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Statistical Properties of a Linear Stochastic System*

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In this paper a random linear system of the form of $y(t; \omega = \int_{0}^{t} K(t, \tau; \omega) x(\tau; \omega) d\tau$ is studied, where the kernel is a stochastic process defined on a probability space. The concept of the modified characteristic function for the output process is introduced. These characteristic functions are used to identify the distribution of the output process over certain subsets of the probability space, Ω, in order to study the statistical properties of the process. Several examples are given to illustrate the usefulness of the resulting theory. These results extend the previous theory of random linear systems, in that until now, the kernel was deterministic in nature.

1. INTRODUCTION

In this paper we shall study a random linear system of the form

$$y(t; \omega) = \int_0^t K(t, \tau; \omega)\, x(\tau; \omega)\, d\tau, \tag{1.1}$$

where $K(t, \tau; \omega)$ is a stochastic impulse transfer function for $\omega \in \Omega$, a discrete probability space, $x(t; \omega)$ is the a.s. continuous random input, $y(t; \omega)$ is the output process, and the integration is on sample paths. Such a system is discribed by Fig. 1. Utilizing the concept of the modified characteristic functions, we shall obtain or approximate the form of the characteristic function, $\phi_y(s)$, for the output process, $y(t; \omega)$. Kuznetsov *et al.* (1965) have recently studied a similar system; however, the kernel was deterministic in nature. Several examples will be given to illustrate the usefulness of the resulting theory.

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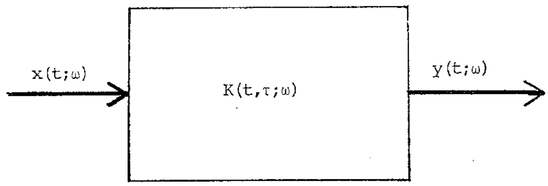

FIGURE 1

In order to begin the theoretical development for the above problem, we shall first present some definitions and lemmas that are essential in fulfilling the aims of this paper.

As is well known in statistics, the moments are a set of descriptive constants of a distribution which are useful in measuring its properties. Another set of constants, called cumulants (semi-invariants, generalized correlation functions) has properties which are sometimes more useful from the theoretical standpoint (Kendall and Stuart, 1943).

DEFINITION 1.1. The *cumulants* $R_1^{(1)}, R_2^{(1)},..., R_r^{(1)},...,$ are defined by the identity in s,

$$\phi_X(s) = \exp\left\{iR_1^{(1)}s + \frac{i^2}{2!}R_2^{(1)}s^2 + \cdots + \frac{i^r}{r!}R_r^{(1)}s^r + \cdots\right\}, \tag{1.2}$$

or equivalently,

$$\ln \phi_X(s) = iR_1^{(1)}s + \frac{i^2}{2!}R_2^{(1)}s^2 + \cdots + \frac{i^r}{r!}R_r^{(1)}s^r + \cdots, \tag{1.3}$$

where $\phi_X(s)$ is the characteristic function of the random variable X. Paley and Winer (1934) have shown that if $\phi_X(s)$ diminishes more quickly with increasing $|X|$ than the exponential $e^{-\lambda|X|}$, then the series in (1.3) converges for $|s| < \lambda$. The above discussion can be extended to the multivariate case easily. For example, in the bivariate case, if the decrease in the joint probability density function is no slower than the exponential function $e^{\lambda_1|X_1|+\lambda_2|X_2|}$, the logarithm of the characteristic function of the random vector (X_1, X_2) can be expanded as

$$\ln \phi_{X_1X_2}(s_1, s_2) = i[R_{10}^{(2)}s_1 + R_{01}^{(2)}s_2] + (i^2/2!)[R_{20}^{(2)}s_1{}^2 + 2R_{11}^{(2)}s_1s_2 + R_{02}^{(2)}s_2{}^2] + \cdots. \tag{1.4}$$

We now state a lemma concerning the characteristic function, the proof of which can be found in Tsokos (1972).

LEMMA 1.1. *Given the n-dimensional random vector $(X_1, X_2,..., X_n)$ with the joint probability density function $f(x_1, x_2,..., x_n)$ and the characteristic function*

$\phi_{X_1 \cdots X_n}(s_1, s_2, \ldots, s_n)$, the multivariate random variable $(Y_1, Y_2, \ldots, Y_m)$ defined by

$$Y_i = \sum_{j=1}^{n} \alpha_{ij} X_j, \qquad i = 1, 2, \ldots, m,$$

where the α_{ij}'s are constants has a characteristic function

$$\phi_{Y_1 Y_2 \cdots Y_m}(s_1', s_2', \ldots, s_m') = \phi_{X_1 X_2 \cdots X_n}\left(\sum_{i=1}^{m} s_i' \alpha_{i1}, \sum_{i=1}^{m} s_i' \alpha_{i2}, \ldots, \sum_{i=1}^{m} s_i' \alpha_{in}\right).$$

2. The Modified Characteristic Function

Given the system (1.1) with an input process $x(t; \omega)$ whose statistical properties are known, we wish to obtain knowledge of the statistical behavior of the output process $y(t; \omega)$.

To the knowledge of the authors in previous research on the subject, the kernel has been treated as a deterministic function. We shall here consider the random kernel $K(t, \tau; \omega)$ to be defined on a probability space Ω in the following manner. Ω is partitioned into a finite number of disjoint measurable sets $A_1, A_2, \ldots, A_q$ such that $\bigcup_{r=1}^{q} A_r = \Omega$, $P(A_r) > 0$, and for $\omega \in A_r$, $r = 1, 2, \ldots, q$, $K(t; \tau; \omega)$ behaves deterministically. That is, the sample paths of $K(t, \tau; \omega)$ are the same for all $\omega \in A_r$. We shall denote $K(t, \tau; \omega)$ for $\omega \in A_r$ by $K_r(t, \tau)$.

Kuznetsov *et al.* (1965), used characteristic functions to develop the theory for a nonstochastic kernel. We shall use a modified characteristic function approach for a stochastic case.

Definition 2.1. *A modified characteristic function* $\phi_X^{(r)}(s)$ for the random variable X is defined by

$$\phi_X^{(r)}(s) = E_r\{e^{isX}\} = \int_{A_r} e^{isx(\omega)}\, dP(\omega),$$

for $\omega \in A_r$.

A modified characteristic function for the multivariate case is defined analogously.

A question we must answer is how will the series expansion for the logarithm of $\phi_X^{(r)}(s)$ differ from that of $\ln \phi_X(s)$? To answer this question we proceed as follows:

$$\begin{aligned}\phi_X^{(r)}(s) &= E_r\{e^{isX}\} \\ &= E_r\left\{1 + isX + \frac{i^2}{2!} s^2 X^2 + \frac{i^3}{3!} s^3 X^3 + \cdots\right\} \qquad (2.1) \\ &= E_r\{1\} + isE_r\{X\} + \frac{i^2}{2!} s^2 E_r\{X^2\} + \cdots.\end{aligned}$$

We shall assume that $E\{X^n\} = \mu_n < \infty$ for all n and that the series in (2.1) converges absolutely near $s = 0$. Set $E_r\{1\} = a$, $(0 < a < 1)$. Then Eq. (2.1) takes the form

$$\begin{aligned}\phi_X^{(r)}(s) &= a + isE_r\{X\} + \frac{i^2}{2!} s^2 E_r\{X^2\} + \cdots \\ &= a\left[1 + \frac{iE_r\{X\}}{a} s + \frac{i^2}{s!} \frac{E_r\{X^2\}}{a} s^2 + \cdots\right],\end{aligned} \tag{2.2}$$

and Eq. (2.2) yields

$$\ln \phi_X^{(r)}(s) = \ln a + \ln \left\{1 + \left(\frac{iE_r\{X\}}{a} s + \frac{i^2 E_r\{X^2\}}{2!a} s^2 + \cdots\right)\right\}. \tag{2.3}$$

Let

$$z = \frac{iE_r\{X\}}{a} s + \frac{i^2}{2!} \frac{E_r\{X^2\}}{a} s^2 + \cdots. \tag{2.4}$$

Then restricting s close enough to the origin so that $|z| < 1$, we obtain

$$\ln \phi_X^{(r)}(s) = \ln a + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots. \tag{2.5}$$

Collecting powers of s upon expanding the powers of z in Eq. (2.5) we obtain

$$\ln \phi_X^{(r)}(s) = \ln a + iR_{r,1}^{(1)} s + \frac{i^2}{2} R_{r,2}^{(1)} s^2 + \cdots. \tag{2.6}$$

Thus, we state

DEFINITION 2.2. The coefficients $R_{r,1}^{(1)}, R_{r,2}^{(1)}, \ldots, R_{r,n}^{(1)}, \ldots$, of the powers of s in Eq. (2.6) are the *modified cumulants* of the probability distribution of the random variable X. We shall denote "$\ln a$" appearing in Eq. (2.6) by $R_{r,0}^{(1)}$. For each $t \in R^+$, the modified cumulants of the random variable $X(t; \omega)$ will be denoted by $R_{r,0}^{(1)}(t)$, $R_{r,1}^{(1)}(t)$, etc. For the multivariate case, modified correlation of the random vectors $X(t_1; \omega)$, $X(t_2; \omega), \ldots$, $X(t_n; \omega)$ will be denoted by $R_{r,0}^{(1)}(t_1, t_2, t_r)$, etc. Expressions for the modified correlations are obtained in Section 3.

3. MAIN RESULTS

Our goal in this section is to answer the following questions:

(i) Given the modified correlations $R_{r,0}$, $R_{r,1}(t)$, $R_{r,2}(t_1, t_2)$, $R_{r,3}(t_1, t_2, t_3), \ldots$ of the input process $x(t; \omega)$ of Eq. (1.1) how can one find the modified cumulants of the random input process $y(t;\omega)$, namely, $\tilde{R}_{r,0}$, $\tilde{R}_{r,1}(t)$, $\tilde{R}_{r,2}(t_1, t_2)$, $\tilde{R}_3(t_1, t_2, t_3),\ldots$ for $\omega \in A_r$, $r = 1, 2,\ldots, q$?

(ii) Once $\tilde{R}_{r,0}$, $\tilde{R}_{r,1}(t)$, $\tilde{R}_{r,2}(t_1, t_2),\ldots$ are known, for $r = 1, 2,\ldots, q$, how can one identify or approximate the modified characteristic function $\phi_Y^{(r)}(s_1{}^1, s_2{}^1,\ldots, s_n{}^1)$ for $\omega \in A_r$?

(iii) Knowing the form of $\phi_Y^{(r)}(s_1{}^1, s_2{}^1,\ldots, s_n{}^1)$, $r = 1, 2,\ldots, q$, how can we identifyfthe probability density function of the resulting process of the system, $y(t;\omega)$?

For brevity, we shall suppress the subscript r in the notation for $R_{r,n}(t_1, t_2,\ldots, t_n)$ and the subscript r for the modified characteristic functions $\phi_X^{(r)}(s)$ and $\phi_Y^{(r)}(s)$. It is understood also that $\omega \in A_r$, for some r between 1 and q. That is, we shall denote

$$\begin{aligned} \phi_X^{(r)}(s) &\equiv \phi_X(s), \\ \phi_Y^{(r)}(s) &\equiv \phi_Y(s), \end{aligned} \tag{3.1}$$

and

$$R_{r,n}(t_1, t_2,\ldots, t_n) \equiv R_n(t_1, t_2,\ldots, t_n),$$

where $\phi_X(s)$, $\phi_Y(s)$ are modified characteristic functions and $R(t_1, t_2,\ldots, t_n)$ denotes a modified cumulant. Also, we shall denote, for $\omega \in A_r$,

$$K(t, \tau; \omega) = K_r(t, \tau) \equiv K(t, \tau). \tag{3.2}$$

We shall assume $x(t;\omega)$ and $K(t, \tau)$ to be continuous for $0 \leqslant \tau < t < T$. Our first problem is the determination of the modified cumulants referring to the random function $x(t;\omega)$, namely,

$$\begin{array}{ll} R_0, & \\ R_1(t), & 0 \leqslant t \leqslant T, \\ R_2(t_1, t_2), & 0 \leqslant t_1, t_2 \leqslant T, \\ R_3(t_1, t_2, t_3), & 0 \leqslant t_1, t_2, t_3 \leqslant T. \end{array}$$

It is necessary to consider one by one the modified characteristic functions of $x(t)$, $(x(t_1), x(t_2))$, $(x(t_1), x(t_2), x(t_3))$. In the notation which follows,

$$\begin{aligned} f(x) = f(x(t;\omega)) &\equiv f(x(t)), \\ x(t_i;\omega) &\equiv X_i, \end{aligned}$$

and

$$y(t_i;\omega) \equiv Y_i.$$

For the univariate case for $x(t; \omega) = X$, the results in Section 2.1 yield

$$\ln \phi_X(s) = R_0^{(1)} + iR_1^{(1)}(t)s + \frac{i^2}{2!} R_2^{(1)}(t)s^2 + \cdots. \tag{3.3}$$

By a *first order modified cumulant* we mean a quantity

$$R_1(t) \equiv R_1^{(1)}(t), \qquad 0 \leqslant t \leqslant T.$$

The above can be extended to the multivariate cases. For example, for the bivariate case, we can similarly obtain

$$\begin{aligned} \ln \phi_{X_1X_2}(s_1, s_2) = R_0^{(2)} &+ i \sum_{\lambda=1}^{2} R_{1,\lambda}^{(2)}(t_1, t_2)s_\lambda \\ &+ \frac{i^2}{2!} \sum_{\lambda,\mu=1}^{2} R_{2,\lambda\mu}^{(2)}(t_1, t_2)\, s_\lambda s_\mu \\ &+ \frac{i^3}{3!} \sum_{\lambda,\mu,\nu=1}^{2} R_{3,\lambda\mu\nu}^{(2)}(t_1, t_2)\, s_\lambda s_\mu s_\nu + \cdots. \end{aligned} \tag{3.4}$$

Using these properties of modified characteristic functions, namely,

$$\phi_{X_1X_2}(s_1, 0) = \phi_{X_1}(s_1),$$

$$\phi_{X_1X_2}(0, s_2) = \phi_{X_2}(s_2),$$

and

$$\phi_{X_1X_2}(s_1, s_2) = \phi_{X_2X_1}(s_2, s_1), \tag{3.5}$$

we obtain

$$R_{2,\lambda\mu}^{(2)}(t_1, t_2) = R_{2,\mu\lambda}^{(2)}(t_2, t_1).$$

Thus, the modified correlation function $R_2(t_1, t_2)$ is understood to mean

$$\begin{aligned} R_2(t_\lambda, t_\mu) &= R_2(t_\mu, t_\lambda) \\ &\equiv R_{2,\lambda\mu}^{(2)}(t_1, t_2). \end{aligned}$$

From (3.3) and (3.5) we have

$$R_{1,1}^{(2)}(t_1, t_2) = R_1^{(1)}(t_1) \equiv R_1(t_1),$$

$$R_{1,2}^{(2)}(t_1, t_2) = R_1^{(1)}(t_2) = R_1(t_2),$$

and

$$R_2^{(1)}(t_\lambda) \equiv R_2(t_\lambda, t_\lambda).$$

Thus Eq. (3.4) becomes

$$\ln \phi_{X_1X_2}(s_1, s_2) = R_0^{(2)} + i \sum_{\lambda=1}^{2} R_1(t_\lambda)s_\lambda + \frac{i^2}{2!} \sum_{\lambda,\mu=1}^{2} R_2(t_\lambda, t_\mu)\, s_\lambda s_\mu + \frac{i^3}{3!} \sum_{\lambda,\mu,\nu=1}^{2} R_{3,\lambda\mu\nu}^{(2)}(t_1, t_2)\, s_\lambda s_\mu s_\nu + \cdots. \tag{3.6}$$

Continuing, the three-dimensional modified correlation function $R_3(t_1, t_2, t_3)$, $0 \leqslant t_1, t_2, t_3 \leqslant T$ can be found as the coefficient of a term of order three in the logarithmic expansion of the modified characteristic function into a Maclaurin series.

Imposing symmetry, namely,

$$R_3(t_1, t_2, t_3) = R_3(t_1, t_3, t_2) = \cdots,$$

we write

$$\ln \phi_{X_1X_2X_3}(s_1, s_2, s_3) = R_0^{(3)} + i \sum_{\lambda=1}^{3} R_1(t_\lambda)s_\lambda + \frac{i^2}{2!} \sum_{\lambda,\mu=1}^{3} R_2(t_\lambda, t_\mu)\, s_\lambda s_\mu + \frac{i^3}{3!} \sum_{\lambda,\mu,\nu=1}^{3} R_3(t_\lambda, t_\mu, t_\nu)\, s_\lambda s_\mu s_\nu + \frac{i^4}{4!} \sum_{\lambda,\mu,\nu=1}^{4} R_{4,\lambda\mu\nu\kappa}^{(3)}(t_1, t_2, t_3)\, s_\lambda s_\mu s_\nu s_\kappa + \cdots. \tag{3.7}$$

Generalizing, the modified characteristic function of the multidimensional random variable $(X_1, X_2, \ldots, X_N)$ can be written as

$$\phi_{X_1X_2\cdots X_N}(s_1, s_2, \ldots, s_N) = \exp\left\{R_0^{(N)} + i \sum_{\lambda=1}^{N} R_1(t_\lambda)s_\lambda + \frac{i^2}{2!} \sum_{\lambda,\mu=1}^{N} R_2(t_\lambda, t_\mu)\, s_\lambda s_\mu + \cdots\right\}. \tag{3.8}$$

The combined distribution of $(Y_1, Y_2, \ldots, Y_M)$ has a modified characteristic function which by analogy with Eq. (3.8) has the form

$$\phi_{Y_1Y_2\cdots Y_M}(s_1, s_2, \ldots, s_M) = \exp \tilde{R}_0^{(M)} + i \sum_{\alpha=1}^{M} \tilde{R}_1(t_\alpha)s_\alpha + \frac{i^2}{2!} \sum_{\alpha,\beta=1}^{M} \tilde{R}_2(t_\alpha, t_\beta)\, s_\alpha s_\beta + \cdots. \tag{3.9}$$

We now present a method to find the modified correlation functions $\tilde{R}_0^{(M)}$, $\tilde{R}_1(t)$, $\tilde{R}_2(t_1, t_2)$,... from our knowledge of $R_0^{(N)}$, $R_1(t)$, $R_2(t_1, t_2)$,... .

First, we shall consider the problem of discrete sequences of time. Divide the interval $[0, T]$ into N small intervals, letting $\Delta = T/N$.

We introduce the averaged quantities

$$y_m = \frac{1}{\Delta} \int_{(m-1)\Delta}^{m\Delta} y(t; \omega)\, dt$$

$$= \frac{1}{\Delta} \sum_{n=1}^{N} \int_{(m-1)\Delta}^{m\Delta} dt \cdot \int_{(n-1)\Delta}^{n\Delta} K(t, \tau)\, x(\tau; \omega)\, d\tau.$$

Furthermore, we assume that $K(t, \tau)$ is continuous and its sign does not change in each interval $[(i-1)\Delta, i\Delta]$. With $x(t; \omega)$ continuous in t, we have by the mean value theorem,

$$y_m = \sum_{n=1}^{N} \frac{1}{\Delta} x(c_n; \omega) \int_{(m-1)\Delta}^{m\Delta} \int_{(n-1)\Delta}^{n\Delta} K(t, \tau)\, dt\, d\tau, \qquad \text{a.e.,} \tag{3.10}$$

for $(n-1)\Delta \leqslant c_n \leqslant n\Delta$, $x(c_n; \omega)$ a constant function of t on $[(n-1)\Delta, n\Delta]$. Letting

$$x(c_n; \omega) = x(n),$$

and

$$K_{mn} = \frac{1}{\Delta} \int_{(m-1)\Delta}^{m\Delta} \int_{(n-1)\Delta}^{n\Delta} K(t, \tau)\, dt\, d\tau,$$

Eq. (3.10) becomes

$$y_m = \sum_{n=1}^{N} K_{mn} x(n).$$

Suppose $\phi_{X_i X_k}(s_i, s_k)$ is the modified characteristic function of a two-dimensional random variable (X_i, X_k) having a joint probability density function $f_{ik}(x_i, x_k)$. Define

$$\begin{aligned} &\phi_{X_m X_n}(s_m, s_n) = \phi_{X_m}(s_m)\, \phi_{X_n}(s_n)\, \phi^{(2)}_{X_m X_n}(s_m, s_n), \\ &\phi_{X_1 X_m X_n}(s_1, s_m, s_n) = \phi_{X_1}(s_1)\, \phi_{X_m}(s_m)\, \phi_{X_n}(s_n), \\ &\qquad \phi^{(2)}_{X_1 X_m}(s_1, s_m)\, \phi^{(2)}_{X_1 X_n}(s_1, s_n)\, \phi^{(2)}_{X_m X_n}(s_m, s_n), \\ &\qquad \phi^{(3)}_{X_1 X_m X_n}(s_1, s_m, s_n), \end{aligned} \tag{3.11}$$

etc.

The functions $\phi^{(2)}_{X_1 X+}(s_1, s_m)$, $\phi^{(3)}_{X_2 X+X0}(s_1, s_m, s_n)$, etc., have the properties of:

(i) Symmetry. That is,

$$\phi^{(3)}_{X_1 X_m X_n}(s_1, s_m, s_n) = \phi^{(3)}_{X_m X_n X_1}(s_m, s_n, s_1),$$

etc.

(ii) If at least one of the s_j's is zero then

$$\phi^{(i)}_{X_1 X_m \cdots X_\cdot}(s_1, \ldots, 0, \ldots, s_p) = 1.$$

From (ii), the logarithm of the above vanishes when at least one of the arguments is zero. For notational purposes, we shall denote

$$\ln \phi^{(i)}_{X_1 X_2 \cdots X_n}(s_1, s_2, \ldots, s_N) = \Gamma^{(i)}_{1,2,\ldots,N}(s_1, s_2, \ldots, s_N),$$

and

$$\ln \phi_{X_1 X_2 \cdots X_N}(s_1, s_2, \ldots, s_N) = \Gamma_{1,2,\ldots,N}(s_1, s_2, \ldots, s_N).$$

Hence,

$$\Gamma^{(i)}_{1,2,\ldots,N}(s_1, s_2, \ldots, s_N) \Big|_{\Pi_i s_i = 0} = 0. \tag{3.12}$$

Paley and Weiner (1934) have shown that the above functions $\Gamma^{(i)}$ and Γ can be expanded into a Maclaurin series. From Section 2 we obtain

$$\Gamma_m(s_m) = R_0^{(m)} + iR_1^{(m)} s_m + \frac{i^2}{2!} R_2^{(mm)} s_m{}^2 + \frac{i^3}{3!} R_3^{(mm)} s_m{}^3 + \cdots, \tag{3.13}$$

and

$$\Gamma^{(2)}_{m,n}(s_m, s_n) = \frac{i^2}{2!} R_2^{(mn)} s_m s_n + \frac{i^3}{3!} R_3^{(mmn)} s_m{}^2 s_n + R_3^{(mnn)} s_m s_n{}^2 + \cdots. \tag{3.14}$$

Equation (3.14) has all the terms containing at least one argument s_j by Eq. (3.12).
Hence, using Eqs. (3.11) to (3.14) we obtain

$$\ln[\phi_{X_1 X_2 \cdots X_N}(s_1, s_2, \ldots, s_N)]$$

$$= \sum_{n=1}^{N} \Gamma_n(s_n) + \sum_{m,n=1}^{N} \Gamma^{(2)}_{m,n}(s_m, s_n) + \sum_{1,m,n=1}^{N} \Gamma^{(3)}_{1,m,n}(s_1, s_m, s_n) \cdots. \tag{3.15}$$

Equation (3.15) can be put into the form

$$\ln[\phi_{X_1X_2\cdots X_N}(s_1, s_2, \ldots, s_N)]$$
$$= R_0 + i\sum_{n=1}^{N} R_1^{(n)} s_n + \frac{i^2}{2!}\sum_{m,n=1}^{N} R_2^{(mn)} s_m s_n + \frac{i^3}{3!}\sum_{1,m,n=1}^{N} R_3^{(1mn)} s_1 s_m s_n + \cdots, \tag{3.16}$$

where R_0 is a constant determined from $\sum_{n=1}^{N} \Gamma_n(s_n)$.

The stochastic input process is defined by the joint probability density function $f(x_1, \ldots, x_N)$ or by the modified characteristic function $\phi_{X_1X_2\cdots X_N}(s_1, s_2, \ldots, s_N)$. If $\ln \phi_{X_1X_2\cdots X_N}(s_1, s_2, \ldots, s_N)$ is known, we can find all the necessary statistical properties which underline the stochastic process $x(t; \omega)$. Fıom Eq. (3.16) this logarithmic function is generated by the modified correlation functions R_0, $R_1^{(n)}$, $R_2^{(mn)}$,..., etc. Our problem boils down to determining these cumulants for the output process $y(t; \omega)$ from the given cumulants of the input process $x(t; \omega)$ and the random impulse transfer function $K(t, \tau)$.

Suppose that it is necessary for us to find

$$\phi_{Y_1\cdots Y_M}(s_1, s_2, \ldots, s_M),$$

where

$$y_i = \sum_{n=1}^{N} K_{in} x(n).$$

Lemma 1.1 gives

$$\phi_{Y_1\cdots Y_M}(s_1, s_2, \ldots, s_M) = \phi_{X_1X_2\cdots X_N}\left(\sum_{i=1}^{M} s_i K_{i1}, \ldots, \sum_{i=1}^{M} s_i K_{iN}\right). \tag{3.17}$$

The logarithms are of the form

$$\ln[\phi_{Y_1\cdots Y_M}(s_1, \ldots, s_M)] = \ln\left[\phi_{X_1\cdots X_N}\left(\sum_{m=1}^{M} s_m K_{m1}, \ldots, \sum_{m=1}^{M} s_m k_{mN}\right)\right]. \tag{3.18}$$

From Eq. (3.16) we obtain

$$\ln[\phi_{Y_1\cdots Y_M}(s_1, \ldots, s_M)] = \tilde{R}_0 + i\sum_{m=1}^{M} \tilde{R}_1^{(m)} s_m + \frac{i^2}{2!}\sum_{m,n=1}^{M} \tilde{R}_2^{(mn)} s_m s_n + \cdots. \tag{3.19}$$

Using Eq. (3.18) we can write Eq. (3.19) as

$$\ln[\phi_{Y_1 \cdots Y_M}(s_1, \ldots, s_M)]$$

$$= R_0 + i \sum_{l=1}^{M} s_1 \sum_{i=1}^{N} K_{1i} R_1^{(i)} + \frac{i^2}{2!} \sum_{l,m=1}^{M} \sum_{i,j=1}^{N} s_1 s_m K_{1i} K_{mj} R_2^{(ij)} + \cdots. \tag{3.20}$$

Hence,

$$\begin{aligned} \tilde{R}_0 &= R_0, \\ \tilde{R}_1^{(m)} &= \sum_{i=1}^{N} K_{mi} R_1^{(i)}, \\ \tilde{R}_2^{(mn)} &= \sum_{i,j=1}^{N} K_{mi} K_{nj} R^{(ij)}, \\ \tilde{R}_3^{(mnl)} &= \sum_{i,j,k=1}^{N} K_{mi} K_{nj} K_{lk} R^{(ijk)}, \end{aligned} \tag{3.21}$$

etc.

We now proceed to develop the theory for the case of continuous time. In this case, we shall partition $[0, T]$ into subintervals $\Delta_1, \Delta_2, \ldots, \Delta_N$, where for $i - 1, 2, \ldots, N, \Delta_i \leqslant \delta$.

Equation (1.1) takes the form

$$y(t_\alpha ; \omega) = \sum_{n=1}^{N} \int_{\Delta_n} K(t, \tau)\, x(\tau; \omega)\, d\tau. \tag{3.22}$$

By the mean value theorem, Eq. (3.22) becomes

$$y(t_\alpha ; \omega) = \sum_{n=1}^{N} x(t_n^*; \omega) \int_{\Delta_n} K(t_\alpha, \tau)\, d\tau, \tag{3.23}$$

where $t_n^* \in \Delta_n$ and $x(t_n^*; \omega) = x(t_n^*)$ is a constant function of t over Δ_n.

Using the mean value theorem once more, we obtain

$$y(t_\alpha ; \omega) = \sum_{n=1}^{N} x(t_n^*)\, K(t_\alpha, t_n^{**})\, \Delta_n, \tag{3.24}$$

where $t_n^* \in \Delta_n$, $t_n^{**} \in \Delta_n$, $0 \leqslant t_\alpha \leqslant T$, and $K(t_\alpha, t_n^{**})$ is constant for each n. Employing Lemma 1.1 and Eq. (3.24) the modified characteristic function of the random variable $(Y_1, Y_2, \ldots, Y_M)$ is

$$\phi_{Y_1 Y_2 \cdots Y_M}(s_1, \ldots, s_M) = \phi_{X_1 \cdots X_N}\left(\sum_{\alpha=1}^{M} s_\alpha K(t_\alpha, t_1^{**})\, \Delta_1, \ldots, \sum_{\alpha=1}^{M} s_\alpha K(t_\alpha, t_N^{**})\, \Delta_N\right).$$

Thus Eq. (3.21) yields

$$\begin{aligned}
\tilde{R}_0 &= R_0 ,\\
\tilde{R}_1(t_\alpha) &= \sum_{n=1}^{N} K(t_\alpha , t_n^{**})\, R_1(t_n^*)\, \Delta_n ,\\
\tilde{R}_2(t_\alpha , t_\beta) &= \sum_{n,m=1}^{N} K(t_\alpha , t_n^{**})\, K(t_\beta , t_n^{**})\, R_2(t_n^* , t_m^*)\, \Delta_n \Delta_m ,\\
\tilde{R}_3(t_\alpha , t_\beta , t_\gamma) &= \sum_{n,m,i=1}^{N} K(t_\alpha , t_n^{**})\, K(t_\beta , t_n^{**})\, K(t_\gamma , t_1^{**})\, R_3(t_n^*, t_m^* , t_i^*)\, \Delta_n \Delta_m \Delta_i ,
\end{aligned} \tag{3.25}$$

etc.

We may mention here that the functions K, R_1, R_2,... are Riemann integrable. Let $\delta \to 0$. Then regardless of where the points t_i^* or t_i^{**} are chosen in Δ_i, from Eq. (3.25) (Valle-Poussin, 1965),

$$\begin{aligned}
\tilde{R}_0 &= R_0 ,\\
\tilde{R}_1(t) &= \int_0^T K(t, \tau)\, R_1(\tau)\, d\tau,\\
\tilde{R}_2(t_1 , t_2) &= \int_0^T \int_0^T K(t_1 , \tau_1)\, K(t_2 , \tau_2)\, R_2(\tau_1 , \tau_2)\, d\tau_1\, d\tau_2 ,\\
\tilde{R}_3(t_1 , t_2 , t_3) &= \int_0^T \int_0^T \int_0^T K(t_1 , \tau_1)\, K(t_2 , \tau_2)\, K(t_3 , \tau_3)\, R_3(\tau_1 , \tau_2 , \tau_3)\, d\tau_1\, d\tau_2\, d\tau_3 ,
\end{aligned} \tag{3.26}$$

etc.

This theory is useful in the sense that given the input process $x(t; \omega)$ for the linear system (1.1) we can find its modified cumulants R_0, $R_1(t)$, $R_2(t_1 , t_2)$,... . We may use Eqs. (3.26) to derive the modified semi-invariants $\tilde{R}_0$, $\tilde{R}_1(t)$, $\tilde{R}_2(t_1 , t_2)$,... for the output process $y(t; \omega)$. These cumulants are used to obtain or approximate the characteristic function of the output process $y(t; \omega)$ which enables us to describe the statistical properties of the output process.

We shall now summarize the above theory. Letting $R_0^{(r)}$, $R_1^{(r)}(t)$, $R_2^{(r)}(t_1 , t_2)$,... denote the modified cumulants that arise from the modified characteristic function $\phi^{(r)}_{X_1 \cdots X_N}(s_1 ,..., s_N)$, we obtain for each $\omega \in A$, $r = 1, 2,..., q$,

$$\begin{aligned}
\tilde{R}_0^{(r)} &= R_0^{(r)},\\
\tilde{R}_1^{(r)}(t) &= \int_0^T K(t_1 , \tau_1)\, R_1^{(r)}(\tau)\, d\tau,\\
\tilde{R}_2^{(r)}(t_1 , t_2) &= \int_0^T \int_0^T K_r(t_1 , \tau_1)\, K_r(t_2 , \tau_2)\, R_2^{(r)}(\tau_1 , \tau_2)\, d\tau_1\, d\tau_2 ,
\end{aligned}$$

etc.

We shall now present a few practical examples to help clarify the theory.

4. Examples

Example 4.1. We are given the input process $x(t;\omega)$ is independent of t. Let

$$x(t;\omega) = x(\omega) = \omega, \qquad \omega \in (-\infty, \infty),$$

and assume that X is distributed as $N(\mu, \sigma^2)$. Suppose

$$A_1 = [0, \infty),$$

and

$$A_2 = (-\infty, 0).$$

The kernel $K(t, \tau; \omega)$ in Eq. (1.1) is defined as follows for $t, \tau \in [0, 1]$, $t \geqslant \tau$:

$$K(t, \tau; \omega) = \begin{cases} t - \tau, & \text{for} \quad \omega \in A_1 \\ 1 - (t - \tau), & \text{for} \quad \omega \in A_2 . \end{cases}$$

For $\omega \in A_1$, the modified characteristic function $\phi_X^{(1)}(s)$ for X, $\omega \in A_1$ is

$$\begin{aligned} \phi_X^{(1)}(s) &= E_1[e^{isX}] \\ &= \int_0^{\infty} e^{isx} \frac{1}{(2\pi\sigma)^{1/2}} e^{-(x-\mu)^2/2\sigma^2}\, dx. \end{aligned}$$

This restriction yields $\phi_i^{1i}(s)$ in the form

$$\phi_X^{(1)}(s) = \exp\left\{R_0 + iR_1 s + \frac{i^2}{2!} R_2 s^2 + \frac{i^3}{3!} R_3 s^3 + \cdots\right\},$$

where the R_i's are independent of t. For the output process $y(t;\omega)$, $\omega \in A_1$, $t \geqslant \tau$,

$$\begin{aligned} \tilde{R}_0 &= R_0 , \\ \tilde{R}_1(t) &= \int_0^t (t - \tau)\, R_1\, d\tau = R_1 \frac{t^2}{2}, \\ \tilde{R}_2(t) &= \int_0^t \int_0^t (t - \tau_1)(t - \tau_2)\, R_2\, d\tau_1\, d\tau_2 = R_2 \left(\frac{t^2}{2}\right)^2, \end{aligned}$$

. . . .

$$\tilde{R}_n(t) = \int_0^t \int_0^t \int_0^t \cdots \int_0^t (t - \tau_1) \cdots (t - \tau_n)\, R_n\, d\tau_1\, d\tau_2 \cdots d\tau_n = R_n \left(\frac{t^2}{2}\right)^n .$$

Thus we have

$$\phi_Y^{(1)}(s) = \exp\left\{R_0 + iR_1 \frac{t^2}{2} s + \frac{i^2}{2!} R_2 \left(\frac{t^2}{2} s\right)^2 + \cdots\right\}$$
$$= \phi_X^{(1)}\left(s\frac{t^2}{2}\right).$$

Hence (see Tsokos, 1972) $y(t; \omega)$ is distributed normal with a mean of $(t^2/2)\mu$ and a variance of $(t^4/4)\sigma^2$. That is,

$$y(t; \omega) \sim N\left(\frac{t^2}{2}\mu, \frac{t^4}{4}\sigma^2\right). \tag{4.1}$$

For $\omega \in A_2 = (-\infty, 0)$,

$$\begin{aligned}\phi_X^{(2)}(s) &= \int_{-\infty}^{0} e^{isx} \frac{1}{(2\mu\sigma)^{1/2}} e^{-(x-\mu)^2/2\sigma^2}\, dx \\ &= \exp\left\{R_0^* + iR_1^* s + \frac{i^2}{2!} R_2^* s^2 + \cdots\right\}.\end{aligned} \tag{4.2}$$

This gives for the output process $y(t; \omega)$, $t \geqslant \tau$, $\omega \in A_2$,

$$\tilde{R}_0^* = R_0^*,$$

$$\tilde{R}_1^*(t) = \int_0^t [1 - (t - \tau)]\, R_1^*\, d\tau = \frac{(2t - t^2)}{2} R_1^*,$$

$$R_2^*(t) = \int_0^t \int_0^t [1 - (t - \tau_1)][1 - (t - \tau_2)]\, R_2^*\, d\tau_1\, d\tau_2 = \left(\frac{2t - t^2}{2}\right)^2 R_2^*,$$

etc. Hence,

$$\phi_Y^{(2)}(s) = \phi_X^{(2)}\left(\left[\frac{2t - t^2}{2}\right] s\right),$$

and we see that for $\omega \in A_2$,

$$y(t; \omega) \sim N\left(\left[\frac{2t - t^2}{2}\right]\mu, \left[\frac{2t - t^2}{2}\right]^2 \sigma^2\right).$$

That is, $y(t; \omega)$ has a normal distribution with mean of $[(2t - t^2)/2]\mu$ and a variance of $[(2t - t^2)/2]^2\sigma^2$.

The results given by Eqs. (4.1) and (4.2) are verified by noticing that from Eq. (1.1),

$$\begin{aligned} y(t;\omega) &= \int_0^t K(t,\tau;\omega)X\,d\tau \\ &= X\int_0^t K(t,\tau;\omega)\,d\tau \\ &= \begin{cases} X\int_0^t (t-\tau)\,d\tau, & \omega\in A_1 \\ X\int_0^t [1-(t-\tau)]\,d\tau, & \omega\in A_2 . \end{cases} \end{aligned}$$

Therefore, the probability density function for $y(t;\omega)$ is

$$f(y) = \begin{cases} f_1(y) = \left(1\Big/2\pi\dfrac{t^2}{2}\right)\exp\left\{-\left(x-\dfrac{t^2}{2}\mu\right)^2\Big/2\dfrac{t^4}{4}\sigma^2\right\}, & x>0 \\ f_2(y) = 1\Big/2\pi\left(\dfrac{2t-t^2}{2}\sigma\right) & \\ \quad\times\exp\left\{-\left(x-\left[\dfrac{2t-t^2}{2}\right]\mu\right)^2\Big/2\left(\dfrac{2t-t^2}{2}\right)^2\sigma^2\right\}, & x\leqslant 0. \end{cases}$$

We shall create several specific cases. For $t = 0.5$, $\mu = 2$, and $\sigma = .1$, we have, using Eqs. (4.1) and (4.2),

$$Y \sim N(0.25, 0.015625) \qquad \text{for} \quad \omega\in A_1,$$

and

$$Y \sim N(0.75, 0.140625) \qquad \text{for} \quad \omega\in A_2 .$$

The graph of the density $f(y)$ for $t = 0.5$ and $\mu = 2$ is shown by Fig. 2

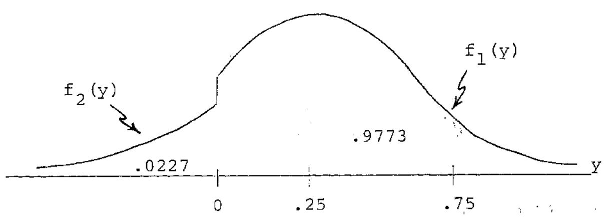

FIGURE 1

EXAMPLE 4.2. Suppose that the input process $x(t;\omega)$ is a standard Weiner process and that A_1, A_2, $K(t,\tau;\omega)$ are defined as in Example 4.1.

Recall that our problem is, given an input process $x(t;\omega)$, to determine the statistical properties of the output process, $y(t;\omega)$. In other words, we must find how the multivariate random variable $(y(t_1; \omega), y(t_2; \omega), \ldots, y(t_n; \omega))$ is distributed for any n, and $t_1, t_2, \ldots, t_n \in [0, 1]$. This example shall show the usefulness of the above theory.

We begin by approximating the distribution of $y(t; \omega)$ for a fixed t. For $\omega \in A_1$, the modified characteristic function $\phi_X^{(1)}(s)$ is

$$\begin{aligned} \phi_X^{(1)}(s) &= E_1[e^{isX}] \\ &= \int_0^\infty \frac{e^{isx}}{(2\pi t)^{1/2}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx. \end{aligned} \tag{4.3}$$

Recall that for a Weiner process,

$$E[x(t; \omega)] = 0$$

and

$$\operatorname{var}[x(t; \omega)] = \sigma_j^2.$$

Without loss of generality, we shall assume that $\alpha^2 = 1$. Thus Eq. (4.3) becomes for $t \in (0, 1]$,

$$\phi_X^{(1)}(s) = \int_0^\infty \frac{e^{isx}}{(2\pi)^{1/2} t} \exp\left[-\frac{x^2}{2t^2}\right] dx. \tag{4.4}$$

Now, Eq. (4.4) can be put in the form

$$\phi_X^{(1)}(s) = \exp\left[-s^2 \frac{t^2}{2}\right] \int_{-ist}^\infty \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} \, dz,$$

and

$$\ln \phi_X^{(1)}(s) = -s^2 \frac{t^2}{2} + \ln\left[\int_{-ist}^\infty \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} \, dz\right]. \tag{4.5}$$

We proceed by calculating $\int_{-ist}^\infty 1/(2\pi)^{1/2} e^{-z^2/2} \, dz$. Let

$$G(s) = \int_{-ist}^\infty \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} \, dz.$$

From the symmetry property of the function $e^{-z^2/2}$, we have

$$\frac{1}{2} = \int_0^\infty \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} \, dz = \int_0^{-ist} \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} \, dz + \int_{-ist}^\infty \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} \, dz.$$

Thus,

$$\int_{-ist}^\infty \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} \, dz = \frac{1}{2} - \int_0^{-ist} \frac{1}{(2\pi)^{1/2}} e^{-z^2/2} \, dz. \tag{4.6}$$

Using the series expansion we can write

$$\int_0^{-ist} \frac{1}{(2\pi)^{1/2}} e^{-z^2/2}\, dz = \frac{1}{(2\pi)^{1/2}} \left[z - \frac{z^3}{6} + \frac{z^5}{40} - \frac{z^7}{336} + \cdots \right] \Big|_0^{-ist}$$
$$= \frac{1}{(2\pi)^{1/2}} \left[(-ist) + \frac{i^3 s^3 t^3}{6} - \cdots \right].$$

Thus Eq. (4.6) becomes

$$\int_{-ist}^{\infty} \frac{1}{(2\pi)^{1/2}} e^{-z^2/2}\, dz = \frac{1}{2} + \frac{its}{(2\pi)^{1/2}} - \frac{i^3 t^3 s^3}{3!(2\pi)^{1/2}} + \frac{i^5 3 t^5 s^5}{5!(2\pi)^{1/2}} - \frac{i^7 3 \cdot 5 t^7 s^7}{7!(2\pi)^{1/2}} + \cdots. \tag{4.7}$$

From Eq. (4.5) we obtain

$$\ln \phi_X^{(1)}(s) = -\frac{s^2 t^2}{2} + \ln \left[\frac{1}{2} + \frac{its}{(2\pi)^{1/2}} - \frac{i^3 t^3 s^3}{3!(2\pi)^{1/2}} + \frac{i^5 3 t^5 s^5}{5!(2\pi)^{1/2}} - \frac{i^7 15 t^7 5^7}{7!(2\pi)^{1/2}} + \cdots \right]$$
$$= -\frac{s^2 t^2}{2} + \ln \frac{1}{2} + \ln \left[1 + \left\{ \frac{2its}{(2\pi)^{1/2}} - \frac{2 i^3 t^3 s^3}{3!(2\pi)^{1/2}} + \frac{2 i^5 \cdot 3 t^5 s^5}{5!(2\pi)^{1/2}} - \cdots \right\} \right]. \tag{4.8}$$

Equation (4.8) reduces to the form

$$\ln \phi_X^{(1)}(s) = -\frac{s^2 t^2}{2} + \ln \frac{1}{2} + \frac{2its}{(2\pi)^{1/2}} - \frac{2 i^3 t^3 s^3}{3!(2\pi)^{1/2}} + \cdots$$
$$- \frac{1}{2} \left\{ \frac{2its}{(2\pi)^{1/2}} - \frac{2 i^3 t^3 s^3}{3!(2\pi)^{1/2}} + \cdots \right\}^2 - \cdots$$
$$+ \frac{1}{3} \left\{ \frac{2its}{(2\pi)^{1/2}} - \frac{2 i^3 t^3 s^3}{3!(2\pi)^{1/2}} + \cdots \right\}^3 - \cdots$$
$$= \ln \frac{1}{2} + i \left[\frac{2t}{(2\pi)^{1/2}} \right] s + \frac{i^2}{2!} \left[t^2 - \frac{2t^2}{\pi} \right] s^2 + \frac{i^3}{3!} \left[-\frac{2t^3}{(2\pi)^{1/2}} + \frac{8t^3}{\pi (2\pi)^{1/2}} \right] s^3 + \cdots.$$

Thus,

$$R_0 = \ln \tfrac{1}{2},$$
$$R_1(t) = \frac{2t}{(2\pi)^{1/2}},$$
$$R_2(t) = \left[1 - \frac{2}{\pi} \right] t^2 = R_2^{(2)}(t, t),$$

and

$$R_3(t) = \left[-1 + \frac{4}{\pi}\right] \frac{2t^3}{(2\pi)^{1/2}} = R_3^{(3)}(t, t, t).$$

Since $t \geqslant \tau$, t, $\tau \in (0, 1]$, we have for the modified characteristic function of the output process $y(t; \omega)$ over A_1, the modified semi-invariants

$$R_0 = \ln \tfrac{1}{2},$$

$$R_1(t) = \int_0^t (t - \tau) \frac{2\tau}{(2\pi)^{1/2}} d\tau = \frac{t^3}{3(2\pi)^{1/2}},$$

$$\tilde{R}_2(t) = \int_0^t \int_0^t (t - \tau_1)(t - \tau_2) \left(1 - \frac{2}{\pi}\right) \tau_1^2 \, d\tau_1 \, d\tau_2 = \left[1 - \frac{2}{\pi}\right] \frac{t^6}{24} = 0.02t^6,$$

and

$$\tilde{R}_3(t) = \int_0^t \int_0^t \int_0^t (t - \tau_1)(t - \tau_2)(t - \tau_3) \, R_3(\tau_1) \, d\tau_1 \, d\tau_2 \, d\tau_3$$
$$= \int_0^t (t - \tau_3) \, d\tau_3 \cdot \int_0^t (t - \tau_2) \, d\tau_2 \cdot \int_0^t (t - \tau_1) \cdot \left[-1 + \frac{4}{\pi}\right] \frac{2\tau_1^3}{(2\pi)^{1/2}} d\tau_1$$
$$< 0.003,$$

because $t \in (0, 1]$.

Similarly, $|R_4(t)|$, $|R_5(t)|$, will become smaller and smaller and thus will have a negligible effect on $\ln \phi_X^{(1)}(s)$. Hence

$$\phi_Y^{(1)}(s) \approx \exp\{\tilde{R}_0 + i\tilde{R}_1(t)s - \tfrac{1}{2}\tilde{R}_2(t)s^2\}.$$

We observe that the form of the modified characteristic function $\phi_X^{(1)}(s)$ over A_1 is the same as that for a normal distribution. Thus, for $\omega \in A_1$, the distribution of the output process $y(t; \omega)$ is approximately $N[R_1(t), R_2(t)]$, that is

$$y(t; \omega) \sim N\left[\frac{t^3}{3(2\pi)^{1/2}}, \left(1 - \frac{2}{\pi}\right) \frac{t^6}{24}\right].$$

Hence, for $\omega \in A_1$, the output process is distributed as normal with a mean of $t^3/3(2\pi)^{1/2}$ and a variance of $(1 - 2/\pi)(t^6/24)$.

When $\omega \in A_2$,

$$\phi_X^{(2)}(s) = \int_{-\infty}^{0} \frac{e^{isx}}{(2\pi)^{1/2}t} \exp\left[-\frac{x^2}{2t^2}\right] dx.$$

Using the same reasoning as above, we obtain

$$R_0 = \ln \tfrac{1}{2},$$

$$R_1(t) = -\frac{2t}{(2\pi)^{1/2}},$$

$$R_2(t) = \left(1 - \frac{2}{\pi}\right) t^2,$$

and

$$R_3(t) = \left(1 - \frac{4}{\pi}\right) \frac{2t^3}{(2\pi)^{1/2}}. \tag{4.9}$$

Equation (4.9) yields

$$\tilde{R}_0 = \ln \tfrac{1}{2},$$

$$\tilde{R}_1(t) = -\frac{t^3}{3(2\pi)^{1/2}},$$

$$\tilde{R}_2(t) = \left(1 - \frac{2}{\pi}\right) \frac{t^6}{24},$$

and

$$\tilde{R}_3(t) = \frac{1}{(2\pi)^{1/2}} \left[1 - \frac{4}{\pi}\right] \frac{t^9}{40}.$$

Performing an analysis identical to the case where $\omega \in A_1$ will yield that $|\tilde{R}_3(t)| < 0.003$. Hence, we do not lose much accuracy in omitting powers of s of order three or more. Therefore, for $\omega \in A_2$, the approximate distribution for $y(t; \omega)$ is normal, with a mean of $-t^3/3(2\pi)^{1/2}$ and variance $(1 - 2/\pi)(t^6/24)$, that is

$$y(t; \omega) \sim N\left[-\frac{t^3}{3(2\pi)^{1/2}}, \left(1 - \frac{2}{\pi}\right) \frac{t^6}{24}\right].$$

Given $t_1, t_2 \in (0, 1]$, we shall now consider the bivariate case where the bivariate random variable (X_1, X_2) is distributed as bivariate normal.

For $\omega \in A_1$, we have

$$\begin{aligned} \phi_{X_1X_2}^{(1)}(s_1, s_2) &= E_1\{e^{i(s_1X_1+s_2X_2)}\} \\ &= \int_0^\infty \int_0^\infty e^{i(s_1x_1+s_2x_2)} \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}(1-\rho^2)^{1/2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\right. \\ &\quad \cdot \left.\left[\left(\frac{x_1}{\sigma_{X_1}}\right)^2 - 2\rho\left(\frac{x_1}{\sigma_{X_1}}\right)\left(\frac{x_2}{\sigma_{X_2}}\right) + \left(\frac{x_2}{\sigma_{X_2}}\right)^2\right]\right\} dx_1\, dx_2. \end{aligned} \tag{4.10}$$

Letting

$$u = \frac{x_1}{\sigma_{X_1}}, \qquad v = \frac{x_2}{\sigma_{X_2}},$$

$$r_1 = \frac{u - \rho v - (1 - \rho^2)\, is_1\sigma_X \mathrm{i}}{(1 - \rho^2)^{1/2}}$$

and

$$r_2 = v - \rho i s_1 \sigma_{X_1} - i s_2 \sigma_{X_2},$$

Eq. (4.10) becomes

$$\phi^{(1)}_{X_1X_2}(s_1, s_2) = \exp\{-\tfrac{1}{2}[s_1{}^2\sigma^2_{X_1} + 2\rho s_1 s_2 \sigma_{X_1}\sigma_{X_2} + s_2{}^2\sigma^2_{X_2}]\}$$
$$\cdot \int_{-(1-\rho^2)^{1/2} i s_1 \sigma_{X_1}}^{\infty} \frac{1}{(2\pi)^{1/2}} e^{-r_1{}^2/2}\, dr_1 \int_{-\rho i s_1 \sigma_{X_1} - i s_2 \sigma_{X_2}}^{\infty} \frac{1}{(2\pi)^{1/2}} e^{-r_2{}^2/2}\, dr_2 .$$

Hence,

$$\ln \phi^{(1)}_{X_1X_2}(s_1, s_2)$$
$$= -\tfrac{1}{2}[s_1{}^2\sigma^2_{X_1} + 2\rho s_1 s_2 \sigma_{X_1}\sigma_{X_2} + s_2{}^2\sigma^2_{X_2}] \tag{4.11}$$
$$+ \ln \int_{-(1-\rho^2)^{1/2} i s_1 \sigma_{X_1}}^{\infty} \tfrac{1}{2} e^{-r_2{}^2/2}\, dr_2 + \ln \int_{-\rho i s_1 \sigma_{X_1} - i s_2 \sigma_{X_2}}^{\infty} \frac{1}{(2\pi)^{1/2}} e^{-r_2{}^2/2}\, dr_2 .$$

Since

$$\int_{\beta}^{\infty} \frac{1}{(2\pi)^{1/2}} e^{-z^2/2}\, dz = \frac{1}{2} - \int_0^{\beta} \frac{1}{(2\pi)^{1/2}} e^{-z^2/2}\, dz,$$

and

$$\int_0^{\beta} \frac{1}{(2\pi)^{1/2}} e^{-z^2/2}\, dz = \frac{1}{(2\pi)^{1/2}} \left[z - \frac{z^3}{6} + \frac{z^{\cdot}}{40} - \frac{z^{\cdot}}{336} + \cdots \right] \Big|_0^{\beta},$$

we obtain

$$\int_{-(1-\rho^2)^{1/2} i s_1 \sigma_{X_1}}^{\infty} \frac{1}{(2\pi)^{1/2}} e^{-r_1{}^2/2}\, dr_1$$
$$= \frac{1}{2} - \left[\frac{(-1-\rho^2)\, i s_1 \sigma_{X_1}}{2\pi} + \frac{(1-\rho^2)^{3/2}\, i^3 s_1{}^3 \sigma^3_{X_1}}{6(2\pi)^{1/2}} + \cdots \right]$$
$$= \frac{1}{2} + \frac{(1-\rho^2)^{1/2}\, i s_1 \sigma_{X_1}}{(2\pi)^{1/2}} - \frac{(1-\rho^2)^{3/2}\, i^3 s_1{}^3 \sigma^3_{X_1}}{6(2\pi)^{1/2}} + \cdots.$$

Also,

$$\int_{-\rho is_1\sigma_{X_1}-is_2\sigma_{X_2}}^{\infty} \frac{1}{(2\pi)^{1/2}} e^{-r_2{}^2/2}\, dr_2$$

$$= \frac{1}{2} + \frac{\rho is_1\sigma_{X_1} + is_2\sigma_{X_2}}{(2\pi)^{1/2}} - \frac{(\rho is_1\sigma_{X_1} + is_2\sigma_{X_2})^3}{6(2\pi)^{1/2}} + \cdots.$$

Thus, Eq. (4.11) becomes

$$\ln \phi_{X_1X_2}^{(1)}(s_1, s_2) = -\tfrac{1}{2}[s_1{}^2\sigma_{X_1}^2 + 2\rho s_1 s_2\sigma_{X_1}\sigma_{X_2} + s_2{}^2\sigma_{X_2}^2]$$
$$+ \ln\left[\frac{1}{2} + \frac{(1-\rho^2)^{1/2}\, is_1\sigma_{X_1}}{(2\pi)^{1/2}} + \cdots\right]$$
$$+ \ln\left[\frac{1}{2} + \frac{\rho is_1\sigma_{X_1} + is_2\sigma_{X_2}}{(2\pi)^{1/2}} + \cdots\right]. \qquad (4.12)$$

Here we note that

$$\ln\left[\frac{1}{2} + \frac{(1-\rho^2)^{1/2}\, is_1\sigma_{X_1}}{(2\pi)^{1/2}} - \frac{(1-\rho^2)^{3/2}\, i^3 s_1\sigma_{X_1}^3}{6(2\pi)^{1/3}} - \cdots\right]$$

$$= \ln \tfrac{1}{2} + \ln\left[1 + \left\{\frac{2(1-\rho^2)^{1/2}\, is_1\sigma_{X_1}}{(2\pi)^{1/2}} - \frac{(1-\rho^2)^{3/2}\, i^3 s_1{}^3\sigma_{X_1}^3}{3(2\pi)^{1/2}} + \cdots\right\}\right] \qquad (4.13)$$

and

$$\ln\left[\frac{1}{2} + \frac{\rho is_1\sigma_{X_1} + is_2\sigma_{X_2}}{(2\pi)^{1/2}} - \frac{(\rho is_1\sigma_{X_1} + is_2\sigma_{X_2})^3}{6(2\pi)^{1/2}} + \cdots\right]$$

$$= \ln \tfrac{1}{2} + \ln\left[1 + \left\{\frac{2\rho is_1\sigma_{X_1} + 2is_2\sigma_{X_2}}{(2\pi)^{1/2}} - \frac{(\rho is_1\sigma_{X_1} + is_2\sigma_{X_2})^3}{3(2\pi)^{1/2}} + \cdots\right\}\right]. \qquad (4.14)$$

From the fact that a characteristic function is analytic in a neighborhood of the origin, we restrict s_1, s_2 near the origin so that the series in Eqs. (4.13) and (4.14) converge. Equation (4.12) may be put in the form

$$\ln \phi_{X_1X_2}^{(1)}(s_1, s_2)$$

$$= \ln \frac{1}{4} + i\left[\left(\frac{2(1-\rho^2)^{1/2}\,\sigma_{X_1}}{(2\pi)^{1/2}} + \frac{2\rho\sigma_{X_1}}{(2\pi)^{1/2}}\right) s_1 + \left(\frac{2\sigma_{X\cdot}}{\pi^{1/2}}\right) s_2\right]$$

$$+ \frac{i^2}{2!}\left[\left(\sigma_{X_1}^2 - \frac{2(1-\rho^2)\,\sigma_{X_1}^2}{\pi} - \frac{2\rho^2\sigma_{X_1}^2}{\pi}\right) s_1{}^2\right.$$

$$\left. + \left(2\rho\sigma_{X_1}\sigma_{X_2} - \frac{4\rho\sigma_{X_1}\sigma_{X_2}}{\pi}\right) s_1 s_2 + \left(\sigma_{X_2}^2 - \frac{2\sigma_{X_2}^2}{\pi}\right) s_2{}^2\right]$$

$$+\frac{i^3}{3!}\left[\left(\frac{(1-\rho^2)^{3/2}\,\sigma_{X_1}^3[-2\pi+8]+\rho^3\phi_{X_1}^3[8-2\pi]}{\pi(2\pi)^{1/2}}\right)s_1{}^3\right.$$

$$+\left(\frac{\rho^2\sigma_{X_1}^2\sigma_{X_2}[24-6\pi]}{\pi(2\pi)^{1/2}}\right)s_1{}^2s_2+\frac{\rho\sigma_{X_1}\sigma_{X_1}^2[24-6\pi]}{\pi(2\pi)^{1/2}}\Bigg)\,s_1s_2{}^2$$

$$\left.+\left(\frac{\sigma_{X_2}^3[8-2\pi]}{\pi(2\pi)^{1/2}}\right)s_2{}^3\right]+\cdots. \tag{4.15}$$

Using the results developed in Section 3, we obtain the necessary cumulants

$$R_2(t_1\,,t_2)=\left(1-\frac{2}{\pi}\right)\rho\sigma_{X_1}\sigma_{X_2}\,,$$

$$\begin{aligned}R_3(t_1\,,t_1\,,t_1)&=R_3^{(2)}(t_1\,,t_1)\\&=\frac{(8-2\pi)\,\sigma_{X_1}^3}{\pi(2\pi)^{1/2}}\,,\\R_3(t_1\,,t_2\,,t_1)&=R_3^{(2)}(t_1^{(2)},t_2)\\&=\frac{(8-2\pi)\,\sigma_{X_1}^2\sigma_{X_2}}{\pi(2\pi)^{1/2}}\,,\\R_3(t_1\,,t_2\,,t_2)&=R_3^{(2)}(t_1\,,t_2^{(2)})\\&=\frac{\rho(8-2\pi)\,\sigma_{X_1}\sigma_{X_2}^2}{\pi(2\pi)^{1/2}}\end{aligned}$$

and

$$\begin{aligned}R_3(t_2\,,t_2\,,t_2)&=R_3^{(2)}(t_2\,,t_2)\\&=\frac{(8-2\pi)\,\sigma_{X_2}^3}{\pi(2\pi)^{1/2}}\,.\end{aligned}$$

For the output process $y(t;\omega)$, $\omega\in A_1$, $R_1(t)$ is known for the case of a fixed t. Furthermore,

$$\begin{aligned}\tilde{R}_2(t_1\,,t_2)&=\int_0^{t_2}\int_0^{t_1}(t_1-\tau_1)(t_2-\tau_2)\,R_2(\tau_1\,,\tau_2)\,d\tau_1\,d\tau_2\\&=\int_0^{t_2}\int_0^{t_1}(t-\tau_1)(t_2-\tau_2)\left[1-\frac{2}{\pi}\right]\rho(\tau_1)^{1/2}(\tau_2)^{1/2}\,d\tau_1\,d\tau_2\\&=\left(1-\frac{2}{\pi}\right)\int_0^{t_2}\int_0^{t_1}(t_1-\tau_1)(t_2-\tau_2)\\&\quad\cdot\frac{\min(\tau_1\,,\tau_2)}{\tau_1^{1/2}\tau_2^{1/2}}\,\tau_1^{1/2}\tau_2^{1/2}\,d\tau\ \ d\tau\ \ ,\end{aligned}\tag{4.16}$$

since for a Weiner process (Doob, 1953),

$$\rho = \frac{\min(t_1, t_2)}{\sigma_{X_1}\sigma_{X_2}}.$$

Equation (4.16) now becomes

$$\begin{aligned} R_2(t_1, t_2) &= \left(1 - \frac{2}{\pi}\right) \int_0^{t_2} (t_2 - \tau_2)\, d\tau_2 \left[\int_0^{\tau_2} (t_1 - \tau_1)\tau_1\, d\tau_1 + \int_{\tau_2}^{t_1} (t_1 - \tau_1)\tau_2\, d\tau_1\right] \\ &= 0.0031 t_2^3[t_2^2 + 10t_1^2 - 5t_1t_2]. \end{aligned}$$

Also, we have

$$\begin{aligned} \tilde{R}_3(t_1, t_2) &= \int_0^{t_2}\int_0^{t_2}\int_0^{t_1} (t_2 - \tau_3)(t_2 - \tau_2)(t_1 - \tau_1) \frac{\rho(8 - 2\pi)}{\pi(2\pi)^{1/2}} \tau_1\tau_2^2\, d\tau_1\, d\tau_2\, d\tau_3 \\ &\approx 0.0024 t_1^{5/2} t_2^6. \end{aligned}$$

Since the magnitude of $\tilde{R}_3(t_1, t_2)$ is of the same order as the magnitude of $\tilde{R}_2(t_1, t_2)$, we cannot just suppress the term

$$\frac{i^3}{3!} \sum_{\lambda,\mu,\nu=1}^{2} \tilde{R}_3(t_1, t_2)\, s_\lambda s_\mu s_\nu$$

plus the remaining terms in the expansion of $\ln \phi^{(1)}_{Y_1Y_2}(s_1, s_2)$. We would need to calculate $\tilde{R}_4(t_1, t_2)$, $\tilde{R}_5(t_1, t_2)$,..., until for some $j \geqslant 3$, the magnitude of $\tilde{R}_{j+1}(t_1, t_2)$ is very small in comparison to that of $\tilde{R}_j(t_1, t_2)$. Thus we would obtain

$$\begin{aligned} \ln \phi^{(1)}_{Y_1Y_2}(s_1, s_2) = R_0 &+ i \sum_{\alpha_1=1}^{2} \tilde{R}_1(t_1, t_2)\, s_{\alpha_1} \\ &+ \frac{i^2}{2} \sum_{\alpha_1,\alpha_2=1}^{2} \tilde{R}_2(t_1, t_2)\, s_{\alpha_1}s_{\alpha_2} \\ &+ \cdots \\ &+ \frac{i^j}{j!} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_j=1}^{2} \tilde{R}_j(t_1, t_2)\, s_{\alpha_1}s_{\alpha_2} \cdots s_{\alpha_j}. \end{aligned}$$

At this stage of the problem, even though we cannot state that the output process is distributed as bivariate normal for $\omega \in A_1$, we have a close approximation to $\ln \phi^{(1)}_{Y_1Y_2}(s_1, s_2)$, and hence we are able to obtain a good approximation to $\phi^{(1)}_{Y_1Y_2}(s_1, s_2)$. This information enables us to study the statistical properties of the output process $y(t; \omega)$. For $\omega \in A_2 = (-\infty, 0]$, we have an analogous situation. This can be seen by considering the modified characteristic function

$$\phi^{(2)}_{X_1X_2}(s_1, s_2) = \int_{-\infty}^{0}\int_{-\infty}^{0} e^{i(s_1x_1+s_2x_2)} f(x_1, x_2)\, dx_1\, dx_2 .$$

Using the identical techniques as in the case for $\omega \in A_1$, a good approximation in $\ln \phi^{(1)}_{Y_1Y_2}(s_1, s_2)$ can be obtained.

EXAMPLE 4.3. Let $A = [0, \alpha)$, $A = [\alpha, \infty)$, and the random impulse function be defined by

$$K(t, \tau; \omega) = \begin{cases} e^{\tau-t}, & \text{for } \omega \in A_1 \\ 1 - e^{(\tau-t)}, & \text{for } \omega \in A_2 \end{cases}$$

where $t \geqslant \tau$, for $t, \tau \in [0, T]$. We shall assume that the input process $x(t; \omega)$ is independent of t and is distributed exponentially, that is, the probability density function of $x(t; \omega) = X$ is

$$f(x) = \begin{cases} \dfrac{1}{\beta} e^{-x/\beta} & \text{for } x \geqslant 0, \ \beta > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

For, $\omega \in A_1$,

$$\phi^{(1)}_X(s) = \int_0^{\alpha} e^{i s x} \frac{1}{\beta} e^{-x/\beta}\, dx,$$

or

$$\phi^{(1)}_X(s) = \exp\left\{R_0 + iR_1 s + \frac{i^2}{2!} R_2 s^2 + \cdots\right\},$$

where $R_0, R_1, R_2, \ldots$ are independent of t. For the output process of $y(t; \omega)$, we have for $t \geqslant \tau$,

$$\begin{aligned}
\tilde{R} &= R_0 ,\\
\tilde{R}_1(t) &= \int_0^t e^{\tau-t} R_1\, d\tau \\
&= R_1 e^{-t}(e^t - 1),\\
\tilde{R}_2(t) &= \int_0^t \int_0^t e^{(\tau_1-t)} e^{(\tau_2-t)} R_2\, d\tau_1\, d\tau_2 \\
&= R_2 e^{-2t}(e^t - 1)^2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\tilde{R}_n(t) &= \int_0^t \int_0^t \cdots \int_0^t e^{(\tau_1-t)} \cdots e^{(\tau_1-t)} R_n\, d\tau_1 \cdots d\tau_n \\
&= R_n e^{-nt}(e^t - 1)^n.
\end{aligned}$$

Thus,

$$\phi_Y^{(1)}(s) = \exp\{R_0 + iR_1 e^{-t}(e^t - 1)s + \cdots\}$$
$$= \phi_X^{(1)}(e^{-t}(e^t - 1)s).$$

Therefore, over A_1,

$$Y = e^{-j}(e^j - 1)X.$$

The probability density function of Y for $\omega \in A_1$ is obtained as follows:
Let

$$k = e^{-j}(e^j - 1).$$

Thus, we have

$$y = kx,$$

which yields

$$x = \frac{y}{k}, \tag{4.17}$$

and the Jacobian of transformation (4.17) is

$$\frac{dx}{dy} = \frac{1}{k} > 0.$$

From this, the probability density function $h(y)$ of $y(t; \omega)$ for $\omega \in A_1$ is

$$h(y) = \begin{cases} f\left(\frac{y}{k}\right)\frac{1}{k}, & 0 \leqslant y < e^{-t}(e^t - 1), \\ 0, & \text{elsewhere,} \end{cases}$$

or

$$h(y) = \begin{cases} \frac{1}{e^{-t}(e^t - 1)\beta} \exp\left\{-\frac{y}{e^{-t}(e^t - 1)\beta}\right\}, & 0 \leqslant y < e^{-t}(e^t - 1), \\ 0, & \text{elsewhere.} \end{cases}$$

We see that over A_1, $y(t; \omega)$ is exponentially distributed, with

$$E\{y(t; \omega)\} = e^{-t}(e^t - 1),$$

and

$$\operatorname{var}[y(t; \omega)] = e^{-2t}(e^t - 1)^2.$$

Over A_2, the approach is identical and $y(t; \omega)$ will be exponentially distributed.

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References

DOOB, J. (1953), "Stochastic Processes," Wiley, New York.

KENDALL, M. G., AND STUART, A. (1943), "The Advanced Theory of Statistics," Vol. I, Hafner, New York.

KUZNETSOV, P., STRATONOVICH, R., AND TIKHONOV, V. (1965), "Nonlinear Transformations of Stochastic Processes," Pergamon, New York.

PALEY, R., AND WEINER, N. (1934), "Fourier Transformas in the Complex Domain," Amer. Math. Soc., New York.

TSOKOS, C. P. (1972), "Probability Distributions: An Introduction to Probability Theory with Applications," Duxbury Press, Duxbury, Mass.

VALLE-POUSSIN, referenced in Chapter I, Kuznetsov *et al.* (1965).

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