), so that the area \( a(L) \) of \( L \) can be defined as \( a(L; w_1, w_2) \), where \((w_1, w_2)\) is any ordered basis of \( L \) over \( \mathbb{Z} \) such that \( \Im(w_1/w_2) > 0 \). The area \( a(L) \) of the lattice \( L \) is a positive real number.

The Legendre eta invariant \( \eta_L \) associated to the complex lattice \( L \) can be defined as follows [8, Chapter 10, §1]. It happens that the series

\[
f(s) = \sum_{w \in L \setminus \{0\}} \frac{1}{w^2 |w|^2s};
\]

is convergent in the half-plane \( \Re(s) > 0 \) and produces by analytic continuation an entire function \( f : \mathbb{C} \to \mathbb{C} \), so that in particular one can define

\[
s_2(L) = f(0).
\]

Then one puts for every \( z \in \mathbb{C} \),

\[
\eta_L(z) = s_2(L)z + \frac{\pi}{a(L)} \bar{z}.
\] (1.1)
The function $\eta_L$ is $\mathbb{R}$-linear, but note that it is not a holomorphic function, since the partial derivative of $\eta_L$ with respect to $\bar{z}$ is nonzero.

Recall that the usual Weierstrass sigma function associated to the complex lattice $L$ [9, Chapter 18, §1] is defined for every $z \in \mathbb{C}$ by the infinite product

$$\sigma_L(z) = z \prod_{w \in L - \{0\}} \left( 1 - \frac{z}{w} \right) e^{\frac{z}{2} (\frac{z}{w} + \frac{1}{2})^2}.$$  

The Weierstrass sigma function $\sigma_L$ is a holomorphic function on $\mathbb{C}$ with simple zeros on $L$ and no other zeros, and it has a transformation formula with respect to the lattice $L$. This transformation formula reads as follows. For every $w \in L$, put $\chi_L(w) = 1$ if $w \in 2L$ and $\chi_L(w) = -1$ if $w \in L - 2L$. Then one has

$$\sigma_L(z + w) = \chi_L(w) \cdot e^{\eta_L(w)(z + \frac{1}{2}w)} \cdot \sigma_L(z)$$  

(1.2)

for every $z \in \mathbb{C}$ and every $w \in L$.

We come now to the Klein functions. The Klein function $K_L$ associated to the complex lattice $L$ is the function defined for every $z \in \mathbb{C}$ by the equality

$$K_L(z) = e^{-\frac{1}{2}z\eta_L(z)} \sigma_L(z).$$  

(1.3)

It is then clear from the definition that, like the sigma function, the Klein function $K_L$ has zeros on $L$ and no other zeros.

Of course, the Klein function $K_L$ is not a holomorphic function as the sigma function is, but nevertheless it has, like the sigma function, a transformation law with respect to the lattice $L$. This law was probably known to Klein (in Lang’s opinion). Let us recall it. Firstly, for every $x \in \mathbb{C}$, let us define

$$e(x) = e^{2\pi i x},$$

and secondly, for every $(u, v) \in \mathbb{C} \times \mathbb{C}$, let us define

$$H_L(u, v) = \frac{\bar{u}v}{a(L)}.$$  

The map $H_L$ is a Hermitian form. If we put $E_L = \Delta \circ H_L$, then for every $(u, v) \in \mathbb{C} \times \mathbb{C}$ we have

$$E_L(u, v) = \frac{1}{2i} \left( \frac{\bar{u}v - u\bar{v}}{a(L)} \right).$$  

(1.4)

The map $E_L$ is a real alternating bilinear form. If $(w_1, w_2)$ is an ordered basis of $L$ over $\mathbb{Z}$ such that $\Delta(w_1/w_2) > 0$, $u$ and $v$ are complex numbers such that $u = aw_1 + bw_2$ and $v = cw_1 + dw_2$ with $a$, $b$, $c$, $d$ real numbers, it is easy to see that

$$E_L(u, v) = bc - ad.$$  

(1.5)

In particular, the values of $E_L$ on $\mathbb{L} \times L$ are rational integers.

The transformation law for the Klein function states that

$$K_L(z + w) = \chi_L(w) \cdot e^{(E_L(w, z)/2)} \cdot K_L(z)$$  

(1.6)

for every $z \in \mathbb{C}$ and every $w \in L$. The proof of (1.6) is immediate from (1.1)–(1.3) ([8, Chapter 10, §1, Theorem 1.3]; note that the minus sign in the exponent of the formula appearing in Theorem 1.3 seems to be a misprint).
Another property of the Klein functions that we shall need in the sequel is the fact that they are homogeneous of degree one, that is, for every \( \lambda \in \mathbb{C}^* \) and every \( z \in \mathbb{C} \) one has
\[
K_{\lambda L}(\lambda z) = \lambda K_L(z).
\] (1.7)

For a proof of (1.7), one can see [8, Chapter 2, §1, p. 27].

Let now \( L \) and \( \Lambda \) be two complex lattices such that \( L \subseteq \Lambda \); of course, this implies that \( [\Lambda : L] \) is finite. The Klein functions satisfy a multiplicative distribution formula, which describes what happens when one changes the lattice. Indeed, for any fixed set of representatives \( R \) of \( \Lambda / L \) and for any \( z \in \mathbb{C} - \Lambda \), one has
\[
K_{\Lambda}(z) = \mathcal{E}(z) \cdot K_L(z) \cdot \prod_{t \in R} \frac{K_L(z + t)}{K_L(t)}
\] (1.8)

where the product \( \prod_{t \in R} \) means as usual the product with \( t \) running through the set of elements of \( R \), except for the element representing the trivial class of \( \Lambda / L \). For a proof of formula (1.8), one can read either [7, Theorems 2.3 and 2.4] or [1, §1, 1.2].

The remainder of this section is devoted to computing the value of the product
\[
\prod_{t \in R} K_L(t)
\]

for certain sets of representatives \( R \) of the quotient \( \Lambda / L \), where \( L \) and \( \Lambda \) are two complex lattices such that \( L \subseteq \Lambda \), and deduce from it and (1.8) a distribution formula for the Klein functions more satisfactory for our purposes than (1.8) (see formula (1.14) below).

Let \( z_1, z_2 \in \mathbb{C}^* \) such that \( \Im(z_1/z_2) > 0 \) and consider the lattice
\[
L(z_1, z_2) = \mathbb{Z}z_1 + \mathbb{Z}z_2.
\]

The Eisenstein series of weight 4 and 6 associated to \( z_1, z_2 \) are defined respectively by
\[
G_2(z_1, z_2) = \sum_{w \in L(z_1, z_2)}' \frac{1}{w^4}, \quad G_3(z_1, z_2) = \sum_{w \in L(z_1, z_2)}' \frac{1}{w^6}.
\]

Let us put
\[
g_2(z_1, z_2) = 60G_2(z_1, z_2), \quad g_3(z_1, z_2) = 140G_3(z_1, z_2)
\]
and define as usual
\[
\Delta(z_1, z_2) = g_2(z_1, z_2)^3 - 27g_3(z_1, z_2)^2.
\]

It is well known that \( \Delta(z_1, z_2) \neq 0 \) for every \( z_1, z_2 \in \mathbb{C}^* \) such that \( \Im(z_1/z_2) > 0 \) (see for example [12, Chapter 7, §2, 2.3]). It is clear from the definition of the delta function that if \( z_1, z_2 \in \mathbb{C}^* \) with \( \Im(z_1/z_2) > 0 \), \( z_1', z_2' \in \mathbb{C}^* \) with \( \Im(z_1'/z_2') > 0 \) and \( z_1, z_2 \) and \( z_1', z_2' \) generates the same complex lattice \( L \), then
\[
\Delta(z_1, z_2) = \Delta(z_1', z_2').
\] (1.9)

The delta function has a development as infinite product given by
\[
\Delta(z_1, z_2) = (2\pi / z_2)^{12} e(z_1/z_2) \prod_{n=1}^{\infty} (1 - e(nz_1/z_2))^{24}.
\] (1.10)
The square of the Dedekind eta function in two variables is defined by the infinite product

\[ \eta^2(z_1, z_2) = (2\pi i/z_2)e(z_1/12z_2)\prod_{n=1}^{\infty}(1-e(nz_1/z_2))^2 \]  (1.11)

for every \( z_1, z_2 \in \mathbb{C}^* \) such that \( \Im(z_1/z_2) > 0 \). Then it is clear from (1.10) and (1.11) that

\[ \eta^2(z_1, z_2)^{12} = \Delta(z_1, z_2). \]  (1.12)

so that we have \( \eta^2(z_1, z_2) \neq 0 \) for every \( z_1, z_2 \in \mathbb{C}^* \) such that \( \Im(z_1/z_2) > 0 \).

We need now to explain some particulars concerning complex lattices and their bases. As before, let \( L \) and \( \Lambda \) be two complex lattices such that \( L \subseteq \Lambda \). Every time one chooses an ordered basis \((w_1, w_2)\) of \( L \) over \( \mathbb{Z} \) so that for some positive integers \( r \) and \( s \), the ordered pair \((w'_1, w'_2)\) defined by

\[ w'_1 = \frac{w_1}{r}, \quad w'_2 = \frac{w_2}{s}, \]

is a basis of \( \Lambda \) over \( \mathbb{Z} \), we shall say that the set \( \mathcal{R} \) of representatives of \( \Lambda/L \) defined by

\[ \mathcal{R} = \{iw'_1+jw'_2; \ 0 \leq i \leq r-1, \ 0 \leq j \leq s-1\} \]

is the set of representatives of \( \Lambda/L \) adapted to the basis \((w_1, w_2)\). As the reader may easily verify, this way of speaking is justified by the fact that, when \((w_1, w_2)\) is an ordered basis of \( L \) over \( \mathbb{Z} \), it can happen that there is not an ordered pair \((r, s)\) of positive integers such that \((w_1/r, w_2/s)\) is a basis of \( \Lambda \) over \( \mathbb{Z} \), but if there is one such pair, it is completely determined by the ordered basis \((w_1, w_2)\) (and moreover \([\Lambda : L] = rs\)).

In any case, the theory of finitely generated abelian groups assures that it is always possible to choose an ordered basis \((w_1, w_2)\) of \( L \) such that there is an ordered pair \((r, s)\) of positive integers such that \((w_1/r, w_2/s)\) is a basis of \( \Lambda \) over \( \mathbb{Z} \) and, if necessary, such that \( \Im(w_1/w_2) > 0 \) (to show this, it suffices to notice that one can change \( w_1 \) by \(-w_1\), without changing neither \( r \) nor \( s \)) and even with \( r \) dividing \( s \) (although this last condition will be ignored in the sequel).

Choose then any ordered basis \( B = (w_1, w_2) \) of \( L \) over \( \mathbb{Z} \) such that there is an ordered couple \((r, s)\) of positive integers such that \((w'_1, w'_2) = (w_1/r, w_2/s)\) is a basis of \( \Lambda \) over \( \mathbb{Z} \). Since the ordered couple \((r, s)\) is determined by \( B \), the 8th root of unity \( \epsilon_B \) defined by

\[ \epsilon_B = e^{\left(\frac{3rs + r - s - 3}{8}\right)}, \]

certainly depends only on \( B \). Let \( \mathcal{R} \) denote the set of representatives of \( \Lambda/L \) adapted to the basis \( B \). Then, when \( \Im(w_1/w_2) > 0 \), one has

\[ \prod_{t \in \mathcal{R}} K_L(t) = \epsilon_B \cdot \eta^2(w'_1, w'_2) \cdot \eta^2(w_1, w_2)^{-[\Lambda : L]]. \]  (1.13)

For a proof of formula (1.13), one can see for example [7, Theorem 2.4, p. 235].

Finally, from (1.8) and (1.13) we have for any \( z \in \mathbb{C} - \Lambda \)

\[ K_\Lambda(z) = \epsilon_B^{-1} \cdot \eta^2(w'_1, w'_2)^{-1} \cdot \eta^2(w_1, w_2)^{[\Lambda : L]} \cdot e\left(E_L\left(z, \sum_{t \in \mathcal{R}} t \right) / 2\right) \cdot \prod_{t \in \mathcal{R}} K_L(z + t) \]  (1.14)

where \( B = (w_1, w_2) \) is any ordered basis of \( L \) over \( \mathbb{Z} \) such that \( \Im(w_1/w_2) > 0 \) and such that there is an ordered couple \((r, s)\) of positive integers such that \((w'_1, w'_2) = (w_1/r, w_2/s)\) is a basis of \( \Lambda \) over \( \mathbb{Z} \) and \( \mathcal{R} \) is the set of representatives of \( \Lambda/L \) adapted to the basis \( B \).
2. A distribution formula for the Siegel functions

Let \( L \) be a complex lattice. If \((w_1, w_2)\) is an ordered basis of \( L \) over \( \mathbb{Z} \) such that \( \Im(w_1/w_2) > 0 \), the Siegel function associated to the ordered basis \((w_1, w_2)\) of the lattice \( L \) is the function defined for every \( z \in \mathbb{C} \) by the equality

\[
\varphi_L(z; w_1, w_2) = K_L(z) \cdot \eta^2(w_1, w_2),
\]

(2.1)

where \( \eta^2 \) is the square of the Dedekind eta function in two variables, defined in (1.11). It is an immediate consequence of the fact that \( \eta^2(w_1, w_2) \neq 0 \) that the function \( \varphi_L(z; w_1, w_2) \), like the Klein function \( K_L \), has zeros on \( L \) and no other zeros. Furthermore, from the transformation formula (1.6) for the Klein function \( K_L \), one obtains that

\[
\varphi_L(z + w; w_1, w_2) = \chi_L(w) \cdot e \left( E_L(w, z) / 2 \right) \cdot \varphi_L(z; w_1, w_2)
\]

(2.2)

for every \( z \in \mathbb{C} \) and every \( w \in L \).

**Theorem 2.1.** Let \( L \) and \( \Lambda \) be complex lattices such that \( L \subseteq \Lambda \); let \( \mathcal{B} = (w_1, w_2) \) be an ordered basis of \( L \) over \( \mathbb{Z} \) such that \( \Im(w_1/w_2) > 0 \) and so that for some positive integers \( r \) and \( s \), the ordered pair \((w'_1, w'_2)\) defined by \( w'_1 = w_1/r \), \( w'_2 = w_2/s \), is a basis of \( \Lambda \) over \( \mathbb{Z} \) and \( \mathcal{R} \) be the set of representatives of \( \Lambda / L \) adapted to the basis \( \mathcal{B} \). Then we have the following distribution formula for the Siegel functions

\[
\varphi_\Lambda(z; w'_1, w'_2) = e_{\mathcal{B}}^{-1} \cdot e \left( E_L \left( z, \sum_{t \in \mathcal{R}} t \right) / 2 \right) \cdot \prod_{t \in \mathcal{R}} \varphi_L(z + t; w_1, w_2),
\]

(2.3)

valid for any \( z \in \mathbb{C} - \Lambda \).

**Proof.** According to definition (2.1), one has

\[
\varphi_\Lambda(z; w'_1, w'_2) = K_\Lambda(z) \cdot \eta^2(w'_1, w'_2).
\]

(2.4)

It follows from (2.4) and (1.14) that

\[
\varphi_\Lambda(z; w'_1, w'_2) = e_{\mathcal{B}}^{-1} \cdot \eta^2(w_1, w_2)^{|\Lambda / L|} \cdot e \left( E_L \left( z, \sum_{t \in \mathcal{R}} t \right) / 2 \right) \cdot \prod_{t \in \mathcal{R}} K_L(z + t).
\]

(2.5)

But again according to definition (2.1) one has

\[
\prod_{t \in \mathcal{R}} K_L(z + t) = \eta^2(w_1, w_2)^{|\Lambda / L|} \cdot \prod_{t \in \mathcal{R}} \varphi_L(z + t; w_1, w_2).
\]

(2.6)

From (2.5) and (2.6) one gets (2.3). \( \Box \)

Our next step will be to prove a version of Theorem 2.1 for divisors of \( \mathbb{C} \). For every divisor \( D = \sum_{a \in \mathbb{C}} n_a [a] \) of \( \mathbb{C} \) and for every \( t \in \mathbb{C} \), denote by \( D \oplus t \) the translated divisor, defined by

\[
D \oplus t = \sum_{a \in \mathbb{C}} n_a [a + t].
\]

(2.7)

Moreover, if \( D = \sum_{a \in \mathbb{C}} n_a [a] \) is a divisor of \( \mathbb{C} \), let us define

\[
\varphi_L(D; w_1, w_2) = \prod_{a \in \mathbb{C} - L} \varphi_L(a; w_1, w_2)^{n_a};
\]

(2.8)
of course, we put \( \varphi_L(D; w_1, w_2) = 1 \) if the support of \( D \) is contained in \( L \). It is an obvious remark that, given two divisors \( D \) and \( D' \) of \( C \), one has

\[
\varphi_L(D + D'; w_1, w_2) = \varphi_L(D; w_1, w_2) \cdot \varphi_L(D'; w_1, w_2).
\]

In other words, the Siegel function \( \varphi_L(z; w_1, w_2) \) produces a homomorphism from the group of divisors of \( C \) to \( C^* \). Finally, if \( D = \sum_{a \in C} n_a(a) \) is a divisor of \( C \), let us call \( \deg(D) \) its degree and \( \sum(D) \) its sum, that is

\[
\deg(D) = \sum_{a \in C} n_a, \quad \sum(D) = \sum_{a \in C} n_a a.
\]

Of course, the map \( \deg \) is a homomorphism from the group of divisors of \( C \) to \( \mathbb{Z} \) and the map \( \sum \) is a homomorphism from the group of divisors of \( C \) to \( C \).

**Theorem 2.2.** Under the same hypotheses as in Theorem 2.1, let \( D \) be a divisor of \( C \) with support disjoint from \( \Lambda \). Then we have

\[
\varphi_A(D; w_1', w_2') = e_B^{-\deg(D)} \cdot e\left(E_L\left(\sum(D), \prod_{t \in R} t / 2\right) \cdot \prod_{t \in R} \varphi_L(D + t; w_1, w_2)\right). \tag{2.9}
\]

**Proof.** Since \( D \) has support disjoint from \( \Lambda \), the same is true for the divisor \( D + t \) for every \( t \in R \). As \( L \subseteq \Lambda \), this implies that the divisor \( D + t \) has support disjoint from \( L \) for every \( t \in R \). Hence, (2.9) is an easy consequence of definitions (2.7) and (2.8), formula (2.3) and the bilinearity of the map \( E_L \). \( \square \)

Indeed, Theorem 2.2 is a big generalization of Theorem 4.1 in [1]. We seize here the opportunity to point out that in that theorem one must replace the divisor \( (x + y) + (x - y) - 2(x - 2(y) + 2(0) \) by the divisor \( (x + y) + (x - y) - 2(x) - 2(y) \) and the condition \( \text{Supp}(D) \cap \Lambda = \{0\} \) by the condition \( \text{Supp}(D) \cap \Lambda = \emptyset \) for the theorem to be true.

For every divisor \( D = \sum_{a \in C} n_a(a) \) of \( C \) and for every \( \alpha \in C^* \), denote by \( \alpha D \) the divisor of \( C \) defined by

\[
\alpha D = \sum_{a \in C} n_a(\alpha a).
\]

The following is a slightly modified version of Theorem 2.2 which will be useful later when dealing with isogenous elliptic curves.

**Theorem 2.3.** Let \( \alpha \) be a nonzero complex number and \( L, \Lambda \) two complex lattices such that \( \alpha \Lambda \subseteq \Lambda \); let \( B = (w_1, w_2) \) be an ordered basis of \( \alpha \Lambda \) over \( \mathbb{Z} \) such that \( \Im(w_1/w_2) > 0 \) and so that for some positive integers \( r \) and \( s \), the ordered pair \( (w_1', w_2') \) defined by \( w_1' = w_1/r, w_2' = w_2/s \), is a basis of \( \Lambda \) over \( \mathbb{Z} \); let \( R \) be the set of representatives of \( \Lambda / \alpha \Lambda \) adapted to the basis \( B \) and let \( D \) be a divisor of \( C \) with support disjoint from \( \alpha^{-1} \Lambda \). Then we have the following distribution formula for the Siegel functions

\[
\varphi_A(\alpha D; w_1', w_2') = e_B^{-\deg(\alpha D)} \cdot e\left(E_L\left(\sum(D), \prod_{s \in S} s / 2\right) \cdot \prod_{s \in S} \varphi_L(D + s; \hat{w}_1, \hat{w}_2)\right) \tag{2.10}
\]

where \( S = \alpha^{-1} R, \hat{w}_1 = \alpha^{-1} w_1 \) and \( \hat{w}_2 = \alpha^{-1} w_2 \).
Proof. If we apply Theorem 2.2 to the divisor \( \alpha D \), the lattices \( \alpha L \) and \( \Lambda \) and the bases \( B \) and \((w_1', w_2')\), we obtain the equality

\[
\varphi_L(\alpha D; w_1', w_2') = e_{\deg(\alpha D)} \cdot e\left(E_{\alpha L}\left(\sum_{t \in \mathcal{R}} t \right)/2\right) \cdot \prod_{t \in \mathcal{R}} \varphi_{\alpha L}(\alpha D \oplus t; w_1, w_2). \tag{2.11}
\]

On the one hand, it is immediate from the definition of the area of a lattice that

\[
a(\alpha L) = \alpha \tilde{a} \cdot a(L),
\]

so that, recalling definition (1.4), for every \( x, y \) in \( \mathbb{C} \) one has

\[
E_{\alpha L}(x, y) = E_L(\alpha^{-1}x, \alpha^{-1}y) \tag{2.12}
\]

and this implies, taking into account the fact that \( \sum(\alpha D) = \alpha \cdot \sum(D) \), that

\[
e\left(E_{\alpha L}\left(\sum_{t \in \mathcal{R}} t \right)/2\right) = e\left(E_L\left(\sum_{s \in \mathcal{S}} s \right)/2\right). \tag{2.13}
\]

On the other hand, it follows from (1.11) that

\[
\eta^2(w_1, w_2) = \alpha^{-1} \eta^2(\tilde{w}_1, \tilde{w}_2); \tag{2.14}
\]

from (2.1), (1.7) and (2.14), one sees that

\[
\varphi_{\alpha L}(\alpha z; w_1, w_2) = \varphi_L(z; \tilde{w}_1, \tilde{w}_2)
\]

for every \( z \) in \( \mathbb{C} \), so that

\[
\prod_{t \in \mathcal{R}} \varphi_{\alpha L}(\alpha D \oplus t; w_1, w_2) = \prod_{s \in \mathcal{S}} \varphi_L(D \oplus s; \tilde{w}_1, \tilde{w}_2). \tag{2.15}
\]

The equality (2.11) together with (2.13) and (2.15) gives (2.10), as we wanted. \( \square \)

3. The distribution formula of Jarvis, revisited

Let \( L \) be a complex lattice. If \((w_1, w_2)\) is an ordered basis of \( L \) over \( \mathbb{Z} \) such that \( \mathbb{N}(w_1/w_2) > 0 \), we have introduced in (2.1) the Siegel function \( \varphi_L(z; w_1, w_2) \), which is a function defined for every \( z \in \mathbb{C} \) and taking values in \( \mathbb{C} \). If now \((w_1', w_2')\) is another ordered basis of \( L \) over \( \mathbb{Z} \) such that \( \mathbb{N}(w_1'/w_2') > 0 \), then thanks to (1.9), (1.12) and (2.1) we have

\[
\varphi_L(z; w_1, w_2)^{12} = \varphi_L(z; w_1', w_2')^{12} \tag{3.1}
\]

for every \( z \in \mathbb{C} \).

Let us consider the multiplicative abelian group \( \mathbb{C}^* \otimes \mathbb{Z} \mathbb{Q} \); it is easy to see that, if \( u \) is a root of unity in \( \mathbb{C} \), then \( u \otimes 1 = 1 \otimes u \) in \( \mathbb{C}^* \otimes \mathbb{Z} \mathbb{Q} \). This means that when one passes from \( \mathbb{C}^* \) to \( \mathbb{C}^* \otimes \mathbb{Z} \mathbb{Q} \), all roots of unity of \( \mathbb{C}^* \) are identified with the neutral element; in other words, this is a natural way of killing the nontrivial torsion of \( \mathbb{C}^* \). Indeed, we have the following lemma.

**Lemma 3.1.** The kernel of the canonical group morphism from \( \mathbb{C}^* \) to \( \mathbb{C}^* \otimes \mathbb{Z} \mathbb{Q} \) defined by \( x \mapsto x \otimes 1 \) is the group of roots of unity in \( \mathbb{C}^* \).

**Proof.** See for example [3, 3.8.13]. \( \square \)
As \( \varphi_L(z; w_1, w_2) = 0 \) if and only if \( z \in L \), equality (3.1) and Lemma 3.1 say to us that we can define a function \( \varphi_L \) from \( C \) to the group \( \mathbb{C}^* \otimes \mathbb{Z} \) by putting for every \( z \in C - L \)

\[
\varphi_L(z) = \varphi_L(z; w_1, w_2) \otimes 1,
\]

where \((w_1, w_2)\) is any ordered basis of \( L \) over \( \mathbb{Z} \) such that \( \Im(w_1/w_2) > 0 \) and

\[ \varphi_L(z) = 1 \otimes 1 \]

for every \( z \in L \). From (2.2) and Lemma 3.1 it is immediate that

\[ \varphi_L(z + w) = (e(E_L(w, z)/2) \otimes 1) \cdot \varphi_L(z) \tag{3.2} \]

for every \( z \in C \) and every \( w \in L \).

If \( D = \sum_{a \in C} \Gamma_0(a) \) is a divisor of \( C \), let us put by definition

\[ \varphi_L(D) = \prod_{a \in C} \varphi_L(a)^{\Gamma_0}. \tag{3.3} \]

We have thus defined a function \( \varphi_L \) on the group \( \text{Div}(C) \) of divisors of \( C \) and taking values in the group \( \mathbb{C}^* \otimes \mathbb{Z} \mathbb{Q} \). It is an obvious remark that, given two divisors \( D \) and \( D' \) of \( C \), one has

\[ \varphi_L(D + D') = \varphi_L(D) \cdot \varphi_L(D'); \tag{3.4} \]

in other words, \( \varphi_L \) is in fact a group homomorphism from \( \text{Div}(C) \) to \( \mathbb{C}^* \otimes \mathbb{Z} \mathbb{Q} \).

Let \( E = C/L \) be the complex elliptic curve defined by the lattice \( L \). It is important to notice that if \( P \in E \), in general we have not a natural way of defining \( \varphi_L(P) \). The reason is that if \( x \) is a lift of \( P \) in \( C \), that is, if \( x \) is a complex number such that \( P = x + L \), then the transformation formula (3.2) for the function \( \varphi_L \) says to us that

\[ \varphi_L(x + w) = (e(E_L(w, x)/2) \otimes 1) \cdot \varphi_L(x) \]

for every \( w \in L \) and it may well happen that \( e(E_L(w, x)/2) \otimes 1 \) is not \( 1 \otimes 1 \) in \( \mathbb{C}^* \otimes \mathbb{Z} \mathbb{Q} \) for some \( w \in L \).

Nevertheless, we have the following lemma and its corollary.

**Lemma 3.2.** Let \( z, z' \in C \). If there are nonzero rational integers \( m, n \) such that \( mz \in L \) and \( nz' \in L \), then \( E_L(z, z') \in \mathbb{Q} \).

**Proof.** Suppose \( m, n \) are nonzero rational integers such that \( mz \in L \) and \( nz' \in L \). Then, since \( mz \) and \( nz' \) are both in \( L \), we have \( E_L(mz, nz') \in \mathbb{Z} \). As \( E_L(mz, nz') = mnE_L(z, z') \), dividing by \( mn \) one deduces that \( E_L(z, z') \in \mathbb{Q} \). \( \square \)

**Corollary 3.3.** Let \( z \in C \). If there is a nonzero rational integer \( n \) such that \( nz \in L \), then \( \varphi_L(z + w) = \varphi_L(z) \) for every \( w \in L \).

**Proof.** It is immediate from (3.2), Lemmas 3.1 and 3.2. \( \square \)

As a consequence of Corollary 3.3, if \( P \) is a torsion point of the elliptic curve \( E \), we can define

\[ \varphi_L(P) = \varphi_L(x), \tag{3.5} \]
where \( x \) is any lift of \( P \) in \( \mathbb{C} \). So, we have found an obvious way of defining a homomorphism \( \varphi_L \) from the subgroup of \( \text{Div}(E) \) consisting of divisors with support contained in the torsion subgroup of \( E \) to \( \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Q} \). Our main task will be now to extend this homomorphism to an interesting bigger subgroup of \( \text{Div}(E) \).

Indeed, we shall be concerned in the sequel with divisors with coefficients in \( \mathbb{Q} \). If \( S \) is any set, let \( \mathbb{Q}[S] \) be the \( \mathbb{Q} \)-vector space of formal sums \( \sum_{s \in S} n_s [s] \) with \( n_s \in \mathbb{Q} \) for each \( s \in S \) and \( n_s = 0 \) for almost all \( s \). In other words,

\[
\mathbb{Q}[S] = \text{Div}(S) \otimes_{\mathbb{Z}} \mathbb{Q}.
\]

If \( D = \sum_{a \in C} n_a [a] \) is a divisor in \( \mathbb{Q}[C] \), we define

\[
\varphi_L(D) = \prod_{a \in C-L} \varphi_L(a; w_1, w_2) \otimes n_a, \tag{3.6}
\]

where \( (w_1, w_2) \) is any ordered basis of \( L \) over \( \mathbb{Z} \) such that \( \Im(w_1/w_2) > 0 \); of course, we put by definition \( \varphi_L(D) = 1 \otimes 1 \) if the support of \( D \) is contained in \( L \). Recall that we are using multiplicative notation for the tensor product \( \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Q} \), but it is important to notice that the group operation is multiplicative in the first component and additive in the second component; intuitively, if \( z \in \mathbb{C}^* \) and \( n/m \in \mathbb{Q} \) with \( n \geq 1 \), then \( z \otimes n/m \) is nothing else than the unique \( n \)th root of \( z^n \otimes 1 \) in the group \( \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Q} \). All this is the well-known extension of scalars.

We have defined in (3.6) a \( \mathbb{Q} \)-linear map \( \varphi_L \) from \( \mathbb{Q}[C] \) to \( \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Q} \) using the Siegel function. If \( E = \mathbb{C}/L \) is the complex elliptic curve defined by the lattice \( L \), recall that our problem now is to define, starting with the Siegel function \( \varphi_L \), a \( \mathbb{Q} \)-linear map from an interesting subspace of \( \mathbb{Q}[E] \) to \( \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Q} \).

Let \( G \) be an abelian group. We have a natural \( \mathbb{Q} \)-linear map

\[
q_2(G) : \mathbb{Q}[G] \rightarrow G \otimes_{\mathbb{Z}} G \otimes_{\mathbb{Z}} \mathbb{Q}
\]

defined by putting \( x \mapsto x \otimes x \otimes 1 \) for every \( x \in G \) and extending linearly. For each subgroup \( H \) of \( G \), we shall call \( q_2(G, H) \) the restriction of \( q_2(G) \) to the subspace \( \mathbb{Q}[G-H] \).

**Lemma 3.4.** Let \( G \) be an abelian group and let \( \text{Tor}(G) \) be its torsion subgroup. The kernel of \( q_2(G) \) is generated as a \( \mathbb{Q} \)-vector space by divisors of the following forms:

(a) Prime divisors \( [a] \) with \( a \in \text{Tor}(G) \).
(b) Divisors \( [a] - [a-b] \) with \( a \in G \), \( b \in \text{Tor}(G) \).
(c) Divisors \( [Na] - N^2[a] \) with \( a \in G \) and \( N \) an integer, \( N \geq 1 \).
(d) Divisors \( [a+b] + [a-b] - 2[a] - 2[b] \) with \( a, b \in G \).

**Proof.** Let \( R \) be the \( \mathbb{Q} \)-subspace of \( \mathbb{Q}[G] \) generated by divisors of the forms (a)-(d). It is an easy exercise to check that \( R \) is contained in the kernel of \( q_2(G) \), so that our task will be to prove that the kernel of \( q_2(G) \) is contained in \( R \).

Consider the extension of scalars \( G \otimes_{\mathbb{Z}} \mathbb{Q} \); it is immediate that any element in \( G \otimes_{\mathbb{Z}} \mathbb{Q} \) may be written in the form \( g \otimes \frac{1}{n_i} \), where \( g \in G \) and \( n_i \in \mathbb{Z}, n_i \geq 1 \). Let \( \{ e_i \otimes \frac{1}{n_i} \}_{i \in I} \) be a basis of \( G \otimes_{\mathbb{Z}} \mathbb{Q} \) as a \( \mathbb{Q} \)-vector space; multiplying each \( e_i \otimes \frac{1}{n_i} \) by \( n_i \) we still have a basis, so that \( B = \{ e_i \otimes 1 \}_{i \in I} \) is also a basis of \( G \otimes_{\mathbb{Z}} \mathbb{Q} \) as a \( \mathbb{Q} \)-vector space.

Suppose \( g \) is any element in \( G \). If we write \( g \) as a \( \mathbb{Q} \)-linear combination of elements of \( B \) and reduce to a common denominator, we find integers \( c_i \) with \( c_i = 0 \) for almost all \( i \in I \) and an integer \( n \geq 1 \) depending on \( g \) such that

\[
g \otimes 1 = \left( \sum_{i \in I} c_i e_i \right) \otimes \frac{1}{n}.
\]
Or equivalently, such that

\[ ng \otimes 1 = \left( \sum_{i \in I} c_i e_i \right) \otimes 1. \]

This implies [3, 3.8.13] the existence of a \( t \in \text{Tor}(G) \) depending on \( g \) and \( n \) such that

\[ ng = \sum_{i \in I} c_i e_i + t. \]  

(3.7)

From the definition of \( R \) and (3.7) one gets

\[ n^2\{g\} \equiv \{ng\} \equiv \{ng - t\} \equiv \sum_{i \in I} c_i e_i \mod R \]

so that

\[ \{g\} \equiv \frac{1}{n^2} \sum_{i \in I} c_i e_i \mod R. \]  

(3.8)

For \( u, v \) in \( G \), it is immediate that

\[ \{u - v\} \equiv -\{u + v\} + 2\{u\} + 2\{v\} \mod R. \]  

(3.9)

Repeated use of (3.9) shows the existence of a finite family of integers \( m_j \) \((1 \leq j \leq s)\) and a family of non-negative integers \( n_{j,i} \) \((1 \leq j \leq s, i \in I)\) with \( n_{j,i} = 0 \) for almost all \( i \in I \) such that

\[ \left\{ \sum_{i \in I} c_i e_i \right\} \equiv \sum_{j=1}^{m} m_j \left\{ \sum_{i \in I} n_{j,i} e_i \right\} \mod R. \]  

(3.10)

For \( u, v \) and \( w \) in \( G \), one has

\[ \{u + v + w\} \equiv -\{u + v - w\} + 2\{u + v\} + 2\{w\} \mod R \]

and

\[ \{u + v + w\} \equiv -\{u - v + w\} + 2\{u + w\} + 2\{v\} \mod R \]

so that

\[ 2\{u + v + w\} \equiv -\{u + v - w\} - \{u - v + w\} + 2\{u + v\} + 2\{u + w\} + 2\{v\} + 2\{w\} \mod R. \]  

(3.11)

But

\[ \{u + v - w\} + \{u - v + w\} \equiv 2\{u\} + 2\{v - w\} \]

\[ \equiv 2\{u\} - 2\{v + w\} + 4\{v\} + 4\{w\} \mod R \]  

(3.12)
where the second congruence comes from (3.9). Combining (3.11) with (3.12) and dividing by 2 one finds

\[ (u + v + w) \equiv (u + v) + (u + w) + (v + w) - (u) - (v) - (w) \mod R. \]  

(3.13)

Observe also that for \( z \) in \( G \), one has

\[ \{2z\} \equiv 4\{z\} \mod R. \]  

(3.14)

Repeated use of (3.13) and (3.14) shows that for any finite linear combination \( \sum_{i \in I} r_i e_i \) of the \( e_i \) with positive integers \( r_i \) as coefficients, \( \{\sum_{i \in I} r_i e_i\} \) is congruent mod \( R \) to a finite linear combination over \( \mathbb{Z} \) of \( \{e_i + e_j\} \) with \( i \neq j \) and \( \{e_i\} \). Hence, recalling (3.10) and (3.8), one sees that \( \{g\} \) is congruent mod \( R \) to a finite linear combination over \( \mathbb{Q} \) of \( \{e_i + e_j\} \) with \( i \neq j \) and \( \{e_i\} \). As this is true for all \( g \in G \), we conclude that any \( \gamma \in \mathbb{Q}[G] \) is congruent mod \( R \) to a finite linear combination over \( \mathbb{Q} \) of \( \{e_i + e_j\} \) with \( i \neq j \) and \( \{e_i\} \).

Let us see now that the kernel of \( q_2(G) \) is contained in \( R \). Take any \( \gamma \) in the kernel of \( q_2(G) \). Then there is a \( \gamma' \in R \), a family of rational numbers \( n_{i,j} \) \((i, j \in I, i \neq j)\) almost all zero and a family of rational numbers \( n_i \) \((i \in I)\) almost all zero such that

\[ \gamma = \gamma' + \sum_{i \neq j} n_{i,j}(e_i + e_j) + \sum_{i \in I} n_i(e_i). \]  

(3.15)

Since \( \gamma \) and \( \gamma' \) are both in the kernel of \( q_2(G) \), (3.15) implies that

\[ \sum_{i \neq j} n_{i,j}(e_i + e_j) + \sum_{i \in I} n_i(e_i) \]  

is also in the kernel of \( q_2(G) \), so that

\[ \sum_{i \neq j} n_{i,j}(e_i \otimes e_j \otimes 1) + \sum_{i \in I} n_i(e_i \otimes e_i \otimes 1) = 0. \]  

(3.16)

Since the \( e_i \otimes 1 \) are linearly independent over \( \mathbb{Q} \), one can deduce from (3.16) that all \( n_{i,j} \) and all \( n_i \) are zero, so that coming back to (3.15) one sees that \( \gamma = \gamma' \) and therefore \( \gamma \in R \), as we wanted. \( \square \)

**Lemma 3.5.** Let \( G \) be an abelian group, \( \text{Tor}(G) \) its torsion subgroup and \( H \) a subgroup of \( \text{Tor}(G) \). The kernel of \( q_2(G, H) \) is generated as a \( \mathbb{Q} \)-vector space by divisors of the following forms:

(a) Prime divisors \([a] \) with \( a \in \text{Tor}(G) - H \).

(b) Divisors \([a] - [a - b] \) with \( a \in G - H \), \( b \in \text{Tor}(G) \), \( a - b \in \text{Tor}(G) - H \).

(c) Divisors \([Na] - N^2[a] \) with \( N \) an integer, \( N \geq 1 \), and \( Na \in G - H \).

(d) Divisors \([a + b] + [a - b] - 2[a] - 2[b] \) with \( a \in G - H \), \( b \in G - H \), \( a + b \in G - H \) and \( a - b \in G - H \).

**Proof.** Let \( S \) be the \( \mathbb{Q} \)-subspace of \( \mathbb{Q}[G - H] \) generated by divisors of the forms (a)-(d). It is very easy to check that \( S \) is contained in the kernel of \( q_2(G, H) \), so that we must prove that the kernel of \( q_2(G, H) \) is contained in \( S \).

Take any \( \gamma \) in the kernel of \( q_2(G, H) \); then \( \gamma \) is also in the kernel of \( q_2(G) \) and so, thanks to Lemma 3.4, one has that \( \gamma \in R \). Consider the canonical \( \mathbb{Q} \)-linear map \( f \) from \( \mathbb{Q}[G] \) to \( \mathbb{Q}[G - H] \) defined by sending to 0 all divisors in \( \mathbb{Q}[G] \) supported on points of \( H \); then one has \( f(\gamma) \in f(R) \). But, since \( \gamma \in \mathbb{Q}[G - H] \), \( f(\gamma) = \gamma \) and so \( \gamma \in f(R) \). Now we leave to the reader the task of checking that \( f(R) \subseteq S \); indeed, it is an entertaining exercise to verify that when one applies \( f \) to any divisor of the form (a), (b), (c) or (d), as described in Lemma 3.4, one always obtains an element in \( S \). Therefore \( \gamma \in S \) and this finishes the proof of the lemma. \( \square \)
Perhaps a commentary about Lemmas 3.4 and 3.5 is in order. Results close to the content of these lemmas are stated with more or less precision in [4, Lemma 4.6], [5, Proposition 0.1], [6,10, Chapter V, Lemma 4.1], [11, Lemma 3.1], [13, Section 1.9] and [14, proposition of Section 1.7]; but among all these references, only in Rolshausen’s thesis [10, Chapter V, Lemma 4.1] one can find a sketch of the proof of the corresponding result.

Let $G$ be an abelian group and $H$ a subgroup of $G$. If $P \in G/H$, let $[P]$ be the set of $a \in G$ such that $P = a + H$. If $D = \sum_{a \in G} n_a[a]$ is a divisor in $\mathbb{Q}[G]$, we define the reduced divisor $D_H$ of $D$ modulo $H$ as

$$D_H = \sum_{P \in G/H} n_P[P],$$

(3.17)

where $n_P = \sum_{a \in [P]} n_a$ for every $P \in G/H$; obviously $D_H \in \mathbb{Q}[G/H]$ and the map $D \mapsto D_H$ defines a $\mathbb{Q}$-linear map from $\mathbb{Q}[G]$ to $\mathbb{Q}[G/H]$. If $D \in \mathbb{Q}[G]$ and $D \in \mathbb{Q}[G/H]$, we shall say that $D$ is a lift of $\mathcal{D}$ in $G$ if and only if $D_H = \mathcal{D}$.

**Lemma 3.6.** Let $M$ be a complex lattice and let $C$ be the complex elliptic curve $C/M$. Any divisor in the kernel of $q_2(C)$ has a lift in the kernel of $q_2(C)$.

**Proof.** We know thanks to Lemma 3.4 that the kernel of $q_2(C)$ is generated as a $\mathbb{Q}$-vector space by divisors of type (a)–(d) as described in that lemma. Since the reduction of divisors modulo $M$ is $\mathbb{Q}$-linear, to prove the lemma it suffices to prove it for divisors of type (a)–(d) as described in Lemma 3.4. We proceed to do this below.

(a) Let $\mathcal{D} = \{P\}$ be a prime divisor of $C$, with $P$ a torsion point of $C$. Suppose that $n$ is an integer, $n > 1$, such that $nP = P$, or equivalently, such that $(n-1)P = 0$. If $x$ is any lift of $P$ in $C$, consider the divisor

$$D = \frac{n^2}{n^2-1}x - \frac{1}{n^2-1}[nx].$$

(b) Let $\mathcal{D} = \{P\} - \{P - Q\}$ be a divisor of $C$, with $P \in C$ and $Q$ a torsion point of $C$. Suppose that $n$ is an integer, $n > 1$, such that $nQ = Q$, or equivalently, such that $(n-1)Q = 0$. If $x$ is any lift of $P$ in $C$ and $q$ is any lift of $Q$ in $C$, consider the divisor

$$D = \{x\} - \frac{n}{n-1}\{x-q\} + \frac{1}{n-1}\{x-nq\} + \frac{n}{n^2-1}\{q\} - \frac{n}{n^2-1}[nq].$$

(c) Let $\mathcal{D} = \{NP\} - N^2\{P\}$ be a divisor of $C$, with $P \in C$ and $N$ an integer, $N \geq 1$. If $x$ is any lift of $P$ in $C$, consider the divisor

$$D = \{Nx\} - N^2\{x\}.$$  

(d) Let $\mathcal{D} = \{P + Q\} + \{P - Q\} - 2\{P\} - 2\{Q\}$ be a divisor of $C$, with $P \in C$ and $Q \in C$. If $x$ is any lift of $P$ in $C$ and $y$ is any lift of $Q$ in $C$, consider the divisor

$$D = \{x + y\} + \{x - y\} - 2\{x\} - 2\{y\}.$$  

In all cases (a)–(d), one easily checks that $D$ is in the kernel of $q_2(C)$ and that $D_M = \mathcal{D}$. □

**Lemma 3.7.** Let $M$ be a complex lattice, let $C$ be the complex elliptic curve $C/M$ and let $M'$ be a complex lattice such that $M \subseteq M'$. Any divisor in the kernel of $q_2(C, M'/M)$ has a lift in the kernel of $q_2(C, M')$.

**Proof.** We can argue exactly as in the proof of Lemma 3.6 but using now Lemma 3.5 instead of Lemma 3.4. □
Lemma 3.8. Let $M$ be a complex lattice, let $C$ be the complex elliptic curve $\mathbb{C}/M$, let $D$ be a divisor in the kernel of $q_2(C)$ and let $D, D'$ be lifts of $D$ in the kernel of $q_2(\mathbb{C})$. Then $\varphi_M(D) = \varphi_M(D')$.

Proof. The fact that $D_M = D'_M$ implies that $(D - D')_M = 0$ and following definition (3.17), this implies that $D - D'$ is of the form

$$D - D' = \sum_{l=1}^{m} n_l((z_l) - (z_l + \gamma_l))$$

where $m$ is an integer, $m \geq 1$, and $n_l \in \mathbb{Q}$, $z_l \in \mathbb{C}$ and $\gamma_l \in M$ for every $l, 1 \leq l \leq m$. Thus, from (3.18) and (3.2), one has

$$\varphi_M(D)\varphi_M(D')^{-1} = e\left(-\sum_{l=1}^{m} n_l E_l(\gamma_l, z_l)/2\right) \otimes 1.$$  

(3.19)

Let $\{w_1, w_2\}$ be an ordered basis of $L$ over $\mathbb{Z}$ such that $\Delta(w_1/w_2) > 0$ and write the $z_l$ and the $\gamma_l$ as linear combinations of $w_1$ and $w_2$, that is

$$z_l = A_l w_1 + B_l w_2, \quad \gamma_l = C_l w_1 + D_l w_2$$

(3.20)

with $A_l, B_l$ in $\mathbb{R}$ and $C_l, D_l$ in $\mathbb{Z}$ for every $l, 1 \leq l \leq m$. According to (1.5), the equalities (3.20) imply that

$$E_l(\gamma_l, z_l) = A_l D_l - B_l C_l$$

for every $l, 1 \leq l \leq m$, so that

$$\sum_{l=1}^{m} n_l E_l(\gamma_l, z_l) = \sum_{l=1}^{m} n_l (A_l D_l - B_l C_l).$$

(3.21)

Since $D$ and $D'$ are both in the kernel of $q_2(\mathbb{C}, M')$, the difference $D - D'$ is also in the kernel of $q_2(\mathbb{C}, M')$. This means, taking into account (3.18), that

$$\sum_{l=1}^{m} n_l (z_l \otimes \gamma_l \otimes 1 + \gamma_l \otimes z_l \otimes 1 + \gamma_l \otimes \gamma_l \otimes 1) = 0.$$  

(3.22)

From (3.20) and (3.22), one has

$$\sum_{l=1}^{m} n_l \left[ C_l((A_l w_1) \otimes w_1 \otimes 1) + D_l((A_l w_1) \otimes w_2 \otimes 1) + C_l((B_l w_2) \otimes w_1 \otimes 1) \right.$$  

$$+ D_l((B_l w_2) \otimes w_2 \otimes 1) + C_l(w_1 \otimes (A_l w_1) \otimes 1) + C_l(w_1 \otimes (B_l w_2) \otimes 1)$$  

$$+ D_l(w_2 \otimes (A_l w_1) \otimes 1) + D_l(w_2 \otimes (B_l w_2) \otimes 1) + C_l^2(w_1 \otimes w_1 \otimes 1)$$  

$$+ C_l D_l(w_1 \otimes w_2 \otimes 1) + C_l D_l(w_2 \otimes w_1 \otimes 1) + D_l^2(w_2 \otimes w_2 \otimes 1) \right] = 0.$$  

(3.23)

Let $\{1\} \cup \{e_i\}_{i \in I}$ be a Hamel basis and write the $A_l$ and the $B_l$ in this basis

$$A_l = \alpha_l + \sum_{i \in I} \alpha_{l,i} e_i, \quad B_l = \beta_l + \sum_{i \in I} \beta_{l,i} e_i.$$  

(3.23)
so that \( \alpha_l \) and the \( \alpha_{i,j} \) (\( i \in I \)) and \( \beta_l \) and the \( \beta_{i,l} \) (\( i \in I \)) are in \( \mathbb{Q} \). Using (3.23), the last equality above becomes

\[
\sum_{l=1}^{m} n_l \prod_{i \in I} C_l \sum_{i \in I} \alpha_{l,i}(e_i w_1) \otimes w_1 \otimes 1 + D_l \sum_{i \in I} \alpha_{l,i}(e_i w_1) \otimes w_2 \otimes 1 \\
+ C_l \sum_{i \in I} \beta_{l,i}(e_i w_2) \otimes w_1 \otimes 1 + D_l \sum_{i \in I} \beta_{l,i}(e_i w_2) \otimes w_2 \otimes 1 \\
+ C_l \sum_{i \in I} \alpha_{l,i} w_1 \otimes (e_i w_1) \otimes 1 + C_l \sum_{i \in I} \beta_{l,i} w_1 \otimes (e_i w_2) \otimes 1 \\
+ D_l \sum_{i \in I} \alpha_{l,i} w_2 \otimes (e_i w_1) \otimes 1 + D_l \sum_{i \in I} \beta_{l,i} w_2 \otimes (e_i w_2) \otimes 1 \\
+ (2\alpha_l C_l + C_l^2) w_1 \otimes w_1 \otimes 1 + (\alpha_l D_l + \beta_l C_l + C_l D_l) w_1 \otimes w_2 \otimes 1 \\
+ (\alpha_l D_l + \beta_l C_l + C_l D_l) w_2 \otimes w_1 \otimes 1 + (2\beta_l D_l + D_l^2) w_2 \otimes w_2 \otimes 1 \right] = 0.
\]

Since \( w_1, w_2 \) together with the \( e_i w_1 \) and the \( e_i w_2 \) constitute a basis of \( \mathbb{C} \) as a \( \mathbb{Q} \)-vector space, the last equality implies in particular that the following linear combinations

\[
\sum_{l=1}^{m} n_l D_l \sum_{i \in I} \alpha_{l,i}(e_i w_1) \otimes w_2 \otimes 1, \quad \sum_{l=1}^{m} n_l C_l \sum_{i \in I} \beta_{l,i}(e_i w_2) \otimes w_1 \otimes 1
\]

are both zero in \( \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C} \). As the obvious map \( \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \) is \( \mathbb{Q} \)-linear, the two linear combinations (3.24) are also zero in \( \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \) and this implies

\[
\left( \sum_{l=1}^{m} n_l D_l \sum_{i \in I} \alpha_{l,i} e_i \right) (w_1 \otimes w_2 \otimes 1) = 0, \quad \left( \sum_{l=1}^{m} n_l C_l \sum_{i \in I} \beta_{l,i} e_i \right) (w_2 \otimes w_1 \otimes 1) = 0,
\]

so that

\[
\sum_{l=1}^{m} n_l D_l \sum_{i \in I} \alpha_{l,i} e_i = 0, \quad \sum_{l=1}^{m} n_l C_l \sum_{i \in I} \beta_{l,i} e_i = 0.
\]

It follows from (3.25) and (3.23) that

\[
\sum_{l=1}^{m} n_l A_l D_l - \sum_{l=1}^{m} n_l \alpha_l D_l = 0, \quad \sum_{l=1}^{m} n_l B_l C_l - \sum_{l=1}^{m} n_l \beta_l C_l = 0
\]

and this shows that

\[
\sum_{l=1}^{m} n_l A_l D_l, \quad \sum_{l=1}^{m} n_l B_l C_l
\]

are both in \( \mathbb{Q} \). Then (3.21), (3.19) and Lemma 3.1 imply that

\[
\varphi_M(D)\varphi_M(D')^{-1} = 1 \otimes 1,
\]

so that \( \varphi_M(D) = \varphi_M(D') \), as we wanted. In connection with this proof, see p. 69 of the paper of Rolshausen and Schappacher [11]. \( \Box \)
Let \( M \) be a complex lattice and let \( C \) be the complex elliptic curve \( \mathbb{C}/M \). Thanks to Lemmas 3.6 and 3.8, if \( D \) is any divisor in the kernel of \( q_2(C) \) we can define

\[
\varphi_M(D) = \varphi_M(D),
\]

(3.26)

where \( D \) is any lift of \( D \) in the kernel of \( q_2(C) \). Since the \( \varphi_M \) defined on divisors of \( C \) and the reduction modulo \( M \) are both \( \mathbb{Q} \)-linear, we have thus defined a new \( \mathbb{Q} \)-linear map \( \varphi_M \) from the kernel of \( q_2(C) \) to \( \mathbb{C}^* \otimes \mathbb{Q} \).

Let \( L \) and \( \Lambda \) be complex lattices and let \( E = \mathbb{C}/L \) and \( F = \mathbb{C}/\Lambda \) be the complex elliptic curves defined by \( L \) and \( \Lambda \), respectively. If \( \psi : E \to F \) is an isogeny from \( E \) to \( F \), then by definition there is a nonzero complex number \( \alpha \) such that

\[
\psi(z + L) = \alpha z + \Lambda
\]

(3.27)

for every \( z \in \mathbb{C} \); we shall say that the isogeny \( \psi \) is determined by \( \alpha \). The isogeny \( \psi \) induces a \( \mathbb{Q} \)-linear map

\[
\psi : \mathbb{Q}[E] \to \mathbb{Q}[F]
\]

defined in the obvious way.

**Lemma 3.9.** Let \( L \) and \( \Lambda \) be complex lattices, let \( E = \mathbb{C}/L \) and \( F = \mathbb{C}/\Lambda \) be the complex elliptic curves defined by \( L \) and \( \Lambda \), respectively, and let \( \psi : E \to F \) be an isogeny from \( E \) to \( F \). Let \( D \) be a divisor in \( \mathbb{Q}[E - \ker \psi] \). The following properties are equivalent:

(1) The divisor \( D \) is in the kernel of \( q_2(E, \ker \psi) \).
(2) The divisor \( \psi(D) \) is in the kernel of \( q_2(F, \{0\}) \).
(3) The divisor \( D \oplus S \) is in the kernel of \( q_2(E, \ker \psi) \) for every \( S \in \ker \psi \).
(4) The divisor \( \sum_{S \in \ker \psi} D \oplus S \) is in the kernel of \( q_2(E, \ker \psi) \).

**Proof.** Lemma 3.5, the \( \mathbb{Q} \)-linearity of \( \psi \) and the fact that \( \psi(\text{Tor}(E)) \subseteq \text{Tor}(F) \) have as an easy consequence that (1) implies (2). Let us prove that (2) implies (1). Take \( D \in \mathbb{Q}[E - \ker \psi] \) such that \( \psi(D) \) is in the kernel of \( q_2(F, \{0\}) \); then, after Lemma 3.5, one can write \( \psi(D) \) as a \( \mathbb{Q} \)-linear combination of divisors of the form (a)–(d) as described in that lemma. Since \( \psi \) is surjective and in fact one has \( \psi(\text{Tor}(E)) = \text{Tor}(F) \), following step by step this linear combination one can easily construct a divisor \( D' \) in the kernel of \( q_2(E, \ker \psi) \) such that \( \psi(D') = \psi(D) \), so that \( \psi(D - D') = 0 \). This implies that \( D - D' \) is of the form

\[
D - D' = \sum_{i=1}^{m} n_i ([P_i] - [P_i - Q_i])
\]

(3.28)

where \( m \) is an integer, \( m \geq 1 \), and \( n_i \in \mathbb{Q} \), \( P_i \in E - \ker \psi \) and \( Q_i \in \ker \psi \) for every \( i, 1 \leq i \leq m \). Since \( \ker \psi \subseteq \text{Tor}(E) \), from (3.28) one deduces, recalling again Lemma 3.5 (see how generators of type (b) are), that \( D - D' \) is in the kernel of \( q_2(E, \ker \psi) \); as \( D' \) is in the kernel of \( q_2(E, \ker \psi) \), \( D \) is certainly in the kernel of \( q_2(E, \ker \psi) \).

We come now to the other equivalences. To simplify notation, write here for a moment \( q_2 \) instead of \( q_2(E) \). It is straightforward \([3, 3.8.13]\) from the definition of \( q_2 \) that

\[
q_2(D \oplus T) = q_2(D)
\]

(3.29)

whenever \( T \) is a torsion point of \( E \). Since \( \ker \psi \subseteq \text{Tor}(E) \), (3.29) assures that

\[
q_2(D \oplus S) = q_2(D)
\]

(3.30)
for every $S \in \ker \psi$, and this proves that (1) implies (3). Obviously (3) implies (4). Finally, if $\ker \psi$ has $n$ elements, it follows from (3.30) that

$$q_2(D) = \frac{1}{n} \cdot q_2 \left( \sum_{S \in \ker \psi} D \oplus S \right)$$

and this proves that (4) implies (1).

The content of the next theorem is the formula of Jarvis.

**Theorem 3.10.** Let $L$ and $\Lambda$ be complex lattices, let $E = C/L$ and $F = C/\Lambda$ be the complex elliptic curves defined by $L$ and $\Lambda$, respectively, and let $\psi : E \to F$ be an isogeny from $E$ to $F$. Let $D$ be a divisor in $\mathbb{Q}[E - \ker \psi]$ satisfying any one of the equivalent conditions of Lemma 3.9. Then we have

$$\varphi_\Lambda(\psi(D)) = \prod_{S \in \ker \psi} \varphi_L(D \oplus S).$$

**Proof.** Recall that $\varphi_\Lambda$ and $\varphi_L$ are both $\mathbb{Q}$-linear, the map $\psi : \mathbb{Q}[E] \to \mathbb{Q}[F]$ is also $\mathbb{Q}$-linear and that

$$(A + B) \oplus S = A \oplus S + B \oplus S$$

for any divisors $A, B$ in $\mathbb{Q}[E]$. Thus, thanks to Lemma 3.5, to prove the theorem it is enough to prove it in case $D$ is of type (a), (b), (c) or (d) as described in that lemma. We proceed to do this below. In any case, the idea of the proof is always the same: apply Theorem 2.3 to an appropriate divisor of $C$.

(a) Let $D = \{P\}$ be a prime divisor of $E$, with $P$ a torsion point of $E$, $P \notin \ker \psi$. Suppose $n$ is an integer, $n > 1$, such that $nP = P$, or equivalently, such that $(n - 1)P = 0$. Let $x$ be any lift of $P$ in $C$ and consider the divisor

$$D = \frac{n^2}{n^2 - 1} \{x\} - \frac{1}{n^2 - 1} \{nx\}. \quad (3.31)$$

We know from the proof of Lemma 3.7 that $D$ is in the kernel of $q_2(C, \alpha^{-1} \Lambda)$ and that $D_L = D$. Then $\alpha D$ is in the kernel of $q_2(C, \Lambda)$ and $(\alpha D)_\Lambda = \psi(D)$. Thus, it follows from definition (3.26) that

$$\varphi_\Lambda(\psi(D)) = \varphi_\Lambda(\alpha D). \quad (3.32)$$

From (3.31), we have

$$\alpha D = \frac{n^2}{n^2 - 1} \{\alpha x\} - \frac{1}{n^2 - 1} \{n\alpha x\}. \quad (3.33)$$

Remark that

$$\varphi_\Lambda(n\alpha x) = \varphi_\Lambda((n - 1)\alpha x + \alpha x) = \varphi_\Lambda(\alpha x). \quad (3.34)$$

the second equality coming from (3.2), the fact that $(n - 1)\alpha x \in \Lambda$, Lemmas 3.1 and 3.2. From (3.33) and (3.34) we have

$$\varphi_\Lambda(\alpha D) = \varphi_\Lambda(\alpha x). \quad (3.35)$$
So finally from (3.32) and (3.35) we have

$$\varphi_A(\psi(D)) = \varphi_A(\alpha x). \quad (3.36)$$

Let now \(B = (w_1, w_2)\) be an ordered basis of \(\alpha L\) over \(\mathbb{Z}\) such that \(\exists (w_1/w_2) > 0\) and so that for some positive integers \(r\) and \(s\), the ordered pair \((w'_1, w'_2)\) defined by \(w'_1 = w_1/r, w'_2 = w_2/s\), is a basis of \(\Lambda\) over \(\mathbb{Z}\), let \(R\) be the set of representatives of \(\Lambda/\alpha L\) adapted to the basis \(B\) and let \(S = \alpha^{-1}R\). Consider the divisor \(\mathcal{D} \oplus S\) for every \(S \in \ker \psi\), let \(x\) be the same lift of \(P\) as above and choose the set \(S\) as a lift of \(\ker \psi = \alpha^{-1}A/L\). Proceeding in the same way as we have proceeded above to obtain (3.36) starting from the divisor \(\psi(D)\), we get the equality

$$\prod_{S \in \ker \psi} \varphi_L(D \oplus S) = \prod_{S \in S} \varphi_L(x + s). \quad (3.37)$$

But applying Theorem 2.3 to the divisor \([x]\) which certainly verifies the crucial hypothesis that its support is disjoint from \(\alpha^{-1}A\), tensoring with 1 and recalling Lemma 3.1, we obtain

$$\varphi_A(\alpha x) = \left(e\left(E_L\left(\sum_{s \in S} s/2\right) \otimes 1\right) \right) \prod_{s \in S} \varphi_L(x + s). \quad (3.38)$$

In view of (3.36)-(3.38), the theorem will be proved for divisors of type \((a)\) provided that we prove

$$e\left(E_L\left(\sum_{s \in S} s/2\right) \otimes 1\right) = 1 \otimes 1. \quad (3.39)$$

Or equivalently

$$e\left(E_L\left(x, \sum_{s \in S} s/2\right) \otimes 1\right) = 1 \otimes 1. \quad (3.40)$$

But \((n - 1)x\) and \([\alpha^{-1}A : L] \sum_{s \in S} s\) are both in \(L\), so that (3.40) is immediate from Lemmas 3.1 and 3.2.

(b) Let \(\mathcal{D} = (P) - \{P - Q\}\) be a divisor of \(E\), with \(P\) a point of \(E\) such that \(P \notin \ker \psi\) and \(Q\) a torsion point of \(E\) such that \(P - Q \notin \ker \psi\). Suppose \(n\) is an integer, \(n > 1\), such that \(nQ = Q\), or equivalently, such that \((n - 1)Q = 0\). Let \(x\) be any lift of \(P\) in \(\mathbb{C}\), let \(q\) be any lift of \(Q\) in \(\mathbb{C}\) and consider the divisor

$$D = [x] - \frac{n}{n - 1} [x - q] + \frac{1}{n - 1} [x - nq] + \frac{n}{n^2 - 1} [q] - \frac{n}{n^2 - 1} [nq]. \quad (3.41)$$

We know from the proof of Lemma 3.7 that \(D\) is in the kernel of \(q_2(\mathbb{C}, \alpha^{-1}A)\) and that \(D_L = \mathcal{D}\). Then \(\alpha D\) is in the kernel of \(q_2(\mathbb{C}, A)\) and \((\alpha D)_L = \psi(D)\). Thus, it follows from definition (3.26) that

$$\varphi_A(\psi(D)) = \varphi_A(\alpha D). \quad (3.42)$$

From (3.41), we have

$$\alpha D = [ax] - \frac{n}{n - 1} [ax - aq] + \frac{1}{n - 1} [ax - naq] + \frac{n}{n^2 - 1} [aq] - \frac{n}{n^2 - 1} [naq]. \quad (3.43)$$

Remark that

$$\varphi_A(\alpha x - naq) = \varphi_A(\alpha x - aq) \cdot e\left(E_A(\alpha x - aq, (n - 1)aq)/2\right) \otimes 1 \quad (3.44)$$
and that
\[
\varphi_A(n\alpha q) = \varphi_A(\alpha q),
\]  
(3.45)
these equalities coming from (3.2), the fact that \((n - 1)\alpha q \in A\), Lemmas 3.1 and 3.2. From (3.43)–(3.45) we have
\[
\varphi_A(\alpha D) = \varphi_A(\alpha x)\varphi_A(\alpha x - \alpha q)^{-1} \cdot e(E_A(\alpha x - \alpha q, \alpha q)/2) \otimes 1.
\]  
(3.46)
Remark that it follows from Lemmas 3.1 and 3.2 that
\[
e(\alpha x - \alpha q, \alpha q)/2) \otimes 1 = e(\alpha x, \alpha q)/2) \otimes 1
\]  
(3.47)
and that it follows from (2.12) that
\[
e(\alpha x, \alpha q)/2) \otimes 1 = e(\alpha^{-1} A(x, q)/2) \otimes 1.
\]  
(3.48)
Thus, from (3.46)–(3.48) we have
\[
\varphi_A(\alpha D) = \varphi_A(\alpha x)\varphi_A(\alpha x - \alpha q)^{-1} \cdot e(\alpha^{-1} A(x, q)/2) \otimes 1.
\]  
(3.49)
So finally from (3.42) and (3.49) we have
\[
\varphi_A(\psi(D)) = \varphi_A(\alpha x)\varphi_A(\alpha x - \alpha q)^{-1} \cdot e(\alpha^{-1} A(x, q)/2) \otimes 1.
\]  
(3.50)
Let now \(B = (w_1, w_2)\) be an ordered basis of \(\alpha L\) over \(\mathbb{Z}\) such that \(\Im(w_1/w_2) > 0\) and so that for some positive integers \(r\) and \(s\), the ordered pair \((w'_1, w'_2)\) defined by \(w'_1 = w_1/r, w'_2 = w_2/s\), is a basis of \(\Lambda\) over \(\mathbb{Z}\), let \(R\) be the set of representatives of \(\Lambda/\alpha L\) adapted to the basis \(B\) and let \(S = \alpha^{-1} R\). Consider the divisor \(D \oplus S\) for every \(S \in \ker \psi\), let \(x\) be the same lift of \(P\) as above, let \(q\) be the same lift of \(Q\) as above and choose the set \(S\) as a lift of \(\ker \psi = \alpha^{-1} \Lambda/L\). Proceeding in the same way as we have proceeded above to obtain (3.50) starting from the divisor \(\psi(D)\), we get the equality
\[
\prod_{S \in \ker \psi} \varphi_L(D \oplus S) = \prod_{s \in S} \varphi_L(x + s)\varphi_L(x + s - q)^{-1} e(E_L(x + s, q)/2) \otimes 1.
\]  
(3.51)
But it follows from Lemmas 3.1 and 3.2 that
\[
\prod_{s \in S} e(E_L(x + s, q)/2) \otimes 1 = e([\alpha^{-1} \Lambda : L]E_L(x, q)/2) \otimes 1
\]  
(3.52)
and it follows from the easy fact that
\[
a(\Lambda) = [\alpha^{-1} \Lambda : L] \cdot a(\alpha^{-1} \Lambda)
\]
that
\[
E_{\alpha^{-1} A}(x, q) = [\alpha^{-1} \Lambda : L] \cdot E_L(x, q).
\]  
(3.53)
From (3.51)–(3.53) we have
\[
\prod_{S \in \ker \psi} \varphi_L(D \oplus S) = \left( \prod_{s \in S} \varphi_L(x + s)\varphi_L(x + s - q)^{-1} \right) \cdot e(E_{\alpha^{-1} A}(x, q)/2) \otimes 1.
\]  
(3.54)
But applying Theorem 2.3 to the divisor \(\{x\} - \{x - q\}\) (which certainly verifies the crucial hypothesis that its support is disjoint from \(\alpha^{-1}A\)), tensoring with 1 and recalling Lemma 3.1, we obtain

\[
\varphi_A(\alpha x)\varphi_A(\alpha x - \alpha q)^{-1} = \left( e \left( E_L \left( \sum_{s \in S} s / 2 \right) \right) \otimes 1 \right) \\
\times \prod_{s \in S} \varphi_L(x+s) \varphi_L(x+s-q)^{-1}.
\] (3.55)

In view of (3.50), (3.54) and (3.55), the theorem will be proved for divisors of type (b) provided that we prove

\[
e \left( E_L \left( \sum_{s \in S} s / 2 \right) \right) \otimes 1 = 1 \otimes 1.
\] (3.56)

Or equivalently

\[
e \left( E_L \left( q \sum_{s \in S} s / 2 \right) \right) \otimes 1 = 1 \otimes 1.
\] (3.57)

But \((n - 1)q\) and \([\alpha^{-1} A : L] \sum_{s \in S} s\) are both in \(L\), so that (3.57) is immediate from Lemmas 3.1 and 3.2.

(c) Let \(D = \{NP\} - N^2\{P\}\) be a divisor of \(E\), with \(P\) a point of \(E\) and \(N\) an integer, \(N \geq 1\), such that \(NP \notin \ker \psi\). Let \(x\) be any lift of \(P\) in \(\mathbb{C}\), and consider the divisor

\[D = \{Nx\} - N^2\{x\}.
\] (3.58)

It is obvious that \(D\) is in the kernel of \(q_2(\mathbb{C}, \alpha^{-1}A)\) and that \(D_L = D\). Then \(\alpha D\) is in the kernel of \(q_2(\mathbb{C}, A)\) and \((\alpha D)_A = \psi(D)\). Thus, it follows from definition (3.26) that

\[\varphi_A(\psi(D)) = \varphi_A(\alpha D).
\] (3.59)

From (3.58) we have

\[\alpha D = \{N\alpha x\} - N^2\{\alpha x\}
\]

so that

\[\varphi_A(\alpha D) = \varphi_A(N\alpha x)\varphi_A(\alpha x)^{-N^2}.
\] (3.60)

From (3.59) and (3.60) we get

\[\varphi_A(\psi(D)) = \varphi_A(N\alpha x)\varphi_A(\alpha x)^{-N^2}.
\] (3.61)

Let \(B = (w_1, w_2)\) be an ordered basis of \(\alpha L\) over \(\mathbb{Z}\) such that \(\Im(w_1/w_2) > 0\) and so that for some positive integers \(r\) and \(s\), the ordered pair \((w'_1, w'_2)\) defined by \(w'_1 = w_1/r\), \(w'_2 = w_2/s\), is a basis of \(A\) over \(\mathbb{Z}\), let \(R\) be the set of representatives of \(\Lambda/\alpha L\) adapted to the basis \(B\) and let \(S = \alpha^{-1}R\).

Let now \(S \in \ker \psi\); then \(D \oplus S = \{NP + S\} - N^2\{P + S\}\). To compute \(\varphi_L(D \oplus S)\) we have to choose a lift of \(D \oplus S\) in the kernel of \(q_2(\mathbb{C}, \ker \psi)\). Let \(n\) be an integer, \(n > 1\), such that \(nS = S\), or equivalently, such that \((n - 1)S = 0\). The reader will easily check that the following divisor
\[ D(S) = \frac{n(N - n)}{1 - n^2} \{Nx + s\} + \frac{Nn(-1 + Nn)}{1 - n^2} \{x + s\} \]
\[ + \frac{1 - Nn}{1 - n^2} \{Nx + ns\} + \frac{N(-N + n)}{1 - n^2} \{x + ns\}. \tag{3.62} \]

where \( s \) is the unique lift of \( S \) in \( S \) and \( x \) is the same lift of \( P \) as above, is a lift of \( D \oplus S \) in the kernel of \( q_2(\mathbb{C}, \ker \psi) \); it has been found by indeterminate coefficients. Thus, it follows from definition (3.26) that

\[ \varphi_L(D \oplus S) = \varphi_L(D(S)). \tag{3.63} \]

Since \( (n - 1)s \in L \), it follows from (3.2) that

\[ \varphi_L(x + ns) = \varphi_L(x + s) \cdot (e(E_L((n - 1)s, x)/2) \otimes 1) \tag{3.64} \]

and that

\[ \varphi_L(Nx + ns) = \varphi_L(Nx + s) \cdot (e(E_L((n - 1)s, Nx)/2) \otimes 1). \tag{3.65} \]

From (3.62), (3.64) and (3.65), using the bilinearity of \( E_L \) we arrive easily at

\[ \varphi_L(D(S)) = \varphi_L(Nx + s)\varphi_L(x + s)^{-N^2} \cdot (e(E_L(s, (N^2 - N)x)/2) \otimes 1). \tag{3.66} \]

It follows from (3.63) and (3.66) that

\[ \varphi_L(D \oplus S) = \varphi_L(Nx + s)\varphi_L(x + s)^{-N^2} \cdot (e(E_L(s, (N^2 - N)x)/2) \otimes 1). \tag{3.67} \]

The equality (3.67) holds for every \( S \in \ker \psi \) and so we have

\[
\prod_{S \in \ker \psi} \varphi_L(D \oplus S) = \left( \prod_{S \in S} \varphi_L(Nx + s)\varphi_L(x + s)^{-N^2} \right) \\
\times \left( e(E_L\left( \sum_{s \in S} s, (N^2 - N)x \right)/2 \right) \otimes 1. \tag{3.68}
\]

But applying Theorem 2.3 to the divisor \( \{Nx\} - N^2\{x\} \) (which certainly verifies the crucial hypothesis that its support is disjoint from \( \alpha^{-1}A \), tensoring with 1 and recalling Lemma 3.1, we obtain

\[ \varphi_A(N\alpha x)\varphi_A(\alpha x)^{-N^2} = \left( \prod_{s \in S} \varphi_L(Nx + s)\varphi_L(x + s)^{-N^2} \right) \cdot \left( e\left( E_L\left( \sum_{s \in S} s, (N^2 - N)x \right)/2 \right) \otimes 1 \right). \]

Or equivalently

\[
\varphi_A(N\alpha x)\varphi_A(\alpha x)^{-N^2} = \left( \prod_{s \in S} \varphi_L(Nx + s)\varphi_L(x + s)^{-N^2} \right) \\
\times \left( e\left( E_L\left( \sum_{s \in S} s, (N^2 - N)x \right)/2 \right) \otimes 1 \right). \tag{3.69}
\]

In view of (3.61), (3.68) and (3.69), the theorem is certainly proved for divisors of type (c).
Let \( D = \{ P + Q \} + \{ P - Q \} - 2P - 2Q \) be a divisor of \( E \), with \( P \) and \( Q \) points of \( E \) such that \( P \not\in \text{ker} \psi \), \( Q \not\in \text{ker} \psi \), \( P + Q \not\in \text{ker} \psi \) and \( P - Q \not\in \text{ker} \psi \). Let \( x \) be any lift of \( P \) in \( \mathbb{C} \) and let \( y \) be any lift of \( Q \) in \( \mathbb{C} \), and consider the divisor

\[
D = \{ x + y \} + \{ x - y \} - 2\{ x \} - 2\{ y \}.
\]  

(3.70)

It is obvious that \( D \) is in the kernel of \( q_2(\mathbb{C}, \alpha^{-1}A) \) and that \( D_L = D \). Then \( \alpha D \) is in the kernel of \( q_2(\mathbb{C}, A) \) and \( (\alpha D)_A = \psi(D) \). Thus, it follows from definition (3.26) that

\[
\varphi_A(\psi(D)) = \varphi_A(\alpha D).
\]  

(3.71)

From (3.70) we have

\[
\alpha D = \{ \alpha x + \alpha y \} + \{ \alpha x - \alpha y \} - 2\{ \alpha x \} - 2\{ \alpha y \}
\]

so that

\[
\varphi_A(\alpha D) = \varphi_A(\alpha x + \alpha y)\varphi_A(\alpha x - \alpha y)\varphi_A(\alpha x)^{-2}\varphi_A(\alpha y)^{-2}.
\]  

(3.72)

Thus, from (3.71) and (3.72) we get

\[
\varphi_A(\psi(D)) = \varphi_A(\alpha x + \alpha y)\varphi_A(\alpha x - \alpha y)\varphi_A(\alpha x)^{-2}\varphi_A(\alpha y)^{-2}.
\]  

(3.73)

Let \( \mathcal{B} = \{ w_1, w_2 \} \) be an ordered basis of \( \alpha L \) over \( \mathbb{Z} \) such that \( \Im(w_1/w_2) > 0 \) and so that for some positive integers \( r \) and \( s \), the ordered pair \( (w'_1, w'_2) \) defined by \( w'_1 = w_1/r \), \( w'_2 = w_2/s \), is a basis of \( A \) over \( \mathbb{Z} \), let \( \mathcal{R} \) be the set of representatives of \( A/\alpha L \) adapted to the basis \( \mathcal{B} \) and let \( S = \alpha^{-1}\mathcal{R} \).

Let now \( S \in \ker \psi \); then \( D \oplus S = \{ P + S + Q \} + \{ P + S - Q \} - 2\{ P + S \} - 2\{ Q + S \} \). To compute \( \varphi_L(D \oplus S) \) we have to choose a lift of \( D \oplus S \) in the kernel of \( q_2(\mathbb{C}, \ker \psi) \). Let \( n \) be an integer, \( n > 1 \), such that \( nS = S \), or equivalently, such that \( (n - 1)S = 0 \). If we write \( D \oplus S \) in the following way

\[
D \oplus S = \{ P + S + Q \} + \{ P + S - Q \} - 2\{ P + S \} - 2\{ Q + S \} - 2\{ Q \} - 2\{ Q + S \},
\]

it is clear, recalling the proof above for divisors of type (b), that the following divisor

\[
D(S) = \{ x + y + s \} + \{ x - y + s \} - 2\{ x + s \} - 2\{ y \}
\]

\[
+ 2\{ y \} - \frac{2n}{n-1}\{ y + s \} + \frac{2}{n-1}\{ y + ns \} + \frac{2n}{n^2-1}\{ s \} - \frac{2n}{n^2-1}\{ ns \},
\]  

(3.74)

where \( s \) is the unique lift of \( S \) in \( \mathcal{S} \), \( x \) is the same lift of \( P \) as above and \( y \) is the same lift of \( Q \) as above, is a lift of \( D \oplus S \) in the kernel of \( q_2(\mathbb{C}, \ker \psi) \). Thus, it follows from definition (3.26) that

\[
\varphi_L(D \oplus S) = \varphi_L(D(S)).
\]  

(3.75)

To compute \( \varphi_L(D(S)) \) we can proceed exactly as in the proof above for divisors of type (b) and we get

\[
\varphi_L(D(S)) = \varphi_L(x + y + s)\varphi_L(x - y + s)\varphi_L(x + s)^{-2}\varphi_L(y + s)^{-2} \cdot (E_L(s, y) \otimes 1).
\]  

(3.76)

It follows from (3.75) and (3.76) that

\[
\varphi_L(D \oplus S) = \varphi_L(x + y + s)\varphi_L(x - y + s)\varphi_L(x + s)^{-2}\varphi_L(y + s)^{-2} \cdot (E_L(s, y) \otimes 1).
\]  

(3.77)
The equality (3.77) holds for every $S \in \ker \psi$ and so we have
\[
\prod_{S \in \ker \psi} \varphi_L(D \oplus S) = \left( \prod_{s \in S} \varphi_L(x + y + s)\varphi_L(x - y + s)\varphi_L(x + s)^{-2}\varphi_L(y + s)^{-2} \right) 
\times e\left( E_L\left( \sum_{s \in S} s, y \right) \right) \otimes 1.
\]
(3.78)

But applying Theorem 2.3 to the divisor \{x + y\} + \{x - y\} - 2\{x\} - 2\{y\} (which certainly verifies the crucial hypothesis that its support is disjoint from $\alpha^{-1}A$), tensoring with 1 and recalling Lemma 3.1, we obtain
\[
\varphi_A(\alpha x + \alpha y)\varphi_A(\alpha x - \alpha y)\varphi_A(\alpha x)^{-2}\varphi_A(\alpha y)^{-2} 
= \left( \prod_{s \in S} \varphi_L(x + y + s)\varphi_L(x - y + s)\varphi_L(x + s)^{-2}\varphi_L(y + s)^{-2} \right) 
\times e\left( E_L\left( \sum_{s \in S} (x + y + \{x - y\} - 2\{x\} - 2\{y\}), \sum_{s \in S} s \right)/2 \right) \otimes 1.
\]

Or equivalently
\[
\varphi_A(\alpha x + \alpha y)\varphi_A(\alpha x - \alpha y)\varphi_A(\alpha x)^{-2}\varphi_A(\alpha y)^{-2} 
= \left( \prod_{s \in S} \varphi_L(x + y + s)\varphi_L(x - y + s)\varphi_L(x + s)^{-2}\varphi_L(y + s)^{-2} \right) \cdot e\left( E_L\left( \sum_{s \in S} s, y \right) \right) \otimes 1.
\]
(3.79)

In view of (3.73), (3.78) and (3.79), the theorem is certainly proved for divisors of type (d). This finishes the proof of the theorem. □

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