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Tri-bimaximal neutrino mixing from A_4 and $\theta_{13} \sim \theta_C$

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Abstract

It is a common believe that, if the tri-bimaximal mixing (TBM) pattern is explained by vacuum alignment in an A_4 model, only a very small reactor angle, say $\theta_{13} \sim O(\lambda_C^2)$ being $\lambda_C \equiv \theta_C$ the Cabibbo angle, can be accommodated. This statement is based on the assumption that all the flavon fields acquire VEVs at a very similar scale and the departures from exact TBM arise at the same perturbation level. From the experimental point of view, however, a relatively large value $\theta_{13} \sim O(\lambda_C)$ is not yet excluded by present data. In this paper, we propose a seesaw A_4 model in which the previous assumption can naturally be evaded. The aim is to describe a $\theta_{13} \sim O(\lambda_C)$ without conflicting with the TBM prediction for θ_{12} which is rather close to the observed value (at λ_C^2 level). In our model the deviation of the atmospherical angle from maximal is subject to the sum-rule: $\sin^2\theta_{23} \approx 1/2 + \sqrt{2}/2 \cos\delta \sin\theta_{13}$ which is a next-to-leading order prediction of our model.

1. Introduction

The present data [1], at 1σ, on solar and atmospherical angles:

$$\theta_{12} = (34.5 \pm 1.4)^\circ, \qquad \theta_{23} = \left(42.3^{+4.4}_{-3.5}\right)^\circ, \tag{1}$$

are fully compatible with the TBM matrix:

$$U_{\rm TB} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix}, \tag{2}$$

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which corresponds to $\sin^2\theta_{12} = 1/3$ ($\theta_{12} = 35.3°$) and $\sin^2\theta_{23} = 1/2$. The TBM pattern also predicts $\theta_{13} = 0$, however, recent analysis based on global fits of the available data leads to hints for $\theta_{13} > 0$ [2,3]. Differently from solar or atmospherical mixing angles, the reactor one is less constrained and its value can still be relatively large, even at 1σ level, say $\sim \lambda_C$:

$$\sin^2\theta_{13} = 0.016 \pm 0.010 \quad [2], \qquad \sin^2\theta_{13} = 0.010^{+0.016}_{-0.011} \quad [3].$$

For this reason the future experimental sensitivity on the reactor angle is fundamental for theoretical understanding of the TBM, which, due to its highly symmetric structure, strongly suggests an underlying non-Abelian flavour symmetry. A natural and economical class of models based on A_4 flavour symmetry [4–8] has been proposed in describing TBM pattern. There are also more involved models based on other discrete groups [9,10] or continuous flavour symmetries [11]. However, if the TBM pattern results from a spontaneously broken flavour symmetry, higher order corrections should introduce deviation from exact TBM, generally all of the same order. Since the experimental departures of θ_{12} from its tri-bimaximal value are at most of order λ_C^2, a future observed value of θ_{13} near its present upper bound should impose severe constraints on model buildings [12]. If this is the case, one apparently has to renounce the nice symmetric nature of θ_{12} and simply imagines that its tri-bimaximal value may be completely accidental [13].

In this paper we will show that the TBM pattern can be explained by A_4 symmetry without necessarily implying a very small value for θ_{13}. The fundamental new ingredient of our construction is to allow a moderate hierarchy, of order λ_C, between the VEVs of flavon fields of the charged lepton and the neutrino sectors. The paper is organized as follows. In the next section, we characterize some general conditions under which a hierarchy between VEVs of flavon fields of different sectors can be accommodated without fine tuning. In Section 3, we introduce the field content of our seesaw model based on $A_4 \times Z_3 \times Z_4$ flavour group with great emphasis on vacuum alignment and its stability. In Section 4, we explain how the charged lepton hierarchy can be reproduced by a particular symmetry breaking pattern of A_4. In Section 5, we describe the neutrino mass at leading order by seesaw mechanism and obtain an exact TBM at this level. Then in Section 6 we include all subleading corrections up to terms suppressed by $1/\Lambda^2$ to our model and analyze possible deviation from TBM. In the end, in Section 7, we comment on other possible phenomenological consequences of our model and conclude.

2. General consideration

The difficulty in the standard formulation of A_4 models [4,5] in generating a relatively large value of θ_{13} is related to the vacuum alignment problem which plays a fundamental rule in order to naturally describe the TBM pattern from spontaneously broken flavour symmetries. The group A_4 (see Appendix A) has two important subgroups: G_S, which is a reflection subgroup generated by S and G_T, which is the group generated by T, isomorphic to Z_3. A_4 can be spontaneously broken by VEVs of two sets of flavon fields, Φ for the neutrino sector and Φ' for the charged lepton sector. The direction of $\langle\Phi\rangle$ should leave the subgroup G_S unbroken leading to the TBM. However one generally has two options for the alignment of Φ'. $\langle\Phi'\rangle$ can be such that G_T is preserved leading to diagonal charged lepton masses but their hierarchy is usually generated by an independent Froggatt–Nielsen (FN) mechanism [14]. The second option is to consider a vacuum alignment of Φ' which entirely breaks A_4 and in this case the mass hierarchy is directly related to $\langle\Phi'\rangle/\Lambda$, being Λ the cut-off scale, without an extra FN component [6–8,15]. A natural mechanism for the vacuum alignment of Φ and Φ' in different directions requires the existence of an Abelian factor G_A in addition to A_4. The aim of G_A is to guarantee the following decomposition of the scalar potential as:

$$V(\Phi, \Phi') = V_\nu(\Phi) + V_e(\Phi') + V^{\text{NLO}}(\Phi, \Phi') + \cdots, \tag{3}$$

where we see that the interaction term between Φ and Φ' appears from next-to-leading order (NLO). We will refer this situation as a "partial" separation in the scalar potential which is tightly related on the fact that only one of the sets Φ and Φ' is charged under G_A, a standard choice in the literature [4,5,8]. At leading order, the two scalar sectors are actually separated, however, the vacuum alignments are affected by NLO corrections encoded in $V^{\text{NLO}}(\Phi, \Phi')$. The order of magnitude of the corrections to the VEVs $\langle\Phi\rangle$ and $\langle\Phi'\rangle$ depends on $\langle\Phi\rangle/\Lambda$ and $\langle\Phi'\rangle/\Lambda$ and they are subject to some conditions. First of all, the corrections to the tri-bimaximal value of θ_{12} are at most of order λ_C^2. Furthermore, the corrections to $\langle\Phi'\rangle$ are required to be smaller than $m_\mu/m_\tau \sim O(\lambda_C^2)$ or more restrictively smaller than $m_e/m_\mu \sim O(\lambda_C^3)$; otherwise, the generated charged lepton hierarchy should not be stable. These conditions shall translate to upper bounds on the scale of flavour symmetry breaking with respect of the cut-off scale:

$$\langle\Phi\rangle/\Lambda, \langle\Phi'\rangle/\Lambda \lesssim \lambda_C^2 \tag{4}$$

$\langle\Phi'\rangle/\Lambda \lesssim \lambda_C^2$. In conclusion, a value of θ_{13} near its present experimental bound cannot be described if the scalar potential is "partially" separated as quoted in (3).

In this paper we will exploit the possibility of a "fully" separated scalar potential which corresponds to (3) with $V^{\text{NLO}}(\Phi, \Phi') = V^{\text{NLO}}(\Phi)$ or $V^{\text{NLO}}(\Phi, \Phi') = V^{\text{NLO}}(\Phi')$. The "fully" separated scalar potential can be obtained if G_A is a direct product of two Abelian factors G_A^ν and G_A^e which separately acts on Φ and Φ'. In this case, since $V_\nu(\Phi)$ and $V_e(\Phi')$ can be minimized in a completely independent way, even including NLO corrections, we are not necessarily subject to the strict condition (4). In fact, it is possible to construct a completely natural model for TBM based on the A_4 symmetry in which $\langle\Phi'\rangle/\Lambda \sim O(\lambda_C^2)$ and $\langle\Phi\rangle/\Lambda \sim O(\lambda_C)$ can be compatible with all experimental constraints. The model belongs the constrained A_4 models considered in [6,7] in which the leading order neutrino TBM and the charged lepton mass hierarchy are simultaneously reproduced by the vacuum alignment. Our choice for G_A in order to guarantee a "fully" separated scalar potential is given by $Z_3 \times Z_4$. We are particularly interested in analyzing the possibility to have a relatively large value of θ_{13} without fine-tuning. We will show indeed that θ_{13} can be of order λ_C while θ_{12} is corrected by subleading effects arising at order λ_C^2. Furthermore, deviations from TBM can be more intriguing since they obey a definite sum-rule which can be in principle tested.

3. Field content and vacuum alignment

In this section we introduce the field content of the model and analyze the most general scalar potential which is invariant under the flavour symmetry $A_4 \times Z_3 \times Z_4$. The lepton SU(2) doublets l_i $(i = e, \mu, \tau)$ are assigned to the triplet A_4 representation, while the lepton singlets e^c, μ^c and τ^c are all invariant under A_4. The neutrino sector is described by seesaw mechanism with 3 heavy right-handed neutrinos ν_i^c which also form an A_4 triplet. The symmetry breaking sector consists of the scalar fields neutral under the SM gauge group, divided in two sets as advanced before: $\Phi = \{\varphi_S, \xi, \zeta\}$ and $\Phi' = \{\varphi_T, \xi'\}$. As anticipated before, in addition to A_4, we also have an Abelian symmetry $G_A = Z_3 \times Z_4$ which is a distinguishing feature of our construction. All the fields of the model, together with their transformation properties under the flavour group, are listed in Table 1. We observe that Φ is charged under Z_3 while Φ' is charged under Z_4.

Table 1
The transformation properties of leptons, electroweak Higgs doublets and flavons under $A_4 \times Z_3 \times Z_4$.

Field	l	e^c	μ^c	τ^c	ν^c	h_u	h_d	φ_T	ξ'	φ_S	$\xi, \tilde{\xi}$	ζ
A_4	3	1	1	1	3	1	1	3	$1'$	3	1	1
Z_3	1	1	1	1	ω	1	1	1	1	ω	ω	ω^2
Z_4	1	-1	$-i$	1	1	1	$-i$	i	i	1	1	1

The vacuum alignment problem of the model can be solved by the supersymmetric driving field method introduced in [5]. This approach exploits the continuous $U(1)_R$ symmetry in the superpotential w under which matter fields have $R = +1$, while Higgses and flavons have $R = 0$. The spontaneous breaking of A_4 can be employed by adding to fields already present in Table 1 a new set of multiplets, called driving fields, with $R = 2$. We introduce a driving field ξ_0, fully invariant under A_4, and two driving fields φ_0^T and φ_0^S, triplet of A_4. The driving fields ξ_0 and φ_0^S, which are responsible for the alignment of φ_S, have a charge ω under Z_3 and are invariant under Z_4. φ_0^T has a charged -1 under Z_4, invariant under Z_3, and drives a non-trivial VEV of φ_T. The most general driving superpotential w_d invariant under $A_4 \times G_A$ with $R = 2$ is a sum of two independent parts $w_d = w_d^\nu(\xi_0, \varphi_0^S, \Phi) + w_d^e(\varphi_0^T, \Phi')$ where

$$w_d^\nu = g_1 \varphi_0^S \varphi_S^2 + g_2 \tilde{\xi}\left(\varphi_0^S \varphi_S\right) + g_3 \xi_0 (\varphi_S \varphi_S) + g_4 \xi_0 \xi^2 + g_5 \xi_0 \xi \tilde{\xi} + g_6 \xi_0 \tilde{\xi}^2 + g_7 M_\zeta \xi_0 \zeta, \quad (5)$$

$$w_d^e = h_1 \xi' \left(\varphi_0^T \varphi_T\right)'' + h_2 \left(\varphi_0^T \varphi_T \varphi_T\right). \quad (6)$$

The "fully" separated superpotential is guaranteed by $G_A = Z_3 \times Z_4$. Eqs. (5) and (6) gives two decoupled sets of F-terms for driving fields which characterize the supersymmetric minimum. In other words, w_d^ν and w_d^e independently determine the vacuum alignment of Φ and Φ', respectively. From (5) we have:

$$\begin{aligned}
\frac{\partial w}{\partial \varphi_{01}^S} &= g_2 \tilde{\xi} \varphi_{S1} + 2 g_1 \left(\varphi_{S1}^2 - \varphi_{S2}\varphi_{S3}\right) = 0,\\
\frac{\partial w}{\partial \varphi_{02}^S} &= g_2 \tilde{\xi} \varphi_{S3} + 2 g_1 \left(\varphi_{S2}^2 - \varphi_{S1}\varphi_{S3}\right) = 0,\\
\frac{\partial w}{\partial \varphi_{03}^S} &= g_2 \tilde{\xi} \varphi_{S2} + 2 g_1 \left(\varphi_{S3}^2 - \varphi_{S1}\varphi_{S2}\right) = 0,\\
\frac{\partial w}{\partial \xi_0} &= g_4 \xi^2 + g_5 \xi \tilde{\xi} + g_6 \tilde{\xi}^2 + g_7 M_\zeta \zeta + g_3 \left(\varphi_{S1}^2 + 2\varphi_{S2}\varphi_{S3}\right) = 0. \quad (7)
\end{aligned}$$

In a finite portion of the parameter space, we find the following stable solution

$$\begin{aligned}
&\langle \tilde{\xi} \rangle = 0, \qquad \langle \xi \rangle = u, \qquad \langle \zeta \rangle = v,\\
&\langle \varphi_S \rangle = (v_S, v_S, v_S), \qquad v_S^2 = -\frac{g_4 u^2 + g_7 M_\zeta v}{3 g_3}, \quad (8)
\end{aligned}$$

with u and v undetermined. Since $\langle\tilde{\xi}\rangle = 0$,[1] we have ignored the existence of $\tilde{\xi}$ in the rest of the paper. Setting to zero the F-terms from Eq. (6), we obtain:

$$\frac{\partial w}{\partial \varphi_{01}^T} = h_1 \xi' \varphi_{T3} + 2h_2\big(\varphi_{T}{}_1^2 - \varphi_{T2}\varphi_{T3}\big) = 0,$$
$$\frac{\partial w}{\partial \varphi_{02}^T} = h_1 \xi' \varphi_{T2} + 2h_2\big(\varphi_{T}{}_2^2 - \varphi_{T1}\varphi_{T3}\big) = 0,$$
$$\frac{\partial w}{\partial \varphi_{03}^T} = h_1 \xi' \varphi_{T1} + 2h_2\big(\varphi_{T}{}_3^2 - \varphi_{T1}\varphi_{T2}\big) = 0,$$

and the stable solution to these four equations is:

$$\langle\xi'\rangle = u' \neq 0, \qquad \langle\varphi_T\rangle = (0, v_T, 0), \qquad v_T = -\frac{h_1 u'}{2h_2}, \tag{9}$$

with u' undetermined. The flat directions can be removed by the interplay of radiative corrections to the scalar potential and soft SUSY breaking terms. It is worth to observe that, thanks to G_A, the VEV alignments (8) and (9) are independent even at NLO.

Since the VEVs of the scalar fields in Φ (Φ') are related each other by adimensional constants of order one, we should expect that they have a common scale indicated by $\langle\Phi\rangle$ ($\langle\Phi'\rangle$). However, $\langle\Phi\rangle/\Lambda$ and $\langle\Phi'\rangle/\Lambda$ can be in principle different and they are subject to phenomenological constraints. As we will see in the next section, $\langle\Phi'\rangle$ is responsible for charged lepton hierarchy so we have to require

$$\frac{m_e}{m_\mu} \sim \lambda_C^3 \lesssim \frac{\langle\Phi'\rangle}{\Lambda} \lesssim \lambda_C^2 \sim \frac{m_\mu}{m_\tau}.$$

The superpotential w_d^e is affected by non-renormalizable terms (see Appendix B for the detail) from the neutrino sector Φ suppressed by $1/\Lambda^2$. Requiring that the sub-leading corrections to $\langle\Phi'\rangle$ are smaller than $m_\mu/m_\tau \sim O(\lambda_C^2)$, we obtain the condition

$$\frac{\langle\Phi\rangle}{\Lambda} \lesssim \lambda_C.$$

The vacuum alignment with a "fully" separated scalar potential allows a hierarchy between the VEVs of the scalars in different sectors $\langle\Phi'\rangle \ll \langle\Phi\rangle$.

Differently from w_d^e, w_d^ν receives NLO corrections which are suppressed only by $1/\Lambda$ but don't depend on the charged lepton sector Φ':

$$\delta w_d^\nu = \frac{1}{\Lambda}\big[\big(\varphi_0^S \varphi_S\big)\zeta^2 + \xi_0 \xi \zeta^2\big].$$

One may wonder if a large VEV of Φ with $\langle\Phi\rangle/\Lambda \sim \lambda_C$ could introduce a too large correction to the leading order vacuum alignment (8) destroying the stability of the TBM prediction. Fortunately, this is not the case. Since there is no fundamental distinction between ζ^2 and ξ the NLO correction δw_d^ν should induce terms which have the same form of those already present in w_d^ν. In fact, including δw_d^ν in the minimization, one easily find that the $\langle\varphi_S\rangle$ receives only a small shift

[1] Since there is no fundamental distinction between the singlets ξ and $\tilde{\xi}$ we have defined $\tilde{\xi}$ as the combination that couples to $(\varphi_0^S \varphi_S)$ in the superpotential w_d. The introduction of an additional singlet is essential to recover a non-trivial solution.

in the same direction of the leading order alignment. For this reason we will no longer consider VEV shifts of φ_S in the following.

4. Charged lepton hierarchy

In the present section, we illustrate how a fully broken A_4 symmetry can generate the charged lepton hierarchy. The key ingredient is the alignment $\langle\varphi_T\rangle \sim (0,1,0)$. Such a VEV breaks the permutation symmetry of the second and third generation of neutrinos in a maximal way in the sense that

$$\langle\varphi_T\rangle^t S_{2-3}\langle\varphi_T\rangle = 0,$$

where

$$S_{2-3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The A_4 group is fully broken[2] in the charged lepton sector by Φ' with the vacuum structure quoted in (9) and only the tau mass is generated at leading order. The muon and electro masses are generated respectively by $\langle\varphi_T\rangle^2 \propto (0,0,1)$ and $\langle\varphi_T\rangle^3 \propto (1,0,0)$. Then the correct hierarchy between the charged lepton masses $m_e \ll m_\mu \ll m_\tau$ is reproduced if we assume $\lambda_C^2 \lesssim \langle\Phi'\rangle/\Lambda \lesssim \lambda_C^3$.

Since Φ' carries a charge i under Z_4 we have to assign different Z_4 charges for lepton singlets. Considering only insertions of Φ', the charged lepton masses are described by w_e, given by, up to $1/\Lambda^3$:

$$\begin{aligned} w_e = {} & \alpha_1 \tau^c(l\varphi_T)h_d/\Lambda \\ & + \beta_1\mu^c\xi'(l\varphi_T)''h_d/\Lambda^2 + \beta_2\mu^c(l\varphi_T\varphi_T)h_d/\Lambda^2 \\ & + \gamma_1 e^c(\xi')^2(l\varphi_T)'h_d/\Lambda^3 + \gamma_2 e^c\xi'(l\varphi_T\varphi_T)''h_d/\Lambda^3 + \gamma_3 e^c(l\varphi_T\varphi_T\varphi_T)h_d/\Lambda^3. \end{aligned}$$

After electroweak symmetry breaking, $\langle h_{u,d}\rangle = v_{u,d}$, given the specific orientation of $\langle\varphi_T\rangle \propto (0,1,0)$, w_e give rise to diagonal and hierarchical mass terms for charged leptons. Defining the expansion parameter $v_T/\Lambda \equiv \lambda^2 \ll 1$ (it is not restrictive to consider v_T to be positive) and the Yukawa couplings y_l ($l = e, \mu, \tau$) as

$$\begin{aligned} y_\tau &= |\alpha_1|, \\ y_\mu &= |\beta_1 u'/v_T + 2\beta_2|\lambda^2, \\ y_e &= \left|\gamma_1(u'/v_T)^2 - \gamma_2 u'/v_T - 2\gamma_3\right|, \end{aligned}$$

the charged lepton masses are given by

$$m_l = y_l\lambda^2 v_d \,(l = e, \mu, \tau). \tag{10}$$

As already pointed out in the previous section and analyzed in detail in Appendix B, the vacuum alignment for φ_T receives correction of order $\langle\Phi\rangle^2/\Lambda^2 \sim \lambda_C^2$ different for each component:

$$\varphi_T = (\delta_{T1}, v_T + \delta_{T2}, \delta_{T3}).$$

2 Similarly as explained in [6,8], a residual symmetry $A_4 \times Z_3$ from $A_4 \times Z_4$ survives in the charged lepton sector guaranteeing the stability of the vacuum alignment.

Including correction to the vacuum alignment for φ_T, the diagonal form of the charged lepton mass should slightly change and small off-diagonal entries appear:

$$m_e = \begin{pmatrix} m_e & m_e O(\lambda_C^2) & m_e O(\lambda_C^2) \\ m_\mu O(\lambda_C^2) & m_\mu & m_\mu O(\lambda_C^2) \\ m_\tau O(\lambda_C^2) & m_\tau O(\lambda_C^2) & m_\tau \end{pmatrix}. \tag{11}$$

The transformation needed to diagonalize m_e is $V_e^T m_e U_e = \mathrm{diag}(m_e, m_\mu, m_\tau)$ and the unitary matrix U_e is given by

$$U_e = \begin{pmatrix} 1 & O(\lambda_C^2) & O(\lambda_C^2) \\ O(\lambda_C^2) & 1 & O(\lambda_C^2) \\ O(\lambda_C^2) & O(\lambda_C^2) & 1 \end{pmatrix}. \tag{12}$$

Another source of off-diagonal correction to charged leptons comes from the interaction with the neutrino sector. In fact, the products $\xi\zeta$ and $\varphi_S\zeta$ are invariant combination under G_A and we can include them on top of each term in w_e. However, we find that the introduction of these additional terms changes the charged lepton mass m_e exactly in the same way as the corrections induced by VEV shifts of φ_T, i.e. (11). Then (12) is the most general structure of the charged lepton contribution to TB mixing.

5. A seesaw realization of the constrained A_4 model

The masses of light neutrinos of our model is described by seesaw superpotential with 3 heavy right-handed neutrinos ν_i^c, triplet of A_4. Terms in the superpotential which contain ν^c invariant under the flavour group are given by:

$$w_\nu = y(\nu^c l)\zeta h^u/\Lambda + x_a \xi(\nu^c\nu^c) + x_b(\varphi_S \nu^c\nu^c) + \text{h.c.} + \cdots. \tag{13}$$

In the heavy neutrino sector $A_4 \times Z_3$ is broken by $\langle\varphi_S\rangle = (v_S, v_S, v_S)$ and $\langle\xi\rangle = u$ down to G_S (with Z_4 unbroken) with an accidental extra $G_{2\text{–}3}$ symmetry. Then the residual symmetry of the right-handed neutrino masses is $G_{\text{TB}} = G_S \times G_{2\text{–}3}$. G_{TB} can be transfered to the light neutrino sector if the Dirac neutrino mass commute its generators. This is in fact the case. After electroweak and A_4 symmetry breaking from (13) we obtain the following leading contribution to the Dirac and Majorana masses:

$$m_0^D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} y v_u \frac{v}{\Lambda}, \qquad M = \begin{pmatrix} a+2b & -b & -b \\ -b & 2b & a-b \\ -b & a-b & 2b \end{pmatrix} u, \tag{14}$$

where

$$a \equiv x_a, \qquad b \equiv x_b \frac{v_S}{u}. \tag{15}$$

We immediately see that $[m_0^D, S] = 0$. The leading order lepton mixing matrix is entirely encoded in the right-handed neutrino mass matrix M which is diagonalized by the transformation:

$$U_0^\dagger M U_0^* = \mathrm{diag}(|a+3b|, |a|, |a-3b|)u, \tag{16}$$

with $U_0 = U_{\text{TB}}\Omega$, where $\Omega = \mathrm{diag}\{e^{i\phi_1/2}, e^{i\phi_2/2}, i e^{i\phi_3/2}\}$ and ϕ_1, ϕ_2, ϕ_3 are respectively phases of $a+3b$, a, $a-3b$. Naturally ϕ_1 and ϕ_3 depend on ϕ_2 and Δ, the relative phase between a and b.

The light neutrino masses are given by the type I seesaw mechanism: $m_\nu = (m_0^D)^T M^{-1} m_0^D$ which is invariant under G_{TB} and then also diagonalized by U_0.[3] Denoting the physical masses of ν_i^c as $M_1 = |a + 3b|$, $M_2 = |a|$ and $M_3 = |a - 3b|$, we obtain

$$U_0^T m_\nu^0 U_0 = \left|\frac{y v_u v}{\Lambda}\right|^2 \mathrm{diag}\left\{\frac{1}{M_1}, \frac{1}{M_2}, \frac{1}{M_3}\right\} = \mathrm{diag}\{m_1, m_2, m_3\}.$$

$m_2 > m_1$ implies $t \equiv |3b|/|a| > -2\cos\Delta$ and in principle both normal and inverted hierarchies in the neutrino spectrum can be reproduced. The normal hierarchy is realized for $t/2 \leqslant \cos\Delta \leqslant 1$ whereas an inverted spectrum requires $-t/2 < \cos\Delta \leqslant 0$. The ratio $r = \Delta m_{\mathrm{sun}}^2/\Delta m_{\mathrm{atm}}^2$ (where $\Delta m_{\mathrm{sun}}^2 = m_2^2 - m_1^2$ and $\Delta m_{\mathrm{atm}}^2 = |m_3^2 - m_1^2|$) is given in our model by:

$$r = \frac{(t + 2\cos\Delta)(1 + t^2 - 2t\cos\Delta)}{4\cos\Delta}. \tag{17}$$

One can show that for the normal hierarchy, a small value of $r \approx 1/30$ can be reproduced only for $\cos\Delta \approx t \approx 1$. In particular, a normal ordered spectrum can never be degenerate. Then we can expand $t = 1 + \delta t$ with $\delta t \ll t$ obtaining the following approximate spectrum:

$$m_1 \approx \sqrt{\Delta m_{\mathrm{sum}}^2/3}, \qquad m_2 \approx 2m_1, \qquad m_3 \approx \sqrt{\Delta m_{\mathrm{atm}}^2 - \Delta m_{\mathrm{sum}}^2/3}. \tag{18}$$

The inverted hierarchy can be realized only for $t \approx -2\cos\Delta$ and in this case we can expand $\cos\Delta = -t/2 + \delta t'$ with $\delta t' \ll t$. Expressing δt in function of r we obtain

$$m_1^2 = \Delta m_{\mathrm{atm}}^2\left[1 + \frac{1}{2t^2} + \left(\frac{1}{t^2} - \frac{1}{1+2t^2}\right)r\right],$$
$$m_2^2 = \Delta m_{\mathrm{atm}}^2\left[1 + \frac{1}{2t^2} + \left(1 + \frac{1}{t^2} - \frac{1}{1+2t^2}\right)r\right],$$
$$m_3^2 = \Delta m_{\mathrm{atm}}^2\left[\frac{1}{2t^2} + \left(\frac{1}{t^2} - \frac{1}{1+2t^2}\right)r\right].$$

In principle, the previous expansion is valid also for a degenerate spectrum realized by $t \ll 1$ which is, however, parametrically fine-tuned[4] in our model.

Before going beyond the leading order result obtained in this section, we can estimate the natural mass scale of the lightest right-handed neutrino ν_3^c considering, for simplicity, a normal hierarchy for light neutrinos. In this case, the right-handed neutrinos are also hierarchical according to $M_3 \approx \sqrt{r/3}M_1$ and $M_2 \approx (1/2)M_1$. By taking neutrino mass scale as $\sqrt{\Delta m_{\mathrm{atm}}^2} \sim 0.05$ eV and the scale of m^D as $v_u\lambda_C$ with $v_u = 174$ GeV one obtains $M_3 \sim 3 \times 10^{13}$ GeV. From Eq. (16) we see that the right-handed neutrinos have a same mass scale as $\langle\Phi\rangle$. Then the hierarchy among

[3] The overall phase appearing in the Dirac neutrino mass m_0^D can be absorbed by the redefinition of ϕ_2 and there are only two independent Majorana phases.

[4] The fine-tuning required in order to reproduce a small r becomes more severe if we include in w_ν also the five-dimensional operator $lh^u lh^u/\Lambda'$ which leads to a mass matrix structure similar to the term $\xi\nu^c\nu^c$. Indeed, if the Weinberg operator has a cutoff scale $\Lambda' \sim \Lambda$, its contribution becomes larger than the seesaw one. This situation is equivalent to go to the limit $a \gg b$ and then it is disfavored. In order to avoid this problem we will assume that the lepton number is violated only by Majorana mass term up to Λ. In other words, we require $\Lambda' \gg \Lambda$ and a direct five-dimensional operator can be neglected.

the scales is

$$\langle \Phi' \rangle \sim \langle \Phi \rangle \lambda_C \sim \Lambda \lambda_C^2$$

with $\langle \Phi \rangle \sim M_3$–$M_1$. Correspondingly the cut-off scale Λ will range between about 10^{14} GeV and 10^{15} GeV. Beyond this energy scale, new physics like grand unified theories should come into play.

6. Deviation from TBM and $\theta_{13} \sim \lambda_C$

In this section we show how a relatively large reactor angle, say $\theta_{13} \sim \theta_C$, can naturally arise in our model, without conflicting with the precise value of θ_{12} predicted by TBM. The neutrino mass described in the previous section predicts an exact TBM. Including sub-leading contributions dictated by higher-dimensional operators, the leading order lepton mixing matrix should be modified. As we shall see in a moment, not all deviations from TBM arise at the same perturbation level, this is one of the most important feature of the model. We find that the NLO corrections generate a non-vanishing reactor angle which is correlated with deviation of atmospherical angle from maximal. While the corrections to solar angle appear only at next-to-next to leading order (NNLO).

First of all we focus on higher order corrections to the right-handed Majorana neutrino mass up to terms suppressed by $1/\Lambda^2$. At NLO, there is only one additional contribution to heavy Majorana mass: $\zeta^2 \nu^c \nu^c/\Lambda$. Since ζ^2 has exactly the same property of ξ, this term can be absorbed by a redefinition of a. The NNLO contributions arise from adding the products $\xi\zeta$ and $\varphi_S \zeta$, invariant combination under G_A, on top of the leading order terms. In this case, not all the corrections have the same structure of the terms already present in w_ν and consequently cannot be regarded as small shifts of a and b, for example $(\nu^c\nu^c)'(\varphi_S\varphi_S)''$ and $(\nu^c\nu^c)''(\varphi_S\varphi_S)'$. However, these terms can be absorbed by parameters y_1 and y_2 in the NLO correction to the Dirac mass δm^D as will be clear in a moment.

Now we move to consider the correction to Dirac neutrino mass: δm^D beginning with terms suppressed by $1/\Lambda^2$. There are many independent terms of the type $(\nu^c l \varphi\varphi)h^u$, with $\varphi \in \{\varphi_S, \xi\}$, invariant of A_4 which contribute to δm^D at this order:

$$\begin{aligned}\delta w_\nu = {} & h^u \frac{y_1}{\Lambda^2}(\nu^c l)'(\varphi_S\varphi_S)'' + h^u \frac{y_2}{\Lambda^2}(\nu^c l)''(\varphi_S\varphi_S)' + h^u \frac{y_3}{\Lambda^2}\nu^c(l\varphi_S)_A\xi \\ & + h^u \frac{y'}{\Lambda^2}(\nu^c l)_1(\varphi_S\varphi_S)_1 + h^u \frac{y''}{\Lambda^2}(\nu^c l)\xi^2 + h^u \frac{y_2'}{\Lambda^2}\nu^c(l\varphi_S)_S\xi. \end{aligned} \tag{19}$$

Observe that the operators with coefficients y', y'', y_2' give contribution to Dirac mass matrix in a form invariant under G_{TB} exactly as right-handed neutrino mass. Then these corrections can be adsorbed into a redefinition of the leading-order coefficients. The relevant correction to the Dirac mass comes from the first three terms in Eq. (19) and has the following form:

$$\delta m^D = \begin{pmatrix} 0 & y_1 + \tilde{y}_3 & y_2 - \tilde{y}_3 \\ y_1 - \tilde{y}_3 & y_2 & \tilde{y}_3 \\ y_2 + \tilde{y}_3 & -\tilde{y}_3 & y_1 \end{pmatrix} v_u \frac{v_S^2}{\Lambda^2}, \tag{20}$$

where $y_1, y_2, \tilde{y}_3 \equiv y_3 u/v_S$ are generally complex number of order 1. Before discussing the important consequence when we include the NLO correction to the Dirac neutrino mass, we comment possible NNLO effects on m^D. Here the NNLO contributions are suppressed by $1/\Lambda^3$

and they are of the type $(\nu^c l\zeta^2\varphi)h^u$. All these terms can be absorbed by a redefinition of y_3 and y'_2, then we can forget them in the following analysis.

In order to find the correction to the leading neutrino mixing matrix $U_0 = U_{\mathrm{TB}}\Omega$, it is convenient to define

$$\hat{m}^D = U_0^\dagger m^D U_0,$$

where $m^D = m_0^D + \delta m^D$. The light neutrino mass is then formally given by

$$m_\nu = U_0 \hat{m}_\nu U_0^T$$

where $\hat{m}_\nu \equiv (\hat{m}^D)^T M_{\mathrm{diag}}^{-1} \hat{m}^D$ with $M_{\mathrm{diag}}^{-1} = \mathrm{diag}\{1/M_1, 1/M_2, 1M_3\}$. If $\hat{m}_\nu$ can be diagonalized by the unitary matrix $\delta U \sim I$ as

$$\delta U \hat{m}_\nu \delta U^T = \mathrm{diag}\{\hat{m}_1, \hat{m}_2, \hat{m}_3\},$$

where $\hat{m}_i \approx m_i$, the full PMNS mixing matrix will be given by

$$U_{\mathrm{PMNS}} = U_e^\dagger U_0 \delta U. \tag{21}$$

In our case, the matrix $\hat{m}^D$ has a very simple expression:

$$\hat{m}^D \approx \begin{pmatrix} 1 & 0 & e^{i\phi_{31}} c_+ \epsilon \\ 0 & 1 & 0 \\ e^{i\phi_{31}} c_- \epsilon & 0 & -1 \end{pmatrix} y v_u \frac{v}{\Lambda}, \tag{22}$$

where $\phi_{31} = (\phi_3 - \phi_1)/2$, $c_{+(-)} = i\sqrt{3}/2(y_2 - y_1 + (-)2\tilde{y}_3)$ and $\epsilon = v_S^2/(v\Lambda) \sim \lambda_C$. Then we get

$$\hat{m}_\nu = \begin{pmatrix} m_1 & 0 & e^{i\phi_{31}}(c_+ m_1 + c_- m_3)\epsilon \\ 0 & m_2 & 0 \\ e^{i\phi_{31}}(c_+ m_1 + c_- m_3)\epsilon & 0 & m_3 \end{pmatrix} + O(\epsilon^2). \tag{23}$$

This result means that a correction $(\delta U)_{13} \sim \lambda_C$ can be present and we can expect that a deviation of θ_{12} from it tri-bimaximal value arises only at order λ_C^2. However, observe that if $m_1 \approx m_3$ i.e. the spectrum becomes degenerate, a fine-tuning will be required in order to reproduce a small $(\delta U)_{13}$. From this viewpoint, a degenerate spectrum is disfavored if we require that the deviation from TBM is naturally small.

Forgetting for a moment U_e which arises only at NNLO, from Eq. (21), one find that

$$U_{e3} = \sqrt{\frac{2}{3}} e^{i\phi_{13}} (\delta U)_{13}, \qquad U_{\mu 3} = -\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{6}} e^{i\phi_{13}} (\delta U)_{13}, \tag{24}$$

and U_{l2}, $l = e, \mu, \tau$, remain unchanged. As a result, the solar angle θ_{12} remains rather close to its tri-bimaximal value. However, $(\delta U)_{13}$ simultaneously induces a departure of θ_{13} and of $\theta_{23} - \pi/4$ from zero. Defining δ' as the phase of $(\delta U)_{13}$, the CP-violating Dirac phase is given by $-\delta = \delta' + \phi_{13}$. Since $\sin\theta_{13} = \sqrt{2/3}|(\delta U)_{13}|$, the deviation of the atmospherical angle from maximal is subject to the following sum-rule:

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2} \approx \frac{1}{2} + \frac{\sqrt{2}}{2} \cos\delta \, \sin\theta_{13} + O(\theta_{13}^2), \tag{25}$$

this is a prediction of our model. This is a special feature of the present seesaw A_4 model. The presence of the Abelian factor G_A in our model, not only allows a relatively large value of θ_{13}, at θ_C level, also strongly suppresses possible higher order contributions giving rise correlation between them.

Independently from the seesaw sector, TBM and in particular the solar angle receives corrections from charged lepton sector. Adopting the standard parametrization of U_{PMNS}, from (21) and (12) one finds that all the mixing angles receive a correction of order λ_C^2. Then we in particular obtain

$$\sin^2\theta_{12} = \frac{1}{3} + O\big(\lambda_C^2\big).$$

As claimed in the beginning, θ_{13} can be of order λ_C since it arises from corrections at NLO in the neutrino sector while θ_{12} receives corrections only of order λ_C^2 which are subleading effects at NNLO.

7. Conclusion and discussion

In this paper we have addressed one of the most important issues in the A_4 realization of TBM, i.e. if a $\theta_{13} \sim \theta_C$ can be allowed without fine tuning. We have discussed a framework, referred as constrained A_4 model, in which the vacuum alignment is realized by a fully separated scalar potential. The model is based on the $A_4 \times Z_3 \times Z_4$ flavour symmetry and (Type I) seesaw mechanism. In the charged lepton sector, the A_4 group is entirely broken by the set of scalar field $\Phi' = \{\varphi_T, \xi'\}$. The symmetry breaking parameter $\langle\Phi'\rangle/\Lambda \sim \lambda_C^2$ directly controls the charged lepton mass hierarchy without requiring a $U(1)_{\text{FN}}$ symmetry. In the neutrino sector, the set of scalar fields $\Phi = \{\varphi_S, \xi, \zeta\}$ breaks the A_4 group to its subgroup G_S guaranteeing the TBM at leading order. The symmetry breaking parameter $\langle\Phi\rangle/\Lambda$, however, can be chosen at order of the Cabibbo angle λ_C without altering the required vacuum alignment for Φ'. Moreover, a non-vanishing θ_{13} and a deviation of θ_{23} from $\pi/4$ are simultaneously generated at order $O(\lambda_C)$ leaving θ_{12} unchanged. Subsequently, a deviation of the solar angle from its TBM value is generated at order $O(\lambda_C^2)$ which just corresponds to its 1σ experimental sensitivity.

The model is called constrained A_4 model because, differently from its standard formulation widely studied in literature, the NLO corrections are also dictated by A_4 symmetry itself. This is another interesting feature of our model. There is, indeed, a correlation between the deviation of θ_{23} from maximal and the value of generated θ_{13}: $\sin^2\theta_{23} \approx 1/2 + \sqrt{2}/2\cos\delta\,\sin\theta_{13} + O(\theta_{13}^2)$ which can be in principle tested by future experiments. Concerning the neutrino spectrum, it can be either of normal hierarchy or inverted one. However, a degenerated spectrum is parametrically fine tuned and is disfavored requiring that the deviation from TBM is naturally small. For this reason, we should also expect that the effect of running on mixing angles is negligible. Since the solar angle has been measured more precisely than the others, its running can be potentially important if the neutrino spectrum were degenerate.

The corrections beyond the leading order are important not only in describing deviations from TBM, but also give rise other interesting phenomenology. For example, the same breaking pattern for charged lepton sector can be easily extended to the quark sector. In this case, the V_{CKM} arises when the correction to the vacuum alignment φ_T is taken into account. Then the resulting V_{CKM} should have the same form of the unitary matrix diagonalizing charged leptons U_e given in (12). The inclusion of the sub leading corrections can also play an important role in explaining the baryon asymmetry of the universe (BAU) through leptogenesis [16]. As pointed out in [7], the generated BAU can be indeed directly trigged by low energy phases appearing U_{e3}. Moreover, the structure of A_4 symmetry breaking pattern can be revealed by other physical effects [17], not directly related to neutrino properties, such as lepton flavour violating process as well as the anomalous magnetic moments and the electric dipole moments of charged leptons. Such a possibility becomes realistic if there is new physics at a much lower energy scale around 1–10 TeV. All these issues merit a further and more detailed study.

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Appendix A. The group A_4

The group A_4 has 12 elements and four non-equivalent irreducible representations: one triplet and three independent singlets 1, $1'$ and $1''$. Elements of A_4 are generated by the two generators S and T obeying the relations:

$$S^2 = (ST)^3 = T^3 = 1. \tag{26}$$

We will consider the following unitary representations of T and S:

$$\begin{aligned}
&\text{for } 1: && S = 1, \quad T = 1, \\
&\text{for } 1': && S = 1, \quad T = e^{i4\pi/3} \equiv \omega^2, \\
&\text{for } 1'': && S = 1, \quad T = e^{i2\pi/3} \equiv \omega,
\end{aligned} \tag{27}$$

and for the triplet representation

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \qquad S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \tag{28}$$

The tensor product of two triplets is given by $3 \times 3 = 1 + 1' + 1'' + 3_S + 3_A$. From (27) and (28), one can easily construct all multiplication rules of A_4. In particular, for two triplets $\psi = (\psi_1, \psi_2, \psi_3)$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ one has:

$$\begin{aligned}
&\psi_1\varphi_1 + \psi_2\varphi_3 + \psi_3\varphi_2 \sim 1, \\
&\psi_3\varphi_3 + \psi_1\varphi_2 + \psi_2\varphi_1 \sim 1', \\
&\psi_2\varphi_2 + \psi_3\varphi_1 + \psi_1\varphi_3 \sim 1'', \\
&\begin{pmatrix} 2\psi_1\varphi_1 - \psi_2\varphi_3 - \psi_3\varphi_2 \\ 2\psi_3\varphi_3 - \psi_1\varphi_2 - \psi_2\varphi_1 \\ 2\psi_2\varphi_2 - \psi_1\varphi_3 - \psi_3\varphi_1 \end{pmatrix} \sim 3_S, \qquad \begin{pmatrix} \psi_2\varphi_3 - \psi_3\varphi_2 \\ \psi_1\varphi_2 - \psi_2\varphi_1 \\ \psi_3\varphi_1 - \psi_1\varphi_3 \end{pmatrix} \sim 3_A.
\end{aligned} \tag{29}$$

Appendix B. Correction to alignment of φ_T and φ_S

In this appendix we will study correction to the leading order alignment of φ_S and φ_T when we include higher dimensionality operators up to the order $1/\Lambda^2$.

In our model, the correction to the driving superpotential for φ_S, depends only on Φ at NNLO, then the obtained vacuum alignment $\langle\varphi_S\rangle \propto (1,1,1)$ is always stable since it preserves the subgroup G_S of A_4. However a relative large $\langle\Phi\rangle/\Lambda \sim \lambda_C$ may have some effects on the leading order alignment for $\varphi_T \propto (0,1,0)$. The products $\xi\zeta$ and $\varphi_S\zeta$ are invariant combination under G_A, then we can include them on top of each term in w_d^e. With the introduction of these higher dimensionality operators, w_d^e should be modified into $w_d^e + \delta w_d^e$ where[5]

$$\delta w_d^e = \frac{1}{\Lambda^2}\big[t_1\zeta\xi\xi'\big(\varphi_0^T\varphi_T\big)'' + t_2\zeta\xi\big(\varphi_0^T\varphi_T\varphi_T\big) + t_3\zeta\xi'\big(\varphi_0^T\varphi_T\varphi_S\big)'' + t_4\zeta\big(\varphi_0^T\varphi_S\big)'(\varphi_T\varphi_T)''\big].$$

The alignment for φ_T should be shifted (the shift in ξ' is needless) and we can look for a solution that perturbs $\langle\varphi_T\rangle$ to second order in the $1/\Lambda$ expansion:

$$\langle\xi'\rangle = u', \qquad \langle\varphi_T\rangle = (\delta_{T1}, v_T + \delta_{T2}, \delta_{T3}).$$

The minimum conditions from $w_d^e + \delta w_d^e$ become equations in the shifts δv_{Ti}:

$$-4h_2 v_T\delta v_{T3} + \left(t_4 - t_3\frac{4h_2}{h_1}\right)\frac{vv_S}{\Lambda^2}v_T^2 = 0,$$
$$2h_2 v_T\delta v_{T2} + \left(t_4 + t_3\frac{4h_2}{h_1}\right)\frac{vv_S}{\Lambda^2}v_T^2 + \left(2t_2 - t_1\frac{2h_2}{h_1}\right)\frac{vu}{\Lambda^2}v_T^2 = 0,$$
$$-4h_2 v_T\delta v_{T1} + \left(t_4 + t_3\frac{4h_2}{h_1}\right)\frac{vv_S}{\Lambda^2}v_T^2 = 0.$$

These equations are linear in δv_{Ti} and can be easily solved by:

$$\frac{\delta v_{T3}}{v_T} = \left(\frac{t_4}{4h_2} - \frac{t_3}{h_1}\right)\frac{vv_S}{\Lambda^2},$$
$$\frac{\delta v_{T2}}{v_T} = -\left(\frac{t_4}{2h_2} + \frac{2t_3}{h_1}\right)\frac{vv_S}{\Lambda^2} + \left(\frac{t_1}{h_1} - \frac{t_2}{h_2}\right)\frac{vu}{\Lambda^2},$$
$$\frac{\delta v_{T1}}{v_T} = \left(\frac{t_4}{4h_2} + \frac{t_3}{h_1}\right)\frac{vv_S}{\Lambda^2}.$$

Observe that the shifts in three components are different but all of the same order of magnitude, as claimed in the text:

$$\frac{\delta v_{Ti}}{v_T} \sim O\big(\lambda_C^2\big).$$

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