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Reduction of m-regular noncrossing partitions

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Abstract

In this paper, we present a reduction algorithm which transforms m-regular partitions of $[n] = \{1, 2, \ldots, n\}$ to $(m-1)$-regular partitions of $[n-1]$. We show that this algorithm preserves the noncrossing property. This yields a simple explanation of an identity due to Simion–Ullman and Klazar in connection with enumeration problems on noncrossing partitions and RNA secondary structures. For ordinary noncrossing partitions, the reduction algorithm leads to a representation of noncrossing partitions in terms of independent arcs and loops, as well as an identity of Simion and Ullman which expresses the Narayana numbers in terms of the Catalan numbers.

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Keywords: Partition; Noncrossing partition; m-regular partition; RNA secondary structure; Davenport–Schinzel sequence; Narayana number; Catalan number

1. Introduction

A *partition* P of $[n] = \{1, 2, \ldots, n\}$ is a collection $\{B_1, B_2, \ldots, B_k\}$ of nonempty disjoint subsets of $[n]$, called *blocks* such that $B_1 \cup \cdots \cup B_k = [n]$. We may assume that $\{B_1, B_2, \ldots, B_k\}$ are listed in the increasing order of their minimum elements. The set of all partitions of $[n]$ with k blocks is denoted by $\mathcal{P}(n, k)$. The cardinality of $\mathcal{P}(n, k)$ is the well-known Stirling number of the second kind [15].

A partition $P \in \mathcal{P}(n, k)$ is called *m-regular*, $m \geq 1$, if for any two distinct elements x, y in the same block, we have $|x-y| \geq m$. If all blocks of P are singletons (of cardinality one) we set $m = \infty$. The set of m-regular partitions in $\mathcal{P}(n, k)$ is denoted by $\mathcal{P}(n, k, m)$, and

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its cardinality is denoted by $p(n,k,m)$. When $m=1$, a 1-regular partition is an ordinary partition. A partition is called *poor* if each block contains at most two elements. The set of all poor partitions in $\mathcal{P}(n,k,m)$ is denoted by $\mathcal{P}_2(n,k,m)$, and its cardinality is denoted by $p_2(n,k,m)$.

Any partition P can be expressed by its *canonical sequential form* $P=a_1a_2\cdots a_n$ where $a_i=j$ if the element i is in the block B_j. For instance, 1231242 is the canonical sequential form of $P=(1,4)(2,5,7)(3)(6)\in\mathcal{P}(7,4,2)$. In fact, one can use a sequence on any set of k symbols to represent a partition of k blocks, where the symbols are linearly ordered. If we use the alphabet $\{a,b,c,d\}$ of four letters with the order $a<b<c<d$, then the corresponding canonical sequential form for P becomes $abcabdb$. Note that if $a_1a_2\cdots a_n$ is a canonical sequential form of a partition with k blocks, then each of $1,2,\ldots,k$ appears at least once and the first occurrence of i precedes that of j if $i<j$. The sequence $a_1a_2\cdots a_n$ is also called the *restricted growth function* of a partition P [18]. These two requirements are the normalization conditions of the Davenport–Schinzel sequences [2, 8], as noted by Klazar [6, 7].

We say that $P\in\mathcal{P}(n,k,m)$ is $abab$*-free* if its canonical sequential form does not contain any subsequence (not necessarily a consecutive segment) of the form $\cdots a\cdots b\cdots a\cdots b\cdots$, which is often written as $abab$. Equivalently, P is $abab$-free if there do not exist four elements $x,y,u,v\in[n]$ with $x<u<y<v$ such that x,y belong to the same block and u,v belong to another block. The set of $abab$-free partitions in $\mathcal{P}(n,k,m)$ is denoted by $\mathcal{P}(abab,n,k,m)$. An $abab$-free partition is also called a *noncrossing partition*. For example, the partition $P=(1,4)(2,5,7)(3)(6)$ is not $abab$-free because of the violation of the four elements $1<2<4<5$.

Regular partitions also arise in the enumeration of RNA secondary structures. In biology, an RNA sequence can be viewed as a sequence of molecules A (adenine), C (cytosine), G (guanine) and U (uracil); these single-stranded molecules fold onto themselves by the so-called *Watson–Crick rules*: A forms base pairs with U and C forms base pairs with G. A helical structure can be formed based on the sequence of molecules and the rules. If such a helical structure can be realized as a planar graph, then it is called an *RNA secondary structure*. In the mathematical modelling of RNA secondary structures, one may disregard what the molecules are and consider a helical structure as a sequence of numbers $1,2,\ldots,n$ along with some base pairs, where we have the restriction that all base pairs are allowed except for any two adjacent numbers i and $i+1$, and there do not exist two base pairs (i,j) and (k,l) with $i<k<j<l$. In this setting, the set $\mathcal{R}(n,k)$ of all RNA secondary structures with length n and k base pairs can be viewed as the set $\mathcal{P}_2(abab,n,n-k,2)$ of noncrossing poor partitions. A formula for $\mathcal{R}(n,k)$ is obtained by Schmitt and Waterman [12] in terms of the Narayana numbers.

However, a further biological consideration suggests a generalization of the above mathematical modelling of RNA secondary structures. As pointed out by Hofacker, Schuster and Stadler [5], within each matching bracket (or base pair) there should be at least three elements. In the language of partitions, that is to say that if i and j are in the same block, then we have $|i-j|\geq 4$. Equivalently, this is the notion of 4-regular partitions. Thus, we are led to the study of m-regular noncrossing poor partitions. It is known that for $m=1$, such partitions correspond to Motzkin paths, for $m=2$, there is a correspondence

with Motzkin paths without peaks [9, 17]. In general, Klazar [6] gives a formula for the number of m-regular noncrossing poor partitions.

The main result of this paper is a reduction algorithm that transforms a partition in $\mathcal{P}(n, k, m)$ to a partition in $\mathcal{P}(n-1, k-1, m-1)$. We show that the algorithm preserves the noncrossing property or the $abab$-free property. This leads to a quick explanation of the following identity:

$$p(abab, n, k, m) = p_2(abab, n-1, k-1, m-1). \tag{1.1}$$

An earlier version of this relation was first obtained by Simion and Ullman [14], where the relation is stated for $m = 2$. Klazar found the above identity in general and gave a generating function proof in [6]. Another bijective proof of (1.1) was found by Klazar [7]. We should note that the notations in [4, 5] are somewhat different. No simple explanation of (1.1) seems to be known. We hope that our algorithm may have served this purpose.

It is worth noting that ordinary noncrossing partitions can be further reduced into independent arcs and loops (defined subsequently, just before Theorem 3.2). Essentially, this gives a correspondence between noncrossing partitions and 2-Motzkin paths, and leads to an identity expressing the Narayana numbers in terms of the Catalan numbers due to Simion and Ullman [14].

2. The reduction algorithm

We begin with a bijective understanding of an identity of Yang [19] concerning the number of m-regular partitions of $[n]$.

Theorem 2.1. *For $m \geq 2$, we have*

$$p(n, k, m) = p(n-1, k-1, m-1). \tag{2.1}$$

For the case $m = 2$, a 2-regular partition is called a "restricted partition" and (2.1) was obtained by Prodinger [10]. Bijective proofs of (2.1) for 2-regular partitions are given by many people including Prodinger [10] and Yang [19]. However, these proofs do not seem to apply to general m-regular partitions.

In this paper, we find a simple reduction algorithm for m-regular partitions. The key idea is to use a digraph to represent a partition, which is called *the linear representation*. Given a partition $P = \{B_1, B_2, \ldots, B_k\}$ of $[n]$, we draw a digraph $D(P)$, or D for short, on the vertex set $[n]$. For each block B_i, we associate it with a directed path P_i starting with the minimum element in B_i, and going through elements in B_i in the increasing order. Note that when a block B_i has only one element, the corresponding path is an isolated vertex. The digraph D can be drawn on a line such that the vertices $1, 2, \ldots, n$ are arranged in the increasing order and the arcs always have the direction from left to right. For this reason, one does not really need to display the direction of each arc (see Fig. 1). An undirected version of the linear representation of a partition was used by Simion [13]. As we will see, the directions are useful to clarify the argument for the reduction algorithm.

The reduction algorithm. For a partition $P \in \mathcal{P}(n, k, m)$, where $n, k, m \geq 1$, we may reduce it to a partition in $\mathcal{P}(n-1, k-1, m-1)$:

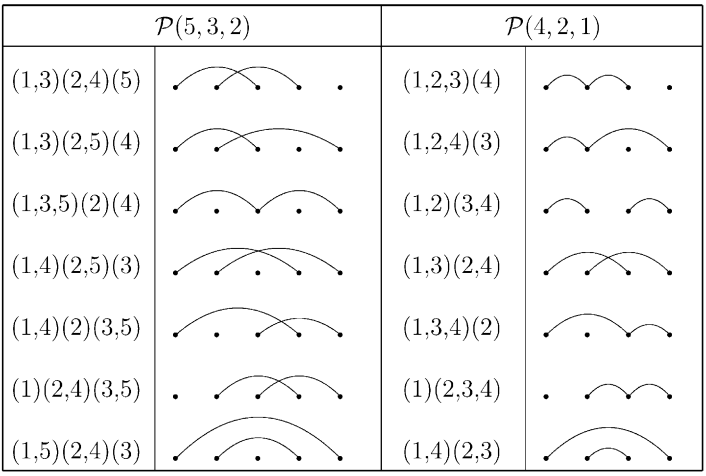

Fig. 1. Correspondence between $\mathcal{P}(5, 3, 2)$ and $\mathcal{P}(4, 2, 1)$.

1. For each arc (i, j) in the linear representation of P, replace it by the arc $(i, j-1)$.
2. Delete the vertex n.

Theorem 2.2. *When $m \geq 2$, the reduction algorithm gives a bijection between $\mathcal{P}(n, k, m)$ and $\mathcal{P}(n-1, k-1, m-1)$.*

Proof. Suppose that P is a partition in $\mathcal{P}(n, k, m)$. Let D be the linear representation of P, and let D' be the digraph obtained from D by reducing every arc (replacing (i, j) by $(i, j-1)$). Since $m \geq 2$, it is clear that in D' every arc has the direction from the smaller vertex to the bigger vertex, and for each vertex j in D', neither its indegree nor outdegree is greater than 1. Thus, each component of D' is a directed path from the minimum vertex to the maximum vertex in the increasing order. In other words, D' is also a linear representation of an $(m-1)$-regular partition of $[n]$. It is easy to see that D' has the same number of arcs as D. It follows that D' and D have the same number of connected components. Let H be the digraph obtained from D' by deleting the isolated vertex n. Then H is the linear representation of the desired partition.

By reversing the above procedure, one may show that the reduction algorithm yields a bijection. □

An example is given in Fig. 1, which illustrates the bijection between $\mathcal{P}(5, 3, 2)$ and $\mathcal{P}(4, 2, 1)$.

3. Reduction of noncrossing partitions

In this section, we show that the reduction algorithm preserves the noncrossing property. This gives a simple explanation of the following identity due to Simion and Ullman [14] and Klazar [6].

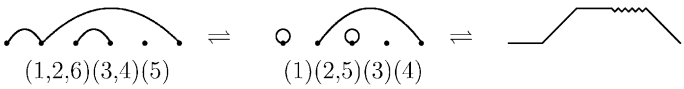

Fig. 2. A noncrossing partition and the corresponding 2-Motzkin path.

Theorem 3.1. *For $m \geq 2$, we have*

$$p(abab, n, k, m) = p_2(abab, n-1, k-1, m-1). \tag{3.1}$$

Proof. Suppose P is a partition in $\mathcal{P}(abab, n, k, m)$ and D is the linear representation of P. Let Q be the partition of $[n-1]$ obtained from P by applying the reduction algorithm, and D' the linear representation of Q. Suppose D' has a path of length two, $i \to j \to k$, say, then $(i, j+1)$ and $(j, k+1)$ are two arcs in D. Since D is a linear representation, these two arcs $(i, j+1)$ and $(j, k+1)$ belong to different components, which contradicts the assumption that P is $abab$-free. It follows that Q is a poor partition.

By the reduction algorithm we see that Q has $k-1$ blocks and is $(m-1)$-regular. It remains to show that Q is noncrossing. Suppose that there are four elements $x < u < y < v$ such that $B_i = \{x, y\}$ and $B_j = \{u, v\}$, where B_i and B_j are distinct blocks of Q. Then in the linear representation D', (x, y) and (u, v) are two crossing arcs, it follows that $(x, y+1)$ and $(u, v+1)$ are two crossing arcs in D, which is a contradiction to the assumption that P is noncrossing. The converse can be justified in the same manner. Therefore, we have established the desired one-to-one correspondence. □

In Fig. 1, there are only two partitions in $\mathcal{P}(abab, 5, 3, 2)$: $(1, 3, 5)(2)(4)$ and $(1, 5)(2, 4)(3)$; after applying the reduction algorithm, we get $(1, 2)(3, 4)$ and $(1, 4)(2, 3)$, which are the two partitions in $\mathcal{P}_2(abab, 4, 2, 1)$.

Theorem 3.1 is useful for the enumeration of m-regular noncrossing partitions [6]. We recall that the notion of m-regular noncrossing poor partitions coincides with that of general RNA secondary structures. The reduction algorithm can be used even for ordinary noncrossing partitions. In a digraph D, we say that two arcs are *independent* if they have no vertex in common, and a *loop* is an arc from a vertex to itself.

Theorem 3.2. *There is a one-to-one correspondence between noncrossing partitions of $[n]$ with k blocks and digraphs on $[n-1]$ consisting of $n-k$ independent noncrossing arcs or loops.*

Digraphs described in Theorem 3.2 are related to 2-Motzkin paths introduced by Barcucci, del Lungo, Pergola and Pinzani [1]. Roughly speaking, if we consider loops and singletons as straight and wavy level steps respectively, then we obtain

Theorem 3.3. *There is a one-to-one correspondence between noncrossing partitions of $[n]$ with k blocks and 2-Motzkin paths of length $n-1$ with $n-k$ straight level steps or up steps.*

Fig. 2 is an illustration of the above bijection.

In [4], Deutsch and Shapiro established a bijection between ordered trees and 2-Motzkin paths, and derived many important consequences regarding combinatorial structures such as Dyck paths, bushes, $\{0, 1, 2\}$-trees, Schröder paths, RNA secondary structures, noncrossing partitions, fine paths, etc. The above theorem on the reduction of noncrossing partitions to 2-Motzkin paths can be viewed as a simpler version of the Deutsch–Shapiro correspondence, since there are easy bijections between ordered trees and noncrossing partitions [3, 11].

Theorem 3.3 leads to an identity of Simion and Ullman [14] expressing the Narayana numbers by the Catalan numbers. Recall that the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the number of plane trees with $n+1$ vertices, and the Narayana number $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ is the number of plane trees with $n+1$ vertices and k leaves, which also counts the number of noncrossing partitions of $[n]$ with k blocks [3, 11].

Corollary 3.4 (Simion and Ullman [14], Corollary 3.2). *For all $n \geq 1$ and $1 \leq k \leq n$, we have the following relation:*

$$N_{n,k} = \sum_{i=0}^{n-k} \binom{n-1}{2i} \binom{n-2i-1}{n-i-k} C_i. \tag{3.2}$$

Proof. Suppose P is a noncrossing partition on $[n]$ with k blocks. Let H be the digraph on $[n-1]$ with independent noncrossing arcs, namely the linear representation of the partition obtained from P by applying the reduction algorithm. Suppose i is the number of loops in H. Removing the loops, we get a digraph H' on $n-i-1$ vertices with $k-1$ components consisting of $n-k-i$ independent arcs and $2k+i-n-1$ isolated vertices. The digraph H' corresponds to a noncrossing poor partition on $[n-i-1]$ with $k-1$ blocks (namely $2k+i-n-1$ singletons and $n-k-i$ blocks of size two). Hence we get

$$N_{n,k} = \sum_{i=0}^{n-k} \binom{n-1}{i} p_2(abab, -i-1, k-1, 1). \tag{3.3}$$

There is a bijection between noncrossing poor partitions of $[2n-2k-2i]$ without singletons and Dyck paths of length $2n-2k-2i$ (see [16], p. 222, Exercise 6.19 (n) and its solution on p. 258), and it is well-known that the number of such Dyck paths equals the $(n-k-i)$-th Catalan number. In view of the number of ways to choose the singletons, we obtain

$$p_2(abab, -i-1, k-1, 1) = \binom{n-i-1}{2n-2i-2k} C_{n-i-k}.$$

By replacing $n-i-k$ with i and taking summation, we get (3.2). □

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