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A Multiplicity Result for Strongly Nonlinear Perturbations of Elliptic Boundary Value Problems*

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1. INTRODUCTION

Let Ω be a bounded domain in $\mathbf{R}^n$, for $n \geq 2$, with smooth boundary $\partial\Omega$, and denote by $\{\lambda_m\}$ the increasing sequence of eigenvalues of $-\Delta$ over Ω with Dirichlet boundary conditions. Let $g\colon \Omega \times \mathbf{R} \to \mathbf{R}$ be a C^1 function satisfying: (i) $g(x, \xi)$ is bounded for ξ large and negative, (ii) $g(x, \xi)$ may grow superlinearly in ξ for ξ large and positive. Denote by φ_1 a positive eigenfunction corresponding to λ_1, and let $h \in C^\alpha(\Omega)$, for some $\alpha > 0$, be orthogonal to φ_1 with respect to the $L^2(\Omega)$ inner product. In this paper we are concerned with the existence of multiple solutions of the semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u - \lambda u - g(x, u) = t\varphi_1 + h, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases} \qquad (P_t, \lambda)$$

where λ and t are real parameters.

The case $\lambda < \lambda_1$ is known as a superlinear Ambrosetti–Prodi problem, see, for example, [1]. In this case, it is proved in [1] that under certain conditions on g, for t large and negative, problem (P_t, λ) has at least two solutions.

It is interesting to consider the question of existence of solutions of (P_t, λ) for t large and positive. For the case $\lambda < \lambda_1$ one can show that problem (P_t, λ) has no solutions for t large and positive (see Proposition 2 under Preliminary Results). Hence, if we are looking for solutions of (P_t, λ), for t large and positive, we must require that $\lambda \geq \lambda_1$.

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In an interesting paper of Ruf and Srikanth [7] the following problem is considered

$$\begin{cases} -\Delta u - \lambda u - (u^+)^p = t\varphi_1 + h, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $u^+ = \max\{u, 0\}$, $1 < p < (n+2)/(n-2)$ if $n \geq 3$ and $p > 1$ if $n = 2$, and h is orthogonal to φ_1 with respect to the $L^2(\Omega)$ inner product. In [7] it is shown that if $\lambda > \lambda_1$ and $\lambda \neq \lambda_k$ for all $k > 1$, then for t large and positive the above problem has at least two solutions.

Motivated by the results in [7, 1], in this paper we present a multiplicity result for (P_t, λ) in the case $\lambda > \lambda_1$. Suppose that

$$\lim_{\xi \to -\infty} \frac{\partial}{\partial \xi} g(x, \xi) = 0, \qquad \text{uniformly in } x \in \Omega, \tag{1}$$

and

$$\liminf_{\xi \to +\infty} \frac{g(x, \xi)}{\xi} > 0, \qquad \text{uniformly in } x \in \Omega, \tag{2}$$

where the limit on the left hand side of (2) could be ∞ on Ω, or on a subset of Ω with positive measure. In addition, assume that $(\partial/\partial\xi) g(x, \xi)$ satisfies the growth condition

$$|g_\xi(x, \xi)| \leq C_1 + C_2 |\xi|^{p-1} \qquad \text{for } \xi \in \mathbf{R} \text{ and } x \in \Omega, \tag{3}$$

where $1 < p < (n+2)/(n-2)$ if $n \geq 3$ and $p > 1$ if $n = 2$, for some constants C_1 and C_2. Suppose also that

$$\liminf_{\xi \to +\infty} \frac{\xi g(x, \xi) - 2G(x, \xi)}{|g(x, \xi)|^{1+1/p}} > 0, \qquad \text{uniformly in } x \in \Omega, \tag{4}$$

where $G(x, \xi) = \int_0^\xi g(x, t)\, dt$. In Section 4 we prove that under these assumptions for $\lambda > \lambda_1$ (with $\lambda \neq \lambda_k$ for all $k \geq 2$), there exists a positive number $T = T(h)$, such that, if $t > T$, then problem (P_t, λ) has at least two solutions.

The existence of the first solution u_t to problem (P_t, λ) is proved using the method of sub- and super-solutions (see, for example, [8]), while the second solution is obtained by means of the Mountain Pass Theorem of Ambrosetti and Rabinowitz [6, Theorem 2.2]. Although the arguments in this paper are close to those of De Figueiredo in [1], the techniques used here are also similar to those used by Lazer and McKenna in [4] for the case of jumping nonlinearities which cross a finite number of eigenvalues (see also the related works [3, 5]). The multiplicity results in [3–5] are obtained by means of a combination of degree theoretic calculations and critical point theory techniques via a reduction method. The main difference between this work and the work of Lazer and McKenna [4] is that in this paper the nonlinearity $g(\xi)$ is allowed to grow superlinearly in ξ for $\xi > 0$ (and consequently the nonlinearity could cross infinitely many eigenvalues), while in [4] it is assumed that $\lim_{\xi \to +\infty} g'(\xi)$ exists and is finite, and that $\lambda + \lim_{\xi \to +\infty} g'(\xi)$ is not an eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions. This allows one to obtain the *a priori* bounds needed for the degree theoretic computations. If g is allowed to grow superlinearly, those bounds are harder to obtain.

2. NOTATION, DEFINITIONS, AND SOME BASIC FACTS

Denote by $H_o^1(\Omega)$ the completion of $C_c^\infty(\Omega)$ with respect to the norm given by $\|u\|^2 = \int_\Omega |\nabla u|^2$. $H_o^1(\Omega)$ is then a real Hilbert space with inner product $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v$ for all u and v in $H_o^1(\Omega)$.

By a solution of the problem (P_t, λ) we mean a function $u \in H_o^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v - \lambda \int_\Omega uv - \int_\Omega g(x,u)v - t\int_\Omega \varphi_1 v - \int_\Omega hv = 0, \qquad \forall v \in H_o^1(\Omega).$$

Since g is assumed to be continuously differentiable, standard regularity arguments imply that any solution of (P_t, λ) is, in fact, in $C^2(\Omega) \cap C(\overline{\Omega})$.

For each $t \in \mathbf{R}$ define a functional $J_t\colon H_o^1(\Omega) \to \mathbf{R}$ by

$$J_t(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega u^2 - \int_\Omega G(x,u) - t\int_\Omega \varphi_1 u - \int_\Omega hu \qquad \forall u \in H_o^1(\Omega),$$

where $G(x,\xi) = \int_0^\xi g(x,t)\,dt$ for $\xi \in \mathbf{R}$ and all $x \in \Omega$. The growth condition (3) can be used to prove that $J_t \in C^2(H_o^1(\Omega), \mathbf{R})$ with Frechet derivatives at $u \in H_o^1(\Omega)$ given by

$$J_t'(u)v = \int_\Omega \nabla u \cdot \nabla v - \lambda\int_\Omega uv - \int_\Omega g(x,u)v - t\int_\Omega \varphi_1 v - \int_\Omega hv, \qquad \forall v \in H_o^1(\Omega),$$

and

$$J_t''(u)(v,w) = \int_\Omega \nabla v \cdot \nabla w - \lambda \int_\Omega vw - \int_\Omega g_\xi(x,u) vw$$

$$\text{for all } v, w \in H_o^1(\Omega).$$

For each $u \in H_o^1(\Omega)$ let $A_t(u)$ denote the linear self-adjoint operator induced on $H_o^1(\Omega)$ by setting

$$J_t''(u)(v,w) = \langle A_t(u) v, w \rangle \qquad \text{for all } v, w \in H_o^1(\Omega).$$

A critical point u of J_t is said to be nondegenerate if $A_t(u)$ is invertible; or equivalently, if there is no $v \neq 0$ such that $\langle A_t(u)v, w\rangle = 0$ for all $w \in H_o^1(\Omega)$.

The functional J_t is said to satisfy the Palais–Smale condition if

$$(\mathbf{PS}) \quad \begin{cases} \text{every sequence } \{u_n\} \subset H_o^1(\Omega) \text{ satisfying:} \\ \text{(i) } J_t(u_n) \text{ is bounded, and} \\ \text{(ii) } J_t'(u_n) \to 0 \text{ in norm as } n \to \infty, \\ \text{has a strongly convergent subsequence.} \end{cases}$$

3. PRELIMINARY RESULTS

For the sake of comparison with the superlinear Ambrosetti–Prodi problem, in this section we shall temporarily replace (2) with

$$\liminf_{\xi \to +\infty} \frac{g(x,\xi)}{\xi} > \max\{\lambda_1 - \lambda, 0\}, \tag{2'}$$

where the inequality holds uniformly in Ω.

We first observe that conditions (1) and (2′) imply the existence of positive constants μ and C_o such that $\lambda + \mu \neq \lambda_k$ for any k, and

$$g(x,\xi) \geq \mu\xi - C_o \qquad \text{for } x \in \Omega, \text{ and } \xi \in \mathbf{R}. \tag{5}$$

We will now see how the condition in (5) implies the existence of a lower bound for solutions on (P_t, λ) with $t \geq a$ for some $a \in \mathbf{R}$ (see Lemma 4 in [1]). Recall that a function $u \in C^{2,\alpha}(\Omega)$ is said to be a subsolution of (P_t, λ), if

$$-\Delta u - \lambda u \leq g(x,u) + t\varphi_1 + h \qquad \text{in } \Omega$$

and $u \leq 0$ on $\partial\Omega$, and $v \in C^{2,\alpha}(\Omega)$ is a supersolution if it satisfies the above inequalities with $\leq$ replaced by $\geq$.

LEMMA 1. *Assume* (1) *and* (2′). *Given* $a \in \mathbf{R}$, *there exists a function* $w \in C^{2,\alpha}(\Omega)$ *with* $w = 0$ *on* $\partial\Omega$, *such that* w *is a subsolution of* (P_t, λ) *for all* $t \geq a$, *and, if* $v \in C^{2,\alpha}(\Omega)$ *is a supersolution of* (P_t, λ) *for* $t \geq a$, *then* $w \leq v$ *in* Ω.

Proof. Let w be the unique solution of the linear Dirichlet problem

$$\begin{cases} -\Delta w - \lambda w = \mu w - C_o + a\varphi_1 + h, & \text{in } \Omega; \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

then, by (5), w is a subsolution of (P_t) for any $t \geq a$. The maximum principle yields the second part of the lemma. ∎

In the case $\lambda < \lambda_1$ in (P_t, λ) and g satisfying (2′), one can choose μ so that $\mu > \lambda_1 - \lambda$. In this case we can get an upper bound for the values of $t \geq 0$ for which (P_t, λ) has a solution. This result is essentially due to Kazdan and Warner [2]; see also Lemma 3 in [1].

PROPOSITION 2. *Assume* (1) *and* (2′) *for* $\lambda < \lambda_1$. *There exists a* $\tau \in \mathbf{R}$ *such that* (P_t, λ) *has no solutions for* $t > \tau$.

Proof. Let u be a solution of (P_t, λ) for $t \geq 0$. Apply Lemma 1 with $a = 0$ to obtain a function $w \in C^{2,\alpha}(\Omega)$ satisfying $w \leq u$ in Ω. Green's identity yields

$$\begin{aligned} 0 &= \int_\Omega [(\Delta\varphi_1)u - (\Delta u)\varphi_1] \\ &= -(\lambda_1 - \lambda)\int_\Omega u\varphi_1 + \int_\Omega g(x,u)\varphi_1 + t\int_\Omega \varphi_1^2. \end{aligned}$$

Using (5) we then obtain

$$0 \geq [\mu - (\lambda_1 - \lambda)]\int_\Omega u\varphi_1 + t\int_\Omega \varphi_1^2 - C_o\int_\Omega \varphi_1,$$

and since $\mu - (\lambda_1 - \lambda) > 0$ in this case, we obtain

$$0 \geq [\mu - (\lambda_1 - \lambda)]\int_\Omega w\varphi_1 + t\int_\Omega \varphi_1^2 - C_o\int_\Omega \varphi_1,$$

from which the result follows. ∎

In view of Proposition 2, if we are looking for solutions of (P_t, λ) for t large and positive, we must require that $\lambda \geq \lambda_1$. In this paper we will consider the case $\lambda > \lambda_1$ and $\lambda \neq \lambda_k$ for any $k > 1$. We shall establish the existence of at least one solution of (P_t, λ) for t large and positive by considering the following modified version of (P_t, λ):

$$\begin{cases} -\Delta u - \lambda u = g\left(x, u - a\varphi_1 + \tilde{h}\right), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases} \qquad \left(\tilde{P}_a, \lambda\right)$$

where $a \in \mathbf{R}$, and $\tilde{h} = (-\Delta - \lambda)^{-1}h$, so that $\tilde{h} \in C^{2,\alpha}(\Omega) \cap C^1(\overline{\Omega})$ by elliptic regularity theory.

Proposition 3. *Let $\lambda > \lambda_1$ and suppose that g satisfies* (1) *and* (2). *Assume also that $\lambda \neq \lambda_k$ for any $k > 1$. Given $h \in C^\alpha(\Omega)$ such that $\int_\Omega h\varphi_1 = 0$, there exists a constant $A = A(h)$ such that for $a > A$ problem $(\tilde{P}_a, \lambda)$ has at least one solution.*

Proof. By condition (1), there exists $K \in \mathbf{R}$ such that $g(x, \xi) \leq K$ for all $x \in \Omega$ and $\xi \leq 0$. Let $V \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be the unique solution to the Dirichlet problem

$$\begin{cases} -\Delta V - \lambda V = K, & \text{in } \Omega; \\ V = 0, & \text{on } \partial\Omega. \end{cases}$$

Choose a so large and positive that $V - a\varphi_1 + \tilde{h} < 0$ in Ω. Then

$$-\Delta V - \lambda V = K \geq g\left(x, V - a\varphi_1 + \tilde{h}\right) \qquad \text{in } \Omega,$$

and so V is a supersolution of $(\tilde{P}_a, \lambda)$.

The argument in the proof of Lemma 1 yields now a subsolution w of $(\tilde{P}_a, \lambda)$ satisfying $w \leq V$ in Ω. The result then follows by the method of monotone iterations, see for instance [8]. ∎

Remark. Observe that if u_a is a solution of $(\tilde{P}_a, \lambda)$, then $u = u_a - a\varphi_1 + \tilde{h}$ is a solution of (P_t, λ) for $t = (\lambda - \lambda_1)a$. Thus Proposition 3 establishes the existence of $T = (\lambda - \lambda_1)A$ such that, for $t > T$, problem (P_t, λ) has at least one solution $u_t = u_a - a\varphi_1 + \tilde{h}$. By virtue of the method of monotone iterations, such a solution is minimal in the sense that, if u is any other solution of (P_t, λ), then $u_t \leq u$ in Ω. Furthermore, for each $t > T$,

$$u_t \leq V - \frac{t}{\lambda - \lambda_1}\varphi_1 + \tilde{h} < 0 \qquad \text{in } \Omega;$$

hence, $u_t(x) \to -\infty$ as $t \to \infty$ for every $x \in \Omega$. We collect these facts in the following

THEOREM 4. *Let $\lambda > \lambda_1$ and suppose that g satisfies* (1) *and* (2). *Assume also that $\lambda \neq \lambda_k$ for any $k > 1$. Given $h \in C^\alpha(\Omega)$ with $\int_\Omega h\varphi_1 = 0$, there exists a constant $T = T(h)$ such that for $t > T$ problem (P_t, λ) has a minimal solution u_t satisfying $u_t < 0$ in Ω. Furthermore, for each $x \in \Omega$, $u_t(x) \to -\infty$ as $t \to \infty$.*

Our next task is to prove that, for t sufficiently large, the solution u_t given by Theorem 4 is a nondegenerate critical point of J_t. We start out by stating the following lemma whose proof is similar to that of Lemma 7 in [1, p. 659], and is therefore omitted.

LEMMA 5. *Let $\lambda > \lambda_1$ and assume that g satisfies* (1) *and* (2). *Suppose also that $\lambda \neq \lambda_k$ for any $k \geq 2$. Let $t > T$ and u_t be the minimal solution of (P_t, λ) given by Theorem* 4. *Then the first eigenvalue γ_1 of the problem*

$$\begin{cases} -\Delta v - \big(\lambda + g_\xi(x, u_t)\big)v = \gamma v, & \text{in } \Omega; \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

is non-negative.

Next we show as in [4, Lemma 4.7, p. 145] that, for t sufficiently large and positive, u_t is a nondegenerate critical point of J_t.

PROPOSITION 6. *Let $\lambda > \lambda_1$ and assume that g satisfies* (1) *and* (2). *Suppose also that $\lambda \neq \lambda_k$ for any $k \geq 2$. There exists $T_2 \in \mathbf{R}$ such that, for $t > T_2$, the critical point u_t of J_t given by Theorem* 4 *is nondegenerate.*

Proof. Assume by way of contradiction that there exist $t_n \to \infty$, and a corresponding sequence of functions $v_n \in H_o^1(\Omega)$ with $\int_\Omega v_n^2 = 1$ for all n, such that, for $u_n = u_{t_n}$, $\langle A(u_n)v_n, w\rangle = 0$ for all $w \in H_o^1(\Omega)$; i.e.,

$$\int_\Omega \nabla v_n \cdot \nabla w - \int_\Omega \big(\lambda + g_\xi(x, u_n)\big)v_n w = 0, \qquad \text{for all } w \in H_o^1(\Omega). \quad (6)$$

Since $u_n < 0$ in Ω for all n, and $|g_\xi(x, \xi)| \leq K$ for all $\xi \leq 0$ and $x \in \Omega$, replacing w by v_n in (6) we conclude that $\{v_n\}$ is bounded in $H_o^1(\Omega)$. Hence, passing to a subsequence if necessary, we may assume that there exists $v \in H_o^1(\Omega)$ such that $v_n \rightharpoonup v$ weakly in $H_o^1(\Omega)$ and $v_n \to v$ strongly in $L^2(\Omega)$; so that $\int_\Omega v^2 = 1$. The fact that $u_n(x) \to -\infty$ for all $x \in \Omega$ as $n \to \infty$, condition (1), and the Dominated Convergence Theorem can now be used in (6) to get that

$$\int_\Omega \nabla v \cdot \nabla w - \lambda \int_\Omega vw = 0, \qquad \text{for all } w \in H_o^1(\Omega);$$

i.e., v is a nontrivial solution of

$$\begin{cases} -\Delta v - \lambda v = 0, & \text{in } \Omega: \\ v = 0, & \text{on } \partial\Omega; \end{cases}$$

but $\lambda \neq \lambda_k$ for all k, thus, $v \equiv 0$, which is a contradiction. Hence, there must be a $T_2 > T$ such that, for $t > T_2$, u_t is a nondegenerate critical point of J_t. ∎

Remark. As a consequence of Proposition 6 we can conclude that the γ_1 in Lemma 5, for $t > T_2$, is in fact positive. Indeed, if $\gamma_1 = 0$, then there exists $v_1 \neq 0$ in Ω satisfying

$$\begin{cases} -\Delta v_1 - \big(\lambda + g_\xi(x, u_t)\big) v_1 = 0, & \text{in } \Omega; \\ v_1 = 0, & \text{on } \partial\Omega, \end{cases}$$

so that

$$\int_\Omega \nabla v_1 \cdot \nabla w - \int_\Omega \big(\lambda + g_\xi(x, u_n)\big) v_1 w = 0, \qquad \text{for all } w \in H_o^1(\Omega),$$

which contradicts the fact that u_t is nondegenerate for $t > T_2$. We therefore have, for $t > T_2$,

$$\int_\Omega |\nabla v|^2 - \int_\Omega \big(\lambda + g_\xi(x, u_t)\big) v^2 \geq \gamma_1 \int_\Omega v^2, \qquad \text{for all } v \in H_o^1(\Omega), \quad (7)$$

where $\gamma_1 > 0$. We also have that

$$\int_\Omega |\nabla v|^2 - \int_\Omega \big(\lambda + g_\xi(x, u_t)\big) v^2 \geq \int_\Omega |\nabla v|^2 - (\lambda + K) \int_\Omega v^2, \qquad \forall v \in H_o^1(\Omega), \quad (8)$$

where K is such that $|g(x, \xi)| \leq K$ for all $\xi \leq 0$. Combining (7) and (8) we obtain

$$\int_\Omega |\nabla v|^2 - \int_\Omega \big(\lambda + g_\xi(x, u_t)\big) v^2 \geq c \int_\Omega |\nabla v|^2, \qquad \text{for all } v \in H_o^1(\Omega), \quad (9)$$

where $c = 1/(1 + (\lambda + K)/\gamma_1)$.

4. EXISTENCE OF A SECOND SOLUTION

A solution of (P_t, λ) can also be obtained by means of the mountain pass theorem of Ambrosetti and Rabinowitz (Theorem 2.2 in [6, p. 7]). For this purpose we will need to verify that the functional J_t satisfies the Palais−Smale condition.

Lemma 7. *Let* $\lambda > \lambda_1$ *be such that* $\lambda \neq \lambda_k$ *for any* $k \geq 2$, *and suppose that g satisfies conditions* (1) *through* (4), *then* J_t *satisfies* (PS).

Proof. Let $\{u_n\}$ be a sequence in $H_o^1(\Omega)$ satisfying $|J_t(u_n)| \leq M$ for all n, and $J_t'(u_n) \to 0$ in norm as $n \to \infty$. Define $\nabla J_t\colon H_o^1(\Omega) \to H_o^1(\Omega)$ by $J_t'(u)v = \langle \nabla J_t(u), v\rangle$ for all $v \in H_o^1(\Omega)$. By virtue of the growth condition in (3) and the Sobolev embedding theorem, ∇J_t is of the form $I - N$, where N is a completely continuous operator on $H_o^1(\Omega)$. Thus, it suffices to prove that $\{u_n\}$ is bounded in $H_o^1(\Omega)$ (see [9, Proposition 2.2, p. 71]). Suppose to the contrary that, for a subsequence if necessary, $\|u_n\| \to \infty$ as $n \to \infty$.

From the condition $\nabla J_t(u_n) \to 0$ as $n \to \infty$ we get that, for all n and all $v \in H_o^1(\Omega)$.

$$\int_\Omega \nabla u_n \cdot \nabla v - \lambda \int_\Omega u_n v - \int_\Omega g(x, u_n) v - \int_\Omega (t\varphi_1 + h) v = o(1)\|v\|, \quad (10)$$

where $o(1) \to 0$ as $n \to \infty$. Thus, putting $v = u_n$ in (10),

$$\int_\Omega |\nabla u_n|^2 - \lambda \int_\Omega u_n^2 - \int_\Omega g(x, u_n) u_n - \int_\Omega (t\varphi_1 + h) u_n = o(1)\|u_n\|. \quad (11)$$

Using the condition $|J_t(u_n)| \leq M$ we get

$$\begin{aligned} 2M - J_t(u_n)u_n &\geq 2J_t(u_n) - J_t(u_n)u_n \\ &\geq \int_\Omega (g(x, u_n)u_n - 2G(x, u_n)) - C_3\|u_n\| \end{aligned}$$

for some constant $C_3 > 0$. So that, using (11),

$$2M + C_4\|u_n\| \geq \int_\Omega (g(x, u_n)u_n - 2G(x, u_n)), \quad (12)$$

for some constant $C_4 > 0$.

From $|g(x, \xi)| \leq K$ for all $\xi \leq 0$ we get that

$$\left| \int_{u_n \leq 0} (g(x, u_n)u_n - 2G(x, u_n)) \right| \leq C_5\|u_n\|,$$

for some $C_5 > 0$. Combining this with (12) we get that

$$\int_{u_n > 0} (g(x, u_n)u_n - 2G(x, u_n)) \leq 2M + C_6 \|u_n\|,$$

for some $C_6 > 0$. Hence

$$\lim_{n \to \infty} \int_{u_n > 0} \frac{(g(x, u_n)u_n - 2G(x, u_n))}{\|u_n\|^{1+1/p}} = 0.$$

Here we have used the fact that, by (4), $\xi g(x, \xi) - 2G(x, \xi) > 0$ for ξ large and positive. Hence, by (4) again

$$\int_{u_n > 0} \frac{|g(x, u_n)|^{1+1/p}}{\|u_n\|^{1+1/p}} \to 0 \qquad \text{as } n \to \infty.$$

Now, since $|g(x, \xi) \leq K$ for all $\xi \leq 0$, we also get that

$$\int_{u_n \leq 0} \frac{|g(x, u_n)|^{1+1/p}}{\|u_n\|^{1+1/p}} \to 0 \qquad \text{as } n \to \infty.$$

We therefore conclude that

$$\frac{g(x, u_n)}{\|u_n\|} \to 0 \qquad \text{in } L^{1+1/p}(\Omega) \text{ as } n \to \infty. \tag{13}$$

Thus, by the Sobolev embedding theorem,

$$\frac{g(x, u_n)}{\|u_n\|} \to 0 \qquad \text{in } H^{-1}(\Omega) \text{ as } n \to \infty, \tag{14}$$

where $H^{-1}(\Omega)$ denotes the dual of $H_o^1(\Omega)$.

Let $w_n = u_n / \|u_n\|$ for all n. Then $\|w_n\| = 1$ for all n, and so we may assume that there exists $w \in H_o^1(\Omega)$ such that $w_n \rightharpoonup w$ weakly in $H_o^1(\Omega)$ and $w_n \to w$ strongly in $L^2(\Omega)$. Dividing (10) by $\|u_n\|$, letting $n \to \infty$, and using (14) we obtain

$$\int_\Omega \nabla w \cdot \nabla v - \lambda \int_\Omega wv = 0, \qquad \text{for all } v \in H_o^1(\Omega);$$

i.e., w is a solution of

$$\begin{cases} -\Delta w - \lambda w = 0, & \text{in } \Omega; \\ w = 0, & \text{on } \partial\Omega; \end{cases}$$

but $\lambda \neq \lambda_k$ for all k, thus $w \equiv 0$. We then have that $w_n \to 0$ in $L^2(\Omega)$. Now, from the assumption that $J_t'(u_n) \to 0$ in norm and the fact that $\{w_n\}$ is bounded, we get that $J_t'(u_n)w_n \to 0$ as $n \to \infty$. So that

$$\frac{J_t'(u_n)w_n}{\|u_n\|} \to 0 \qquad \text{as } n \to \infty,$$

or, dividing (10) (with $v = w_n$) by $\|u_n\|$, and letting $n \to \infty$,

$$\int_\Omega |\nabla w_n|^2 - \lambda \int_\Omega w_n^2 - \int_\Omega \frac{g(x, u_n)w_n}{\|u_n\|} \to 0 \qquad \text{as } n \to \infty, \tag{15}$$

where, by Hölder's inequality,

$$\left| \int_\Omega \frac{g(x, u_n)w_n}{\|u_n\|} \right| \leq \left\| \frac{g(\cdot, u_n)}{\|u_n\|} \right\|_{1+1/p} \|w_n\|_{p+1}. \tag{16}$$

Using (13), the Sobolev embedding theorem, and the fact that $\{\|w_n\|\}$ is uniformly bounded, we obtain from (16) that

$$\int_\Omega \frac{g(x, u_n)w_n}{\|u_n\|} \to 0 \qquad \text{as } n \to \infty,$$

which, in combination with (15), yields

$$\lim_{n\to\infty} \int_\Omega |\nabla w_n|^2 = \lambda \lim_{n\to\infty} \int_\Omega w_n^2 = 0.$$

But this is in contradiction with the fact that $\|w_n\| = 1$ for all n. Hence $\{u_n\}$ must be bounded and the lemma is proved. ∎

THEOREM 8. *Let $\lambda > \lambda_1$ be such that $\lambda \neq \lambda_k$ for any $k \geq 2$, and suppose that g satisfies conditions* (1) *through* (4). *Let T_2 be as given by Proposition* 6. *Then, for $t > T_2$, (P_t, λ) has at least two solutions.*

Proof. One solution u_t is given by Theorem 4. By Proposition 6, u_t is a nondegenerate critical point of J_t. Moreover, there exists a constant $c > 0$ such that (9) holds:

$$\int_\Omega |\nabla v|^2 - \int_\Omega \big(\lambda + g_\xi(x, u_t)\big) v^2 \geq c \int_\Omega |\nabla v|^2, \qquad \text{for all } v \in H_o^1(\Omega).$$

We will apply the Ambrosetti–Rabinowitz mountain pass theorem [6] to obtain another critical point of J_t distinct from u_t. We have already established in Lemma 7 that J_t satisfies (PS); thus, it remains to show that

(i) there exist $\rho > 0$ and $\beta > J_t(u_t)$ such that $J_t(u) \geq \beta$ for all $u \in H_o^1(\Omega)$ with $\|u - u_t\| = \rho$, and

(ii) there is $w \in H_o^1(\Omega)$ such that $\|w - u_t\| > \rho$ and $J_t(w) < J_t(u_t)$.

To prove (i) we proceed as in De Figueiredo [1]. Using the fact that u_t is a critical point of J_t we obtain

$$J_t(u_t + v) - J_t(u_t) = \frac{1}{2}\int_\Omega (|\nabla v|^2 - \lambda v^2) - \int_\Omega [G(x, u_t + v) - G(x, u_t) - g(x, u_t)v]$$

for all $v \in H_o^1(\Omega)$. By Taylor's theorem we have

$$G(x, u_t + \xi) - G(x, u_t) - g(x, u_t)\xi = \tfrac{1}{2} g_\xi(x, u_t)\xi^2 + r(x, \xi)$$

where $|r(x, \xi)|/\xi^2 \to 0$ as $\xi \to 0$, for all $x \in \Omega$.

Choose $\varepsilon > 0$ so small that $\varepsilon < \lambda_1 c/2$, where c is the constant in (9). Then there exists $\delta > 0$ such that $|\xi| \leq \delta$ implies

$$|r(x, \xi)| \leq \varepsilon |\xi|^2 \qquad \text{for all } x \in \Omega.$$

On the other hand, using (3) we obtain a constant $C_\delta > 0$ such that

$$|r(x, \xi)| \leq C_\delta |\xi|^{p+1} \qquad \text{for } |\xi| \geq \delta \text{ and all } x \in \Omega.$$

Hence

$$|r(x, \xi)| \leq \varepsilon |\xi|^2 + C_\delta |\xi|^{p+1} \qquad \text{for all } \xi \in \mathbf{R} \text{ and } x \in \Omega,$$

and therefore

$$\int_\Omega |r(x, v)| \leq \varepsilon \int_\Omega v^2 + C_\delta \int_\Omega |v|^{p+1}$$

which, by the Sobolev embedding theorem, yields

$$\int_\Omega |r(x, v)| \leq \frac{\varepsilon}{\lambda_1}\|v\|^2 + C_7 \|v\|^{p+1}$$

for some constant $C_7 > 0$.

We then have that

$$J_t(u_t + v) - J_t(u_t) \geq \frac{1}{2}\int_\Omega\Big(|\nabla v|^2 - \big(\lambda + g_\xi(x, u_t)\big)v^2\Big) - \frac{\varepsilon}{\lambda_1}\|v\|^2 - C_7\|v\|^{p+1}$$

and, using (9),

$$J_t(u_t + v) - J_t(u_t) \geq \left(\frac{c}{2} - \frac{\varepsilon}{\lambda_1} - C_7\|v\|^{p-1}\right)\|v\|^2.$$

Now choose $\rho > 0$ so small that $c/2 - \varepsilon/\lambda_1 - C_7\rho^{p-1} > 0$. Then for all $v \in H_o^1(\Omega)$ with $\|v\| = \rho$,

$$J_t(u_t + v) \geq \beta,$$

where

$$\beta = J_t(u_t) + \left(\frac{c}{2} - \frac{\varepsilon}{\lambda_1} - C_7\rho^{p-1}\right)\rho^2 > J_t(u_t),$$

which yields (i).

To see (ii), consider

$$J_t(r\varphi_1) = \frac{1}{2}(\lambda_1 - \lambda)r^2\int_\Omega \varphi_1^2 - \int_\Omega G(x, r\varphi_1) - tr\int_\Omega \varphi_1^2$$

for $r > 0$. Using (5) we obtain

$$G(x, \xi) \geq \frac{\mu}{2}\xi^2 - C_o\xi \qquad \text{for } \xi \geq 0,$$

and some $\mu > 0$. Then

$$J_t(r\varphi_1) \leq \frac{1}{2}(\lambda_1 - \lambda - \mu)r^2\int_\Omega \varphi_1^2 - r\int_\Omega\big(C_o\varphi_1 + t\varphi_1^2\big),$$

which shows that $J_t(r\varphi_1) \to -\infty$ as $r \to \infty$. Therefore, taking $w = r\varphi_1$ for r large enough, we obtain (ii).

The proof now follows by a straightforward application of the mountain pass theorem of Ambrosetti and Rabinowitz. ∎

Remark. Observe that if we assume that $G(x, \xi) > 0$ for ξ sufficiently large and positive and all $x \in \Omega$, then condition (2) actually follows from

(4). Consequently, one can obtain the same multiplicity result of Theorem 8 with (2) replaced by this assumption, or by the condition

$$\liminf_{\xi \to +\infty} G(x, \xi) > 0, \qquad \text{uniformly in } x \in \Omega. \tag{2''}$$

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REFERENCES

1. D. G. De Figueiredo, On the superlinear Ambrosetti-Prodi problem, *Nonlinear Anal*. **8** (1984), 655–665.
2. J. Kazdan and F. Warner, Remarks on some quasilinear elliptic equations, *Comm*. *Pure Appl*. *Math*. **28** (1978), 567–597.
3. A. C. Lazer and P. J. McKenna, On the number of solutions of a nonlinear Dirichlet problem, *J*. *Math*. *Anal*. *Appl*. **84** (1981), 282–294.
4. A. C. Lazer and P. J. McKenna, Critical point theory and boundary value problems with nonlinearities crossing multiple eigenvalues, *Comm*. *Partial Differential Equations* **10** (1985), 107–150.
5. A. C. Lazer and P. J. McKenna, Critical point theory and boundary value problems with nonlinearities crossing multiple eigenvalues, II, *Comm*. *Partial Differential Equations* **11** (1986), 1653–1676.
6. P. Rabinowitz, Minimax methods for critical point theory with applications to differential equations, *in* ‘‘CBMS Regional Conf. Ser. in Math.,’’ Vol. 65, Amer. Math. Soc., Providence, RI, 1986.
7. B. Ruf and P. N. Srikanth, Multiplicity results for superlinear elliptic problems with partial interference with the spectrum, *J*. *Math*. *Anal*. *Appl*. **118** (1986), 15–23.
8. D. H. Sattinger, Topics in stability and bifurcation theory, *in* ‘‘Lecture Notes in Mathematics,’’ Vol. 309, Springer-Verlag, Berlin/New York, 1973.
9. M. Struwe, ‘‘Variational Methods,’’ Springer-Verlag, Berlin/New York, 1990.