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Pointwise Blow-up of Sequences Bounded in L^1

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Given a sequence of functions bounded in $L^1([0,1])$, is it possible to extract a subsequence that is pointwise bounded almost everywhere? The main objective of this note is to present an example showing that this is not possible in general. We will also prove a pair of positive results. We show that if the sequence of functions consists of multiples of characteristic functions of measurable sets, the answer is yes. We also show that it is always possible to extract a subsequence that is pointwise bounded on a countable, dense set of points. © 2001 Academic Press

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Given a sequence of functions $\{f^n\}$ bounded in $L^1([0,1])$, is it possible to find a subsequence f^{n_k} and a "large" set $E \subseteq [0,1]$ such that $\sup_k f^{n_k}(x) < \infty$ for every $x \in E$? We will see that if "large" means with positive measure, the answer is no, but if "large" means dense, then the answer is yes. We will also prove that in the special case where $f^n(x) = c^n \chi_{E^n}(x)$, where c^n is constant and E^n is measurable, we can find a subsequence that is pointwise bounded almost everywhere.

The basic problem under consideration arose very naturally from a problem in nonlinear partial differential equations. We will start by outlining the issues in differential equations that lead to our measure theory problem.

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We consider the problem of existence of weak solutions to the incompressible 2D Euler equations having as initial vorticity a bounded Radon measure ω_0 of compact support in the Sobolev $H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$. Existence for the problem above was established in 1991 by J.-M. Delort (see [2, 7]), where the vorticity is the sum of a nonnegative measure as above with an arbitrary function in $L^1_c \cap H^{-1}_{\mathrm{loc}}$. In [7], Schochet described Delort's result in terms of the absence of concentrations in a sequence of approximate vorticities. From the point of view of this work, the key result in Schochet's work is Lemma 3.7, which states that the weak $* - \mathcal{BM}$ limit $\omega = \omega(x, t)$ of a sequence of smooth approximations $\{\omega^n = \omega^n(x, t)\}$ is a weak solution of the 2D incompressible Euler equations (in a suitable sense) if a certain nonconcentration condition is satisfied. This nonconcentration condition is that, for all $x \in \mathbb{R}^2$ and for almost all $0 < t < \infty$,

$$\lim_{\delta \to 0} \sup_n \left| \int_{B(x,\delta)} |\omega^n(x,t)|\, dx \right| = 0. \tag{1}$$

If $\omega_0 \geq 0$, the approximate solution ω^n is also nonnegative. In this case, the integral that appears in (1) is the integral of ω^n itself in a ball of radius δ, which is $\mathcal{O}(|\log \delta|^{-1/2})$ as $\delta \to 0$, uniformly in x and n, for almost all time, a well-known consequence of the conservation of energy.

In [4] the authors, with another collaborator, prove an extension of Delort's result for flows with initial vorticity odd with respect to x_1, nonnegative at $x_1 > 0$. Lopes Filho *et al.* took the same basic approach as Schochet in [7], but the coexistence of positive and negative vorticities made proving nonconcentration harder. We managed to prove a new a priori estimate of the form

$$\left| \int_{B(x,\delta)} |\omega^n(x,t)|\, dx \right| \leq f^n(t) |\log\, \delta|^{-1/2},$$

where the sequence f^n is nonnegative and bounded in $L^2([0, T])$. Clearly, if one could extract a subsequence of $\{f^n\}$ pointwise bounded for almost all t, the existence of a weak solution would be a consequence of Lemma 3.7 of [7]. As the present article shows, this is not possible, so that a new approach to nonconcentration had to be developed for Lopes Filho *et al.* to prove their result. The main motivation for the present note is to justify the introduction of the notion of time-averaged nonconcentration in [4].

A few trivial observations about our problem should be made right away. First, we can substitute the L^1 bound for bounds in any Orlicz space. Renormalizing the sequence by composing with the appropriate nonlinear function reduces the problem back to L^1. If the sequence f^n converges strongly in L^1 to something, then there exists a subsequence f^{n_k} which converges almost everywhere (see [6, Theorem 3.12]), and convergence almost everywhere trivially implies pointwise boundedness almost everywhere. Also, if one has a little additional regularity, such as a $W^{s,p}$ bound for some $s > 0$, the compactness of the Sobolev imbedding implies that a subsequence converging strongly in L^q, for $q < np/(n - sp)$, can be extracted, and the question is again answered in the affirmative.

For simplicity, we will assume hereafter that all functions are nonnegative. There is no loss of generality in this assumption.

One can easily find a counterexample for our original contention if the passage to subsequence is dropped. Consider the intervals $I^{n,j} = [j/n, (j+1)/n]$, with $j = 0, 1, \ldots, n-1$ and $f^{n,j} \equiv n\chi_{I^{n,j}}$. Clearly this sequence has infinite supremum everywhere, but it is also easy to extract a subsequence pointwise bounded almost everywhere: fix $a \in [0,1]$ and consider $f^{n,j}$ such that $a \in I^{n,j}$. Clearly the only point where such a subsequence is unbounded is a itself.

Our first result is the existence of an a.e. pointwise bounded subsequence if the sequence of functions consists of constant multiples of characteristic functions of measurable subsets of $[0,1]$.

Proposition 1. *Let $f^n = f^n(x) = c^n\chi_{E^n}(x)$, where $c^n \in \mathbb{R}$ and $E^n \subseteq [0,1]$ are measurable. Assume that $\{f^n\}$ is bounded in $L^1([0,1])$. Then, there exists a subsequence f^{n_k} of f^n and a set $E \subseteq [0,1]$ such that for any $x \in E$, $\sup_k f^{n_k}(x) < \infty$ and $|E| = 1$.*

Proof. If the sequence of numbers c^n is bounded, there is nothing to prove. If the sequence is not bounded, consider a subsequence c^{n_k} such that $c^{n_k} > k^2$, and n_k is increasing in k. This can always be done for an unbounded sequence. Since $\{f^n\}$ is bounded in L^1, we have that, for some $C > 0$, $|E^{n_k}| \leq C/k^2$, so that

$$\sum_{k=1}^{\infty} |E^{n_k}| < \infty.$$

The set of points x in $[0,1]$ for which the $\sup_k |f^{n_k}(x)| = \infty$ is precisely the set of points contained in an infinite number of the E^{n_k}, which has measure zero by the Borel–Cantelli lemma. ∎

The main result of this note is the construction of a sequence of functions f^n, bounded in $L^1([0,1])$, such that any subsequence f^{n_k} of f^n has supremum a.e. infinite. The argument we will present below is due to C. S. Isnard and B. F. Svaiter (personal communication). We begin the construction with a definition.

Definition 1. Let $\{E^n\}$ be a sequence of measurable subsets of $[0,1]$ and $0 < \alpha < 1$. We will say that $\{E^n\}$ is an α-essential cover of $[0,1]$ if $|E^n| \leq \alpha$, for all n, and, for any subfamily $\{E^{n_k}\}$ of $\{E^n\}$,

$$\left|\bigcup_k E^{n_k}\right| = 1.$$

LEMMA 1. *For any* $0 < \alpha < 1$ *there exists an* α*-essential cover of* $[0, 1]$.

Proof. The proof is based on the classical construction of Cantor sets. We assume w.l.o.g. that $\alpha = 1/N$ for some integer N since an α-essential cover is also a β-essential cover for any $\beta > \alpha$. Points $a \in (0, 1)$ are described in base N as an infinite sequence of digits $a = 0.a_1 a_2 \ldots$, with each $a_i = 0, 1, \ldots, N - 1$. This map from $(0, 1)$ into the functions of $\mathbb{N} \to \mathbb{Z}_N$ is not a bijection, because certain rational numbers, those which have finite expansion in base N, are associated with the two distinct expansions $0.a_1 a_2 \ldots a_n 00 \ldots$ and $0.a_1 a_2 \ldots (a_n - 1)(N - 1)(N - 1) \ldots$. From now on, when we refer to the interval $(0, 1)$, we mean the interval with this set of measure zero removed, so to each point in $(0, 1)$ is associated a unique expansion in base N.

We define E^n as the set of points a such that the nth digit in the expansion of a is zero. The Lebesgue measure of E^n is $1/N$, since one could partition $(0, 1)$ into N sets, where $a_n = 0, 1, 2, \ldots, N - 1$, which can all be transformed into one another by a translation. This means that they all have the same Lebesgue measure, which is then $1/N$. An infinite subfamily of the E^n corresponds to the choice of an infinite number of digits $a_{n_1}, a_{n_2}, \ldots$. The union of the E^{n_k} is the set of points for which at least one of these digits a_{n_k} is zero, and hence, the set of points which are not contained in $\bigcup E^{n_k}$ is the set of points for which a_{n_k} is nonzero for all k. The Lebesgue measure is the density associated to the uniform probability distribution on $[0, 1]$, so that the probability of a subset is equal to its measure. The key point is that, as random variables, the digits a_n are independent and uniformly distributed in $\mathbb{Z}_{\mathbb{N}}$ (this is proved for the binary expansion in [1] (see [1, Sections 1.1 and 1.4]), and the discussion there can be easily extended for the base-N expansion). Hence, the probability of the set where all a_{n_k} are nonzero is zero. ∎

The important point for what follows is not the statement of Lemma 1, but the fact that for each N, the corresponding family $\{E^n\}$ constructed in the proof above is a $1/N$-essential cover of $(0, 1)$.

THEOREM 1. *There exists a sequence of functions* f^n *bounded in* $L^1([0, 1])$ *such that for any subsequence* f^{n_k} *of* f^n, *the pointwise supremum* $\sup_k f^{n_k}(x) = \infty$ *for almost all* $x \in (0, 1)$.

Proof. For each $k = 1, 2, \ldots$ let $E^{n,k}$ be the nth set in the construction above, taking $N = 4^k$. Consider the functions

$$f^n = f^n(x) = \sum_{k=1}^{\infty} 2^k \chi_{E^{n,k}}(x).$$

Note first that

$$\int_0^1 f^n\,dx \le \sum_{k=1}^{\infty} 2^k |E^{n,k}| = \sum_{k=1}^{\infty} 2^k 4^{-k} = 1.$$

Next, observe that $E^{n,k} \subseteq \{x \in (0,1)\colon f^n(x) > 2^k\}$, so that for any subsequence $f^{n(l)}$ of f^n,

$$\bigcup_{l=1}^{\infty} E^{n(l),k} \subseteq \left\{x \in (0,1)\colon \sup_l f^{n(l)}(x) > 2^k\right\},$$

which, by the fact that $E^{n,k}$ is a 4^{-k}-essential cover of $(0,1)$, implies that the set $\{x \in (0,1)\colon \sup_l f^{n(l)}(x) > 2^k\}$ has total measure. However,

$$\left\{x \in (0,1)\colon \sup_l f^{n(l)}(x) = \infty\right\} = \bigcap_{k=1}^{\infty} \left\{x \in (0,1)\colon \sup_l f^{n(l)}(x) > 2^k\right\},$$

so that the set of points where the supremum of $f^{n(l)}$ is infinite is written as a countable intersection of sets of total measure, which then has total measure. ∎

The example in the result above shows that it is not possible, in general, to extract from a bounded sequence in L^1 a subsequence pointwise bounded on a set of positive measure. There are several refinements of the Lebesgue measure which allow us to distinguish the relative size of sets of measure zero. We will see below that the sequence $\{f^n\}$ above is pointwise bounded on a large (measure zero) set.

THEOREM 2. *Any subsequence of the sequence $\{f^n\}$ constructed in Theorem 1 is pointwise bounded on a set of Hausdorff dimension one.*

Proof. It is enough to prove that the whole sequence is pointwise bounded on a set of Hausdorff dimension one.

For each integer M, let $\{E^{n,M}\}$ be the $1/M$-essential cover of $(0,1)$ introduced in Lemma 1. Let

$$F^M \equiv \left(\bigcup_{n=1}^{\infty} E^{n,M}\right)^c.$$

The set F^M is a Cantor-like set, which closely resembles a uniform Cantor set (see [3, Example 4.5]), with $m = M - 1$ and $\lambda = 1/M$. The calculation of the Hausdorff dimension of uniform Cantor sets can be adapted in a straightforward way to F^M, and we obtain that

$$\dim_H F^M = \frac{\log(M-1)}{\log M}.$$

In the notation above, the sequence $\{f^n\}$ is written as

$$f^n(x) = \sum_{k=1}^{\infty} 2^k \chi_{E^{n,4^k}}.$$

We have that, for fixed n,

$$\{x: f^n(x) \geq 2^k\} = \bigcup_{j\geq k} E^{n,4^j}.$$

Indeed, if $x \in \cup_{j\geq k} E^{n,4^j}$ then there is $j_0 \geq k$ such that $x \in E^{n,4^{j_0}}$, so that $f^n(x) \geq 2^{j_0} \geq 2^k$. On the other hand, if $x \notin \cup_{j\geq k} E^{n,4^j}$ then $f^n(x) \leq \sum_{l=1}^{k-1} 2^l = 2^k - 2$.

Next we observe that

$$\left\{x: \sup_n f^n(x) \geq 2^k\right\} = \bigcup_n \bigcup_{j\geq k} E^{n,4^j}.$$

The fact that the left-hand set contains the set on the right-hand side is immediate. On the other hand, x does not belong to the set in the right-hand side; then, for every n, $f^n(x) \leq 2^k - 2$, as we deduced above. This implies that $\sup_n f^n(x) < 2^k$, which proves the equality of the sets.

From what we have seen, taking first the union over n, and then over j, we have

$$\left\{x: \sup_n f^n(x) = \infty\right\} = \bigcap_k \bigcup_{j\geq k} \bigcup_n E^{n,4^j}.$$

Hence, taking complements, we get

$$\left\{x: \sup_n f^n(x) < \infty\right\} = \bigcup_k \bigcap_{j\geq k} F^{4^j}. \tag{2}$$

We claim that if $2k \leq j+1$ then $F^{4^k} \subseteq F^{4^j}$. To see this, fix $a \in (0,1)$ and let a_i, b_i, and c_i be the ith digit in the expansion of a in bases 4, 4^j, and 4^k, respectively. Note that, for any $i = 1, 2, \ldots,$

$$b_i = \sum_{\ell=1}^{j} a_{(i-1)j+\ell} 4^{j-\ell} \quad \text{and} \quad c_i = \sum_{\ell=1}^{k} a_{(i-1)k+\ell} 4^{k-\ell}.$$

If $b_{i_0} = 0$, for some index i_0, then, by the relation above, there exists a sequence of j consecutive zeroes in the expansion of a in base 4, namely, $a_{(i_0-1)j+1}$ up to $a_{i_0 j}$. The condition $2k \leq j+1$ is designed precisely to guarantee that if one partitions the expansion in base 4 of a into consecutive groups of k digits, then there will be at least one group of k digits falling entirely within the range $a_{(i_0-1)j+1}$ to $a_{i_0 j}$. This group of k zeroes corresponds to a single digit in the expansion of a in base 4^k, which, from the relation above, is zero. Therefore, since the set F^{4^k} is the set of numbers in $(0, 1)$ whose expansion in base 4^k contains no zeroes, the argument above shows that this set is contained in F^{4^j}, as we wished.

Finally, by the claim above and (2),

$$\left\{x\colon \sup_n f^n(x) < \infty\right\} \supseteq \bigcup_k F^{4^{[(k+1)/2]}} \supseteq F^{4^\ell},$$

for all $\ell = 1, 2, \ldots$. This means that $\{x\colon \sup_n f^n(x) < \infty\}$ contains subsets of Hausdorff dimension $\log(4^\ell - 1)/\log(4^\ell)$, which can be made arbitrarily close to 1. ∎

We conclude with the best general result we have found.

THEOREM 3. *Let f_n be a sequence of smooth functions, uniformly bounded in $L^1([0, 1])$. There exists a subsequence $\{f_{n_k}\}$ and a countable, dense set of points $\{x_m\}$ in $[0, 1]$ such that $\sup_k f_{n_k}(x_m) < \infty$ for every $m = 1, 2, \ldots$.*

Proof. By Fatou's lemma we have $\int \liminf |f_n| < +\infty$, so $\liminf |f_n|(x) < +\infty$ for almost all $x \in [0, 1]$. For those x, there clearly exists a subsequence $\{f_{n_k}\}$ (which depends on x) such that $\sup_k f_{n_k}(x) < \infty$.

Next consider the family of intervals in $[0, 1]$ of the form $(i/2^n - 1/2^{n+1}, i/2^n + 1/2^{n+1})$, and choose an arbitrary enumeration of this set of intervals where the mth interval is denoted by I_m. Restrict the sequence f_n to I_1 and use the argument above to obtain a subsequence $f_{n_k,1}$ and a point $x_1 \in I_1$ such that $\sup_k f_{n_k,1}(x_1) < \infty$. Next, restrict the subsequence $f_{n_k,1}$ to I_2 and choose the $f_{n_k,2}$ subsequence of $f_{n_k,1}$ and a point $x_2 \in I_2$ such that $\sup_k f_{n_k,2}(x_2) < \infty$. We proceed in such a way for all m. The diagonal subsequence $f_{n_k,k}$ and the set $\{x_k\}$ possess the desired property. ∎

Due to Theorem 2, there is space between Theorems 1 and 3 for the following question: Given a sequence $\{f_n\}$, bounded in $L^1([0, 1])$, can one find a subsequence f_{n_k} and a set E, with Hausdorff dimension 1 (or even greater than zero), such that f_{n_k} is pointwise bounded on E? An affirmative answer might have interesting applications in partial differential equations.

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