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Nonzero solutions of Hammerstein integral equations with discontinuous kernels

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Abstract

Using the theory of fixed point index, we establish new results for the existence of nonzero solutions of integral equations of the form $u(t) = \int_G k(t,s) f(s,u(s))\, ds$, where G is a compact set in $\mathbb{R}^n$ and k changes sign, so positive solutions may not exist, f satisfies Carathéodory conditions and k may be discontinuous. We apply our results to prove the existence of nontrivial solutions of some nonlocal boundary value problems.

Keywords: Fixed point index; Cone; Nonzero solution

1. Introduction

One approach to finding solutions to a semilinear boundary value problem (BVP) for some differential equation is to write the BVP as an equivalent Hammerstein integral equation

$$u(t) = \int_0^1 k(t,s) f\big(s,u(s)\big)\, ds := Tu(t) \tag{1.1}$$

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and find a solution as a fixed point of the operator T. In particular, it is possible to use the classical theory of fixed point index in cones to establish the existence of positive solutions.

Lan and Webb [8] proved that at least one positive solution existed for some boundary conditions of separated type, under some conditions on f which strictly included f being either sublinear or superlinear. These results have been improved by Lan [6] to yield existence of multiple positive solutions under suitable conditions on f for the separated BCs.

Webb [11] used Lan's results for the Hammerstein integral equation to establish the existence of multiple positive solutions for some nonlocal BCs, known as three-point BCs, when a parameter α satisfies $0 < \alpha < 1$. Webb's results improved some of Ma's [9] who dealt with the existence of one positive solution for one of the BVPs studied in [11] in the sublinear and superlinear cases only, by different methods.

Recently, Infante and Webb [5] have studied the same three-point BVPs but for other values of the parameter, by an extension of the methods of Lan [6]. Existence theory in these cases had been given in a number of papers by Gupta and some co-authors; see, for example, [2,3] and references therein. In some of these cases positive solutions do not exist but, by considering a suitable cone, it was proved in [5] that there exist one or multiple *nonzero* solutions that change sign, under suitable conditions of f.

In the present paper we extend [5] to allow for discontinuities in the kernel and more general functions f. One motivation is that certain nonlocal boundary value problems lead to precisely this situation. We shall study in detail the problem

$$u''(t) + f\big(t, u(t)\big) = 0 \quad (0 < t < 1), \tag{1.2}$$

with boundary condition

$$u(1) = \alpha u'(\eta), \quad u(0) = 0, \quad 0 < \eta < 1. \tag{1.3}$$

In this case the kernel of the corresponding integral equation has a discontinuity. We shall use our theory to show that multiple nonzero (but not necessarily positive) solutions exist, under suitable conditions on f, when either $0 \leqslant \alpha < 1 - \eta$ or $\alpha < 0$. These results are completely new.

2. Existence of nontrivial solutions of Hammerstein integral equations

We begin by giving some new results for the following Hammerstein integral equation:

$$u(t) = \int_G k(t, s) f\big(s, u(s)\big)\, ds := Tu(t), \tag{2.1}$$

where G is a compact set in $\mathbb{R}^n$ of positive measure. We will work in the space $C(G)$ of continuous functions endowed with the usual supremum norm. We shall make the following assumptions on f, g and the kernel k. Recall that f is said to satisfy the Carathéodory conditions if for each u, $s \mapsto f(s,u)$ is measurable and for almost every s, $u \mapsto f(s,u)$ is continuous.

(C_1) Suppose that for every $r > 0$, $f : G \times [-r,r] \to [0,\infty)$ satisfies Carathéodory conditions on $G \times [-r,r]$ and there exists a measurable function $g_r : G \to [0,\infty)$ such that

$$f(s,u) \leqslant g_r(s) \quad \text{for almost all } s \in G \text{ and all } u \in [-r,r].$$

(C_2) $k : G \times G \to \mathbb{R}$ is measurable, and for every $\tau \in G$ we have

$$\lim_{t\to\tau} \int\limits_G \bigl|k(t,s) - k(\tau,s)\bigr| g_r(s)\,ds = 0.$$

(C_3) There exist a closed subset $G_0 \subset G$ with $\operatorname{meas}(G_0) > 0$, a measurable function $\Phi : G \to [0,\infty)$ and a constant $c \in (0,1]$ such that

$$\bigl|k(t,s)\bigr| \leqslant \Phi(s) \quad \text{for } t \in G \text{ and almost every } s \in G,$$
$$c\Phi(s) \leqslant k(t,s) \quad \text{for } t \in G_0 \text{ and almost every } s \in G.$$

(C_4) For each r there is $M_r < \infty$ such that $\int_G \Phi(s) g_r(s)\,ds \leqslant M_r$.

The hypothesis (C_3) means finding upper bounds for $|k|$ on G and lower bounds of the same form for k for $t \in G_0$. In applications we have some freedom of choice in determining G_0 but we are constrained by needing $k(t,s)$ to be positive for $t \in G_0$ and almost every $s \in G$.

These hypotheses allow us to work in the cone

$$K = \bigl\{u \in C(G)\colon\ \min\bigl\{u(t)\colon\ t \in G_0\bigr\} \geqslant c\|u\|\bigr\}.$$

This is similar to but larger than the cone used by Lan [7], which type of cone is apparently due to Guo [4]. Note that functions in K are positive on the subset G_0 but may change sign on G.

In order to use the well-known fixed point index for compact maps, we need to prove that $T : K \to K$ is compact; that is, T is continuous and $\overline{T(Q)}$ is compact for each bounded subset $Q \subset K$.

We write $K_r = \{u \in K\colon\ \|u\| < r\}$ and $\overline{K}_r = \{u \in K\colon\ \|u\| \leqslant r\}$.

Theorem 2.1. *Assume that (C_1)–(C_4) hold for some $r > 0$. Then T maps $\overline{K}_r$ into K and is compact.*

Proof. The compactness of T follows from Proposition 3.1 of [10] since, as G is compact, the limit in (C_2) is readily shown to be uniform in $\tau \in G$. To see that $T : \overline{K}_r \to K$, for $u \in \overline{K}_r$ and $t \in G$, we have

$$\left|Tu(t)\right| \leqslant \int_G \left|k(t,s)\right| f\big(s,u(s)\big)\,ds$$

so that

$$\|Tu\| \leqslant \int_G \Phi(s) f\big(s,u(s)\big)\,ds.$$

Also

$$\min_{t\in G_0}\big\{Tu(t)\big\} \geqslant c\int_G \Phi(s) f\big(s,u(s)\big)\,ds.$$

Hence $Tu \in K$ for every $u \in \overline{K}_r$. □

Remark 2.2. In Theorem 2.1, if the hypotheses hold for each $r > 0$, then T maps K into K and is compact. We shall only consider this case in this paper.

We require some knowledge of the classical fixed point index for compact maps; see, for example, [1] or [4] for further information.

Let K be a cone in a Banach space X. If Ω is a bounded open subset of K (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K. When D is an open bounded subset of X we write $D_K = D \cap K$, an open subset of K. The following result is a well-known consequence of fixed point index theory.

Lemma 2.3. *Let D be an open bounded set with $D_K \neq \emptyset$ and $\overline{D}_K \neq K$. Assume that $T : \overline{D}_K \to K$ is a compact map such that $x \neq Tx$ for $x \in \partial D_K$. Then the fixed point index $i_K(T, D_K)$ has the following properties*:

(1) *If there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(T, D_K) = 0$.*
(2) *If $\|Tx\| \leqslant \|x\|$ for $x \in \partial D_K$, then $i_K(T, D_K) = 1$.*
(3) *Let D^1 be open in X with $\overline{D^1} \subset D_K$. If we have $i_K(T, D_K) = 1$ and $i_K(T, D^1_K) = 0$, then T has a fixed point in $D_K \setminus \overline{D^1_K}$. The same result holds if $i_K(T, D_K) = 0$ and $i_K(T, D^1_K) = 1$.*

Let $q : C(G) \to \mathbb{R}$ denote the continuous function $q(u) = \min\{u(t)\colon\ t \in G_0\}$. Following Lan [6], for $\rho > 0$, we shall use the set $\Omega_\rho = \{u \in K\colon\ q(u) < c\rho\}$.

Lemma 2.4. *Ω_ρ defined above has the following properties*:

(a) *Ω_ρ is open relative to K.*
(b) *$K_{c\rho} \subset \Omega_\rho \subset K_\rho$.*
(c) *$u \in \partial\Omega_\rho$ if and only if $q(u) = c\rho$.*
(d) *If $u \in \partial\Omega_\rho$, then $c\rho \leqslant u(t) \leqslant \rho$ for $t \in G_0$.*

We omit the simple proof as it is exactly similar to the one in [6].

We now prove a lemma which implies the index is zero.

Lemma 2.5. *Assume that there exists $\rho > 0$ such that $u \neq Tu$ for $u \in \partial\Omega_\rho$ and the following conditions hold*:

$(H_\rho^{\geqslant})$ *There exists a measurable function $\psi_\rho : G_0 \to \mathbb{R}_+$ such that*

$$f(s,u) \geqslant c\rho\psi_\rho(s) \quad \textit{for all } u \in [c\rho, \rho] \textit{ and almost all } s \in G_0,$$

and $\inf_{t\in G_0} \int_{G_0} k(t,s)\psi_\rho(s)\,ds \geqslant 1$.

Then $i_K(T, \Omega_\rho) = 0$.

Proof. Let $e(t) \equiv 1$ for $t \in G$. Then $e \in K$. We prove that

$$u \neq Tu + \lambda e \quad \text{for } u \in \partial\Omega_\rho \text{ and } \lambda > 0.$$

In fact, if not, there exist $u \in \partial\Omega_\rho$ and $\lambda > 0$ such that $u = Tu + \lambda e$. By $(H_\rho^{\geqslant})$, we have for $t \in G_0$

$$\begin{aligned} u(t) &= \int_G k(t,s) f\big(s, u(s)\big)\,ds + \lambda \geqslant \int_{G_0} k(t,s) f\big(s, u(s)\big)\,ds + \lambda \\ &\geqslant c\rho \int_{G_0} k(t,s)\psi_\rho(s)\,ds + \lambda \geqslant c\rho + \lambda. \end{aligned}$$

This implies $q(u) \geqslant c\rho + \lambda > c\rho$, contradicting (c) of Lemma 2.4. Hence (1) of Lemma 2.3 implies $i_K(T, \Omega_\rho) = 0$. □

Note that if strict inequality holds in $(H_\rho^{\geqslant})$, taking $\lambda = 0$ we see that $u \neq Tu$ for $u \in \partial\Omega_\rho$.

Remark 2.6. A commonly used assumption in place of (1) of Lemma 2.3 is $\|Tu\| \geqslant \|u\|$ for $\|u\| = \rho$. We observe that this follows from a somewhat different version of $(H_\rho^{\geqslant})$, namely $f(s,u) \geqslant \rho\psi_\rho(s)$ for $c\rho \leqslant u \leqslant \rho$, where

$\sup_{t\in G_0}\int_{G_0} k(t,s)\psi_\rho(s)\,ds \geqslant 1$. Indeed, for $t \in G_0$ and $u \in K$ with $\|u\| = \rho$ we have

$$\left|Tu(t)\right| = \int_G k(t,s) f\bigl(s,u(s)\bigr)\,ds \geqslant \int_{G_0} k(t,s)\rho\psi_\rho(s)\,ds \geqslant \rho = \|u\|.$$

This remark shows that using the open set Ω_ρ and (1) of Lemma 2.3 gives a stronger result.

We now give a result which implies the index is 1.

Lemma 2.7. *Assume that there exists* $\rho > 0$ *such that* $u \neq Tu$ *for* $u \in \partial K_\rho$ *and* f *satisfies the following condition*:

$(H_\rho^{\leqslant})$ *There exists a measurable function* $\phi_\rho : G \to \mathbb{R}_+$ *such that*

$$f(s,u) \leqslant \rho\phi_\rho(s) \quad \textit{for all } u \in [-\rho,\rho] \textit{ and almost all } s \in G,$$

and $\sup_{t\in G}\int_G |k(t,s)|\phi_\rho(s)\,ds \leqslant 1$.

Then $i_K(T, K_\rho) = 1$.

Proof. By $(H_\rho^{\leqslant})$ we have for $u \in \partial K_\rho$ and $t \in G$

$$\begin{aligned}\left|Tu(t)\right| &= \left|\int_G k(t,s) f\bigl(s,u(s)\bigr)\,ds\right| \leqslant \int_G \left|k(t,s)\right| f\bigl(s,u(s)\bigr)\,ds \\ &\leqslant \rho\int_G \left|k(t,s)\right|\phi_\rho(s)\,ds \leqslant \rho = \|u\|.\end{aligned}$$

This implies $\|Tu\| \leqslant \|u\|$ for $u \in \partial K_\rho$. By (2) of Lemma 2.3, we have $i_K(T, K_\rho) = 1$. $\square$

Note that if strict inequality holds in $(H_\rho^{\leqslant})$, then $u \neq Tu$ for $u \in \partial K_\rho$.

We now give our new result which asserts that Eq. (2.1) has at least one or at least two nonzero solutions which are positive on the subset G_0 of G.

Theorem 2.8. *The integral equation Eq.* (2.1) *has a nonzero solution in* K *if either of the following conditions hold*:

(H_1) *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1 < c\rho_2$ *such that* $(H_{\rho_1}^{\leqslant})$, $(H_{\rho_2}^{\geqslant})$, $u \neq Tu$ *for* $u \in \partial\Omega_{\rho_2}$.

(H_2) *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1 < \rho_2$ *such that* $(H_{\rho_1}^{\geqslant})$, $(H_{\rho_2}^{\leqslant})$, $u \neq Tu$ *for* $u \in \partial K_{\rho_2}$.

Eq. (2.1) *has two nonzero solutions in K if one of the following conditions hold*:

(S_1) *There exist ρ_1, ρ_2, ρ_3 with $\rho_1 < c\rho_2$ and $\rho_2 < \rho_3$ such that $(H^{\leqslant}_{\rho_1})$, $(H^{\geqslant}_{\rho_2})$, $u \neq Tu$ for $u \in \partial\Omega_{\rho_2}$ and $(H^{\leqslant}_{\rho_3})$ hold.*

(S_2) *There exist ρ_1, ρ_2, ρ_3 with $\rho_1 < \rho_2 < c\rho_3$ such that $(H^{\geqslant}_{\rho_1})$, $(H^{\leqslant}_{\rho_2})$, $u \neq Tu$ for $u \in \partial K_{\rho_2}$ and $(H^{\geqslant}_{\rho_3})$ hold.*

Moreover, if in (S_1), strict inequality holds in $(H^{\leqslant}_{\rho_1})$, then Eq. (2.1) *has a third solution $u_0 \in K_{\rho_1}$.*

Proof. Assume that (S_1) holds. We show that either T has a fixed point u_1 in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$ or on its boundary. If $u \neq Tu$ for u in the boundary, by Lemmas 2.5 and 2.7, we have $i_K(T, K_{\rho_1}) = 1$, $i_K(T, \Omega_{\rho_2}) = 0$. By (b) of Lemma 2.4, we have $\overline{K}_{\rho_1} \subset K_{c\rho_2} \subset \Omega_{\rho_2}$ since $\rho_1 < c\rho_2$. It follows from (3) of Lemma 2.3 that T has a fixed point u_1 in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$. Similarly, T has a fixed point u_2 in $K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$ or on its boundary. When strict inequality holds then $u \neq Tu$ for $u \in \partial K_{\rho_1}$, so $i_K(T, K_{\rho_1}) = 1$ and T has a fixed point u_0 in K_{ρ_1}. The other assertions are proved similarly. □

Remark 2.9. It is possible to give results for more than two solutions by merely adding more conditions of the same type to the list in (S_1) or (S_2). We do not do state such results leaving them to the reader who may refer to [6] for the type of result that may be stated.

Remark 2.10. Note that the third solution $u_0 \in K_{\rho_1}$ might be zero. Although the statement and proof of Theorem 2.8 is almost identical to the similar result in [7] which deals with positive solutions, our new result allows solutions that are only positive on a subset and may change sign, and indeed this happens in the differential equation we consider below.

In the particular case when $f(t,u) = g(t)h(u)$, where $\Phi g \in L^1$ and h is continuous, it is possible to give conditions that are more easily verified.

Definition 2.11. We define the following numbers:

$$m = \left(\max_{t \in G} \int_G \left| k(t,s) \right| g(s)\, ds \right)^{-1}, \qquad M = \left(\min_{t \in G_0} \int_{G_0} k(t,s) g(s)\, ds \right)^{-1}$$

$$h^{-\rho,\rho} = \sup_{u \in [-\rho,\rho]} \frac{h(u)}{\rho}, \qquad h^0 = \limsup_{u \to 0} \frac{h(u)}{|u|}, \qquad h^\infty = \limsup_{|u| \to \infty} \frac{h(u)}{|u|},$$

$$h_{c\rho,\rho} = \inf_{u \in [c\rho,\rho]} \frac{h(u)}{\rho}, \qquad h_0 = \liminf_{u \to 0+} \frac{h(u)}{u}, \qquad h_\infty = \liminf_{u \to \infty} \frac{h(u)}{u}.$$

Lemma 2.12. *We have the following implications*:

(1) $h^0 < m$ *implies* $h^{-\rho,\rho} < m$ *for some* ρ (*small*) *and* $h^{-\rho,\rho} \leqslant m$ *implies* $(H_\rho^{\leqslant})$.
(2) $h^\infty < m$ *implies* $h^{-\rho,\rho} < m$ *holds for some* ρ (*large*).
(3) $h_0 > M$ *implies* $h_{c\rho,\rho} > cM$ *for some* ρ *and* $h_{c\rho,\rho} \geqslant cM$ *implies* $(H_\rho^{\geqslant})$.
(4) $h_\infty > M$ *implies* $h_{c\rho,\rho} > cM$ *holds for some* ρ.

Proof. (1) For $\varepsilon > 0$ there is $\rho_\varepsilon > 0$ such that $h(u)/|u| < h^0 + \varepsilon$ for $|u| \leqslant \rho_\varepsilon$ which implies there is $\rho > 0$ such that $h^{0,\rho} < m$ when $h^0 < m$. Also $h^{0,\rho} \leqslant m$ implies $h(u)g(s) \leqslant m\rho g(s)$ so that $(H_\rho^{\leqslant})$ holds with $\phi_\rho(s) = mg(s)$. (2) Let $\beta > m$. There is r such that $h(u)/|u| < \beta$ for $|u| \geqslant r$. As h is continuous there exists γ such that $h(u) < \beta|u| + \gamma$ for all u. Let $\rho = \gamma/(m-\beta)$; then $h(u) < m\rho$ for $|u| \leqslant \rho$. The proofs of (3) and (4) are straightforward. □

We now give a more easily checked version of Theorem 2.8.

Theorem 2.13. *Let* $f(t,u) = g(t)h(u)$ *be as above and assume that* $\int_{G_0} \Phi(s) \times g(s)\,ds > 0$. *Then Eq.* (2.1) *has a nonzero solution in* K *if one of the following conditions hold*:

(H_1') *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1 < c\rho_2$ *such that*

$$h^{-\rho_1,\rho_1} \leqslant m \quad \textit{and} \quad h_{c\rho_2,\rho_2} \geqslant cM.$$

(H_2') *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1 < \rho_2$ *such that*

$$h_{c\rho_1,\rho_1} \geqslant cM \quad \textit{and} \quad h^{-\rho_2,\rho_2} \leqslant m.$$

Eq. (2.1) *has two nonzero solutions in* K *if there is* $\rho > 0$ *such that either of the following conditions hold*:

(S_1') $0 \leqslant h^0 < m$, $h_{c\rho,\rho} \geqslant cM$, $u \neq Tu$ *for* $u \in \partial\Omega_\rho$ *and* $0 \leqslant h^\infty < m$.
(S_2') $M < h_0 \leqslant \infty$, $h^{-\rho,\rho} \leqslant m$, $u \neq Tu$ *for* $u \in \partial K_\rho$ *and* $M < h_\infty \leqslant \infty$.

Theorem 2.13 generalises Theorem 2.9 of [5] by allowing discontinuous kernels and generalises Theorem 2.2 of [7] by allowing kernels that are not positive everywhere hence giving existence of solutions that change sign.

3. Multiple nonzero solutions of Eq. (1.2)

We now investigate the BVP

$$u'' + f\big(t, u(t)\big) = 0, \quad \text{a.e. on } [0,1], \tag{3.1}$$

with boundary conditions

$$u(1) = \alpha u'(\eta), \quad u(0) = 0, \quad 0 < \eta < 1, \ \alpha < 1 - \eta. \tag{3.2}$$

By a solution of this BVP we will mean a solution of the corresponding Hammerstein integral equation

$$u(t) = \int_0^1 k(t,s) f\big(s, u(s)\big)\, ds. \tag{3.3}$$

The kernel (Green's function) in (3.3) is

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - \begin{cases} \frac{\alpha t}{1-\alpha}, & s \leqslant \eta, \\ 0, & s > \eta, \end{cases} - \begin{cases} t-s, & s \leqslant t, \\ 0, & s > t. \end{cases}$$

Note that the kernel is discontinuous on the line $s = \eta$ but does satisfy (C_2). We shall study separately the cases $\alpha > 0$ and $\alpha < 0$. In the special case $\alpha = 0$, existence of one positive solution is covered by the results of [8]. The results we obtain are new.

3.1. The case $\alpha > 0$

In this case we shall suppose that $0 < \alpha < 1 - \eta$. This is necessary for our method in order to obtain appropriate lower bounds. We have to exhibit $\Phi(s)$, a subinterval $[a,b] \subset [0,1]$ and a constant $c < 1$ such that

$$\big|k(t,s)\big| \leqslant \Phi(s) \quad \text{for every } t \in [0,1] \text{ and almost every } s \in [0,1],$$
$$k(t,s) \geqslant c\Phi(s) \quad \text{for every } t \in [a,b] \text{ and almost every } s \in [0,1].$$

We show that we may take

$$\Phi(s) = \max\left\{1, \frac{\alpha}{\eta}\right\} \frac{s(1-s)}{1-\alpha}.$$

Upper bounds

Case 1. $s > \eta$. If $t < s$ then $k(t,s) \geqslant 0$ and

$$k(t,s) = \frac{t}{1-\alpha}(1-s) \leqslant \frac{s(1-s)}{1-\alpha}.$$

If $t \geqslant s$ then

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - (t-s) = \frac{s(1-\alpha) + t(\alpha - s)}{1-\alpha}.$$

The minimum/maximum occur when $t = 1$ or $t = s$. Thus $k \geqslant 0$. If $s > \alpha$ then

$$k(t,s) = \frac{s(1-\alpha) + t(\alpha - s)}{1-\alpha} \leqslant \frac{s(1-\alpha) + s(\alpha - s)}{1-\alpha} = \frac{s(1-s)}{1-\alpha}.$$

If $s \leqslant \alpha$ then

$$k(t,s) = \frac{s(1-\alpha)+t(\alpha-s)}{1-\alpha} \leqslant \frac{s(1-\alpha)+\alpha-s}{1-\alpha} = \frac{\alpha(1-s)}{1-\alpha} < \frac{\alpha}{\eta}\frac{s(1-s)}{(1-\alpha)}.$$

Case 2. $s \leqslant \eta$. If $t < s$

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - \frac{\alpha t}{1-\alpha} = \frac{t(1-s-\alpha)}{1-\alpha}.$$

When $s \leqslant 1-\alpha$ we have $k(t,s) \geqslant 0$ and

$$k(t,s) \leqslant \frac{s(1-s-\alpha)}{1-\alpha} \leqslant \frac{s(1-s)}{1-\alpha}.$$

The case $\eta \geqslant s > 1-\alpha$ cannot occur since we have $0 < \alpha < 1-\eta$.

If $t \geqslant s$ then

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - \frac{\alpha t}{1-\alpha} - (t-s) = \frac{s(1-t-\alpha)}{1-\alpha}.$$

If $t \leqslant 1-\alpha$ then $k(t,s) \geqslant 0$ and

$$k(t,s) \leqslant \frac{s(1-t)}{1-\alpha} \leqslant \frac{s(1-s)}{1-\alpha}.$$

If $t > 1-\alpha$ then $k(t,s) \leqslant 0$ and

$$-k(t,s) = \frac{s(-1+t+\alpha)}{1-\alpha} \leqslant \frac{\alpha s}{1-\alpha} < \frac{s(1-\eta)}{1-\alpha} \leqslant \frac{s(1-s)}{1-\alpha}.$$

Lower bounds

We show that we may take an arbitrary $[a,b] \subset (0,1-\alpha)$.

Case 1. $s > \eta$. If $t < s$ then

$$k(t,s) = \frac{t}{1-\alpha}(1-s) \geqslant a\frac{(1-s)}{1-\alpha} \geqslant a\frac{s(1-s)}{1-\alpha}.$$

If $t \geqslant s$

$$k(t,s) = \frac{s+\alpha t-\alpha s-st}{1-\alpha} \geqslant s\frac{(1-b)}{1-\alpha} \geqslant (1-b)\frac{s(1-s)}{1-\alpha}.$$

Case 2. $s \leqslant \eta$. If $t < s$ then

$$k(t,s) = t\frac{1-s-\alpha}{1-\alpha} \geqslant a\frac{(1-\eta-\alpha)}{1-\alpha} \geqslant a(1-\eta-\alpha)4\frac{s(1-s)}{1-\alpha}.$$

If $t \geqslant s$ then

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - \frac{\alpha t}{1-\alpha} - (t-s) = \frac{s-st-\alpha s}{1-\alpha} \geqslant \frac{s(1-b-\alpha)}{1-\alpha} \geqslant (1-b-\alpha)\frac{s(1-s)}{1-\alpha}.$$

The conclusion is that we may take

$$c = \frac{\min\{4a(1-\eta-\alpha), (1-b-\alpha)\}}{\max\{1, \alpha/\eta\}}.$$

We state a result when $f(t,u) = g(t)h(u)$; of course, there is a more general result analogous to Theorem 2.8.

Theorem 3.1. *Let $[a,b] \subset (0, 1-\alpha)$ and suppose that $\int_a^b \Phi(s)g(s)\,ds > 0$. Let c be as given above. Let m, M be as defined previously. Then for $0 < \alpha < 1-\eta$ the BVP* (3.1), (3.2) *has at least one nonzero solution, positive on $[a,b]$, if either*

(h_1') *$0 \leqslant h^0 < m$ and $M < h_\infty \leqslant \infty$, or*
(h_2') *$0 \leqslant h^\infty < m$ and $M < h_0 \leqslant \infty$,*

and has two nonzero solutions, positive on $[a,b]$, if there is $\rho > 0$ such that either

(S_1') *$0 \leqslant h^0 < m$, $h_{c\rho,\rho} \geqslant cM$, $u \neq Tu$ for $u \in \partial\Omega_\rho$, and $0 \leqslant h^\infty < m$, or*
(S_2') *$M < h_0 \leqslant \infty$, $h^{-\rho,\rho} \leqslant m$, $u \neq Tu$ for $u \in \partial K_\rho$, and $M < h_\infty \leqslant \infty$.*

We give a simple example to illustrate the theorem.

Example 3.2. Set $f(t,u) \equiv 2$. In this case the solution is

$$u(s) = -s\left(s - \frac{1-2\alpha\eta}{1-\alpha}\right).$$

For $\eta \leqslant 1/2$ and $\eta + \alpha < 1$, the solution is actually positive on all of $[0,1]$. For $\eta > 1/2$ the solution is negative for $t > t_0 = (1-2\alpha\eta)/(1-\alpha)$, but is positive on $(0, 1-\alpha)$.

3.2. *The case $\alpha < 0$*

To simplify the calculations we write $-\beta$ in place of α, so that $\beta > 0$. We show that for these BCs we can take

$$\Phi(s) = \max\left\{\frac{(1-\eta+\beta)}{1-\eta}, \frac{\beta}{\eta}\right\}\frac{s(1-s)}{1+\beta}.$$

Upper bounds

Case 1. $s > \eta$. If $t < s$ then $k(t,s) \geqslant 0$ and

$$k(t,s) = \frac{t}{1+\beta}(1-s) \leqslant \frac{s(1-s)}{1+\beta}.$$

If $t \geqslant s$ then

$$k(t,s) = \frac{s - \beta t + \beta s - st}{1+\beta} \leqslant \frac{s - \beta s + \beta s - st}{1+\beta} = \frac{s(1-t)}{1+\beta} \leqslant \frac{s(1-s)}{1+\beta}.$$

If $t \leqslant s(1+\beta)/(\beta+s)$, $k(t,s) \geqslant 0$ and we are done. If $t > s(1+\beta)/(\beta+s)$ we have

$$-k(t,s) = \frac{ts+\beta t - s - s\beta}{1+\beta} \leqslant \frac{s+\beta-s-s\beta}{1+\beta} \leqslant \frac{\beta}{\eta}\frac{s(1-s)}{1+\beta}.$$

Case 2. $s \leqslant \eta$. Note that in this case $(1-s)/(1-\eta) \geqslant 1$. If $t < s$ then $k(t,s) \geqslant 0$ and

$$\begin{aligned} k(t,s) &= \frac{t}{1+\beta}(1-s) + \frac{\beta t}{1+\beta} = t\frac{1-s+\beta}{1+\beta} \\ &\leqslant \frac{s(1-s+\beta)}{1+\beta} \leqslant \frac{s(1-s+\beta(1-s)/(1-\eta))}{1+\beta} \\ &\leqslant \frac{(1-\eta+\beta)}{1-\eta}\frac{s(1-s)}{1+\beta}. \end{aligned}$$

If $t \geqslant s$ then $k(t,s) \geqslant 0$ and

$$\begin{aligned} k(t,s) &= \frac{t}{1+\beta}(1-s) + \frac{\beta t}{1+\beta} - (t-s) = \frac{s-st+\beta s}{1+\beta} \\ &\leqslant \frac{s(1-s+\beta)}{1+\beta} \leqslant \frac{(1-\eta+\beta)}{1-\eta}\frac{s(1-s)}{1+\beta}. \end{aligned}$$

Lower bounds

We show that we may take an arbitrary $[a,b] \subset (0,\eta]$.

Case 1. $s > \eta$. If $t < s$ then

$$k(t,s) = \frac{t}{1+\beta}(1-s) \geqslant a\frac{(1-s)}{1+\beta} \geqslant a\frac{s(1-s)}{1+\beta}.$$

Since we take $b \leqslant \eta$ the (awkward) case $t \geqslant s$ does not occur.

Case 2. $s \leqslant \eta$. If $t < s$ then

$$k(t,s) = \frac{t-st+\beta t}{1+\beta} = t\frac{1-s+\beta}{1+\beta} \geqslant a\frac{s(1-s)}{1+\beta}.$$

If $t \geqslant s$ then

$$k(t,s) = \frac{s-st+\beta s}{1+\beta} \geqslant \beta\frac{s}{1+\beta} \geqslant \beta\frac{s(1-s)}{1+\beta}.$$

The conclusion is that we may take

$$c = \frac{\min\{a,\beta\}}{\max\{(1-\eta+\beta), \beta/\eta\}}.$$

Remark 3.3. In this case it is possible to take a somewhat larger b, namely any $b < b_0$ where $b_0 := \eta(1+\beta)/(\eta+\beta)$, but the corresponding c is more complicated.

For the case when $f(t,u) = g(t)h(u)$ we have the following result.

Theorem 3.4. *Let $[a,b] \subset (0,\eta]$ and suppose that $\int_a^b \Phi(s)g(s)\,ds > 0$. Let c be as given above. Let m, M be as defined previously. Then for $\alpha < 0$ the BVP* (3.1), (3.2) *has at least one nonzero solution, positive on $[a,b]$, if either (h_1') or (h_2') of Theorem* 3.1 *is satisfied. There are two nonzero solutions, positive on $[a,b]$, if there is $\rho > 0$ such that either (S_1') or (S_2') of Theorem* 3.1 *holds.*

The following example illustrates the result.

Example 3.5. Let $g(t) = 1$ and

$$h(u) = \begin{cases} 2 & \text{if } |u| \leqslant 3/m, \\ u^p & \text{for } u \text{ very large,} \end{cases}$$

where $p > 1$. Then $h_0 = \infty$ and $h_\infty = \infty$ and choosing ρ with $2/m < \rho < 3/m$ we have $h^{-\rho,\rho} < m$. Hence (S_2') holds and the BVP has two nonzero solutions which are positive on $(0,\eta]$, the 'small' solution being as written in Example 3.2.

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