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European Journal of Combinatorics 28 (2007) 214–227
European Journal of Combinatorics
www.elsevier.com/locate/ejc

A q-analog of dual sequences with applications

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Received 5 March 2005; accepted 21 July 2005
Available online 22 August 2005

Abstract

In the present paper combinatorial identities involving q-dual sequences or polynomials with coefficients that are q-dual sequences are derived. Further, combinatorial identities for q-binomial coefficients (Gaussian coefficients), q-Stirling numbers and q-Bernoulli numbers and polynomials are deduced.

MSC: primary 05A30; secondary 33D99

1. Introduction

Given a sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of elements of a commutative ring R (for example, the complex numbers, polynomials or rational functions), one usually describes as the Euler–Seidel matrix associated with (a_n) the double sequence (a_n^k) $(n \geq 0, k \geq 0)$ given by the recurrence [7]

$$a_n^0 = a_n, \quad a_n^k = a_n^{k-1} + a_{n+1}^{k-1} \quad (k \geq 1, n \geq 0).$$

The sequence (a_n^0) of the first row of the matrix is the *initial sequence*. The sequence (a_0^n) of the first column of the matrix is the *final sequence*. Such a matrix is equivalent to the table obtained by computing the finite difference of consecutive terms of (a_0^n) and iterating

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0195-6698/$ - see front matter
doi:10.1016/j.ejc.2005.07.007

the procedure. One passes from the initial sequence to the last one and conversely through

$$a_0^n = \sum_{i=0}^{n} \binom{n}{i} a_i^0 \Longleftrightarrow a_n^0 = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a_0^i. \tag{1}$$

If one sets $a_n = (-1)^n a_n^0$ and $a_n^* = (-1)^n a_0^n$, then the above relations can be written as

$$a_n^* = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i \Longleftrightarrow a_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i^*. \tag{2}$$

In [12] the sequence (a_n^*) is called the *dual sequence* of (a_n). It is well known that if $a_n = (-1)^n B_n$, where $(B_n) = (1, -1/2, 1/6, 0, -1/30, \ldots)$ is the sequence of Bernoulli numbers, then $a_n^* = a_n$, that is $((-1)^n B_n)$ is *self-dual*. Generalizing the results of Kaneko [10] and Momiyama [11] on Bernoulli numbers, Sun [12] has recently proved some remarkable identities on dual sequences. Other generalizations of Kaneko's identity have been obtained by Gessel [9] using umbral calculus.

The aim of this paper is to give a q-version of Sun's results in [12]. In the last two decades there has been an increasing interest in generalizing the classical results with a generic parameter q, which is the so-called phenomenon of "q-disease". As regards the Euler–Seidel matrix, Clarke et al. [6] have given a q-analog of (1) with application to q-enumeration of derangements.

We shall need some standard q-notation, which can be found in Gasper and Rahman's book [8]. The q-shifted factorial $(a;q)_n$ is defined by $(a;q)_0 = 1$ and

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

if n is a positive integer. For $k \in \mathbb{Z}$ the k-integer $[k]_q$ is defined by $[k]_q = \frac{1-q^k}{1-q}$, so $[-k]_q = -q^{-k}[k]_q$. For integer k, the q-binomial coefficient $\begin{bmatrix}\alpha\\k\end{bmatrix}$ is defined by $\begin{bmatrix}\alpha\\k\end{bmatrix} = 0$ if $k < 0$ and

$$\begin{bmatrix}\alpha\\k\end{bmatrix} = \frac{(1-q^\alpha)(1-q^{\alpha-1})\cdots(1-q^{\alpha-k+1})}{(q;q)_k}$$

if k is a positive integer. Let (a_n) be a sequence of a commutative ring. We call the sequence (a_n^*) given by

$$a_n^* = \sum_{i=0}^{n} \begin{bmatrix}n\\i\end{bmatrix} (-1)^i a_i q^{\binom{i}{2}} \tag{3}$$

the q-*dual sequence* of (a_n). By Gauss inversion [1, p. 96] we get

$$a_n = \sum_{r=0}^{n} \begin{bmatrix}n\\r\end{bmatrix} (-1)^r a_r^* q^{\binom{r+1}{2}-nr}. \tag{4}$$

We will need the following q-analog of the binomial formula [3, p. 36]:

$$(z;q)_n = \sum_{j=0}^{n} \begin{bmatrix}n\\j\end{bmatrix} (-1)^j z^j q^{\binom{j}{2}}, \tag{5}$$

and the q-Chu–Vandermonde formula [8, p. 354]:

$$ {}_2\Phi_1\left[\begin{matrix} q^{-n}, a \\ c \end{matrix}; q, q\right] := \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k (a;q)_k}{(c;q)_k} \frac{z^k}{(q;q)_k} = \frac{(c/a;q)_n}{(c;q)_n} a^n. \tag{6} $$

The following is our basic theorem.

Theorem 1. *For $k, l \in \mathbb{N}$ the following identities hold true:*

$$ \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} \frac{(-1)^j a^*_{k+j+1}}{[k+j+1]_q} q^{\binom{j+1}{2} - l(k+j+1)} + \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(-1)^j a_{l+j+1}}{[l+j+1]_q} q^{\binom{j+1}{2}} = \frac{a_0}{[k+l+1]_q \begin{bmatrix} k+l \\ k \end{bmatrix}}, \tag{7} $$

$$ \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} (-1)^j a^*_{k+j} q^{\binom{j+1}{2} - l(k+j)} = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j a_{l+j} q^{\binom{j}{2}}, \tag{8} $$

$$ \sum_{j=0}^{l+1} \begin{bmatrix} l+1 \\ j \end{bmatrix} (-1)^{j+1} [k+j+1]_q a^*_{k+j} q^{\binom{j}{2} - l(k+j) - k} = \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} (-1)^j [l+j+1]_q a_{l+j} q^{\binom{j-1}{2}}. \tag{9} $$

The above theorem is a q-analog of Theorem 2.1 in Sun [12]. Note also that Eq. (8) was also a q-analog of Theorem 7.4 in Gessel [9].

The rest of this paper will be organized as follows: we prove Theorem 1 in Section 2 and present a q-analog of Sun's main theorem in Section 3. In Section 4, we present some interesting examples as applications of our Theorems 1 and 2.

2. Proof of Theorem 1

Plugging (3) into the first sum of the left-hand side of (7), we have

$$ LHS = a_0 B + \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j \frac{a_{l+j+1}}{[l+j+1]_q} q^{\binom{j+1}{2}} + C, \tag{10} $$

where

$$ B = \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} (-1)^j \frac{q^{\binom{j+1}{2} - l(k+j+1)}}{[k+j+1]_q}, $$

and

$$ C = \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} (-1)^j \frac{q^{\binom{j+1}{2} - l(k+j+1)}}{[k+j+1]_q} \sum_{i=1}^{k+j+1} \begin{bmatrix} k+j+1 \\ i \end{bmatrix} (-1)^i a_i q^{\binom{i}{2}}. $$

It is known (see [13] for further applications) that

$$\frac{1}{(x+a_0)(x+a_1)\cdots(x+a_l)}=\sum_{j=0}^{l}\frac{\prod_{\substack{i=0\\ i\neq j}}^{l}(a_i-a_j)^{-1}}{x+a_j}. \tag{11}$$

Setting $x=-q^{-k-1}$ and $a_i=q^i$ $(0\le i\le l)$ in (11) we obtain

$$\sum_{j=0}^{l}(-1)^j\frac{q^{\binom{j+1}{2}-l(k+j+1)}}{(q;q)_j(q;q)_{l-j}(1-q^{k+j+1})}=\frac{1}{(q^{k+1};q)_{l+1}}. \tag{12}$$

It follows that

$$B=\frac{(1-q)(q;q)_l}{(q^{k+1};q)_{l+1}}=\frac{1}{[k+l+1]_q\begin{bmatrix}k+l\\ k\end{bmatrix}}.$$

Exchanging the order of summation we can rewrite C as follows:

$$\begin{aligned}C&=\sum_{i=1}^{k+l+1}(-1)^i\frac{a_i}{[i]_q}q^{\binom{i}{2}}\sum_{j=i-k-1}^{l}(-1)^j\begin{bmatrix}l\\ j\end{bmatrix}\begin{bmatrix}k+j\\ i-1\end{bmatrix}q^{\binom{j+1}{2}-lj-(k+1)l}\\ &=\sum_{i=1}^{k}(-1)^i\frac{a_i}{[i]_q}q^{\binom{i}{2}}\begin{bmatrix}k\\ i-1\end{bmatrix}{}_2\Phi_1\left[\begin{matrix}q^{-l},q^{k+1}\\ q^{k-i+2}\end{matrix};q,q\right]q^{-(k+1)l}.\end{aligned}$$

Applying the q-Chu–Vandermonde formula (6) we obtain

$$\begin{aligned}C&=\sum_{i=1}^{k}(-1)^i\frac{a_i}{[i]_q}q^{\binom{i}{2}}\begin{bmatrix}k\\ i-1\end{bmatrix}\frac{(q^{-i+1};q)_l}{(q^{k-i+2};q)_l}\\ &=\sum_{i=1}^{k}(-1)^{i+l}\begin{bmatrix}k\\ i-l-1\end{bmatrix}\frac{a_i}{[i]_q}q^{\binom{i}{2}+\binom{l+1}{2}-il}\\ &=\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}(-1)^{j+1}\frac{a_{l+j+1}}{[l+j+1]_q}q^{\binom{j+1}{2}}.\end{aligned}$$

Substituting the values of B and C into (10) yields (7).

To derive (8) and (9) from (7) we define the linear operator δ_q by

$$\delta_q(a_n)=-q^{1-n}[n]_q a_{n-1}\qquad\text{for } n\ge 0.$$

Then $\delta_q(a_n^*)=[n]_q a_{n-1}^*$. Indeed,

$$\begin{aligned}\delta_q(a_n^*)&=\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}(-1)^i\delta_q(a_i)q^{\binom{i}{2}}=\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}(-1)^{i+1}q^{1-i}[i]_q a_{i-1}q^{\binom{i}{2}}\\ &=[n]_q\sum_{i=0}^{n}(-1)^i\begin{bmatrix}n-1\\ i-1\end{bmatrix}(-1)^{i-1}a_{i-1}q^{\binom{i-1}{2}}=[n]_q a_{n-1}^*.\end{aligned}$$

Now, applying δ_q to (7) yields (8). Furthermore, replacing k by $k+1$ and l by $l+1$ in (8) then applying δ_q on both sides yields (9).

Remark. We can also prove (8) and (9) directly by using the q-Chu–Vandermonde formula.

3. A q-analog of Sun's main theorem

In this section, we assume that x, y and z are commuting indeterminates. Define $[x, y]^n$ by $[x, y]^0 = 1$ and

$$[x, y]^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} x^i y^{n-i}$$

for positive integer n. So $[x, y]^n = (x + y)^n$ when $q = 1$. Similarly

$$[x, y, z]^n = [x, [y, z]]^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} x^i [y, z]^{n-i} = \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \frac{[n]_q!}{[i]_q![j]_q![k]_q!} x^i y^j z^k,$$

and hence $[x, y, z]^n$ is a symmetric polynomial of x, y, z and $[x, y, z]^n = (x + y + z)^n$ when $q = 1$.

Like the definition of Bernoulli polynomials, we introduce

$$A_n(x) = \sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} a_i q^{\binom{i}{2}} x^{n-i} \quad \text{and} \quad A_n^*(x) = \sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} a_i^* x^{n-i}.$$

The following is our q-analog of the main theorem of Sun [12, Th. 1.1].

Theorem 2. *Let* $k, l \in \mathbb{N}$*; then*

$$\begin{aligned} &(-1)^l \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} x^{l-j} \frac{A_{k+j+1}^*(z)}{[k+j+1]_q} q^{-kj-\binom{k+1}{2}} \\ &\quad + (-1)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} x^{k-j} \frac{A_{l+j+1}([1, -z, -x])}{[l+j+1]_q} q^{\binom{j+1}{2}-k(l+j+1)} \\ &= \frac{a_0(-x)^{k+l+1}}{[k+l+1] \begin{bmatrix} k+l \\ k \end{bmatrix}}. \end{aligned} \tag{13}$$

$$\begin{aligned} &(-1)^l \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} x^{l-j} A_{k+j}^*(z) q^{k(l-j)} \\ &= (-1)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} x^{k-j} A_{l+j}([1, -z, -x]) q^{\binom{k-j}{2}}. \end{aligned} \tag{14}$$

$$\begin{aligned} &(-1)^{l+1} \sum_{j=0}^{l+1} \begin{bmatrix} l+1 \\ j \end{bmatrix} x^{l+1-j} [k+j+1]_q A_{k+j}^*(z) q^{(k+1)(l-j)+1} \\ &= (-1)^k \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} x^{k+1-j} [l+j+1]_q A_{l+j}([1, -z, -x]) q^{\binom{k-j}{2}-j}. \end{aligned} \tag{15}$$

Proof. We derive from (4) and (5) that

$$\begin{aligned}
A_n([1,-x]) &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i a_i [1,-x]^{n-i} q^{\binom{i}{2}} \\
&= \sum_{i,j,s\geq 0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} (-1)^{i-j} a_j^* \begin{bmatrix} n-i \\ s \end{bmatrix} (-x)^s q^{\binom{i-j}{2}} \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} a_j^* \sum_{i,s\geq 0} \begin{bmatrix} n-j \\ s \end{bmatrix} (-1)^s x^s \begin{bmatrix} n-j-s \\ i-j \end{bmatrix} (-1)^{i-j} q^{\binom{i-j}{2}} \\
&= (-1)^n \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j a_j^* x^{n-j} \\
&= (-1)^n A_n^*(x),
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
A_n([1,-z,-x]) &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i a_i [1,-z,-x]^{n-i} q^{\binom{i}{2}} \\
&= \sum_{i,j\geq 0} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n-i \\ j \end{bmatrix} (-1)^{i+j} x^j a_i [1,-z]^{n-i-j} q^{\binom{i}{2}} \\
&= \sum_{i,j\geq 0} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j x^j \begin{bmatrix} n-j \\ i \end{bmatrix} (-1)^i a_i [1,-z]^{n-i-j} q^{\binom{i}{2}} \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j x^j A_{n-j}([1,-z]) \\
&= (-1)^n \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} x^j A_{n-j}^*(z).
\end{aligned} \tag{17}$$

Denote the first sum of the left-hand side in (13) by $\mathcal{C}$. Applying (16) and (17), the left-hand side of (13) is equal to

$$\begin{aligned}
&(-1)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{x^{k-j}}{[l+j+1]_q} q^{\binom{j+1}{2}-k(l+j+1)} \sum_{i=0}^{l+j+1} \begin{bmatrix} l+j+1 \\ i \end{bmatrix} \\
&\quad \times A_i([1,-z])(-x)^{l+j+1-i} + \mathcal{C} = a_0(-x)^{k+l+1}\mathcal{B} + S + \mathcal{C},
\end{aligned} \tag{18}$$

where

$$\mathcal{B} = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j \frac{q^{\binom{j+1}{2}-k(l+j+1)}}{[l+j+1]_q},$$

and

$$S = (-1)^{k+l+1} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}-k(l+j+1)} \sum_{i=1}^{l+j} \begin{bmatrix} l+j \\ i-1 \end{bmatrix} x^{k+l+1-i} \frac{A_i^*(z)}{[i]_q}.$$

Exchanging k and l in (12) yields

$$\mathcal{B} = \frac{1}{[k+l+1]_q \begin{bmatrix} k+l \\ k \end{bmatrix}}.$$

Now, we show that $S = -\mathcal{C}$. Exchanging the order of summation we have

$$\begin{aligned} S &= (-1)^{k+l+1} \sum_{i=1}^{l} \begin{bmatrix} l \\ i-1 \end{bmatrix} \frac{A_i^*(z)}{[i]_q} x^{k+1+l-i} \; {}_2\Phi_1 \left[\begin{matrix} q^{-k}, q^{l+1} \\ q^{l-i+2} \end{matrix} ; q, q \right] q^{-k(l+1)} \\ &= (-1)^{k+l+1} \sum_{i=1}^{l} \begin{bmatrix} l \\ i-1 \end{bmatrix} \frac{A_i^*(z)}{[i]_q} x^{k+l-i} \frac{(q^{-i+1};q)_k}{(q^{l-i+2};q)_k} \\ &\quad \text{(by } q\text{-Chu–Vandemonde)} \\ &= (-1)^{l+1} \sum_{i=1}^{l} \begin{bmatrix} l \\ i-k-1 \end{bmatrix} \frac{A_i^*(z)}{[i]_q} x^{k+1+l-i} q^{-ik+\binom{k+1}{2}} \\ &= (-1)^{l+1} \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} \frac{A_{k+j+1}^*(z)}{[k+j+1]_q} x^{l-j} q^{-jk-\binom{k+1}{2}} = -\mathcal{C}. \end{aligned}$$

Next, the right-hand side of (14) is equal to

$$\begin{aligned} &(-1)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} x^{k-j} \sum_{i=0}^{l+j} \begin{bmatrix} l+j \\ i \end{bmatrix} (-x)^{l+j-i} A_i([1,-z]) q^{\binom{k-j}{2}} \\ &= (-1)^{k+l} \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix} x^{k+l-i} A_i^*(z) \; {}_2\Phi_1 \left[\begin{matrix} q^{-k}, q^{l+1} \\ q^{l-i+1} \end{matrix} ; q, q \right] q^{\binom{k}{2}} \\ &= (-1)^{k+l} \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix} x^{k+l-i} A_i^*(z) \frac{(q^{-i};q)_k}{(q^{l-i+1};q)_k} q^{\binom{k}{2}+k(l+1)} \\ &= (-1)^{l} \sum_{i=0}^{l} \begin{bmatrix} l \\ i-k \end{bmatrix} x^{k+l-i} A_i^*(z) q^{-ik+k^2+kl} \\ &= (-1)^{l} \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} x^{l-j} A_{k+j}^*(z) q^{k(l-j)}, \end{aligned}$$

which is exactly the left-hand side of (14).

Finally, exchanging the order of summation, the right-hand side of (15) can be written as

$$\begin{aligned} R = (-1)^k \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} x^{k+1-j} [l+j+1]_q \\ \times \sum_{i=0}^{l+j} \begin{bmatrix} l+j \\ i \end{bmatrix} (-x)^{l+j-i} A_i([1,-z]) q^{\binom{k-j}{2}-j} \end{aligned}$$

$$= (-1)^{k+l} \sum_{i,j\geq 0} \begin{bmatrix} k+1 \\ j \end{bmatrix} x^{k+l+1-i}[i+1]_q (-1)^j \begin{bmatrix} l+j+1 \\ i+1 \end{bmatrix} A_i^*(z) q^{\binom{k-j}{2}-j}$$

$$= (-1)^{k+l} \sum_{i=0}^{l+1} \begin{bmatrix} l+1 \\ i \end{bmatrix} x^{k+l+1-i}[i+1]_q A_i^*(z)\, {}_2\Phi_1 \left[\begin{matrix} q^{-k-1}, q^{l+2} \\ q^{l-i+1} \end{matrix} ; q, q \right] q^{\binom{k}{2}}.$$

By the q-Chu–Vandermonde formula we have

$$R = (-1)^{k+l} \sum_{i=0}^{l+1} \begin{bmatrix} l+1 \\ i \end{bmatrix} x^{k+l+1-i}[i+1]_q A_i^*(z) \frac{(q^{-i-1};q)_{k+1}}{(q^{l-i+1};q)_{k+1}} q^{\binom{k}{2}+(k+1)(l+2)}$$

$$= (-1)^{l+1} \sum_{i=0}^{l+1} \begin{bmatrix} l+1 \\ i-k \end{bmatrix} x^{k+l+1-i}[i+1]_q A_i^*(z) q^{(k+1)(l+2-i)+k^2-k-1}$$

$$= (-1)^{l+1} \sum_{j=0}^{l+1} \begin{bmatrix} l+1 \\ j \end{bmatrix} x^{l+1-j}[k+j+1]_q A_{k+j}^*(z) q^{(k+1)(l-j)+1},$$

which is exactly the left-hand side of (15). □

Remark. When $q = 1$, Theorems 1 and 2, which correspond to Theorems 2.2 and 1.1 of Sun [12], are actually equivalent. Indeed, in such a case, we have

$$(-1)^n A_n^*(1-x) = A_n(x), \tag{19}$$

which can be verified as follows:

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} a_i^*(x-1)^{n-i} &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \sum_{j=0}^{i} \binom{i}{j} (-1)^j a_j (1-x)^{n-i} \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j a_j \sum_{i=j}^{n} \binom{n-j}{i-j} (x-1)^{n-i} \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j a_j x^{n-j}. \end{aligned}$$

Now, taking $a_n = (-1)^{l+k+n} x^{k+l-n} A_n(y)$ with $q = 1$,

$$\begin{aligned} a_n^* &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{l+k} x^{k+l-i} A_i(y) \\ &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{l+k} x^{k+l-i} \sum_{j=0}^{i} \binom{i}{j} (-1)^j a_j y^{i-j} \\ &= (-1)^{l+k+n} x^{k+l-n} \sum_{j=0}^{n} \binom{n}{j} a_j (-1)^{n-j} \sum_{i=j}^{n} \binom{n-j}{i-j} x^{n-i} y^{i-j} \\ &= (-1)^{l+k} x^{k+l-n} A_n(x+y). \end{aligned}$$

It follows from (19) that

$$a_n^* = (-1)^{k+l+n} x^{k+l-n} A_n^*(1-x-y).$$

Substituting the above values of a_n and a_n^* in Theorem 1 we obtain Theorem 2. Conversely, it is easy to see that Theorem 1 is a special case of Theorem 2 because

$$A_n(0) = (-1)^n a_n, \qquad A_n(1) = a_n^*.$$

Hence we have proved that Theorems 1 and 2 are actually equivalent when $q = 1$.

4. Some applications

In this section we derive some examples from our main theorem; most of them are q-analogs of results in Sun [12].

Example 1. For any fixed integer $i \geq 0$ let $a_n = (-1)^n \begin{bmatrix} n \\ i \end{bmatrix} t^{n-i} q^{\binom{i}{2}}$; then it follows from (3) and (5) that

$$\begin{aligned} a_n^* &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} t^{k-i} q^{\binom{i}{2}+\binom{k}{2}} \\ &= \begin{bmatrix} n \\ i \end{bmatrix} q^{i^2-i} \sum_{k=i}^{n} \begin{bmatrix} n-i \\ k-i \end{bmatrix} (tq^i)^{k-i} q^{\binom{k-i}{2}} \\ &= \begin{bmatrix} n \\ i \end{bmatrix} (-tq^i; q)_{n-i} q^{i^2-i}. \end{aligned}$$

Substituting the above values in (8) of Theorem 1 yields

$$\sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} \begin{bmatrix} k+j \\ i \end{bmatrix} (-1)^{l-j} (-q^i t; q)_{k+j-i} q^{j(j+1)/2 - lj + \binom{i}{2}} = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} l+j \\ i \end{bmatrix} t^{l+j-i} q^{kl+j(j-1)/2}. \tag{20}$$

For variations of methods, we will give two more proofs of (20). Note that when $q = 1$ Eq. (20) reduces to a crucial result of Sun [12, Lemma 3.1], which was proved by using a derivative operator.

We first q-generalize Sun's proof by using a q-derivative operator. For any polynomial $f(t)$ in t, let D_q be the q-derivative operator with respect to t:

$$D_q f(t) = \frac{f(tq) - f(t)}{(q-1)t}.$$

Clearly we have

$$D_q t^n = \frac{q^n - 1}{q-1} t^{n-1}, \qquad D_q((-t; q)_n) = [n]_q (-qt; q)_{n-1}.$$

For integer $i \geq 0$ define $[i]_q! = \prod_{j=0}^{i} [j]_q$; then

$$D_q^i (t^n) = [i]_q! \begin{bmatrix} n \\ i \end{bmatrix} t^{n-i}, \tag{21}$$

$$D_q^i ((-t; q)_n) = q^{i(i-1)/2} [i]_q! \begin{bmatrix} n \\ i \end{bmatrix} (-q^i t; q)_{n-i}. \tag{22}$$

By Gauss inversion, the q-binomial formula (5) is equivalent to

$$z^n = \sum_{j=0}^{n} (-1)^j q^{\binom{j+1}{2}-nj} \begin{bmatrix} n \\ j \end{bmatrix} (z;q)_j.$$

Replacing z by $-tq^k$ we get

$$(tq^k)^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^{n-j} q^{j(j+1)/2-nj} (-tq^k;q)_j. \tag{23}$$

Now, using the q-derivative operator and (21)–(23), we can write the difference of the two sides of (20) as follows:

$$\begin{aligned} &\frac{1}{[i]_q!} D_q^i \Bigg((-t;q)_k \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} (-1)^{l-j} q^{j(j+1)/2-lj} (-tq^k;q)_j \\ &\qquad - (tq^k)^l \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{j(j-1)/2} t^j \Bigg) \\ &\quad = \frac{1}{[i]_q!} D_q^i \left((-t;q)_k (tq^k)^l - (tq^k)^l (-t;q)_k \right), \end{aligned}$$

which is clearly equal to 0.

Our second proof of (20) uses the machinery of basic hypergeometric functions. Rewriting (20) in terms of basic hypergeometric functions, we have

$$\begin{aligned} &\begin{bmatrix} k \\ i \end{bmatrix} (-1)^l (-q^i t;q)_{k-i} q^{\binom{i}{2}} {}_3\Phi_2 \left[\begin{matrix} q^{-l}, \; q^{k+1}, \; -tq^k \\ q^{k-i+1}, \; 0 \end{matrix} ; q, q \right] \\ &\quad = \begin{bmatrix} l \\ i \end{bmatrix} t^{l-i} q^{kl} {}_2\phi_1 \left[\begin{matrix} q^{-k}, \; q^{l+1} \\ q^{l-i+1} \end{matrix} ; q, -tq^k \right]. \end{aligned} \tag{24}$$

A standard proof of (24) goes then as follows:

$$\begin{aligned} &\begin{bmatrix} k \\ i \end{bmatrix} (-1)^l (-tq^i;q)_{k-i} q^{\binom{i}{2}} (-tq^{k-i})^l {}_3\Phi_2 \left[\begin{matrix} q^{-l}, q^{-i}, (-tq^{i-1})^{-1} \\ q^{k-i+1}, 0 \end{matrix} ; q, q \right] \\ &\quad \text{(by [8, p. 241(III.11)])} \\ &\quad = \begin{bmatrix} k \\ i \end{bmatrix} t^l q^{(k+i)l} q^{\binom{i}{2}} (-tq^i;q)_{k-i} \frac{(q^{l-i+1};q)_i}{(q^{-k};q)_i} (tq^{k+l})^{-i} \\ &\qquad \times {}_2\Phi_1 \left[\begin{matrix} q^{-i}, q^{k+l+1-i} \\ q^{l-i+1} \end{matrix} ; q, -tq^i \right] \quad \text{(by [8, p. 241(III.6)])} \\ &\quad = \begin{bmatrix} l \\ i \end{bmatrix} q^{kl} t^{l-i} (-tq^i;q)_{k-i} \frac{(-tq^k;q)_\infty}{(-tq^i;q)_\infty} {}_2\Phi_1 \left[\begin{matrix} q^{-k}, q^{l-1} \\ q^{l-i+1} \end{matrix} ; q, -tq^k \right] \\ &\quad \text{(by [8, p. 241(III.3)])} \end{aligned}$$

which is equal to the right-hand side of (24).

Example 2. Let $a_n = \begin{bmatrix} x+n \\ m \end{bmatrix} q^{-mn}$ for $n \in \mathbb{N}$. By the notation (3), we have

$$a_n^* = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i \begin{bmatrix} x+i \\ m \end{bmatrix} q^{\binom{i}{2}-im}$$
$$= \begin{cases} (-1)^n \begin{bmatrix} x \\ m-n \end{bmatrix} q^{-mn+\binom{n}{2}} & \text{if } m \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1 implies that

$$\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(-1)^j}{[l+j+1]_q} \begin{bmatrix} x+l+j+1 \\ m \end{bmatrix} q^{-m(l+j+1)+\binom{j}{2}}$$
$$= (-1)^k \sum_{k \leq j \leq m} \frac{1}{[j]_q} \begin{bmatrix} l \\ j-k-1 \end{bmatrix} \begin{bmatrix} x \\ m-j \end{bmatrix} q^{\binom{j-k}{2}+\binom{j}{2}-l(j-1)-mj}$$
$$+ \frac{\begin{bmatrix} x \\ m \end{bmatrix}}{[k+l+1]_q \begin{bmatrix} k+l \\ k \end{bmatrix}}.$$

Example 3. Let $c_n = \begin{bmatrix} y \\ n \end{bmatrix} / \begin{bmatrix} x \\ n \end{bmatrix}$ for $n \in \mathbb{N}$. Then $c_n^* = \begin{bmatrix} x-y \\ n \end{bmatrix} q^{ny} / \begin{bmatrix} x \\ n \end{bmatrix}$. In fact,

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^k \begin{bmatrix} -n+k-1 \\ k \end{bmatrix} q^{nk-\binom{k}{2}}.$$

$$\begin{bmatrix} x \\ n \end{bmatrix} c_n^* = \sum_{i=0}^{n} \begin{bmatrix} x-i \\ n-i \end{bmatrix} (-1)^i \begin{bmatrix} y \\ i \end{bmatrix} q^{\binom{i}{2}}$$
$$= (-1)^n \sum_{i=0}^{n} \begin{bmatrix} x-n+1 \\ n-i \end{bmatrix} \begin{bmatrix} y \\ i \end{bmatrix} q^{(x-i)(n-i)-\binom{n-i}{2}+\binom{i}{2}}$$
$$= (-1)^n q^{xn-\binom{n}{2}} \begin{bmatrix} n-x-1+y \\ n \end{bmatrix} = \begin{bmatrix} x-y \\ n \end{bmatrix} q^{ny}.$$

By the identities in Theorem 1, we obtain

$$\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(-1)^j \begin{bmatrix} y \\ l+j+1 \end{bmatrix}}{\begin{bmatrix} x-1 \\ l+j \end{bmatrix}} q^{\binom{j+1}{2}} + \sum_{j=0}^{l} \frac{(-1)^j \begin{bmatrix} x-y \\ k+j+1 \end{bmatrix}}{\begin{bmatrix} x-1 \\ k+j \end{bmatrix}} q^{(k+j+1)y+\binom{j+1}{2}-l(k+j+1)}$$
$$= \frac{[x]_q}{[k+l+1] \begin{bmatrix} k+l \\ k \end{bmatrix}}$$

and

$$\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j \frac{\begin{bmatrix} y \\ l+j \end{bmatrix}}{\begin{bmatrix} x \\ l+j \end{bmatrix}} q^{\binom{j}{2}} = \sum_{j=0}^{l} (-1)^j \frac{\begin{bmatrix} x-y \\ k+j \end{bmatrix}}{\begin{bmatrix} x \\ k+j \end{bmatrix}} q^{(k+j)y+\binom{j+1}{2}-l(k+j)}.$$

Example 4. Carlitz [4, (3.1)] defined the *q-Stirling numbers of the second kind* $\left\{ {m \atop n} \right\}_q$ by

$$[n]_q^m = \sum_{i=0}^{m} \left\{ {m \atop i} \right\}_q [i]_q! \begin{bmatrix} n \\ i \end{bmatrix} q^{\binom{i}{2}}.$$

By Gauss inversion we get

$$\begin{aligned} \left\{ {m \atop n} \right\}_q &= \frac{q^{-\binom{n}{2}}}{[n]_q!} \sum_{i=0}^{n} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix} [n-i]_q^m \\ &= \frac{1}{[n]_q!} \sum_{i=0}^{n} (-1)^{n-i} q^{\binom{i+1}{2}-ni} \begin{bmatrix} n \\ i \end{bmatrix} [i]_q^m. \end{aligned}$$

So we have the following q-dual sequences:

$$a_n = (-1)^n [n]_q! \left\{ {m \atop n} \right\}_q, \qquad a_n^* = [n]_q^m.$$

Substituting these values in Theorem 1 yields corresponding identities. For example, applying (8) we obtain

$$\begin{aligned} &\frac{1}{[l]_q!} \sum_{j=0}^{l} (-1)^{l-j} q^{\binom{j+1}{2}-l(k+j)} \begin{bmatrix} l \\ j \end{bmatrix} [k+j]_q^m \\ &\quad = \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} l+j \\ j \end{bmatrix} \frac{[k]_q!}{[k-j]_q!} \left\{ {m \atop l+j} \right\}_q. \end{aligned}$$

The left-hand side of the above identity is called a *non-central q-Stirling number of the second kind*, with non-centrality parameter k, by Charalambides [5]. This number was first discussed by Carlitz [4, (3.8)] and recently by Charalambides [5, (3.5)]. Note that for $k = 0$ these numbers reduce to the usual q-Stirling numbers of the second kind, while for $k \neq 0$ the above identity connects the non-central to the usual q-Stirling numbers of the second kind.

Example 5. Taking $e(t) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$ as a q-analog of the exponential function e^x, Al-Salam [2, 2.1] defined a q-analog of Bernoulli numbers B_n by

$$\frac{1}{e(t)-1} = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} B_n.$$

These q-Bernoulli numbers B_n satisfy the following recurrence relation (see [2, 4.3]):

$$[1, B]^n = \begin{cases} B_n & n > 1, \\ 1 + B_1 & n = 1. \end{cases}$$

Now, if $a_0^* = B_0 = 1$ and, for $n \geq 1$, $a_n^* = B_n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} B_i$, then $a_0 = 1$ and for $n \geq 1$

$$a_n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i q^{\binom{i+1}{2}-in} \sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix} B_j$$

$$= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} B_j \sum_{i \geq j} \begin{bmatrix} n-j \\ i-j \end{bmatrix} (-1)^i q^{\binom{i+1}{2}-ni}$$
$$= (-1)^n B_n q^{-\binom{n}{2}}.$$

Theorem 1 infers then the following identities, which are q-analogs of the identities of Kaneko [10] and Momiyama [11] on Bernoulli numbers.

Proposition 1. *For* $k, l \in \mathbb{N}$,

$$\sum_{j=0}^{k} \begin{bmatrix} l \\ j \end{bmatrix} \frac{(-1)^j B_{k+j+1}}{[k+j+1]_q} q^{\binom{j+1}{2}-l(k+j+1)} + \sum_{j=0}^{l} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(-1)^{l+1} B_{l+j+1}}{[l+j+1]_q} q^{-\binom{l+1}{2}-lj}$$
$$= \frac{1}{[k+l+1]_q \begin{bmatrix} k+l \\ k \end{bmatrix}},$$

$$\sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} (-1)^j B_{k+j} q^{\binom{l-j}{2}} = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^l B_{l+j} q^{l(k-j)},$$

$$\sum_{j=0}^{l+1} \begin{bmatrix} l+1 \\ j \end{bmatrix} (-1)^{j+1} [k+j+1]_q B_{k+j} q^{\binom{l-j}{2}-j}$$
$$= \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} (-1)^l [l+j+1]_q B_{l+j} q^{(k-j)(l+1)+1}.$$

Example 6. Al-Salam [2] also defined the q-Bernoulli polynomials $B_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} B_k x^{n-k}$. By Example 5, if $a_n = (-1)^n B_n q^{-\binom{n}{2}}$ then $a_n^* = B_n$. Therefore

$$A_n(x) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} B_i x^{n-i} = B_n(x) \text{ and } A_n^*(x) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^i B_i x^{n-i} = B_n^*(x).$$

If we replace $A_n(x)$ and $A_n^*(x)$ by $B_n(x)$ and $B_n^*(x)$, respectively, we get the following result.

Proposition 2. *For* $k, l \in \mathbb{N}$,

$$(-1)^l \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} x^{l-j} \frac{B_{k+j+1}^*(z)}{[k+j+1]_q} q^{-kj-\binom{k+1}{2}}$$
$$+ (-1)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} x^{k-j} \frac{B_{l+j+1}([1,-x,-z])}{[l+j+1]_q} q^{\binom{j+1}{2}-k(l+j+1)}$$
$$= \frac{a_0(-x)^{k+l+1}}{[k+l+1] \begin{bmatrix} k+l \\ k \end{bmatrix}},$$

$$(-1)^l \sum_{j=0}^{l} \begin{bmatrix} l \\ j \end{bmatrix} x^{l-j} B_{k+j}^*(z) q^{k(l-j)}$$

$$= (-1)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} x^{k-j} B_{l+j}([1, -x, -z]) q^{\binom{k-j}{2}},$$

$$(-1)^{l+1} \sum_{j=0}^{l+1} \begin{bmatrix} l+1 \\ j \end{bmatrix} x^{l+1-j} [k+j+1]_q B^*_{k+j}(z) q^{(k+1)(l-j)+1}$$

$$= (-1)^k \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} x^{k+1-j} [l+j+1]_q B_{l+j}([1, -x, -z]) q^{\binom{k-j}{2}-j}.$$

Remark. It is easy to see that $B_n(0) = B_n$ and $B_n(0)^* = (-1)^n B_n$. Hence Proposition 1 can be derived from Proposition 2 by taking $x = 1$ and $z = 0$.

Acknowledgements

This work was done under the auspices of the National Science Foundation of China. The second author thanks Sun Zhi-Wei for asking a q-question about the results in [12], and was also supported by the EC's IHRP Programme, within Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272.

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