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ALGEBRAIC CATEGORIES WITH FEW MONOIDAL BICLOSED STRUCTURES OR NONE

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1. Introduction

We show that the categories of magmas, of semigroups, of magmas with identity, of monoids, of groups, of rings, and of commutative rings, admit no monoidal biclosed structures whatsoever; that the category of abelian groups admits none but the classical one, and similarly for abelian monoids; and that the category of small categories admits exactly two, each symmetric, one being the classical cartesian closed structure.

The cases of groups and monoids have already been treated by the first two authors, using a different method, in [7]. The absence of a *symmetric* monoidal closed structure on these two categories, and on the categories of magmas-with-identity and rings, was shown by Rosický [17], using methods somewhat similar to ours below.

Linton [14] showed that an equational variety of algebras admits a symmetric monoidal closed structure, *in which the tensor product represents the bi-homomorphisms*, if and only if the theory is *commutative* – meaning that the operations on an algebra are themselves homomorphisms, so that every algebra is a *double algebra*. This does not prevent the algebras for a non-commutative theory from admitting some *other* symmetric monoidal closed structure: for instance, such a category has a *cartesian* closed structure if the operations are all unary.

The methods used by Day in [1], [2], and [3] to obtain monoidal biclosed structures on certain categories of functors, including such algebraic categories as those above, do not claim to find *all* such structures, except in the case of a full functor category $[\mathscr{B}, \mathscr{V}]$ with $\mathscr{V}$ closed. A general method for finding such structures on an *algebraic* category in the sense of Ehresmann [5] has been discussed by the first two authors in [7] and [8], the latter summarized in [13]. Although this full machinery is not needed for the proof of the "negative" results below, that investigation was instrumental in the discovery of them. A similar investigation, in the restricted case where the unit for the tensor product is a generator, is contained in the unpublished 1973 manuscript [16] of A. Pultr.

2. Monoidal biclosed structures on algebraic categories

In order to include, besides equational varieties of algebras, such categories as **Cat**, we are using "algebraic" in the extended sense – Freyd in [10] calls it "essentially algebraic" – to include structures where the operations are defined, not necessarily on finite products, but on finite limits: the axioms still being equational. For accounts of algebraic categories in this wider sense, see Ehresmann [5, 6], Gabriel–Ulmer [11], Isbell [12], and Diers [4]. We recall only the most basic facts.

For our purposes, a *sketch* $\mathbf{S}$ is a small category $\mathcal{S}$ in which is distinguished a small set of finite cones. A *model* of $\mathbf{S}$ in a finitely-complete category $\mathcal{B}$ is a functor $\mathcal{S} \to \mathcal{B}$ sending the given cones to limit-cones; with natural transformations as maps, these form a category $\mathcal{B}^{\mathbf{S}}$. A model of $\mathbf{S}$ in **Set** is simply called a *model of* $\mathbf{S}$; categories of the form $\mathbf{Set}^{\mathbf{S}}$ (to within equivalence) are said to be *algebraic*, and they coincide with the *locally finitely presentable* categories of [11]. They clearly include all the examples of our Introduction.

If $\mathbf{T}$ is a second sketch, it is clear that $(\mathcal{B}^{\mathbf{S}})^{\mathbf{T}}$ is of the form $\mathcal{B}^{\mathbf{S}\otimes\mathbf{T}}$ for an evident sketch $\mathbf{S}\otimes\mathbf{T}$, which is moreover isomorphic to $\mathbf{T}\otimes\mathbf{S}$. A model of $\mathbf{S}\otimes\mathbf{T}$ may be called an $(\mathbf{S},\mathbf{T})$-model: the point is that the $\mathbf{T}$-operations are homomorphisms for the $\mathbf{S}$-structure, and vice-versa. A model of $\mathbf{T}\otimes\mathbf{T}$ may be called a *double* model of $\mathbf{T}$.

A fundamental result, due to Freyd [9] in the case of equational varieties and extended by Isbell [12] even beyond the algebraic case, is the following: for an algebraic category $\mathcal{A} = \mathbf{Set}^{\mathbf{S}}$ and any complete and locally-small category $\mathcal{B}$, the category $\mathcal{B}^{\mathbf{S}}$ is equivalent to the category of adjunctions $Q \dashv P: \mathcal{B}^{\mathrm{op}} \to \mathcal{A}$. Thus to "solve" for P and Q the equation $\mathcal{A}(A, PB) \cong \mathcal{B}(B, QA)$ is just to give a model of $\mathbf{S}$ in $\mathcal{B}$. Observe the consequence that the notion of such a model depends only on the category $\mathcal{A}$, not on the particular "presentation" $\mathbf{S}$ of the theory: which justifies the common practice of calling the model an *$\mathcal{A}$-object in $\mathcal{B}$*. It may equally be called a $\mathcal{B}$-object in $\mathcal{A}$, when $\mathcal{B}$ too is an algebraic category $\mathbf{Set}^{\mathbf{T}}$; it is just an $(\mathbf{S},\mathbf{T})$-model in **Set**.

The above fundamental result has the following easy extension. If $\mathcal{A} = \mathbf{Set}^{\mathbf{S}}$ and $\mathcal{B} = \mathbf{Set}^{\mathbf{T}}$ are algebraic categories as above, and $\mathcal{C}$ is any cocomplete and locally-small category, the category $\mathcal{C}^{\mathbf{S}\otimes\mathbf{T}}$ is equivalent to the category whose objects are triples of functors $P: \mathcal{B}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$, $Q: \mathcal{C}^{\mathrm{op}} \times \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$, $R: \mathcal{A}^{\mathrm{op}} \times \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$, together with adjunction isomorphisms

$$\mathcal{A}(A, P(B, C)) \cong \mathcal{B}(B, Q(C, A)) \cong \mathcal{C}(C, R(A, B)).$$

Accordingly an $(\mathbf{S},\mathbf{T})$-model in $\mathcal{C}$ may be called an $(\mathcal{A}, \mathcal{B})$-object in $\mathcal{C}$.

It follows that to give a tensor product $\otimes: \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ on an algebraic category, with both right adjoints as in

$$\mathscr{A}(A \otimes B, C) \cong \mathscr{A}(A, [B, C]) \cong \mathscr{A}(B, \{A, C\}), \tag{2.1}$$

is precisely to give an $(\mathscr{A}, \mathscr{A})$-object (a *double $\mathscr{A}$-object*) in $\mathscr{A}^{\mathrm{op}}$. This is the starting-point of the treatment in [8]; the associativity and unit conditions, and those of coherence, impose progressive restrictions on the double-$\mathscr{A}$-object in $\mathscr{A}^{\mathrm{op}}$, and the calculations are the more manageable in proportion as the sketch **S** is small. We give no more details here in this generality, having already gone further than is necessary for our negative results below.

3. The single-sorted case

In all the examples of algebraic categories $\mathscr{A} = \mathbf{Set}^{\mathbf{S}}$ that we are to consider, the structure is single-sorted, so that a model of **S** in $\mathscr{B}$ is carried by a single object, and an $\mathscr{A}$-object in $\mathscr{B}$ really is an object of $\mathscr{B}$ with some extra structure. In such cases it is convenient to use that familiar concrete language which mentions the carrier and understands the structure.

Syntactically, we have a single-sorted structure if **S** contains some object for which the corresponding evaluation $U: \mathbf{Set}^{\mathbf{S}} \to \mathbf{Set}$ reflects isomorphisms; then the object E of $\mathscr{A} = \mathbf{Set}^{\mathbf{S}}$ representing U is a strong generator. Semantically, and independently of the presentation **S** of the theory, an algebraic category $\mathscr{A}$ is said to be *single-sorted* if it contains a strong generator E; whereupon the "forgetful functor" may be taken as $U = \mathscr{A}(E, -): \mathscr{A} \to \mathbf{Set}$. In the varietal examples of our Introduction, we take for E the free algebra on one generator in the usual sense, so that U is the usual forgetful functor; in the case of **Cat** we take for E the arrow-category **2**, so that U is "set of morphisms".

There is now yet a third way of describing an $\mathscr{A}$-object in $\mathscr{B}$: namely as an object B of $\mathscr{B}$ together with a lifting, through $U: \mathscr{A} \to \mathbf{Set}$, of the representable functor $\mathscr{B}(-, B): \mathscr{B}^{\mathrm{op}} \to \mathbf{Set}$.

Since $U = \mathscr{A}(E, -)$ lifts through itself, E has a *canonical* structure of $\mathscr{A}$-object in $\mathscr{A}^{\mathrm{op}}$ (cf. [9]); this corresponds of course to the identity adjunction $\mathscr{A} \to \mathscr{A}$. For any A in $\mathscr{A}$, the left-exact functor $\mathscr{A}(-, A): \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ carries the $\mathscr{A}$-object E of $\mathscr{A}^{\mathrm{op}}$ to the $\mathscr{A}$-object $\mathscr{A}(E, A) = UA$ of **Set** whose structure is just that of A as an algebra.

If we now have a tensor product on $\mathscr{A}$, with both right adjoints as in (2.1), the cocontinuous functors $A \otimes -$ and $- \otimes B$ carry the coalgebra E (meaning co-$\mathscr{A}$-object in $\mathscr{A}$, or $\mathscr{A}$-object in $\mathscr{A}^{\mathrm{op}}$) to coalgebras $A \otimes E$ and $E \otimes A$. This gives two coalgebra structures (right and left respectively) on $E \otimes E$, each of which is given by homomorphisms for the other, so that $E \otimes E$ is a double coalgebra: this is the double-$\mathscr{A}$-object in $\mathscr{A}^{\mathrm{op}}$ of Section 2. Of course any copower of $E \otimes E$ is also a double coalgebra.

Suppose now that $\otimes$ is to have a two-sided unit I (to within coherent isomorphisms). Because E is a strong generator, there is a strong epimorphism $p: k \cdot E \to I$ from some copower $k \cdot E$ of E. Applying to p the left-adjoint functors $E \otimes -$ and $- \otimes E$, and using the isomorphisms $E \otimes I \cong E$ and $I \otimes E \cong E$, we get strong epimorphisms

$$q_1, q_2 : k \cdot (E \otimes E) \to E, \tag{3.1}$$

which are homomorphisms of coalgebras for the left and right structures, respectively, on $k \cdot (E \otimes E)$.

Even the existence of a *right* unit I for $\otimes$, with no other conditions, places heavy restrictions on the possibilities for $\mathscr{A}$. For then we still have the coalgebra epimorphism q_1 of (3.1); applying to this the left-exact $\mathscr{A}(-, A): \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ for any algebra A, we get a monomorphism of algebras $\mathscr{A}(E, A) \to \mathscr{A}(k \cdot (E \otimes E), A)$. But the domain here is A as an algebra, while the codomain is a double algebra. Hence

Proposition 1. *If a one-sorted algebraic category admits a monoidal biclosed structure, every algebra admits an embedding into (one of the structures of) a double algebra.*

What Rosický showed in [17], under the extra hypothesis of *symmetry* for the tensor product, was that every algebra must embed into a double one with a special property: namely that the two structures are isomorphic under an involutory permutation of the underlying set.

4. The varietal examples

In an equational variety, all nullary operations must coincide in a double algebra; hence by Proposition 1 such a category $\mathscr{A}$ can have no monoidal biclosed structure if there is more than one nullary operation. This cuts out the varieties of rings and commutative rings.

Again, by a simple classical argument, in a double magma-with-identity, the two multiplications coincide and are associative and commutative. So Proposition 1 also cuts out the varieties of magmas-with-identity, of monoids, and of groups.

Since every magma or semigroup does admit a double structure, with the second multiplication given by $xy = x$, something more than Proposition 1 is needed to show the non-existence of a monoidal biclosed structure on these varieties. Call an algebra A in a variety $\mathscr{A}$ *idempotent* if, for every operation ω and every element $a \in A$, we have $\omega(a, a, \ldots, a) = a$; of course non-trivial idempotent algebras can exist only if there are no nullary operations. Call the algebra A *self-commuting* if each of the operations $A^n \to A$ on A is a homomorphism: so that A is a double algebra in which the two structure coincide. Then we have:

Proposition 2. *If an equational variety admits a monoidal biclosed structure, every idempotent algebra is self-commuting.*

Proof. The empty algebra 0 (if it exists) is the initial object, so that it is preserved by the left-exact $A\otimes -$, giving $A\otimes 0 = 0$. Thus the unit I for $\otimes$ is not empty, and there is at least one homomorphism $f: E\to I$. From this we get a coalgebra map

$$E\otimes E \xrightarrow[1\otimes f]{} E\otimes I \xrightarrow[r]{\cong} E$$

with respect to the left structure on $E\otimes E$; and applying $\mathscr{A}(-, A)$ turns this into an algebra map

$$\mathscr{A}(E, A)\to \mathscr{A}(E\otimes E, A). \tag{4.1}$$

Now suppose A is idempotent. Then any constant function from an algebra to A is a homomorphism; and since a homomorphism $E\to A$ is uniquely determined by the image of the generator of E, the set $\mathscr{A}(E, A)$ consists exactly of the constant functions. It follows that (4.1) is the canonical bijection of $\mathscr{A}(E, A)$ onto the subset of $\mathscr{A}(E\otimes E, A)$ consisting of the constant functions. Hence this subset is a subalgebra, canonically isomorphic to A, of the left structure on $\mathscr{A}(E\otimes E, A)$.

This used the fact that I is a right unit for $\otimes$; a similar argument using that it is a *left* unit gives the corresponding conclusion, but now with the *right* structure on $\mathscr{A}(E\otimes E, A)$. Hence A is self-commuting. □

A self-commuting magma, if it has an identity, is necessarily commutative, by the classical argument mentioned above. But the semigroup $\{e, x, y\}$, in which e is an identity, x and y are idempotents, $xy = x$, and $yx = y$, is not commutative. Thus by Proposition 2 the varieties of magmas and of semigroups do not admit monoidal biclosed structures.

We turn now to the case of abelian groups; that of abelian monoids is entirely similar. Here E is $\mathbf{Z}$; let us write $*$ for the classical tensor product. Since every abelian group is canonically a double algebra, we can use *any* object for $\mathbf{Z}\otimes\mathbf{Z}$ to get a tensor product satisfying (2.1). The value of $A\otimes B$ can then be written explicitly by two applications of the Eilenberg–Watts theorem, according to which any left-adjoint functor $T:\mathbf{Ab}\to\mathbf{Ab}$ is of the form $T(\mathbf{Z})*-$; we get $A\otimes B = A*(\mathbf{Z}\otimes\mathbf{Z})*B$. If I is a right unit for $\otimes$, we have in particular $\mathbf{Z}\cong\mathbf{Z}\otimes I\cong(\mathbf{Z}\otimes\mathbf{Z})*I$. It is easy, however, to show that, if $\mathbf{Z}\cong C*D$, then $C\cong\mathbf{Z}$; thus $\mathbf{Z}\otimes\mathbf{Z}\cong\mathbf{Z}$, and $A\otimes B\cong A*B$. Hence:

Proposition 3. *The variety of abelian groups admits no monoidal biclosed structure other than the classical one; and similarly for abelian monoids.*

5. The example of categories

Let $\mathscr{A}$ be any category with a bifunctor $\otimes:\mathscr{A}\times\mathscr{A}\to\mathscr{A}$, an object I, and natural isomorphisms $l_A: I\otimes A\cong A$ and $r_A: A\otimes I\cong A$, satisfying the "coherence" condition

$l_I = r_I : I \otimes I \to I$. (It is in fact always possible to force this last condition by changing one of l and r.) Consider the monoid End $1_{\mathscr{A}}$ of endomorphisms of the identity functor $1_{\mathscr{A}}: \mathscr{A} \to \mathscr{A}$, and the monoid End I of endomorphisms of I in $\mathscr{A}$. There is a monoid-homomorphism End $1_{\mathscr{A}} \to$ End I sending $\alpha : 1_{\mathscr{A}} \to 1_{\mathscr{A}}$, with components $\alpha_A : A \to A$, to its component $\alpha_I : I \to I$. There is also a monoid-homomorphism End $I \to$ End $1_{\mathscr{A}}$ sending $f : I \to I$ to f^*, with A-component f_A^* given by

$$A \xrightarrow[r_A^{-1}]{} A \otimes I \xrightarrow[1 \otimes f]{} A \otimes I \xrightarrow[r_A]{} A.$$

The fact that $r_I = l_I$, together with the naturality of l, then gives $f_I^* = f$. Hence End $I \to$ End $1_{\mathscr{A}}$ is a coretration of monoids, and in particular a monomorphism. (For closely related arguments, cf. Pultr [15].)

In the case $\mathscr{A} = \mathbf{Cat}$, End $1_{\mathscr{A}}$ consists of the identity alone. For if $\alpha \in$ End $1_{\mathscr{A}}$, the naturality of α with respect to functors $\mathbf{1} \to A$ implies that α_A is the identity on objects; and then its naturality with respect to functors $\mathbf{2} \to A$ implies that it is the identity on maps.

Now suppose that $\otimes$ gives a monoidal biclosed structure on **Cat**. It follows from the above that *the unit I for $\otimes$ must be the unit category* $\mathbf{1}$; for the empty category $\mathbf{0}$ is initial, and so (as in the proof of Proposition 2) cannot be the unit, while every category other than $\mathbf{0}$ and $\mathbf{1}$ has at least two endomorphisms (the identity and the constant functor at some object).

The canonical cocategory structure in **Cat** on $E = \mathbf{2}$ is of course that whose object of objects is $\mathbf{1}$, whose object of morphisms is $\mathbf{2}$, whose object of composable pairs is $\mathbf{3}$, and whose structural maps constitute the full subcategory $\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ of **Cat**. Since $I = \mathbf{1}$, the left cocategory structure on $\mathbf{2} \otimes \mathbf{2}$ has $\mathbf{2}$ for its object of objects; so that, if the object of morphisms $\mathbf{2} \otimes \mathbf{2}$ is the category M, the object of composable pairs is the fibred coproduct $N = M \amalg_{\mathbf{2}} M$. The syntactic requirement, that the "composition" operation $M \to N$ be right inverse to each of the operations $N \to M$ derived from the "identity" operation $M \to \mathbf{2}$, places heavy restrictions on M; it is in fact a simple exercise to show that M is either the cartesian product $\mathbf{2} \times \mathbf{2}$ in **Cat**, or else the full subcategory of this obtained by omitting the unique map $(0, 0) \to (1, 1)$. That is, M is either the free-living commutative square or the free-living non-commutative square.

Each of these possibilities does give a double cocategory structure $\mathbf{2} \otimes \mathbf{2}$; and only one, to within isomorphism. We conclude that:

Proposition 4. **Cat** *has exactly two monoidal biclosed structures, each symmetric. One is the cartesian closed structure, for which the internal hom is the category of functors and natural transformations; for the other, the internal hom is the category of functors and "transformations"* $\{\alpha_A : TA \to SA\}$ *with no naturality requirement.*

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