The weak maximum principle for a class of strongly coupled elliptic differential systems

Xu Liu \(^{a,\ast}\), Xu Zhang \(^{b,1}\)

\(^a\) School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
\(^b\) Yangtze Center of Mathematics, Sichuan University, Chengdu 610064, China

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Abstract

A classical counterexample due to E. De Giorgi, shows that the weak maximum principle does not remain true for general linear elliptic differential systems. Since then, there were some efforts to establish the weak maximum principle for special elliptic differential systems, but the existing works are addressing only the cases of weakly coupled systems, or almost-diagonal systems, or even some systems coupling in various lower order terms. In this paper, by contrast, we present maximum modulus estimates for weak solutions to some coupled elliptic differential systems with different principal parts, under some mild assumptions. The systems under consideration are strongly coupled in the second order terms and other lower order terms, without restrictions on the size of ratios of the different principal part coefficients, or on the number of equations and space variables.
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\(^\ast\) Corresponding author.
E-mail addresses: liuxu@amss.ac.cn (X. Liu), zhang_xu@scu.edu.cn (X. Zhang).

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1. Introduction

Let \( m, n \in \mathbb{N}\setminus\{0\} \), and let \( \Omega \subset \mathbb{R}^m \) be a bounded domain with an \( C^1 \) boundary \( \Gamma \) and having the cone property. We consider the following nonhomogeneous, isotropic elliptic differential system of second order:

\[
\begin{align*}
- \text{div}(a^{11}\nabla y_1) & - \text{div}(a^{12}\nabla y_2) - \cdots - \text{div}(a^{1n}\nabla y_n) + \sum_{i=1}^{n} C^{1i} \cdot \nabla y_i + D^1 \cdot y = f^1 & \text{in } \Omega, \\
- \text{div}(a^{21}\nabla y_1) & - \text{div}(a^{22}\nabla y_2) - \cdots - \text{div}(a^{2n}\nabla y_n) + \sum_{i=1}^{n} C^{2i} \cdot \nabla y_i + D^2 \cdot y = f^2 & \text{in } \Omega, \\
\vdots & \\
- \text{div}(a^{n1}\nabla y_1) & - \text{div}(a^{n2}\nabla y_2) - \cdots - \text{div}(a^{nn}\nabla y_n) + \sum_{i=1}^{n} C^{ni} \cdot \nabla y_i + D^n \cdot y = f^n & \text{in } \Omega, \\
y_1 = g^1, & y_2 = g^2, & \ldots, & y_n = g^n & \text{on } \Gamma,
\end{align*}
\]

(1.1)

and the following general nonhomogeneous elliptic differential system of second order:

\[
\begin{align*}
- \sum_{p,q=1}^{m} \left[ (a_{pq}^{11} y_1 x_p x_q) + (a_{pq}^{12} y_2 x_p x_q) + \cdots + (a_{pq}^{1n} y_n x_p x_q) \right] & \text{in } \Omega, \\
+ \sum_{i=1}^{n} C^{1i} \cdot \nabla y_i + D^1 \cdot y = f^1 & \text{in } \Omega, \\
- \sum_{p,q=1}^{m} \left[ (a_{pq}^{21} y_1 x_p x_q) + (a_{pq}^{22} y_2 x_p x_q) + \cdots + (a_{pq}^{2n} y_n x_p x_q) \right] & \text{in } \Omega, \\
+ \sum_{i=1}^{n} C^{2i} \cdot \nabla y_i + D^2 \cdot y = f^2 & \text{in } \Omega, \\
\vdots & \\
- \sum_{p,q=1}^{m} \left[ (a_{pq}^{n1} y_1 x_p x_q) + (a_{pq}^{n2} y_2 x_p x_q) + \cdots + (a_{pq}^{nn} y_n x_p x_q) \right] & \text{in } \Omega, \\
+ \sum_{i=1}^{n} C^{ni} \cdot \nabla y_i + D^n \cdot y = f^n & \text{in } \Omega, \\
y_1 = g^1, & y_2 = g^2, & \ldots, & y_n = g^n & \text{on } \Gamma.
\end{align*}
\]

(1.2)

In both (1.1) and (1.2), \( y = (y^1, \ldots, y^n)^\top \) is unknown, while \( a^{ij}, a_{pq}^{ij}, C^{ij}, D^i, f^i \) and \( g^i \) (\( i, j = 1, \ldots, n; p, q = 1, \ldots, m \)) are suitable given functions (see the next section for the assumptions on these functions). The main purpose of this paper is to study the weak maximum principle, or the boundedness of weak solutions, to the systems (1.1) and (1.2) with suitable measurable principal part coefficients.

It is well known that the weak maximum principle is one of the basic issues in the theory of partial differential equations and it plays an essential role in the study of many other problems.
For example, a central problem in the calculus of variations is the regularity of stationary points for functionals of the type

\[ J(u) = \int_{\Omega} F(x, u(x), \nabla u(x)) \, dx, \]

where \( u(\cdot) = (u^1(\cdot), u^2(\cdot), \ldots, u^n(\cdot))^T \) is a vector-valued function defined on \( \Omega \), and \( F(\cdot, \cdot, \cdot) \) is a suitable function defined on \( \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \). Research in this area has been stimulated by D. Hilbert’s Problem 19, which can be reduced to the regularity of weak solutions to nonlinear elliptic equations or systems. This problem was successfully solved by C. Morrey [16] in two dimensions and the general case with \( n = 1 \) was finally solved by E. De Giorgi [6] and J. Nash [18], and refined by J. Moser [17]. We refer to [1,2,10–12] and the references cited therein for more details in this respect. It is well known that a fundamental step of the De Giorgi–Nash–Moser approach in solving the scalar Hilbert’s Problem 19 (i.e., \( n = 1 \)) is to establish maximum modulus estimates for single linear elliptic equations.

In many physical and geometrical applications, \( u \) may be a vector function, and therefore, the corresponding Euler–Lagrange equation is a system. Naturally, one expects to extend the De Giorgi–Nash–Moser approach to the case of systems. However, in 1968, E. De Giorgi [7] gave a surprising counterexample of an unbounded solution to a second order linear elliptic system with bounded coefficients. This means that the weak maximum principle fails for general second order linear elliptic systems, and therefore, the De Giorgi–Nash–Moser estimates are no longer valid for general elliptic systems.

In order to establish maximum modulus estimates for weak solutions to elliptic differential systems, as a consequence of the above mentioned De Giorgi’s counterexample, one has to impose some restrictions on the structure of the system. There exist a few works in this direction. In [12], a weak maximum principle was proved for a class of special elliptic systems with variable coefficients, in which the principal operator in each equation takes the same form, and it is acting only on one component of the solution vector. In [3,4,13], some weak maximum principles were discussed in the framework of Campanato’s space for linear or quasilinear elliptic systems under some additional conditions, say, \( 2 \leq m \leq 4 \) in [3], the coefficients matrix being constant in [4], and a dispersion assumption on the eigenvalues of the principal part coefficients matrix in [13] (and hence the system is almost-diagonal in high space dimensions).

In this paper, we choose the usual Sobolev space as the working space and derive the weak maximum principle for two classes of strongly coupled elliptic systems with different principal parts, in the spirit of the classical framework for single equations. We emphasize that our systems are strongly coupled, i.e., the (second order) terms of the principal parts are coupled to each other. Therefore, when establishing the desired \( a \text{ priori} \) estimate, it is necessary to get rid of some undesired terms generated by different principal operators and/or different solution components appeared in the same equation. This goal is achieved by choosing delicately suitable weighted test functions. As far as we know, this is the first result on the weak maximum principle (in the classical sense) for strongly coupled elliptic systems. Nevertheless, the structure conditions imposed for the system (1.1) and the system (1.2) in this paper imply certain regularity requirement for the coefficients. Therefore, it seems difficult to apply the results of this paper to general quasilinear elliptic systems (see Remark 2.4 for a detailed explanation).

The rest of this paper is organized as follows. Section 2 is devoted to stating the main results of this work. In Section 3, we collect some preliminary results which will be useful later. Sections 4
and 5 are addressed to the proof of our main results, i.e., the boundedness of weak solutions to the systems (1.1) and (1.2), respectively. Finally, in Section 6, we give an example in which the assumptions for proving the boundedness of weak solution to the system (1.2) are satisfied.

2. Statement of the main results

To begin with, we introduce some assumptions. Suppose that, for $i,j = 1,\ldots,n$,

$$ a^{ij} \in L^\infty(\Omega) $$

and

$$ \begin{cases} 
C^{ij}(\cdot) \in L^\theta(\Omega; \mathbb{R}^m) & \text{and} & D^i(\cdot) \in L^q(\Omega; \mathbb{R}^n) & \text{for some } \theta > m, \\
f = (f^1, \ldots, f^n) \top \in H^{-1}(\Omega; \mathbb{R}^n), \\
g = (g^1, \ldots, g^n) \top \in H^1(\Omega; \mathbb{R}^n), 
\end{cases} $$

(2.2)

and for $i,j = 1,\ldots,n$ and $p,q = 1,\ldots,m$,

$$ a^{ij}_{pq} \in L^\infty(\Omega), \quad a^{ij}_{pq} = a^{ji}_{qp}. $$

(2.3)

Moreover, we assume that, for some positive constant $\rho$,

$$ \sum_{i,j=1}^n a^{ij}_{\xi^i \xi^j} \geq \rho |\xi|^2, \quad \forall (x, \xi) = (x, \xi^1, \ldots, \xi^n) \in \Omega \times \mathbb{R}^n, $$

(2.4)

and

$$ \sum_{i,j=1}^n \sum_{p,q=1}^m a^{ij}_{pq}(x) \xi^i_p \xi^j_q \geq \rho |\xi|^2, $$

$$ \forall (x, \xi) = (x, \xi^1_1, \ldots, \xi^1_m, \ldots, \xi^n_1, \ldots, \xi^n_m) \in \Omega \times \mathbb{R}^{nm}. $$

(2.5)

Furthermore, as mentioned before, the weak maximum principle for general elliptic systems was proved in the case of $m = 2$ [16]. Therefore, we assume that $m > 2$ in the sequel.

It is not difficult to show that conditions (2.4) and (2.5) mean that both systems (1.1) and (1.2) are elliptic (see [5, Section 1 of Chapter 8]). Clearly, the system (1.1) is a special case of the system (1.2). The weak solution to (1.2) is understood in the following sense.

**Definition 2.1.** We call $y = (y^1, \ldots, y^n) \top \in H^1(\Omega; \mathbb{R}^n)$ to be a weak solution to the system (1.2) if for any $\varphi = (\varphi^1, \ldots, \varphi^n) \top \in H^1_0(\Omega; \mathbb{R}^n)$,

$$ \sum_{i,j=1}^n \sum_{p,q=1}^m \int_{\Omega} a^{ij}_{pq}(x) y^i_p \varphi^i_q \, dx + \sum_{i=1}^n \left[ \sum_{j=1}^n C^{ij}(x) \cdot \nabla y^j \varphi^j + D^i(x) \cdot y \varphi^i \right] \, dx = \langle f, \varphi \rangle_{H^{-1}(\Omega; \mathbb{R}^n), H^1_0(\Omega; \mathbb{R}^n)}, $$

and $y^i - g^i \in H^1_0(\Omega), i = 1,\ldots,n$.

Similar to the proof of [5, Theorem 2.3 in Chapter 1], it is easy to show the following well-posedness result for the system (1.2).
Lemma 2.1. Let conditions \((2.2)\), \((2.3)\) and \((2.5)\) be fulfilled. Then, there exists a constant \(\nu_0 = \nu_0(n, m, \theta) > 0\) such that the system \((1.2)\) admits a unique weak solution \(y \in H^1(\Omega; \mathbb{R}^n)\) whenever one of the following conditions is satisfied:

(i) \(C^{ij}(\cdot) \equiv 0\) for \(i, j = 1, \ldots, n\), and the function matrix \((D^1(x), \ldots, D^n(x))\) is semipositive definite; or

(ii) The following inequality holds

\[
\sum_{i=1}^n (D^i(x) \cdot \mu)^i \geq \nu_0 \rho \frac{m+\theta}{m} \left[ \sum_{i,j=1}^n |C^{ij}|_{L^\theta(\Omega; \mathbb{R}^m)} \right]^{\frac{2\theta}{m}} |\mu|^2,
\]

\(\forall (x, \mu) = (x, \mu^1, \mu^2, \ldots, \mu^n) \in \Omega \times \mathbb{R}^n\).

Moreover, the following estimate (for the weak solution \(y\)) holds

\[
|y|_{H^1(\Omega; \mathbb{R}^n)} \leq C \left( n, m, \theta, \Omega, \rho, |a_{pq}^{ij}|_{L^\infty(\Omega)}, |C^{ij}|_{L^\theta(\Omega; \mathbb{R}^m)}, |D^i|_{L^2(\Omega; \mathbb{R}^n)} \right)
\]

\[
\times \left( |f|_{H^{-1}(\Omega; \mathbb{R}^n)} + |g|_{H^1(\Omega; \mathbb{R}^n)} \right).
\]

The proof of Lemma 2.1 is standard and therefore we omit the details.

Next, we put

\[
A = \begin{pmatrix}
a_{11} & a_{21} & \cdots & a_{n1} \\
a_{12} & a_{22} & \cdots & a_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & \cdots & a_{nn}
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
a_{22} & a_{32} & \cdots & a_{n2} \\
a_{23} & a_{33} & \cdots & a_{n3} \\
\vdots & \vdots & \ddots & \vdots \\
a_{2n} & a_{3n} & \cdots & a_{nn}
\end{pmatrix}.
\]

Also, denote by \(\det A\) the determinant of matrix \(A\) and by \(B^{ij}(i, j = 1, \ldots, n)\) the \((i, j)\)th cofactor of \(A\). It is easy to see that \(B^{11} = \det B\). Moreover, under condition \((2.4)\), it is easy to show that \(B^{11} \neq 0\).

The first main result in this paper is the following boundedness estimate on the weak solution to \((1.1)\).

Theorem 2.1. Suppose that conditions \((2.1)\), \((2.2)\) and \((2.4)\) are fulfilled, and one of the conditions (i) and (ii) in Lemma 2.1 is satisfied. Suppose that \(f \in L^\theta(\Omega; \mathbb{R}^n)\) and

\[
\frac{B^{ij}}{\det B} \in W^{1,\infty}(\Omega), \quad i, j = 1, \ldots, n.
\]

Then the weak solution \(y \in H^1(\Omega; \mathbb{R}^n)\) to \((1.1)\) satisfies

\[
\sup_{\Omega} |y| \leq C \left( m, n, \theta, \Omega, \rho, |a_{pq}^{ij}|_{L^\infty(\Omega)}, |C^{ij}|_{L^\theta(\Omega; \mathbb{R}^m)}, |D^i|_{L^2(\Omega; \mathbb{R}^n)}, \left| \frac{B^{ij}}{\det B} \right|_{W^{1,\infty}(\Omega)} \right)
\]

\[
\times \left( |g|_{H^1(\Omega; \mathbb{R}^n)}, |f|_{L^\theta(\Omega; \mathbb{R}^n)} \sup_{\Omega} |y| \right).
\]
The proof of Theorem 2.1 will be given in Section 4.

**Remark 2.1.** We conjecture that the assumption (2.7) in Theorem 2.1 is a technical condition, and therefore it is not really necessary. However, we do not know how to drop this assumption at this moment. We shall explain this assumption more in Remark 4.1.

**Remark 2.2.** Since almost all of the natural materials are isotropic, Theorem 2.1 suffices for many physical applications. For example, consider the following stationary drift–diffusion model for the flow of electrons and holes in semiconductor devices (see [8]):

\[
\begin{align*}
\text{div}(\mu_1(x, \nabla V)(\nabla N - N\nabla V)) &= G_1(x, N, P) \quad \text{in } \Omega, \\
\text{div}(\mu_2(x, \nabla V)(\nabla P - P\nabla V)) &= G_2(x, N, P) \quad \text{in } \Omega, \\
\text{div}(\nu(x, \nabla V)\nabla V) &= N - P - M \quad \text{in } \Omega,
\end{align*}
\]  

(2.8)

where \( \Omega \) denotes the domain occupied by a semiconductor, \( N \) denotes the electron density, \( P \) denotes the hole density and \( V \) denotes the electric potential. Also, \( \mu_1(\cdot, \cdot) \) and \( \mu_2(\cdot, \cdot) \) are two known functions of \( x \in \Omega \) and \( \nabla V \), which stand for the mobilities. Further, \( G_1(\cdot, \cdot, \cdot) \) and \( G_2(\cdot, \cdot, \cdot) \) denote the generation–recombination terms, which are lower order terms in (2.8), and \( M(\cdot) \) is a known function on \( \Omega \). Moreover, a little different from [8], we assume that \( \nu(\cdot, \cdot) \), the permittivity, is a known function of \( x \in \Omega \) and \( \nabla V \) (rather than a positive constant). Therefore, the linearized model of the quasilinear system (2.8) is the following linear partial differential system:

\[
\begin{align*}
\text{div}(a_1(x)\nabla N - b_1(x)\nabla V) &= c_1(x)N + d_1(x)P + g_1(x) \quad \text{in } \Omega, \\
\text{div}(a_2(x)\nabla P - b_2(x)\nabla V) &= c_2(x)N + d_2(x)P + g_2(x) \quad \text{in } \Omega, \\
\text{div}(\varepsilon(x)\nabla V) &= N - P - M(x) \quad \text{in } \Omega,
\end{align*}
\]  

(2.9)

where \( a_i, b_i, \varepsilon, c_i, d_i, g_i \in L^\infty(\Omega) \) (\( i = 1, 2 \)), \( a_i \geq \rho \) in \( \Omega \) (\( i = 1, 2 \)), \( \varepsilon \geq \rho \) in \( \Omega \) for some positive constant \( \rho \), \( a_1\varepsilon \geq 2b_1^2 \) in \( \Omega \), and \( a_2\varepsilon \geq 2b_2^2 \) in \( \Omega \). Then, (2.9) is a strongly coupled linear elliptic differential system. Moreover, it is easy to check that the condition (2.7) in Theorem 2.1 means that \( \frac{a_1}{a_2}, \frac{b_2}{b_1, \varepsilon}, \frac{b_1}{\varepsilon} \in W^{1, \infty}(\Omega) \).

Nevertheless, from the mathematical point of view, it would be quite interesting to extend Theorem 2.1 to more general anisotropic systems such as (1.2), in which the scalar functions \( a^{ij}(i, j = 1, 2, \ldots, n) \) that appeared in the system (1.1) are replaced by the \( \mathbb{R}^{\times m} \) matrix-valued functions \( (a^{ij}_{pq})_{1 \leq p, q \leq m} \). Note however that, by the above mentioned De Giorgi’s counterexample [7], this seems to be highly nontrivial in the general case. In the rest of this section, we shall extend Theorem 2.1 to the system (1.2) under some technical assumptions.

In order to treat the system (1.2), we put

\[
M_{pq} = \begin{pmatrix}
\frac{a^{11}_{pq}}{a^{12}_{pq}} & \frac{a^{21}_{pq}}{a^{22}_{pq}} & \cdots & \frac{a^{n1}_{pq}}{a^{nn}_{pq}} \\
\frac{a^{12}_{pq}}{a^{11}_{pq}} & \frac{a^{22}_{pq}}{a^{21}_{pq}} & \cdots & \frac{a^{n2}_{pq}}{a^{nn}_{pq}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a^{1n}_{pq}}{a^{11}_{pq}} & \frac{a^{2n}_{pq}}{a^{21}_{pq}} & \cdots & \frac{a^{nn}_{pq}}{a^{nn}_{pq}}
\end{pmatrix}
\quad \text{and} \quad L_{pq} = \det M_{pq} \quad (p, q = 1, \ldots, m).
\]  

(2.10)
We assume that

\[ L_{pq} \neq 0, \quad \forall p, q = 1, \ldots, m. \]  

(2.11)

Also, we denote by \( v_{pq}^{ij} \) \((i, j = 1, \ldots, n; p, q = 1, \ldots, m)\) the \((i, j)\)th cofactor of \( M_{pq} \).

Further, let us introduce the following assumption:

\textbf{(H)} There exist functions \( f_{pq}, \ h^{ij} \in W^{1, \infty}(\Omega) \) \((i, j = 1, \ldots, n; p, q = 1, \ldots, m)\) such that

1. \( h^{11} \equiv 1 \), \( h^{ij} = h^{ji} \), and the following matrix is uniformly positive definite:

\[
V := \begin{pmatrix}
1 & h^{12} & \cdots & h^{1n} \\
h^{21} & h^{22} & \cdots & h^{2n} \\
\vdots & \vdots & \ddots & \vdots \\
h^{n1} & h^{n2} & \cdots & h^{nn}
\end{pmatrix},
\]

i.e., \( V \geq \rho_1 I_{n \times n} \) for some positive number \( \rho_1 \);

2. The function \( E^{ij} := \frac{f_{pq}}{F_{pq}} \sum_{l=1}^{n} h^{lj} v_{pq}^{li} \) is independent of \( p \) and \( q \), and \( E^{ij} \in W^{1, \infty}(\Omega) \) for any \( i, j = 1, \ldots, n \);

3. The following matrix is uniformly positive definite:

\[
F := \begin{pmatrix}
F_{11} & F_{12} & \cdots & F_{1m} \\
F_{21} & F_{22} & \cdots & F_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
F_{m1} & F_{m2} & \cdots & F_{mm}
\end{pmatrix},
\]

i.e., \( F \geq \rho_2 I_{m \times m} \) for some positive number \( \rho_2 \), where \( F_{pq} := \sum_{l=1}^{n} a^{l1}_{pq} E^{1l} \) for any \( p, q = 1, \ldots, m \);

4. The following matrix is uniformly positive definite:

\[
M := \begin{pmatrix}
F & h^{12} F & \cdots & h^{1n} F \\
h^{21} F & h^{22} F & \cdots & h^{2n} F \\
\vdots & \vdots & \ddots & \vdots \\
h^{n1} F & h^{n2} F & \cdots & h^{nn} F
\end{pmatrix} \quad \text{for } nm \times nm,
\]

i.e., \( M \geq \rho_3 I_{nm \times nm} \) for some positive number \( \rho_3 \).

Now, we can state another main result in this article as the following boundedness result for the weak solution to (1.2).

\[ \text{Remark:} \quad \text{The condition (2.11) is not satisfied for the system (1.1). Indeed, it is easy to check that for the system (1.1), the corresponding } L_{pq} = 0 \text{ for any } p \neq q \text{ and } p, q = 1, \ldots, m. \text{ Hence, Theorem 2.2 below does not subsume Theorem 2.1.} \]
Theorem 2.2. Suppose that conditions (2.2), (2.3), (2.5) and (2.11) are fulfilled, and one of the conditions (i) and (ii) in Lemma 2.1 is satisfied. Suppose that \( f \in L^\theta_2(\Omega; \mathbb{R}^n) \) and the assumption (H) holds. Then the weak solution \( y \in H^1(\Omega; \mathbb{R}^n) \) to (1.2) satisfies the following estimate:

\[
\sup_{\Omega} |y| \leq C \left( m, n, \theta, \Omega, \rho, \rho_1, \rho_2, \rho_3, \|a^{ij}_{pq}\|_{L^\infty(\Omega)}, \|C^{ij}\|_{L^\theta(\Omega; \mathbb{R}^m)}, \|D^j\|_{L^\theta_2(\Omega; \mathbb{R}^n)}, \|E^{ij}\|_{W^{1,\infty}(\Omega)}, \|h^{ij}\|_{W^{1,\infty}(\Omega)}, \|g\|_{H^1(\Omega; \mathbb{R}^n)}, \|f\|_{L^\theta_2(\Omega; \mathbb{R}^n)} \right). 
\]

The proof of Theorem 2.2 will be given in Section 5. Also, in Remark 5.1, we shall explain why conditions (1)–(4) in the assumption (H) are introduced. Moreover, in Section 6, we shall give an illustrative example, in which all of the assumptions in Theorem 2.2 are satisfied.

Remark 2.3. It is well known that one of the classical topics in partial differential equations is the strong maximum principle for elliptic differential equations, which has many applications ([9,19–21] and so on). However, the existing results on strong maximum principle are mainly focusing on single elliptic equations, although one can find some works on weakly coupled elliptic systems [1,14,22] and the references therein. It would be quite interesting to establish a strong maximum principle for the system (1.1) or even for the system (1.2), but this remains to be done and it seems to be far from easy.

Remark 2.4. Notice that the condition (2.7) in Theorem 2.1 and the second condition of the assumption (H) in Theorem 2.2 imply certain regularity requirement for the coefficients of the system (1.1) and the system (1.2), respectively. Therefore, this may affect their applications to general strongly coupled quasilinear elliptic differential systems.

Remark 2.5. It seems natural to expect that a similar weak maximum principle holds for some strongly coupled parabolic differential systems, say the parabolic counterpart of (1.1) or even (1.2). Indeed, it is well known that, in the scalar case (i.e. \( n = 1 \)), there is no essential difference between the proof of weak maximum principle for elliptic equations and that for parabolic equations. However, we do not succeed to extend our approach developed in this paper to the parabolic case. We refer to [15] and the references cited therein for the maximum principle for strongly coupled parabolic differential systems in the setting of Campanato’s space.

3. Some preliminaries

In this section, we collect some known preliminary results which will be useful later. The first one is the following interpolation result.

Lemma 3.1. (See [12, Theorem 2.1 in Chapter 2].) For any \( u \in W^{1,t}_0(\Omega), \ t \geq 1 \) and \( \tau \geq 1 \), it holds that

\[
|u|_{L^{p*}(\Omega)} \leq \beta |\nabla u|^{\alpha}_{L^t(\Omega)}|u|^{1-\alpha}_{L^\tau(\Omega)},
\]

where \( \alpha = \left( \frac{1}{t} - \frac{1}{p^*} \right) \left( \frac{1}{\tau} - \frac{1}{t^*} \right)^{-1}, \ t^* = \frac{tm}{m-t}, \) and \( \beta \) is a constant depending only on \( m, t, p^*, \tau \) and \( \alpha \). Moreover, if \( t < m, \ p^* \) can be any number between \( \tau \) and \( t^* \); if \( t \geq m, \ p^* \) can be any number larger than \( \tau \).
For any real-valued Lebesgue measurable function \( v \) defined on \( \Omega \) and \( k \in \mathbb{R} \), we put
\[
A_k = \{ x \in \Omega ; v(x) > k \},
\]
and denote by \( |A_k| \) the Lebesgue measure of the set \( A_k \). The next lemma is quite useful in deriving the supremum of the function \( v \).

**Lemma 3.2.** (See [12, Theorem 5.1 in Chapter 2].) Suppose that \( v \in W^{1,m_0}(\Omega) \cap L^{q_0}(\Omega) \) for some \( m_0 \in [1,m] \) and some \( q_0 \geq 1 \). If for some \( k_0 \geq 0 \) and any fixed \( k \geq \text{esssup}_\Gamma v + k_0 \), the function \( v \) satisfies the following inequality:
\[
\int_{A_k} |\nabla v|^{m_0} \, dx \leq \gamma \left[ \int_{A_k} (v - k)^{l_0} \, dx \right]^{m_0 / q_0} + \gamma k^\sigma |A_k|^{1 - m_0 / m + \varepsilon_0},
\]
where \( \gamma, l_0, \sigma \) and \( \varepsilon_0 \) are positive constants satisfying \( l_0 < \frac{mm_0}{m-m_0} \) and \( m_0 \leq \sigma < \varepsilon_0 q_0 + m_0 \), then
\[
\text{esssup}_\Omega v \leq C \left( \Omega, m_0, q_0, \gamma, l_0, \sigma, \varepsilon_0, k_0, \text{esssup}_\Gamma v, |v|_{L^{q_0}(\Omega)} \right).
\]
Moreover, when \( \sigma = m_0 \), \( |v|_{L^{q_0}(\Omega)} \) appeared in \( C \) can be replaced by \( |v|_{L^1(\Omega)} \).

The last lemma is a result on comparison of the determinants between a matrix and its symmetrizing matrix.

**Lemma 3.3.** (See [23, Theorem 3.7.1].) For any real square matrix \( E \), if \( H(E) = E + E^\top \) is positive definite, then
\[
\det H(E) \leq \det E.
\]

4. Proof of Theorem 2.1

The goal of this section is to prove the first main result in this paper, i.e., Theorem 2.1.

**Proof of Theorem 2.1.** In view of Lemma 3.2, in order to derive the desired maximum modulus estimate for the weak solution \( y = (y^1, y^2, \ldots, y^n)^\top \) to (1.1), the point is to establish estimate (3.1) for \( v := |y|^2 \). To this aim, we need to choose suitably some weighted test function, as described as follows. For any fixed \( k \geq \text{esssup}_\Gamma |y|^2 + 1 \) and \( r > 0 \), put
\[
\phi_r(x) = \min \{ (|y(x)|^2 - k)_+, r \},
\]
where \( s_+ := \max \{ s, 0 \} \) (for any \( s \in \mathbb{R} \)). Take \( \varphi = (\varphi^1, \varphi^2, \ldots, \varphi^n)^\top \) as the desired test function, where
\[
\varphi^i = (y^1 T^{1i} + y^2 T^{2i} + \cdots + y^n T^{ni}) \phi_r,
\]
while \( T^{ij} \in W^{1,\infty}(\Omega) \) (\( i, j = 1, 2, \ldots, n \)) are suitable functions to be specified later. Note that the weighted functions \( T^{ij} \) (in the test function) will be used to eliminate some undesired terms generated by different principal operators and different solution components in the same equation for the system (1.1). Meanwhile, \( \phi_r \) (in the test function) is chosen to derive certain estimates for \( \int_{\{v>k\}} |\nabla v|^2 \, dx \), which implies the desired estimate (3.1) for \( v \). We divide the rest of the proof into several steps.

**Step 1.** In order to establish the desired estimates for \( \int_{\{v>k\}} |\nabla v|^2 \, dx \) (see (4.17) and (4.18)), first, we derive an estimate for \( \int_{\Omega} |\nabla \phi_r|^2 \, dx \). Note that \( \phi \in H^1_0(\Omega; \mathbb{R}^n) \). Then, by Definition 2.1, it follows that

\[
\sum_{i=1}^{n} \int_{\Omega} \left( a^{1i} \nabla y^1 + a^{2i} \nabla y^2 + \cdots + a^{ni} \nabla y^n \cdot \nabla \left[ \left( y^1 T^{1i} + y^2 T^{2i} + \cdots + y^n T^{ni} \right) \phi_r \right] \right) \, dx \\
+ \sum_{i=1}^{n} \int_{\Omega} \left( \sum_{j=1}^{n} C^{ij} \cdot \nabla y^j + D^i \cdot y \right) \left( y^1 T^{1i} + y^2 T^{2i} + \cdots + y^n T^{ni} \right) \phi_r \, dx \\
= \sum_{i=1}^{n} \int_{\Omega} f^i \left( y^1 T^{1i} + y^2 T^{2i} + \cdots + y^n T^{ni} \right) \phi_r \, dx.
\]

This implies that, for any \( \varepsilon > 0 \),

\[
\sum_{i=1}^{n} \int_{\Omega} \left( a^{1i} |\nabla y^1|^2 + a^{2i} |\nabla y^2|^2 + \cdots + a^{ni} |\nabla y^n|^2 \right) \phi_r \\
+ \sum_{i=1}^{n} \int_{\Omega} \left( a^{ij} T^{li} \nabla y^j \cdot \nabla \phi_r + a^{ij} T^{li} y^j \nabla y^j \cdot \nabla \phi_r \right) \\
+ \frac{1}{2} \left[ a^{1i} T^{1i} \nabla (y^1)^2 + a^{2i} T^{2i} \nabla (y^2)^2 + \cdots + a^{ni} T^{ni} \nabla (y^n)^2 \right] \cdot \nabla \phi_r \right) \, dx \\
= \sum_{i=1}^{n} \int_{\Omega} f^i \left( y^1 T^{1i} + y^2 T^{2i} + \cdots + y^n T^{ni} \right) \phi_r - \sum_{i=1}^{n} \int_{\Omega} a^{ij} y^j \nabla y^j \cdot \nabla T^{ji} \phi_r \\
- \left( \sum_{j=1}^{n} C^{ij} \cdot \nabla y^j + D^i \cdot y \right) \left( y^1 T^{1i} + y^2 T^{2i} + \cdots + y^n T^{ni} \right) \phi_r \, dx \\
\leq C \int_{\Omega} |f| |y| \phi_r \, dx + \varepsilon \int_{\Omega} |\nabla y|^2 \phi_r \, dx \\
+ C \int_{\Omega} \left[ \varepsilon^{-1} \left( \sum_{i,j=1}^{n} |C^{ij}|^2 + 1 \right) + \sum_{i=1}^{n} |D^i|^2 \right] |y|^2 \phi_r \, dx.
\]

Here and hereafter \( C \) denotes a generic constant (which may be different from one place to another), depending only on \( n, m, \theta, \rho, |a^{ij}|_{L^\infty(\Omega)} \) and \( |T^{ij}|_{W^{1,\infty}(\Omega)} \) (\( i, j = 1, 2, \ldots, n \)). Also, it is clear that
\[\begin{align*}
[a^1 T_{1i} \nabla (y^1)^2 + a^2 T_{2i} \nabla (y^2)^2 + \cdots + a^n T_{ni} \nabla (y^n)^2] \cdot \nabla \phi_r \\
= a^1 T_{1i} \nabla |y|^2 \cdot \nabla \phi_r + [(a^2 T_{2i} - a^1 T_{1i}) \nabla (y^2)^2 + \cdots + (a^n T_{ni} - a^1 T_{1i}) \nabla (y^n)^2] \cdot \nabla \phi_r.
\end{align*}\]

This, together with (4.1), yields that

\[\begin{align*}
\sum_{i=1}^n \int_\Omega \left\{ (a^1 T_{1i} |\nabla y^1|^2 + a^2 T_{2i} |\nabla y^2|^2 + \cdots + a^n T_{ni} |\nabla y^n|^2) \phi_r + \frac{1}{2} a^1 T_{1i} |\nabla |y||^2 \cdot \nabla \phi_r \right\} dx \\
+ \sum_{i=1}^n \int_\Omega \left\{ \sum_{l,j \in \{1,2,\ldots,n\}, l \neq j} (a^{ij} T^{li} \nabla y^j \cdot \nabla y^l \phi_r + a^{ij} T^{li} y^l \nabla y^j \cdot \nabla \phi_r) \\
+ \frac{1}{2} [(a^2 T_{2i} - a^1 T_{1i}) \nabla (y^2)^2 + \cdots + (a^n T_{ni} - a^1 T_{1i}) \nabla (y^n)^2] \cdot \nabla \phi_r \right\} dx \\
\leq C \int_\Omega |f||y| \phi_r \, dx + \varepsilon \int_\Omega |\nabla y|^2 \phi_r \, dx \\
+ C \int_\Omega \left[ \varepsilon^{-1} \left( \sum_{i,j=1}^n |C^{ij}|^2 + 1 \right) + \sum_{i=1}^n |D^i| \right] |y|^2 \phi_r \, dx.
\end{align*}\]  

(4.2)

In the following, we choose suitable weighted functions $T_{ij} \in W^{1,\infty}(\Omega)$ ($i, j = 1, 2, \ldots, n$) such that

\[\begin{align*}
\sum_{i=1}^n \int_\Omega \left\{ \sum_{l,j \in \{1,2,\ldots,n\}, l \neq j} (a^{ij} T^{li} \nabla y^j \cdot \nabla y^l \phi_r + a^{ij} T^{li} y^l \nabla y^j \cdot \nabla \phi_r) \\
+ \frac{1}{2} [(a^2 T_{2i} - a^1 T_{1i}) \nabla (y^2)^2 + \cdots + (a^n T_{ni} - a^1 T_{1i}) \nabla (y^n)^2] \cdot \nabla \phi_r \right\} dx = 0.
\end{align*}\]

For this purpose, consider the following linear system:

\[\begin{align*}
\sum_{i=1}^n a^{ij} T^{li} &= 0, \quad \forall j, l = 1, 2, \ldots, n \text{ with } j \neq l, \\
\sum_{i=1}^n (a^2 T_{2i} - a^1 T_{1i}) &= 0, \\
\vdots \\
\sum_{i=1}^n (a^n T_{ni} - a^1 T_{1i}) &= 0.
\end{align*}\]  

(4.3)

By Lemma 3.3 and (2.4), it follows that $\det A \geq \rho^n$ and $\det B \geq \rho^{n-1}$. One can check easily that the following is a solution to the system (4.3):
\[
\begin{align*}
T_{11} &= 1, \\
T_{1i} &= (-1)^{1+i} \frac{B_{1i}}{\det B} \quad (i = 2, \ldots, n), \\
T_{ji} &= (-1)^{i+j} \sum_{l=1}^{n} a_{1l}^j T_{1l} B_{ji} \quad (j = 2, 3, \ldots, n, \; i = 1, 2, \ldots, n).
\end{align*}
\]

(4.4)

Also, it is not difficult to check that
\[
\sum_{l=1}^{n} a_{1l}^j T_{1l} = \frac{\det A}{\det B}.
\]

From this fact and noting (4.4), we see that
\[
T_{ji} = (-1)^{i+j} B_{ji} \quad (i, j = 1, 2, \ldots, n).
\]

(4.5)

Moreover, by (4.3), there exists a constant \(\rho^* > 0\), depending only on \(n, \rho\) and \(|a_{ij}|_{L^{\infty}(\Omega)} (i, j = 2, \ldots, n)\), such that
\[
\sum_{l=1}^{n} a_{1l}^j T_{1l} \geq \cdots \geq \sum_{l=1}^{n} a_{1n}^n T_{1n} \geq \frac{\rho^n}{\det B} \geq \rho^*.
\]

(4.6)

Therefore, by (4.2), (4.3) and (4.6), and noting that the term “\(\varepsilon \int_{\Omega} |\nabla y|^2 \phi_r \, dx\)” in the right hand side of (4.2) can be absorbed by choosing \(\varepsilon\) to be small enough, we arrive at
\[
\int_{\Omega} \left( |\nabla y|^2 \phi_r + |\nabla \phi_r|^2 \right) \, dx \\
\leq C \left[ \int_{\Omega} |f| |y| \phi_r \, dx + \int_{\Omega} \left( \sum_{i,j=1}^{n} |C^{ij}|^2 + \sum_{i=1}^{n} |D^i| + 1 \right) |y|^2 \phi_r \, dx \right].
\]

(4.7)

Step 2. We now estimate each term in the right side of (4.7) and show that the left side of this inequality is uniformly bounded with respect to \(r > 0\). First of all, by Hölder’s inequality and Lemma 3.1, we see that
\[
\int_{\Omega} |f| |y| \phi_r \, dx \leq \int_{\Omega} |f| (|y|^2 - k)^{\frac{1}{2}} \phi_r \, dx + k^{\frac{1}{2}} \int_{\Omega} |f| \phi_r \, dx
\leq |f| \frac{\delta}{L^{\frac{1}{2}}(\Omega; \mathbb{R}^n)} \left( (|y|^2 - k)^{\frac{1}{2}} \phi_r \right)_{L^{\frac{1}{2}}(\Omega; \mathbb{R}^n)} + C_1 |f| \frac{\delta}{L^{\frac{1}{2}}(\Omega; \mathbb{R}^n)} |\phi_r|_{L^{\frac{1}{2}}(\Omega; \mathbb{R}^n)}
\leq |f| \frac{\delta}{L^{\frac{1}{2}}(\Omega; \mathbb{R}^n)} \left( (|y|^2 - k) \phi_r \right)_{L^{\frac{1}{2}}(\Omega; \mathbb{R}^n)} + C_1 |f| \frac{\delta}{L^{\frac{1}{2}}(\Omega; \mathbb{R}^n)} |y|^2 |H^1(\Omega; \mathbb{R}^n)}
\leq |f| \frac{\delta}{L^{\frac{1}{2}}(\Omega; \mathbb{R}^n)} \left( (|y|^2 - k) \phi_r \right)_{L^{\frac{1}{2}}(\Omega; \mathbb{R}^n)} + 1 + C_1 |f| \frac{\delta}{L^{\frac{1}{2}}(\Omega; \mathbb{R}^n)} |y|^2 |H^1(\Omega; \mathbb{R}^n)}.
\]

(4.8)
Here and hereafter $C_1$ denotes a positive constant depending only on $k$, $\theta$, $n$, $m$ and $\Omega$. Put

$$
L = \sum_{i,j=1}^{n} |C_{ij}|^2_{L^\theta(\Omega;\mathbb{R}^m)} + \sum_{i=1}^{n} |D_i|^\theta_{L^\infty(\Omega;\mathbb{R}^n)} + 1.
$$

(4.9)

Using Lemma 3.1 again, we find that

$$
\int_\Omega \left( \sum_{i,j=1}^{n} |C_{ij}|^2 + \sum_{i=1}^{n} |D_i| + 1 \right) |y|^2 \phi_r \, dx
\leq C_1 L \left[ \int_\Omega \left( |y|^2 \phi_r \right)^\theta_{L^\infty(\Omega)} dx \right]^{\theta-2} \leq C_1 L \left( |y|^2 - k \right) \phi_r \, dx
\leq C_1 L \left( |y|^2 - k \right) \phi_r \, dx + C_1 L |y|^2_{H^1(\Omega;\mathbb{R}^n)}.
$$

(4.10)

On the other hand, write $u_* = \sqrt{(|y|^2 - k)\phi_r}$. Then, it follows that

$$
\int_\Omega |\nabla u_*|^2 \, dx = \int_{\{|y|^2 > k\}} |\nabla u_*|^2 \, dx = \int_{\{|y|^2 > k\}} \frac{2\phi_r y \nabla y + (|y|^2 - k) \nabla \phi_r}{2\sqrt{(|y|^2 - k)\phi_r}} \, dx
\leq 2 \int_{\{|y|^2 > k\}} \frac{\phi_r y \nabla y}{\sqrt{(|y|^2 - k)\phi_r}} \, dx + \frac{1}{2} \int_{\{k + r \geq |y|^2 > k\}} \frac{(|y|^2 - k) \nabla \phi_r}{\sqrt{(|y|^2 - k)\phi_r}} \, dx
\leq 2 \int_{\{|y|^2 > k\}} (|y|^2 - k)^{-1} \phi_r |y|^2 |\nabla y|^2 \, dx
+ 2 \int_{\{k + r \geq |y|^2 > k\}} (|y|^2 - k) \phi_r^{-1} |y|^2 |\nabla y|^2 \, dx.
$$

(4.11)

Noting $\phi_r \leq |y|^2 - k$, we see that

$$
\int_{\{|y|^2 > k\}} (|y|^2 - k)^{-1} \phi_r |y|^2 |\nabla y|^2 \, dx
= \int_{\{|y|^2 > k\}} \phi_r |\nabla y|^2 \, dx + k \int_{\{|y|^2 > k\}} (|y|^2 - k)^{-1} \phi_r |\nabla y|^2 \, dx
\leq \int_{\Omega} \phi_r |\nabla y|^2 \, dx + k \int_{\{|y|^2 > k\}} |\nabla y|^2 \, dx \leq \int_{\Omega} \phi_r |\nabla y|^2 \, dx + k \int_{\Omega} |\nabla y|^2 \, dx.
$$

(4.12)

Noting that $\phi_r = |y|^2 - k$ whenever $k + r \geq |y|^2$, it is clear that
\[\int_{\{k+r \geq |y|^2 > k\}} \left( |y|^2 - k \right) \phi_r^{-1} |y|^2 |\nabla y|^2 \, dx = \int_{\{k+r \geq |y|^2 > k\}} |y|^2 |\nabla y|^2 \, dx \]
\[= \int_{\{k+r \geq |y|^2 > k\}} \left( |y|^2 - k \right) |\nabla y|^2 \, dx + k \int_{\{k+r \geq |y|^2 > k\}} |\nabla y|^2 \, dx \]
\[= \int_{\{k+r \geq |y|^2 > k\}} \phi_r |\nabla y|^2 \, dx + k \int_{\{k+r \geq |y|^2 > k\}} |\nabla y|^2 \, dx \]
\[\leq \int_{\Omega} \phi_r |\nabla y|^2 \, dx + k \int_{\Omega} |\nabla y|^2 \, dx. \quad (4.13)\]

Therefore, by (4.11)–(4.13), we conclude that
\[\int_{\Omega} |\nabla u_\ast|^2 \, dx \leq 4 \int_{\Omega} \phi_r |\nabla y|^2 \, dx + C_1 \int_{\Omega} |\nabla y|^2 \, dx. \quad (4.14)\]

By (4.14) and Lemma 3.1, for any \(0 < \varepsilon < 1\), we end up with
\[\left| (|y|^2 - k) \phi_r \right|_{L^{\frac{q}{q-2}}(\Omega)} = \left( \int_{\Omega} u_\ast^\frac{q}{q-2} \, dx \right)^{\frac{q-2}{q}} \leq \varepsilon \int_{\Omega} |\nabla u_\ast|^2 \, dx + C_1 \varepsilon^{-1} \left( \int_{\Omega} |u_\ast| \, dx \right)^2 \]
\[\leq 4 \varepsilon \int_{\Omega} \phi_r |\nabla y|^2 \, dx + C_1 \varepsilon |y|_{H^1(\Omega; \mathbb{R}^n)} + C_1 \varepsilon^{-1} \left[ \int_{\Omega} (|y|^2 - k)^{\frac{1}{2}} \phi_r \, dx \right]^2 \]
\[\leq 4 \varepsilon \int_{\Omega} \phi_r |\nabla y|^2 \, dx + C_1 \varepsilon |y|_{H^1(\Omega; \mathbb{R}^n)} + C_1 \varepsilon^{-1} |y|_{L^2(\Omega; \mathbb{R}^n)}^4. \quad (4.15)\]

Therefore, substituting (4.15) into (4.8) and (4.10), respectively, we see that
\[\int_{\Omega} |f| |y| \phi_r \, dx \leq \left| f \right|_{L^{\frac{q}{q-6}}(\Omega; \mathbb{R}^n)} \left[ 4 \varepsilon \int_{\Omega} \phi_r |\nabla y|^2 \, dx + C_1 \varepsilon |y|_{H^1(\Omega; \mathbb{R}^n)} + C_1 \varepsilon^{-1} |y|_{L^2(\Omega; \mathbb{R}^n)}^4 + 1 \right] \]
\[+ C_1 |f|_{L^2(\Omega; \mathbb{R}^n)} |y|_{H^1(\Omega; \mathbb{R}^n)}^2 \]
and
\[\int_{\Omega} \left( \sum_{i,j=1}^{n} |C^{ij}|^2 + \sum_{i=1}^{n} |D^i| + 1 \right) |y|^2 \phi_r \, dx \]
\[\leq 4 \varepsilon C_1 L \int_{\Omega} \phi_r |\nabla y|^2 \, dx + C_1 L (1 + C_1 \varepsilon) |y|_{H^1(\Omega; \mathbb{R}^n)} + C_1^2 L \varepsilon^{-1} |y|_{L^2(\Omega; \mathbb{R}^n)}^4. \]

Combining the above inequalities with (4.7) and taking \(\varepsilon\) sufficiently small such that
\[(4|f|_{L^\theta(\Omega;\mathbb{R}^n)} \varepsilon + 4C_1 L \varepsilon) C < \frac{1}{2},\]

where \(C\) and \(C_1\) are the constants appeared in (4.7) and (4.10), respectively, we arrive at

\[
\int_\Omega (|\nabla y|^2 \phi_r + |\nabla \phi_r|^2) \, dx \leq C_2. \tag{4.16}
\]

Here and hereafter \(C_2\) is a positive constant depending on \(C, C_1, L, |f|_{H^1(\Omega;\mathbb{R}^n)}\) and \(|y|_{H^1(\Omega;\mathbb{R}^n)}\), but independent of \(r\).

Recall that \(v = |y|^2\). Since \(\phi_r \in H^1_0(\Omega)\), by the definition of \(\phi_r\), letting \(r \to +\infty\) in (4.16), for any fixed \(k \geq \text{esssup}_\Omega |y|^2 + 1\), we obtain that

\[
\int_\Omega |\nabla y|^2 (|y|^2 - k)_+ \, dx + \int_\Omega \left[ (|y|^2 - k)_+ \right]^2 \, dx + \int_{\{v > k\}} |\nabla v|^2 \, dx \leq C_2. \tag{4.17}
\]

**Step 3.** In this step, we construct a sequence of inequalities in the form of (3.1) for the function \(v = |y|^2\) with respect to \(A_k\) (recall that \(A_k = \{x \in \Omega; v(x) > k\}\)). Again, by (4.7) and noting that \(\phi_r \leq (|y|^2 - k)_+\), we obtain that

\[
\int_\Omega (|\nabla y|^2 \phi_r + |\nabla \phi_r|^2) \, dx
\leq C \left[ \int_\Omega |f| |y| (|y|^2 - k)_+ \, dx + \int_\Omega \sum_{i=1}^n \left( \sum_{j=1}^n |C_{ij}|^2 + |D_i|^2 + 1 \right) |y|^2 (|y|^2 - k)_+ \, dx \right].
\]

Letting \(r \to +\infty\) in the above inequality, we see that

\[
\int_{A_k} |\nabla y|^2 (|y|^2 - k) \, dx + \int_{A_k} |\nabla v|^2 \, dx
\leq C \int_{A_k} \left[ |f| + \sum_{i=1}^n \left( \sum_{j=1}^n |C_{ij}|^2 + |D_i|^2 + 1 \right) \right] (|y|^4 + 1) \, dx.
\]

For any \(\varepsilon > 0\), by the Hölder inequality and Lemma 3.1, this implies that

\[
\int_{A_k} |\nabla y|^2 (|y|^2 - k) \, dx + \int_{A_k} |\nabla v|^2 \, dx
\leq C \left( |f|_{L^2(\Omega;\mathbb{R}^n)} + L \left( |y|^2 - k \right)_{L^2(\Omega;\mathbb{R}^n)} + k^2 |A_k|^{-\frac{2}{n}} \right)
\]
\[
+ C \left( |f|_{L^2(\Omega;\mathbb{R}^n)} + L \right) |A_k|^{-\frac{2}{n}}
\leq C \left( L + |f|_{L^2(\Omega;\mathbb{R}^n)} (\varepsilon |\nabla y|^2)_{L^2(\Omega;\mathbb{R}^n)} + C(\varepsilon) |y|^2 - k \right)_{L^2(\Omega;\mathbb{R}^n)}
\]
\[ + C \left( L + \|f\|_{L^2(\Omega;\mathbb{R}^n)} \right) k^2 |A_k|^{1 - \frac{2}{\theta}}. \] (4.18)

Taking \( \varepsilon \) sufficiently small, then by (4.17) and (4.18), one derives that
\[ \int_{A_k} |\nabla v|^2 \, dx \leq C_3 \int_{A_k} |v - k|^2 \, dx + C_3 k^2 |A_k|^{1 - \frac{2}{\theta}}, \]
where \( C_3 \) denotes a positive constant depending only on \( C, L \) and \( \|f\|_{L^2(\Omega;\mathbb{R}^n)} \).

**Step 4.** In order to use Lemma 3.2, we choose
\[ m_0 = 2, \quad l_0 = 2, \quad \sigma = 2, \quad \varepsilon_0 = \frac{2}{m} - \frac{2}{\theta}, \quad k_0 = 1, \quad \gamma = C_3. \]

Then, by Lemma 3.2, it follows that
\[ \text{esssup}_{\Omega} |y| \leq C \left( m, n, \theta, \Omega, \rho, \|a^{ij}\|_{L^\infty(\Omega)}, \frac{B^{ij}_{\det B}}{L} \right) \left( \|f\|_{L^2(\Omega;\mathbb{R}^n)}, |y|_{L^2(\Omega;\mathbb{R}^n)}, \text{esssup}_{\Gamma} |y| \right). \] (4.19)

Since \( y \) is the weak solution to (1.1), by Lemma 2.1, we conclude that
\[ |y|_{L^2(\Omega;\mathbb{R}^n)} \leq C \left( m, n, \theta, \Omega, \rho, \|a^{ij}\|_{L^\infty(\Omega)}, \|f\|_{L^2(\Omega;\mathbb{R}^n)} + |g|_{H^1(\Omega;\mathbb{R}^n)} \right). \]

This, combined with (4.19), yields the desired conclusion in Theorem 2.1. \( \square \)

**Remark 4.1.** We now explain why we need the technical condition (2.7) in Theorem 2.1. Recall that \( \frac{B^{ij}_{\det B}}{\det B} \) (\( i, j = 1, \ldots, n \)), are linked with the weighted functions \( T^{ij} \) by (4.5), while these weighted functions are introduced to eliminate some undesired terms that appeared in the inequality (4.2). On the other hand, in order to guarantee that the test function \( \varphi \in H^1_0(\Omega;\mathbb{R}^n) \), we have to require that \( T^{ij} \), and therefore \( \frac{B^{ij}_{\det B}}{\det B} \), belongs to \( W^{1,\infty}(\Omega) \) for any \( i, j = 1, \ldots, n \).

5. **Proof of Theorem 2.2**

Now, let us prove the second main result in this article, i.e., Theorem 2.2.

**Proof of Theorem 2.2.** The main idea is the same as that in the proof of Theorem 2.1. However, in order to drop the undesired terms generated by mixed partial derivatives of second order in each equation of the system (1.2), we have to choose a test function which is quite different from that in the proof of Theorem 2.1.

For the weak solution \( y = (y^1, y^2, \ldots, y^n)^T \) to (1.2), we put
\[ \psi = \sum_{j,l=1}^n h^{jl} y^j y^l, \quad v = \psi^{\frac{s+1}{2}}. \]
where \((h^{ij})_{1 \leq i,j \leq n}\) is the uniformly positive definite matrix appeared in the assumption (H), and \(s\) is a positive constant to be determined later. Our goal is to establish some estimate in the form of (3.1) for \(v\) (see (5.17)). Then, by Lemma 3.2, this suffices to give the desired maximum modulus estimate for \(|y|\).

In what follows, we shall choose \(\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^n)^\top \in H^1_0(\Omega; \mathbb{R}^n)\) as the desired test function, where

\[
\varphi^j = \sum_{l=1}^n E^{il} y^l \zeta_r,
\]

while \(E^{ij}\) \((i,j = 1, 2, \ldots, n)\) are given by the assumption (H), \(\zeta_r\) is a suitable function (of \(\psi\)) to be specified later (for each \(r > 0\)). Note that, as before, the weighted functions \(E^{ij}\) are introduced to eliminate the undesired terms that appear in the sequel.

We divide the rest of the proof into several steps.

**Step 1.** As a preliminary to derive the desired estimates for \(v = \psi^{\frac{r+1}{2}}\), first, we establish an estimate for \(\int_\Omega \sum_{p,q=1}^m F_{pq} \psi_x^p (\zeta_r) x_q \, dx\), where \(F = (F_{pq})_{1 \leq p,q \leq m}\) is the uniformly positive definite matrix given in the assumption (H). Noting that \(\varphi \in H^1_0(\Omega; \mathbb{R}^n)\), by Definition 2.1, we obtain that

\[
\sum_{i,j=1}^n \sum_{p,q=1}^m \sum_{l=1}^n \int_{\Omega} a_{pq} y_q^l (E^{il} y^l \zeta_r) x_q \, dx + \sum_{i,j,l=1}^n \int_{\Omega} C^{ij} \cdot \nabla y^l E^{il} y^l \zeta_r \, dx
\]

\[
+ \sum_{i,l=1}^n \int_{\Omega} D^i \cdot y E^{il} y^l \zeta_r \, dx = \sum_{i,l=1}^n \int_{\Omega} f^i E^{il} y^l \zeta_r \, dx.
\]

This implies that

\[
\sum_{i,j=1}^n \sum_{p,q=1}^m \sum_{l=1}^n \left[ a_{pq} E^{il} y_q^l x_q \zeta_r + a_{pq} E^{il} y_q^l y^l (\zeta_r) x_q + a_{pq} E^{il} x_q y_q^l y^l \zeta_r \right] \, dx
\]

\[
+ \sum_{i,j,l=1}^n \int_{\Omega} C^{ij} \cdot \nabla y^l E^{il} y^l \zeta_r \, dx + \sum_{i,l=1}^n \int_{\Omega} D^i \cdot y E^{il} y^l \zeta_r \, dx
\]

\[
= \sum_{i,l=1}^n \int_{\Omega} f^i E^{il} y^l \zeta_r \, dx.
\]

Therefore,

\[
\sum_{i,j=1}^n \sum_{p,q=1}^m \sum_{l=1}^n \left[ a_{pq} E^{il} y_q^l x_q \zeta_r + a_{pq} E^{il} y_q^l y^l (\zeta_r) x_q \right] \, dx
\]

\[
\leq C_4 \left[ \int_{\Omega} |f||y|\zeta_r + \left( 1 + \sum_{i,j=1}^n |C^{ij}| \right) \int_{\Omega} |\nabla y| |y| \zeta_r + \sum_{i=1}^n \int_{\Omega} |D^i||y|^2 \zeta_r \right] \, dx. \tag{5.1}
\]
Here and henceforth $C_4$ denotes a constant depending only on $n$, $m$, $\rho$, $|a_{ij}|_{L^\infty(\Omega)}$ and $|E^{ij}|_{W^{1,\infty}(\Omega)}$, $(i, j = 1, \ldots, n)$.

Next, we need to estimate the two terms in the left side of (5.1). More precisely, we shall establish two estimates for $\int_\Omega |\nabla y|^2 \zeta_r \, dx$ and $\int_\Omega \sum_{p,q=1}^m F_{pq} \psi_{xp}(\zeta_r) x_q \, dx$ from the first term and the second term in the left side of (5.1), respectively. To this aim, by (2.11), using the condition (2) in the assumption (H) and the Cramer rule, we see that for any $p,q = 1, \ldots, m$, functions $E_{ij}$ $(i,j = 1, \ldots, n)$ (given by the assumption (H)) satisfy

$$\sum_{l=1}^n a_{pq}^{il} E_{il} = f_{pq}^{ij}.$$  

In particular, by $h_{11} = 1$, we find that $f_{pq}^{ij} = \sum_{l=1}^n a_{pq}^{il} E_{il}$. Therefore,

$$\sum_{l=1}^n a_{pq}^{il} E_{il} = h_{1j} \sum_{l=1}^n a_{pq}^{il} E_{il}.$$  \hspace{1cm} (5.2)

This implies that

$$2 \sum_{i,j=1}^n \sum_{p,q=1}^m \sum_{l=1}^n \int_\Omega a_{pq}^{ij} E_{il} y_{xp}^i y_{xq}^l (\zeta_r) x_q \, dx$$

$$= 2 \sum_{p,q=1}^m \sum_{j,l=1}^n \int_\Omega \left( \sum_{i=1}^n a_{pq}^{il} E_{il} \right) h_{ij} y_{xp}^i y_{xq}^l (\zeta_r) x_q \, dx$$

$$= \sum_{p,q=1}^m \int_\Omega \left( \sum_{i=1}^n a_{pq}^{il} E_{il} \right) \left( \sum_{j,l=1}^n h_{ij} y_{xq}^l (\zeta_r) x_q \right) dx$$

$$- \sum_{p,q=1}^m \int_\Omega \left( \sum_{i=1}^n a_{pq}^{il} E_{il} \right) \left[ \sum_{j,l=1}^n (h_{ij}^l y_{xq}^l (\zeta_r) x_q \right] dx.$$

Recalling $\psi = \sum_{j,l=1}^n h_{ij} y_{xq}^l$, we see that

$$\int_\Omega \sum_{p,q=1}^m \left( \sum_{i=1}^n a_{pq}^{il} E_{il} \right) \psi_{xp}(\zeta_r) x_q \, dx$$

$$\leq 2 \sum_{i,j=1}^n \sum_{p,q=1}^m \sum_{l=1}^n \int_\Omega a_{pq}^{ij} E_{il} y_{xp}^i y_{xq}^l (\zeta_r) x_q \, dx + C_5 \int_\Omega |y|^2 |\nabla \zeta_r| \, dx,$$  \hspace{1cm} (5.3)

where $C_5$ is a positive constant depending only on $C_4$, $\rho_3$ and $|h_{ij}|_{W^{1,\infty}(\Omega)}$, $(i, j = 1, 2, \ldots, n)$. On the other hand, by the condition (4) in the assumption (H) and noting the equality (5.2), it is easy to see that the uniformly positive definite matrix $M$ in the assumption (H) can be rewritten in the following form:
\[ M = \begin{pmatrix}
\sum_{l=1}^{n} a_{11}^{l} E^{11} & \cdots & \sum_{l=1}^{n} a_{1m}^{l} E^{1l} & \cdots & \sum_{l=1}^{n} a_{11}^{ln} E^{1n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\sum_{l=1}^{n} a_{m1}^{l} E^{m1} & \cdots & \sum_{l=1}^{n} a_{mm}^{l} E^{m1} & \cdots & \sum_{l=1}^{n} a_{1m}^{ln} E^{1n} \\
\sum_{l=1}^{n} a_{11}^{ln} E^{l1} & \cdots & \sum_{l=1}^{n} a_{1m}^{ln} E^{l1} & \cdots & \sum_{l=1}^{n} a_{11}^{mm} E^{ln} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\sum_{l=1}^{n} a_{m1}^{ln} E^{m1} & \cdots & \sum_{l=1}^{n} a_{mm}^{ln} E^{m1} & \cdots & \sum_{l=1}^{n} a_{1m}^{mm} E^{ln}
\end{pmatrix}_{nm \times nm}. \]

Therefore,

\[
\sum_{i,j=1}^{n} \sum_{p,q=1}^{m} \int_{\Omega} a_{pq}^{l} E^{i} y_{x_{p}} y_{x_{q}} \zeta_{r} dx \geq \rho_{3} \int_{\Omega} |\nabla y|^{2} \zeta_{r} dx. \tag{5.4}
\]

Now, by (5.1), (5.3) and (5.4), we end up with

\[
\int_{\Omega} \left( |\nabla y|^{2} \zeta_{r} + \sum_{p,q=1}^{m} F_{pq} \psi_{x_{p}} (\zeta_{r})_{x_{q}} \right) dx \\
= \int_{\Omega} \left( |\nabla y|^{2} \zeta_{r} + \sum_{p,q=1}^{m} \left( \sum_{i=1}^{n} a_{pq}^{i} E^{i1} \right) \psi_{x_{p}} (\zeta_{r})_{x_{q}} \right) dx \\
\leq C_{5} \int_{\Omega} \left[ |f| |y| \zeta_{r} + \left( 1 + \sum_{i,j=1}^{n} |C_{ij}|^{2} \right) |\nabla y| |y| \zeta_{r} + \sum_{i=1}^{n} |D_{i}| |y|^{2} \zeta_{r} + |y|^{2} |\nabla \zeta_{r}| \right] dx. \tag{5.5}
\]

**Step 2.** We now derive the desired estimate for \( v = \psi^{s+1} \). To this aim, for any \( s, r > 0 \) and \( k > \sup_{\Gamma} \psi^{s} + 1 \), put

\[
A_{k} = \left\{ x \in \Omega \mid \psi^{s}(x) > k \right\} \quad \text{and} \quad A_{k}^{r} = \left\{ x \in \Omega \mid k < \psi^{s}(x) < k + r \right\}.
\]

Moreover, we choose

\[
\zeta_{r} = \min \left\{ r, \left( \psi^{s} - k \right)_{+} \right\}.
\]

Then, by (5.5), and using the third condition in the assumption (H), we conclude that

\[
\int_{A_{k}} |\nabla y|^{2} \zeta_{r} dx + \int_{A_{k}^{r}} \psi^{s-1} |\nabla y|^{2} dx \\
\leq C_{6} \int_{A_{k}} \left[ |f| |y| \zeta_{r} + \left( 1 + \sum_{i,j=1}^{n} |C_{ij}|^{2} + \sum_{i=1}^{n} |D_{i}| \right) |y|^{2} \zeta_{r} + |y|^{2} |\nabla \zeta_{r}| \right] dx, \tag{5.6}
\]
where $C_6$ denotes a positive constant depending only on $\Omega$, $s$, $C_5$ and $\rho_2$. Moreover, using the condition (1) in the assumption (H), we find that

$$\psi \geq \rho_1 |y|^2. \quad (5.7)$$

In the following, we estimate each term in the right side of (5.6) and show that the left side of this equality is uniformly bounded with respect to $r > 0$. First of all, noting that $\zeta_r \leq \psi_s \leq C_6 |y|^{2s}$ in $A_k$, by Hölder’s inequality and noting that $|A_k| \leq |\Omega|$, we obtain that

$$\int_{A_k} |f||y|^{2s+1} dx \leq \int_{A_k} |f||(y|^{2s+2} + 1) dx$$

$$\leq C_6 |f|_{L^\theta(\Omega; \mathbb{R}^n)} (||y|^{2s+2} \|_{L^{\frac{\theta}{\theta-2}}(A_k)} + |A_k|^{\frac{\theta}{\theta-2}})$$

$$\leq C_6 |f|_{L^\theta(\Omega; \mathbb{R}^n)} (||y|^{2s+2} \|_{L^{\frac{\theta}{\theta-2}}(A_k)} + 1). \quad (5.8)$$

Further, by Hölder’s inequality and recalling (4.9), we obtain that

$$\int_{A_k} \left(1 + \sum_{i,j=1}^n |C^{ij}|^2 + \sum_{i=1}^n |D^i| \right) |y|^2 \zeta_r dx$$

$$\leq \int_{A_k} \left(1 + \sum_{i,j=1}^n |C^{ij}|^2 + \sum_{i=1}^n |D^i| \right) |y|^{2s+2} dx \leq C_6 L ||y|^{2s+2} \|_{L^{\frac{\theta}{\theta-2}}(A_k)}. \quad (5.9)$$

Further, for any $\varepsilon > 0$, by Hölder’s inequality, we see that

$$\int_{A_k} |y|^2 |
abla \zeta_r| dx \leq C_6 \int_{A_k} |y|^2 \psi^{s-1} |
abla \psi| dx$$

$$\leq \varepsilon \int_{A_k} \psi^{s-1} |
abla \psi|^2 dx + \varepsilon^{-1} C_6 \int_{A_k} |y|^4 \psi^{s-1} dx$$

$$\leq \varepsilon \int_{A_k} \psi^{s-1} |
abla \psi|^2 dx + \varepsilon^{-1} C_6 \int_{A_k} |y|^{2s+2} dx. \quad (5.10)$$

By $\theta > m$, it follows that $\frac{\theta}{\theta-2} < \frac{m}{m-2}$. Hence, if we take the positive number $s$ sufficiently small such that $s < \min\left\{ \frac{m(\theta-2)}{(m-2)\theta} - 1, \frac{2}{m-2} \right\}$, then it is easy to see that

$$(2s + 2) \frac{\theta}{\theta-2} < \frac{2m}{m-2} \quad \text{and} \quad 2s + 2 < \frac{2m}{m-2}.$$

By $y \in H^1(\Omega; \mathbb{R}^n)$, one obtains that $y \in L^{(2s+2)\theta/(\theta-2)}(\Omega; \mathbb{R}^n)$ and $y \in L^{2s+2}(\Omega; \mathbb{R}^n)$. This implies that
\[
\left| |y|^{2s+2}\right|_{L^{\frac{2}{r-2}}(A_k)} + \int_{A_k} y^{2s+2} \, dx < +\infty. \tag{5.11}
\]

Substituting (5.8)–(5.11) into (5.6) and letting \( r \to +\infty \), and recalling that \( v = \psi^{\frac{s+1}{2}} \), we conclude that
\[
|\nabla v|_{L^2(A_k)}^2 \leq C_6 \int_{A_k} \psi^{s-1} |\nabla \psi|^2 \, dx < +\infty. \tag{5.12}
\]

Now, let us estimate the right side of (5.6) and derive the desired estimate for \( v \). First, notice that
\[
\int_{A_k} |f||y| \zeta_r \, dx \leq \int_{A_k} |f||y|^2 \zeta_r \, dx + \int_{A_k} |f| \zeta_r \, dx. \tag{5.13}
\]

Further, by (5.7) and Hölder’s inequality, we obtain that
\[
\int_{A_k} \left(1 + \sum_{i,j=1}^n |C^{ij}|^2 + \sum_{i=1}^n |D^i| + |f| \right) |y|^2 \zeta_r \, dx
\leq C_7 \int_{A_k} \left(1 + \sum_{i,j=1}^n |C^{ij}|^2 + \sum_{i=1}^n |D^i| + |f| \right) \psi^{s+1} \, dx
\leq C_7 (L + |f|_{L^\infty(\Omega;\mathbb{R}^n)}) \psi^{s+1} + |f|_{L^\infty(\Omega;\mathbb{R}^n)} \psi^{s+1} \, dx
\leq C_7 (L + |f|_{L^\infty(\Omega;\mathbb{R}^n)}) \psi^{s+1} + C_7 (L + |f|_{L^\infty(\Omega;\mathbb{R}^n)}) \psi^{s+1} + k^2 |A_k|^{1-\frac{2}{\theta}}. \tag{5.14}
\]

Here and hereafter \( C_7 \) stands for a positive constant depending only on \( s, C_6 \) and \( \rho_1 \). For any \( \epsilon > 0 \), by Lemma 3.1, this implies that
\[
\int_{A_k} |f| \zeta_r \, dx \leq \int_{A_k} |f| \psi^s \, dx + \int_{A_k} |f| \psi^s \, dx + \int_{A_k} |f| \psi^s \, dx
\leq C_7 |f|_{L^\infty(\Omega;\mathbb{R}^n)} \left(\int_{A_k} v^{\frac{2\theta - 2}{\theta - 1}} \, dx\right)^{\frac{\theta - 2}{\theta}} \leq C_7 |f|_{L^\infty(\Omega;\mathbb{R}^n)} \left(\int_{A_k} v^{\frac{2\theta - 2}{\theta - 1}} \, dx + |A_k|\right)^{\frac{\theta - 2}{\theta}}.
\]
\[\begin{align*}
\leq C_7 & \left| f \right|_{L^\frac{2}{\theta}(\Omega;\mathbb{R}^n)} \left( |v - k|^2_{L^2(\Omega;\mathbb{R}^n)} + k^2 |A_k|^{1-\frac{2}{\theta}} \right) \\
\leq C_7 & \left| f \right|_{L^\frac{2}{\theta}(\Omega;\mathbb{R}^n)} \left( \varepsilon |\nabla v|^2_{L^2(\Omega;\mathbb{R}^n)} + C(\varepsilon) |v - k|^2_{L^2(\Omega;\mathbb{R}^n)} \right) + C_7 \left| f \right|_{L^\frac{2}{\theta}(\Omega;\mathbb{R}^n)} k^2 |A_k|^{1-\frac{2}{\theta}}.
\end{align*}\]

Further, for any \( \varepsilon > 0 \), by Hölder’s inequality and (5.7), we get
\[
\int_{A_k} |\nabla \zeta| |\psi| \, dx \leq C_7 \int_{A'_k} |\nabla \zeta| \psi \, dx
\leq \varepsilon \int_{A'_k} \psi |\nabla \psi| \, dx + \varepsilon^{-1} C_7 \int_{A'_k} |\psi|^2 \, dx
\leq \varepsilon \int_{A'_k} \psi |\nabla \psi| \, dx + \varepsilon^{-1} C_7 \left( \int_{A_k} |v - k|^2 \, dx + k^2 |A_k| \right)
\leq \varepsilon \int_{A'_k} \psi |\nabla \psi| \, dx + \varepsilon^{-1} C_7 \left( \int_{A_k} |v - k|^2 \, dx + k^2 |A_k|^{1-\frac{2}{\theta}} \right). \tag{5.16}
\]

Substituting (5.13)–(5.16) into (5.6), taking \( \varepsilon \) sufficiently small and letting \( r \to +\infty \), and taking (5.12) into account, we end up with
\[
\int_{A_k} |\nabla \zeta|^2 (\psi - k) \, dx + \int_{A_k} |\psi - k|^2 \, dx
\leq C_7 \left( L + \left| f \right|_{L^\frac{2}{\theta}(\Omega;\mathbb{R}^n)} \right) \left( |v - k|^2_{L^2(\Omega;\mathbb{R}^n)} + k^2 |A_k|^{1-\frac{2}{\theta}} \right). \tag{5.17}
\]

**Step 3.** In order to use Lemma 3.2, we choose
\[
v = \psi^{\frac{1}{m}}, \quad m_0 = \sigma = l_0 = 2, \quad k_0 = 1, \quad \gamma = C_7 \left( L + \left| f \right|_{L^\frac{2}{\theta}(\Omega;\mathbb{R}^n)} \right),
\]
\[
\varepsilon_0 = \frac{2}{m} - \frac{2}{\theta}.
\]

Then, by Lemma 3.2 and using Lemma 2.1 again, we end up with
\[
\text{esssup}_{\Omega} |y| \leq C \left( m, n, \Omega, \rho, \rho_1, \rho_2, \rho_3, \left| R_{\rho} \right|_{L^\infty(\Omega)}, \left| C_{ij} \right|_{L^\theta(\Omega;\mathbb{R}^m)}, \left| D^i \right|_{L^\frac{2}{\theta}(\Omega;\mathbb{R}^n)}, \left| E^{ij} \right|_{W^{1, \infty}(\Omega)}, \left| h^{ij} \right|_{W^{1, \infty}(\Omega)}, \left| g \right|_{H^1(\Omega;\mathbb{R}^n)}, \left| f \right|_{L^\frac{2}{\theta}(\Omega;\mathbb{R}^n)}, \text{esssup}_{\Gamma} |y| \right).
\]

This completes the proof of Theorem 2.2. \qed
Remark 5.1. In order to establish the desired maximum modulus estimate for the general linear elliptic system (1.2), we impose the structure assumption (H). Now, we explain why conditions (1)–(4) in the assumption (H) are required in the proof of Theorem 2.2. First, in order to eliminate the undesired terms generated by mixed partial derivatives of second order in each equation of the system (1.2), we introduce the function $\psi = \sum_{j,l=1}^{n} h_{j,l} y_j y_l$ (in the test function), which is much more complicated than $|y|^2$ employed in the proof of Theorem 2.1. Hence, in order to obtain a maximum modulus estimate for $|y|$, we require the matrix $V = (h_{j,l})_{1 \leq j,l \leq n}$ to be uniformly positive definite, which implies the condition (1). Next, in order to establish the desired estimate (5.17), the key is to derive estimates (5.3) and (5.4). Therefore, we require that the coefficients $\sum_{i=1}^{n} a_{ij}^{pq} E_{il}$ (in the second term of the left side of (5.1)) are proportional to each other with respect to $j$ and $l$. To this aim, $E_{ij}$, $1 \leq i, j \leq n$, are given by the condition (2), and these functions are required to belong to $W^{1,\infty}(\Omega)$ (because $E_{ij}$ are the weighted functions in the test function). Further, the condition (3) is used to guarantee that the coefficient matrix in the second term of the left side of (5.5) is uniformly positive definite. Finally, notice that the first term in the left side of (5.1) contains the highest order with respect to $y$. Therefore, we need to obtain an estimate of lower bound for $|y|$ from this term. This can be guaranteed, since the condition (4) implies that $M = (\sum_{i=1}^{n} a_{ij}^{pq} E_{ij})_{1 \leq j,l \leq n}$ is uniformly positive definite.

6. An example

In this section we give an example, in which the coefficients $a_{ij}^{pq}$ ($i,j = 1, 2, \ldots, n; p,q = 1, 2, \ldots, m$) (of the system (1.2)) satisfy all of the assumptions in Theorem 2.2.

For any given functions $b_{ij} \in W^{1,\infty}(\Omega)$ and $g_{pq} \in L^{\infty}(\Omega)$ ($i,j = 1, 2, \ldots, n; p,q = 1, 2, \ldots, m$) such that $g_{pq} > 0$ in $\Omega$ and the following matrix is uniformly positive definite:

$$
G := \begin{pmatrix}
g_{11} & g_{12} & \cdots & g_{1m} \\
g_{21} & g_{22} & \cdots & g_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
g_{m1} & g_{m2} & \cdots & g_{mm}
\end{pmatrix},
$$

we take (recall (2.10) for the definition of $L_{pq}$ in terms of $(a_{ij}^{pq})_{1 \leq i,j \leq n}$)

$$a_{ij}^{pq} = b_{ij} g_{pq}, \quad h_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases} \quad \text{and} \quad f_{pq} = \frac{L_{pq}}{(g_{pq})^{n-1}}.
$$

Furthermore, write

$$B := \begin{pmatrix}
b_{11} & b_{21} & \cdots & b_{n1} \\
b_{12} & b_{22} & \cdots & b_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
b_{1n} & b_{2n} & \cdots & b_{nn}
\end{pmatrix}.
$$

Then it is easy to verify the following assertions:

(i) The condition (1) in the assumption (H) holds (because $V = I_{n \times n}$);
(ii) If $b^{ij} \in W^{1,\infty}(\Omega)$ ($i, j = 1, 2, \ldots, n$) are chosen such that

$$
\begin{cases}
    b^{11} > 0, & b^{i1} = 0, \quad i = 2, 3, \ldots, n, \quad \text{in } \Omega, \\
    b^{22} & b^{23} \quad \cdots \quad b^{2n} \\
    b^{32} & b^{33} \quad \cdots \quad b^{3n} \\
    \vdots & \vdots \quad \cdots \quad \vdots \\
    b^{n2} & b^{n3} \quad \cdots \quad b^{nn}
\end{cases}
\chi := \begin{vmatrix}
    b^{22} & b^{23} & \cdots & b^{2n} \\
    b^{32} & b^{33} & \cdots & b^{3n} \\
    \vdots & \vdots & \ddots & \vdots \\
    b^{n2} & b^{n3} & \cdots & b^{nn}
\end{vmatrix} > 0, \quad \text{in } \Omega,
$$

then, for any $p, q = 1, \ldots, m$, $L_{pq} = \det M_{pq} = (g_{pq})^m \cdot \det B = (g_{pq})^m b^{11} \chi > 0$ in $\Omega$. Therefore, the hypothesis (2.11) holds;

(iii) By the definition of $f_{pq}$ ($p, q = 1, 2, \ldots, m$), we see that for any $i, j = 1, \ldots, n$,

$$E^{ij} = \frac{f_{pq}}{L_{pq}} \sum_{l=1}^n h^{li} v^{lj}_{pq} = \frac{1}{(g_{pq})^{n-1}} v^{ji}_{pq} = \mu^{ji},$$

where $v^{ij}_{pq}$ ($i, j = 1, \ldots, n; \ p, q = 1, \ldots, m$) are defined below (2.11), $\mu^{ji}$ is the $(j, i)$th cofactor of $B$. Hence, the condition (2) in the assumption (H) is satisfied;

(iv) By the fact that $a^{ij}_{pq} = b^{ij} g_{pq}$ ($i, j = 1, \ldots, n; \ p, q = 1, \ldots, m$), one can check that the condition (2.5) is equivalent to that the following matrix is uniformly positive definite:

$$K := \begin{pmatrix}
    b^{11} G & \frac{1}{2} b^{12} G & \cdots & \frac{1}{2} b^{1n} G \\
    \frac{1}{2} b^{12} G & b^{22} G & \cdots & \frac{1}{2} (b^{2n} + b^{n2}) G \\
    \frac{1}{2} b^{13} G & \frac{1}{2} (b^{32} + b^{23}) G & \cdots & \frac{1}{2} (b^{3n} + b^{n3}) G \\
    \vdots & \vdots & \ddots & \vdots \\
    \frac{1}{2} b^{1n} G & \frac{1}{2} (b^{2n} + b^{n2}) G & \cdots & b^{nn} G
\end{pmatrix}_{nm \times nm}.$$

Hence, if $b^{ij} \in W^{1,\infty}(\Omega)$ ($i, j = 1, 2, \ldots, n$) are chosen such that for some constant $\rho^* > 0,

$$b^{ii} \geq \rho^* \quad \text{and} \quad b^{ij} \leq \frac{\rho^*}{n} \quad (i, j = 1, 2, \ldots, n; \ i \neq j), \quad \text{in } \Omega,$$

then the matrix $K$ is uniformly positive definite, and therefore the condition (2.5) holds.

(v) Noting that $E^{11} = \frac{f_{pq}}{L_{pq}} \sum_{l=1}^n h^{1l} v^{1l}_{pq} = \frac{1}{(g_{pq})^{n-1}} v^{11}_{pq} = \chi > 0$ in $\Omega$, we see that $F_{pq} = \sum_{l=1}^n a^{1l}_{pq} E^{1l} = \sum_{l=1}^n b^{1l} g_{pq} E^{l1} = b^{11} g_{pq} E^{11}$ in $\Omega$. Therefore, the matrix $F = b^{11} E^{11} G$ is uniformly positive definite, and hence conditions (3) and (4) in the assumption (H) hold true.

In summary, by the above assertions (i)–(v), suppose that the coefficients $a^{ij}_{pq}$ ($i, j = 1, 2, \ldots, n; \ p, q = 1, 2, \ldots, m$) of the system (1.2) satisfy that

$$a^{ij}_{pq} = b^{ij} g_{pq},$$
where $b^{ij} \in W^{1,\infty}(\Omega)$ ($i, j = 1, 2, \ldots, n$) and $g_{pq} \in L^{\infty}(\Omega)$ ($p, q = 1, 2, \ldots, m$) are chosen such that $g_{pq} > 0$, and $G$ is uniformly positive definite, and (6.1)–(6.2) are satisfied. Then, by Theorem 2.2, we conclude the boundedness of the weak solution to the corresponding system (1.2).

References


