Algebras Almost Commuting with Clifford Algebras

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We consider Banach algebras of operators on real and complex Hilbert spaces which almost commute with Clifford algebras in the sense that the commutator with all but one generator of the Clifford algebra is trivial but with the last, lies in a symmetric ideal of compact operators. We compute the K-groups of these algebras and prove periodicity theorems. We then show that these algebras form "non-commutative classifying spaces" for the Ext-functor on C^*-algebras and suitable modifications of them perform a similar role for KK. © 1990 Academic Press, Inc.

1. INTRODUCTION

In this paper we are concerned with the K-groups of certain Banach algebras of operators on Hilbert space. The most illuminating way to think of these algebras is as non-commutative classifying spaces for the Ext-functor on C^*-algebras and also for KK. Their construction in fact parallels that for commutative K-theory as described in Atiyah and Singer [A-S].

The motivation for considering these algebras begins with a remark in [L-S] where it was noted that the phase transition in the (one-sided) two dimensional Ising model was signalled by a jump in the mod 2 index of a

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580 88 2-4

skew-adjoint Fredholm operator. This remark led to a construction of a mod 2 index for the group of Bogoliubov automorphisms of the infinite dimensional Clifford algebra in [C-O]. Subsequently Araki and Evans [A-E] found the phase transition in the two-sided Ising model to be signalled by a jump in a $\mathbb{Z}_2$ index of a certain space of projections. The relationship between all these manifestations of a mod 2 index was eventually sorted out in [C-E]. The investigations of this last paper motivated the present analysis which is in fact independent of much of this earlier literature.

The basic objects of study here are algebras $\mathscr{A}_k$ defined for $k = 1, 2, ...$, which almost commute with the Clifford algebra on k generators in the sense that the elements of $\mathscr{A}_k$ commute with the first $k-1$ generators of the Clifford algebra and have commutators with the kth generator which lie in a symmetric ideal of compact operators (see [Si] for a definition of the latter). These algebras are periodic in k with period 2 in the complex case and period 8 in the real case. We are able to compute K_0 and K_1 for these algebras and to establish a further periodicity in that a shift by one in k induces a shift by one in K_*. The proofs of the latter periodicity results are inspired by and parallel to the proof of Bott periodicity via Morse theory in [Mi].

The periodicity results suggested that these algebras may posses a "classifying space" role. In fact we show that the Ext-groups of a complex C^*-algebra are realised by homotopy classes of *-homomorphisms into $\mathscr{A}_k$. This fact also applies to KK if the algebras $\mathscr{A}_k$ are suitably modified. We have not pursued the question of whether this provides new information on KK-theory although this would certainly merit further investigation. We conclude our discussion with some remarks on related algebras of operators and on the equivariant case.

2. ALGEBRAS ALMOST COMMUTING WITH CLIFFORD ALGEBRAS

Let E denote a real infinite dimensional separable Hilbert space, H the same space equipped with a complex structure (written i), and let $\mathfrak{S}$ denote a symmetric ideal of compact operators on E or H (no confusion will arise in the sequel).

Let $\mathscr{C}_k$ denote the real Clifford algebra on k generators $J_1, ..., J_k$ satisfying $J_i J_j + J_j J_i = -2\delta_{ij}$, $J_i^* = -J_i$ acting on E (we assume that the multiplicity of each irreducible representation of $\mathscr{C}_k$ is infinite). We let $\mathscr{C}_k \otimes \mathbb{C}$ denote the complex Clifford algebra on the same k generators but now assume they act on H with the same condition on the multiplicity of irreducibles.

Introduce the algebras

$$\mathscr{A}_k(E) = \{A \in B(E) | [A, \mathscr{C}_{k-1}] = 0, [A, J_k] \in \mathfrak{S}\}$$
$$\mathscr{A}_k(H) = \{A \in B(H) | [A, \mathscr{C}_{k-1} \otimes \mathbb{C}] = 0, [A, J_k] \in \mathfrak{S}\}.$$

Let Γ_k denote the group generated by $J_1, ..., J_k$ and define for A in $B(E)$

$$\mu_k(A) = \left(\sum_{\gamma \in \Gamma} \gamma A \gamma^* \right) \bigg/ 2^{k+1}.$$

Then μ_k projects onto the subalgebra of $B(R)$ consisting of operators which commute with $\mathscr{C}_k$.

LEMMA 2.1. *For all A in $B(E)$ such that $[A, \mathscr{C}_k] \in \mathfrak{S}$, $A - \mu_k(A)$ lies in $\mathfrak{S}$.*

Proof. Define a character of Γ_k by $\varepsilon(J_i) = -1$. Then $A - \mu_k(A)$ is the sum of terms of the form

$$(A - \varepsilon(J_{i_1} \cdots J_{i_j}) J_{i_1} \cdots J_{i_j} A J_{i_j} \cdots J_{i_1})/2^{k+1}.$$

We induct on j starting with the observation that $A + J_i A J_i \in \mathfrak{S}$ for all i. Then

$$\begin{aligned}
A - &\varepsilon(J_{i_1} \cdots J_{i_j}) J_{i_1} \cdots J_{i_j} A J_{i_j} \cdots J_{i_1} \\
&= A - \varepsilon(J_{i_1} \cdots J_{i_{j-1}}) J_{i_1} \cdots J_{i_{j-1}} A J_{i_{j-1}} \cdots J_{i_1} \\
&\quad + \varepsilon(J_{i_1} \cdots J_{i_{j-1}}) J_{i_1} \cdots J_{i_{j-1}} A J_{i_{j-1}} \cdots J_{i_1} \\
&\quad - \varepsilon(J_{i_1} \cdots J_{i_j}) J_{i_1} \cdots J_{i_j} A J_{i_j} \cdots J_{i_1}.
\end{aligned}$$

The last two terms may be combined as

$$\varepsilon(J_{i_1} \cdots J_{i_{j-1}}) J_{i_1} \cdots J_{i_{j-1}} [A - \varepsilon(J_{i_j}) J_{i_j} A J_{i_j}] J_{i_{j-1}} \cdots J_{i_1}$$

so that the result follows from the induction hypothesis applied to the first pair of terms and the last pair of terms separately.

Thus each element of $\mathscr{A}_k(E)$ is a sum

$$A = \mu_k(A) + [A - \mu_k(A)]$$

of a term which commutes with $\mathscr{C}_k$ and an element of $\mathfrak{S}$. Now define a norm on $\mathscr{A}_k(E)$ by

$$\|A\|_k = \|A\| + \sum_{\gamma \in \Gamma} \|A\gamma - \gamma A\|_{\mathfrak{S}}. \tag{2.1}$$

Then an easy calculation shows that $\mathscr{A}_k(E)$ is a Banach algebra in this norm.

Let $\mathcal{O}_k$ and $\mathcal{U}_k$ denote the groups of orthogonal and unitary operators in $\mathscr{A}_k(E)$ and $\mathscr{A}_k(H)$, respectively.

The fact that $\mu_k(G) = (G - J_k G J_k)/2$ for G in $\mathcal{O}_k$ implies the relation

$$\mu_k(G)^* \mu_k(G) = 1 - (G^* - \mu_k(G)^*)(G - \mu_k(G)) \tag{2.2}$$

so that $\mu_k(G)$ is Fredholm. Note that G maps $\ker \mu_k(G)$ onto $\ker \mu_k(G)^*$ so that $\mu_k(G)$ has Fredholm index zero. When $\mathscr{C}_k$ is not simple ($k = 4n - 1$) then we introduce $W = J_1 J_2 \cdots J_k$ which is central in $\mathscr{C}_k$ with $W^2 = 1$ and define $P_+ = (1 + W)/2$. Now $P_+ G P_+$ satisfies the relations

$$P_+ G^* P_+ P_+ G P_+ + P_+ G^* P_- P_- G P_+ = P_+$$

$$P_- G^* P_- P_- G P_- + P_- G^* P_+ P_+ G P_- = P_- .$$

It is easy to check that $P_- G P_+$ and $P_+ G P_-$ both lie in $\mathfrak{S}$ so that $P_+ G P_+$ and $P_- G P_-$ are both Fredholm. Moreover these relations imply that G maps $\ker P_+ G P_+$ onto $\ker P_- G^* P_-$ and so the Fredholm index of $P_+ G P_+$ is minus that of $P_- G P_-$. Let $\mathcal{O}_{k,*}$ denote the subgroup of $\mathcal{O}_k$ consisting of those G with $P_+ G P_+$ having Fredholm index zero. In the complex case, for k odd, one similarly introduces the spectral projections $P_\pm$ of W and the analogous subgroup $\mathcal{U}_{k,*}$ of $\mathcal{U}_k$. Finally we remark that

$$P_+ G P_+ + P_- G P_- = \mu_k(G) \qquad \text{for} \quad G \text{ in } \mathcal{O}_k.$$

The following lemma is basic to all the results in this section.

LEMMA 2.2.   *If $\mathscr{C}_k (\mathscr{C}_k \otimes \mathbb{C})$ is simple and G is in $\mathcal{O}_k(\mathcal{U}_k)$ then we can write $G = U(1 + X)$ where U is in $\mathcal{O}_k(\mathcal{U}_k)$ with $\mu_k(U) = U$ and X is in $\mathfrak{S}$ with $\mu_k(1 + X) \geqslant 0$. When $\mathscr{C}_k (\mathscr{C}_k \otimes \mathbb{C})$ is not simple then this decomposition holds only for elements of $\mathcal{O}_{k,*}(\mathcal{U}_{k,*})$.*

Proof.   When $\mathscr{C}_k$ is simple we use $\mu_k(G) = (G - J_k G J_k)/2$ and take U to be the isometry in the polar decomposition of $\mu_k(G)$ initially defined only on the orthogonal complement of $\ker \mu_k(G)$ but then extended to all of E by taking it to be any isometry intertwining the equivalent $\mathscr{C}_k$ modules $\ker \mu_k(G)$ and $\ker \mu_k(G)^*$. The other statements in the lemma follow as in Proposition 2.1 [C-O] (which deals with the case $k = 1$). A similar argument works in the complex case for k even (when the Clifford algebra is simple).

When $\mathscr{C}_k$ is not simple and G is in $\mathcal{O}_{k,*}$, let $G_+(G_-)$ be the isometry in the polar decomposition of $P_+ G P_+(P_- G P_-)$. Now we may extend G_+ all of $P_+ E$ by defining it to be any isometry which intertwines the $\mathscr{C}_{k-1}$ action

on ker $P_+ GP_+$ with that on ker $P_+ G^* P_+$. Similarly extend G_- to all of $P_- E$ by letting it intertwine the $\mathscr{C}_{k-1}$ action on ker $P_- GP_-$ with that on ker $P_- G^* P_-$. Let $U = G_+ + G_-$. Then it follows as in Lemma 3.2 of [C1] that $U^* G$ has the required form $1 + X$ with X in $\mathfrak{S}$. The complex case is similar.

LEMMA 2.3.   *The algebras $\mathscr{A}_k(H)$ and $\mathscr{A}_{k+2}(H)$ are isomorphic as are the algebras $\mathscr{A}_k(E)$ and $\mathscr{A}_{k+8}(E)$.*

Proof. The real and complex cases are essentially the same so we consider only the latter. We construct $\mathscr{A}_{k+2}(H)$ directly from $\mathscr{A}_k(H)$ by letting $\mathbb{C}^2$ carry a representation of $\mathscr{C}_2$ generated by J_1 and J_2 and define the $\mathscr{C}_{k+2}$ action on $\mathbb{C}^2 \otimes H$ by letting the generators be $J_1 \otimes 1$, $J_2 \otimes 1$, $J_1 J_2 \otimes J_3, ..., J_1 J_2 \otimes J_{k+2}$, where $J_3, ..., J_{k+2}$ generate the $\mathscr{C}_k$ action on H. Then as J_1 and J_2 generate the 2×2 matrices on $\mathbb{C}^2$ every operator in $\mathscr{A}_{k+2}(H)$ has the form $1 \otimes A$ with A in $\mathscr{A}_k(H)$. This construction clearly sets up an isomorphism of $\mathscr{A}_{k+2}(H)$ with $\mathscr{A}_k(H)$.

Although it is slightly peripheral to our purpose the groups $\mathscr{O}_k(\mathscr{U}_k)$ (or $\mathscr{O}_{k,*}(\mathscr{U}_{k,*})$ when $k = 4n - 1$) have known homotopy type which we now determine. Let $\mathscr{O}_{k,\mu}$ denote the group of orthogonals in $\mathscr{O}_k$ in the range of μ_k and let $\mathscr{O}_{\mathfrak{S}}$ denote the group of orthogonals differing from the identity by an element of $\mathfrak{S}$. Following Milnor [Mi] introduce the space $\Omega_k(n)$ of all complex structures J on $\mathbb{R}^n$ which anticommute with the generators of $\mathscr{C}_{k-1}$ (which we assume also acts on $\mathbb{R}^n$). When $k = 4n - 1$ we let $\Omega_k(n)$ denote the connected component of maximal dimension. Now let $\Omega_k(\infty)$ denote the inductive limit of the spaces $\Omega_k(n)$ as $n \to \infty$. Similarly define $\Omega_k(\infty, \mathbb{C})$ in terms of the spaces $\Omega_k(n, \mathbb{C})$ consisting of complex structures on $\mathbb{C}^n$ anticommuting with the generators of $\mathscr{C}_{k-1} \otimes \mathbb{C}$.

Consider the space $\mathscr{X}_k(\mathscr{X}_k^{\mathbb{C}})$ of complex structures J on $E(H)$ which anticommute with the generators of $\mathscr{C}_{k-1}(\mathscr{C}_{k-1} \otimes \mathbb{C})$ and differ from J_k by an element of $\mathfrak{S}$. We claim this is the homogeneous space $\mathscr{O}_k / \mathscr{O}_{k,\mu}(\mathscr{U}_k / \mathscr{U}_{k,\mu})$. It is easy to see that $\mathscr{O}_k(\mathscr{U}_k)$ acts on $\mathscr{X}_k(\mathscr{X}_k^{\mathbb{C}})$ by conjugation while transitivity of the action is more difficult although the proof is essentially the same in both cases. We sketch the complex case. Let k be odd so that we may regard H as the direct sum $H_1 \oplus H_1$ with H_1 carrying a representation of $\mathscr{C}_{k-1} \otimes \mathbb{C}$. Then the action of J_k on H is given with respect to this splitting by the matrix

$$\begin{pmatrix} -J_1 J_2 \cdots J_{k-1} & 0 \\ 0 & J_1 J_2 \cdots J_{k-1} \end{pmatrix}.$$

Now if J lies in $\mathscr{X}_k^{\mathbb{C}}$ the operator

$$\begin{pmatrix} J_1 & 0 \\ 0 & J_1 \end{pmatrix} \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix} \cdots \begin{pmatrix} J_{k-1} & 0 \\ 0 & J_{k-1} \end{pmatrix} J$$

has square one or minus one and so also defines a splitting of H into spectral subspaces $H^\pm$ each carrying a representation of $\mathscr{C}_{k-1} \otimes \mathbb{C}$. Now H^+ and H^- are both infinite dimensional (as $J - J_k$ lies in $\mathfrak{S}$) and so these representations are equivalent allowing us to identify H^+ with H^-. With respect to this splitting we have J given by the matrix

$$\begin{pmatrix} -J_1 J_2 \cdots J_{k-1} & 0 \\ 0 & J_1 J_2 \cdots J_{k-1} \end{pmatrix}.$$

Furthermore the representations of $\mathscr{C}_{k-1} \otimes \mathbb{C}$ on $H^\pm$ are equivalent to that on H_1 so that we may let $U^\pm : H^\pm \to H_1$ be unitaries intertwining the $\mathscr{C}_{k-1} \otimes \mathbb{C}$ action. Setting $U = U^+ \oplus U^- : H^+ \oplus H^- \to H_1 \oplus H_1$ we deduce that $UJU^{-1} = J_k$. This last relation implies that U lies in $\mathscr{U}_k$. Now for $\mathscr{U}_{k+1}$ we continue with the preceding notation and introduce J_{k+1} as the operator on $H = H_1 \oplus H_1$ given by the matrix

$$\begin{pmatrix} 0 & J'_{k+1} \\ J'_{k+1} & 0 \end{pmatrix},$$

where J'_{k+1} is a complex structure on H_1 which anticommutes with $J_1, J_2, ..., J_{k-1}$. Now if J is in $\mathscr{X}^{\mathbb{C}}_{k+1}$ then J must intertwine the two spectral subspaces of $J_1 J_2 \cdots J_k$ and so have the form

$$\begin{pmatrix} 0 & J' \\ J' & 0 \end{pmatrix}$$

with respect to the decomposition $H = H_1 \oplus H_1$. Now J_{k+1} and J differ by an element of $\mathfrak{S}$ and anticommute so we can assume by the preceding argument that there is a unitary $U_1 : H_1 \to H_1$ such that

$$U_1 J'_{k+1} U_1^* = J'$$

and U_1 commutes with the $\mathscr{C}_{k-1} \otimes \mathbb{C}$ action. Now define U to be the operator

$$U_1 \oplus U_1 : H_1 \oplus H_1 \to H_1 \oplus H_1.$$

Then U is the required element of $\mathscr{U}_{k+1}$ with $U J_{k+1} U^* = J$.

With this preliminary result out of the way we are now ready to state:

PROPOSITION 2.4. *When $\mathscr{C}_k$ and $\mathscr{C}_k \otimes \mathbb{C}$ are simple then the groups $\mathcal{O}_k$ and $\mathscr{U}_k$ have the homotopy type of the spaces $\Omega_k(\infty)$ and $\Omega_k(\infty, \mathbb{C})$, respectively, while for $\mathscr{C}_k$ and $\mathscr{C}_k \otimes \mathbb{C}$ not simple the corresponding statement holds for $\mathcal{O}_{k,*}$ and $\mathscr{U}_{k,*}$.*

Proof. Note firstly that this result is known in the complex case for $k = 1$ (see [C1]) and is a simple consequence of Lemma 2.2 for $k = 2$. It is also known in the real case for $k = 1$. The proof in the general case is the same so we merely sketch it here. Firstly observe that as the group of orthogonals commuting with $\mathscr{C}_k$ is a Kuiper group [K] it is contractible.

Hence $\mathcal{O}_k$ has the homotopy type of the quotient space $\mathscr{X}_k$, which by virtue of the preceding lemma is also the homogeneous space $\mathcal{O}_2/\mathcal{O}_S \cap \mathcal{O}_{k,\mu}$. Now $\mathscr{X}_k$ has the homotopy type of $\Omega_k(\infty)$ (again this is well known for the cases $k = 1$ and 5 and may be proved for the other cases in the same way as in de la Harpe [H, pp. III.6–7]) completing the proof.

With this preliminary material dispensed with we now introduce the main ideas of this section. Following [A–B–S, A–S] introduce the Grothendieck group $\mathscr{G}_k$ of $\mathscr{C}_k$ modules modulo those extendable to $\mathscr{C}_{k+1}$ modules and the analogous complex group denoted $\mathscr{G}_k^{\mathbb{C}}$. Define maps

$$\mathrm{ind}_k : \mathcal{O}_k \to \mathscr{G}_k, \qquad \mathrm{ind}_k^{\mathbb{C}} : \mathcal{U}_k \to \mathscr{G}_k^{\mathbb{C}}$$

given in both cases by taking an orthogonal G to the element $[\ker \mu_k(G)]$ of $\mathscr{G}_k$ or $\mathscr{G}_k^{\mathbb{C}}$ determined by $\ker \mu_k(G)$. We call $\mathrm{ind}_k(G)$ or $\mathrm{ind}_k^{\mathbb{C}}(G)$ the index of G.

PROPOSITION 2.5. *The maps ind_k and $\mathrm{ind}_k^{\mathbb{C}}$ are continuous.*

Proof. The observation that for G in $\mathcal{O}_k$ or $\mathcal{U}_k$,

$$\mu_k(G) = (G - J_k G J_k)/2$$

means that this result is a corollary of the argument in Section 3 of [C-O]. For the reader's convenience we sketch how this goes. Once again the proof is the same in both the real and complex cases and so we restrict to the real case. Firstly one argues that if G_1 and G_2 are sufficiently close then there is an interval $[0, \varepsilon]$ in $\mathbb{R}$ such that if Q is the spectral projection of $\mu_k(G_2)^* \mu_k(G_2)$ corresponding to this interval, then the projection onto $\ker \mu_k(G_1)$ defines, on restriction to QE, a vector space isomorphism of QE onto $\ker \mu_k(G_1)$ which commutes with the $\mathscr{C}_k$ action on these two spaces. Now $(G_2 - \mu_k(G_2))^* \mu_k(G_2)$ is skew adjoint, anticommutes with J_k, has trivial kernel on $(\ker \mu_k(G_2))^{\perp} \cap QE$, and leaves the latter space invariant. So let J denote the isometry in the polar decomposition of $(G_2 - \mu_k(G_2))^* \mu_k(G_2)$. Then, as an operator on $(\ker \mu_k(G_2))^{\perp} \cap QE$, J satisfies the relations $J^2 = -1$, $J^* = -J$, $JJ_i = J_i J$ for $i < k$ and $JJ_k = -J_k J$. Thus we can set $J_{k+1} = JJ_k$ so that $(\ker \mu_k(G_2))^{\perp} \cap QE$ carries a representation of $\mathscr{C}_{k+1}$. Thus $\mathrm{ind}_k(G_1) - \mathrm{ind}_k(G_2) = [(\ker \mu_k(G))^{\perp} \cap QE]$ is a $\mathscr{C}_{k+1}$ module proving that in fact $\mathrm{ind}_k(G_1) = \mathrm{ind}_k(G_2)$.

Before proceeding to show that the index separates connected com-

ponents let us first interpret following [A-S] what the index means in each case where the group G_k^C is non-trivial. In the complex case for k odd the group G_k^C is just $\mathbb{Z}$. Now a $\mathscr{C}_k$ module ker $\mu_k(G)$ is extendable to a $\mathscr{C}_{k+1}$ module exactly when the spectral subspaces of W have the same multiplicity (since J_{k+1} interchanges them); that is, when $P_+ GP_+$ has Fredholm index zero. Thus we may identify $\text{ind}_k(G)$ with the Fredholm index of $P_+ GP_+$.

In the real case for $k = 1 \pmod 8$ a $\mathscr{C}_k$ module ker $\mu_k(G)$ is extendable to a $\mathscr{C}_{k+1}$ module when the multiplicity of the unique irreducible $\mathscr{C}_k$ module is even so that $\text{ind}_k(G)$ is the multiplicity of this module $(\bmod\ 2)$. For $k = 3, 7 \pmod 8$, as above for the complex case, $\text{ind}_k(G) = $ Fredholm index of $P_+ GP_+$. Finally $\text{ind}_k(G)$ for $k = 0 \pmod 8$ is the multiplicity of the irreducible representation of $\mathscr{C}_k$ $(\bmod\ 2)$.

PROPOSITION 2.6. *Two elements of $\mathcal{O}_k(\mathcal{U}_k)$ with the same index are connected.*

Proof. As usual we concentrate on the real case. So let G_1 and G_2 be in $\mathcal{O}_k$ with the same index. For $k = 4n - 1$ additivity of the Fredholm index shows that we may as well assume the index is zero in that case. By Lemma 2 we can write $G_j = U_j(1 + X_j)$ with U_j commuting with $\mathscr{C}_k$ $(j = 1, 2)$. Using the contractibility of Kuiper groups again we conclude that U_j is connected to the identity. On the orthogonal complement of ker $\mu_k(G_j)$ we may write $1 + X_j$ as $\cos A_j + K_j \sin A_j$, where K_j is the isometry in the polar decomposition of $U_j^*(G_j + J_k G_j J_k)/2$ restricted to (ker $\mu_k(G_j)$) and A_j is positive. Now $1 + X_j$ restricts on ker $\mu_k(G_j)$ to an orthogonal S_j anticommuting with J_k so that $1 + X_j$ is the sum of S_j and $\cos A_j + K_j \sin A_j$ where the latter is taken to be zero on ker $\mu_k(G_j)$. Now consider the path of operators

$$S_j + \cos sA_j + K_j \sin sA_j, \qquad \text{for} \quad s \in [0, 1].$$

This connects $1 + X_j$ with an operator which is the identity on (ker $\mu_k(G_j))^\perp$ and equal to S_j on ker $\mu_k(G_j)$. We write $P_j + S_j$ for this operator where P_j projects onto the orthogonal complement of ker $\mu_k(G_j)$. As $\text{ind}_k(G_1) = \text{ind}_k(G_2)$ there are finite dimensional $\mathscr{C}_{k+1}$ modules V_j contained in $P_j E$ such that ker $\mu_k(G_1) \oplus V_1$ and ker $\mu_k(G_2) \oplus V_2$ are isomorphic $\mathscr{C}_k$ modules. Let

$$T: \ker \mu_k(G_1) \oplus V_1 \to \ker \mu_k(G_2) \oplus V_2$$

set up this isomorphism. Let $J_{k+1}^{(1)}$ denote the extra $\mathscr{C}_{k+1}$ generator on V_1. Extend S_1 to all of ker $\mu_k(G_1) \oplus V_1$ by making it the identity on V_1. Similarly extend $J_{k+1}^{(1)}$ to ker $\mu_k(G_1) \oplus V_1$ by making it the identity on ker $\mu_k(G_1)$. Introduce the operator $S_1 J_{k+1}^{(1)} J_k'$, where J_k' denotes the

restriction of J_k to V_1. Extend $S_1 J_{k+1}^{(1)} J_k'$ to all of E by making it the identity operator on the orthogonal complement of ker $\mu_k(G_1) \oplus V_1$. Similarly introduce $S_2 J_{k+1}^{(2)} J_k''$ on ker $\mu_k(G_1) \oplus V_2$, where $J_{k+1}^{(2)}$ is the extra $\mathscr{C}_{k+1}$ generator on V_2 and J_k'' the restriction of J_k to V_2. Clearly T extends also to a $\mathscr{C}_k$-module isomorphism on all of E. Now the operator $J_k'' J_{k+1}^{(2)} S_2^* T S_1 J_{k+1}^{(2)} J_k' T^*$ commutes with the $\mathscr{C}_k$ action on E and so is connected to the identity since the group of all such is a Kuiper group. On the other hand $T S_1 J_{k+1}^{(1)} J_k' T^*$ is connected to $S_1 J_{k+1}^{(1)} J_k'$ as T lies in the same group. Putting all of the above together proves that $S_1 J_{k+1}^{(1)} J_k'$ and $S_2 J_{k+1}^{(2)} J_k''$ are connected. But now (cf. [Mi]) $J_{k+1}^{(1)} J_k'$ is connected to the identity in the group of all orthogonals commuting with the $\mathscr{C}_{k-1}$ action on V_1 so that G_1 and G_2 are in the same connected component.

The fact that the maps ind_k and $\operatorname{ind}_k^{\mathbb{C}}$ are surjective and homomorphisms now follows from properties of the Fredholm index in the case where $\mathscr{C}_k$ is not simple and from the proof of the preceding two propositions in the other case.

The results may be combined as

THEOREM 2.7. *The maps*

$$\operatorname{ind}_k: \mathcal{O}_k \to \mathscr{G}_k, \qquad \operatorname{ind}_k^{\mathbb{C}}: \mathscr{U}_k \to \mathscr{G}_k^{\mathbb{C}}$$

are surjective continuous homomorphisms which separate the connected components of the respective groups.

PROPOSITION 2.8. *The groups $K_1(\mathscr{A}_k(E))$ and $K_1(\mathscr{A}_k(H))$ are equal to $\mathscr{G}_k$ and $\mathscr{G}_k^{\mathbb{C}}$, respectively.*

Proof. This follows exactly as in Proposition 4.3 of [C2] where the case $k = 1$ is proved.

If Theorem 2.7 were not of independent interest then the shortest computation of the K-groups of our algebras, at least in the complex case, would be to exploit [Co] and use stability under the holomorphic functional calculus of $\mathscr{A}_k(H)$ to show that the K-groups were independent of the ideal $\mathfrak{S}$. Then setting $\mathfrak{S}$ equal to the compacts, $K_*(\mathscr{A}_k(H))$ can be computed by looking at quotients by the compacts and using a K-theory six-term exact sequence. (This is actually done in Theorem 2.2 of [C-K].) In the real case we need a different argument. The proof of the following theorem is similar in both the real and complex cases and provides slightly more information than the method of [C-K].

THEOREM 2.9. *We have the isomorphisms*

$$K_0(\mathscr{A}_k(E)) \simeq K_1(\mathscr{A}_{k-1}(E)) \qquad and \qquad K_0(\mathscr{A}_k(H)) \simeq K_1(\mathscr{A}_{k-1}(H)),$$

and the periodicity relations

$$K_0(\mathscr{A}_k(E)) \simeq K_0(\mathscr{A}_{k+8}(E)) \qquad and \qquad K_0(\mathscr{A}_k(H)) \simeq K_0(\mathscr{A}_{k+2}(H)).$$

Proof. The periodicity relations follow from Lemma 2.3. Thus it remains to establish the isomorphisms. We prove them via a sequence of lemmas restricting to the real case.

LEMMA 2.10. *Let V be a self-adjoint involution in $\mathcal{O}_k$ or $\mathcal{U}_k$ (i.e., $V^* = V$, $V^2 = 1$). Then there is a continuous path of self-adjoint involutions V_t ($t \in [0, 1]$) in the group in question which connects $V = V_0$ and V_1 where V_1 restricted to $\ker \mu_k(V)$ anticommutes with J_k while on restriction to $\ker \mu_k(V)^\perp$ it commutes with the $\mathscr{C}_k$ action.*

Proof. For V a self-adjoint involution the hypotheses of Lemma 2.2 apply. Let S be V restricted to $\ker \mu_k(V)$ then S anticommutes with J_k. Extend S to the whole space by making it the identity on $\ker \mu_k(V)^\perp$. Let the decomposition of Lemma 2.2 be $V = K(1 + X)$ with X in $\mathfrak{S}$ and K chosen to be the identity operator on $\ker \mu_k(V)$. Now let L denote the isometry in the polar decomposition of $(V - \mu_k(V))/2$. The relations (2.2) imply that $KL = -LK$ on $\ker \mu_k(V)^\perp$ and $K^2 = 1$. Thus we may write

$$1 + X = \cos A + KL \sin A$$

on $\ker \mu_k(V)^\perp$, where $\cos A$ is defined to be $|\mu_k(V)|$. The path

$$K_t = \cos tA + KL \sin tA, \qquad 0 \leqslant t \leqslant 1$$

connects K_1 continuously to the projection onto $\ker \mu_k(V)^\perp$. Note that one has $KK_t = K_t^* K$ so that KK_t is a self-adjoint involution on $\ker \mu_k(V)^\perp$ for each t. From this it follows that V is connected continuously to the operator KS which has the properties required by the lemma.

LEMMA 2.11. *The isomorphisms of Theorem 8 hold for $k = 3$ (mod 4).*

Proof. Let P be a projection in $\mathscr{A}_k(E)$. According to Taylor [T, Sect. 6.2] it suffices to show that P is connected to the identity. Let $V = P - (1 - P)$ and let KS be the self-adjoint involution given by Lemma 2.10 to which it is connected.

Let W be the product $J_1 J_2 \cdots J_{k-1}$, then $W^2 = -1$ and let L_k be SW on $\ker \mu_k(V)$ and J_k on $\ker \mu_k(V)^\perp$. Then L_k is a complex structure which anticommutes with $J_1, J_2, ..., J_{k-1}$. Now introduce a new action of $\mathscr{C}_k$ by replacing J_k by L_k. Then KS commutes with this new action and so is connected to the identity in the algebra of operators which commute with the new action. Since L_k and J_k differ by a finite rank operator the latter algebra lies in $\mathscr{A}_k(E)$ completing the proof.

When $k = 0 \pmod 4$ we note by Lemma 2.3 it is sufficient to consider the cases $k = 4, 8$. We will consider $k = 4$ in detail ($k = 8$ is similar). Observe firstly that we can choose our representation of $\mathscr{C}_4$ by writing $H = H_1 \oplus H_1$ and letting J_1, J_2 be a representation of $\mathscr{C}_2$ on H_1. Then we let $\mathscr{C}_4$ act by

$$\begin{pmatrix} J_1 & 0 \\ 0 & -J_1 \end{pmatrix}, \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix}, J_3 = \begin{pmatrix} J_1 J_2 & 0 \\ 0 & -J_1 J_2 \end{pmatrix}, J_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now let P be any projection in $\mathscr{A}_4(E)$ and V be the involution $P - (1 - P)$. Then $V = V^+ + V^-$ with respect to the splitting $H = H_1 \oplus H_1$ and

$$V^+ - V^- \in \mathfrak{S}. \tag{2.3}$$

On multiplying V^+ and V^- by $J_1 J_2$ we obtain two new complex structures J^+ and J^- on H_1 with

$$J^+ - J^- \in \mathfrak{S}. \tag{2.4}$$

Thus J^+ and J^- both lie in the homogeneous space $\mathscr{X}_3$ of $\mathcal{O}_3$ consisting of complex structures which differ from J^- by an element of $\mathfrak{S}$. Since the map ind_3 factors through $\mathscr{X}_3$ we know the connected components of this space. Firstly we show that the projection P is connected to a projection commuting with the action of $\mathscr{C}_4$ if and only if J^+ and J^- are connected. Now it is clear that when P is connected to a projection commuting with the $\mathscr{C}_4$ action, J^+ and J^- become the same operator so it remains only to check the converse. However, if J^+ and J^- are connected then there is a path U_t, $0 \leqslant t \leqslant 1$, connected to the identity at $t = 0$ in $\mathcal{O}_3$ such that the path $U_t J^- U_t^*$ connects J^- to J^+. Then $U_t V^- U_t^*$, $0 \leqslant t \leqslant 1$, is a path of self-adjoint involutions joining V^- to V^+. Introduce then the path of self-adjoint involutions on H given by

$$V^+ \oplus U_t V^- U_t^*, \qquad 0 \leqslant t \leqslant 1,$$

which connects V to $V^+ \oplus V^+$. But the latter operator commutes with J_4 and so V is connected to a self-adjoint involution which commutes with the action of $\mathscr{C}_4$ and hence is connected to the identity. It now follows that the map defined implicitly above by associating to a given projection P in $\mathscr{A}_4(E)$ the connected component of the unitary U in $\mathcal{O}_3$ such that $U J^- U^* = J^+$ induces an injective map from the connected components of the space of self-adjoint involutions into $K_1(\mathscr{A}_3(E))$. This map is surjective because one can follow the steps leading from V through (2.3) to (2.4) in reverse. Finally, using the fact that $\mathscr{A}_4(E) \otimes \mathbb{R}^n$ is isomorphic to $\mathscr{A}_4(E)$ (Proposition 4.3 of [C2]), it is not difficult to show that this map also induces a group isomorphism between $K_0(\mathscr{A}_4(E))$ and $K_1(\mathscr{A}_3(E))$.

It remains only to establish the isomorphism in the real case when

$k = 1, 2, 5,$ or 6. In the first two cases we have to prove that the group $K_0(\mathscr{A}_k(E))$ is $\mathbb{Z}/2\mathbb{Z}$ and is trivial in the last two.

First use Lemma 2.10 to connect a self-adjoint involution V in $\mathscr{A}_k(E)$ to one of the form KS with K commuting with the $\mathscr{C}_{k-1}$ action and S anticommuting with J on $\ker \mu_k(V)$ (notation as in the proof of the lemma).

LEMMA 2.12. *If there is a complex structure J_0 on $\ker \mu_k(V)$ which anticommutes with S and with $J_1, J_2, ..., J_k$ then $(1 - V)/2$ defines the trivial element of K_0.*

Proof. Define a new action of $\mathscr{C}_k$ on E by replacing J_k by the operator J'_k which equals $J_0 J_k$ on $\ker \mu_k(V)$ and J_k on the orthogonal complement. Then V commutes with this new action of $\mathscr{C}_k$ and so is connected to the identity by a path V_s ($s \in [0, 1]$) in the space of self-adjoint involutions commuting with the new $\mathscr{C}_k$ action. But since J_k and J'_k differ by a finite rank operator the path V_s lies in $\mathscr{A}_k(E)$. This proves the result.

We now show how this lemma applies in each case noting again by Lemma 2.3 that we may restrict to $k \leqslant 8$.

For $k = 1$ it is not hard to see that J_0 exists exactly when $\mathrm{ind}_1(V)$ is zero. If $\mathrm{ind}_1(V)$ is not zero then V cannot be path connected to an involution in the commutant of $\mathscr{C}_1$ by continuity of ind_1. Similarly for $k = 2$ a J_0 exists when the multiplicity of the irreducible $\mathscr{C}_2$ representation on $\ker \mu_2(V)$ is zero (mod 2). To see that a V for which this multiplicity is odd cannot be connected to one for which it is even apply the argument of the proof of Proposition 2.5 to two self-adjoint involutions V_1 and V_2 which are close in norm. The isometry in the polar decomposition of $(V_2 - \mu_k(V_2)) \mu_k(V_2)$ intertwines the eigenspace of $\mu_2(V_2)$ corresponding to eigenvalue μ with that corresponding to eigenvalue $-\mu$. Since the eigenspaces of this operator are $\mathscr{C}_2$ modules this implies that $(\ker \mu_2(V_2))^\perp \cap QE$ carries a representation of $\mathscr{C}_2$ in which the irreducible has even multiplicity proving the required result.

For the case of $k = 5$ or 6 we need to refer to results of Milnor [Mi, Lemma 24.6]. As a $\mathscr{C}_5$ module, $\ker \mu_5(V)$ has the following structure. The operator $J_1 J_2 J_3$ has square one and so determines a splitting

$$\ker \mu_5(V) = Y^+ \oplus Y^-$$

into the $+1$ and -1 eigenspaces, respectively, which are each copies of the same $\mathscr{C}_2$ module with J_3 being $-J_1 J_2$ on Y^+ and $J_1 J_2$ on Y^-. Now $J_3 J_4$ defines a $\mathscr{C}_3$ module isomorphism of Y^+ with Y^-. The operator $J_1 J_4 J_5$ commutes with $J_1 J_2 J_3$, has square one, and so defines a splitting $Y^+ = W^+ \oplus W^-$ with W^- being $J_2 W^+$. Now if we write S as $S^+ + S^-$

corresponding to the splitting $Y^+ \oplus Y^-$ then by using $J_3 J_4$ to identify these two spaces we can write J_4 and J_5 as the operators

$$J_4 = \begin{pmatrix} 0 & K_4 \\ K_4 & 0 \end{pmatrix} \qquad J_5 = \begin{pmatrix} 0 & K_5 \\ K_5 & 0 \end{pmatrix}$$

with K_4 and K_5 being anticommuting complex structures on the space Y^+ and Y^-. Since S commutes with J_4 we have

$$S^- = -K_4 S^+ K_4. \tag{2.5}$$

Regard J_1 as an operator on Y^+. Then the fact that J_5 anticommutes with S means that $J_1 K_4 K_5$ commutes with S^+ and so S^+ leaves W^+ invariant. Since S commutes with J_2, S restricted to W^- is determined (in the same manner as in (2.5)) by its restriction to W^+. Now any $\mathscr{C}_5$ module extends to a $\mathscr{C}_6$ module where J_6 is defined by a splitting of W^+ into orthogonal spaces: $X \oplus J_1 X$ (cf. [Mi]) W^+ being regarded as a complex space with complex structure J_1 so that X may be thought of as a real subspace of W^+. Now as S commutes with J_1 we can always choose the real subspace X so that S^+ restricted to W^+ leaves this space invariant. As $J_6 = J_2 J_4$ on X and $-J_2 J_4$ on $J_1 X$, S must commute with J_6. Now define $J_0 = J_1 J_2 \cdots J_6$. Then J_0 satisfies the conditions of Lemma 2.12 and this proves $K_0(\mathscr{A}_5(H))$ is trivial. This completes the argument for $k = 5$.

Finally when $k = 6$ we can always extend a $\mathscr{C}_6$ module to a $\mathscr{C}_7$ module by defining J_7 to be $\pm J_1 J_2 \cdots J_6$ and since S anticommutes with J_6 it anticommutes with J_7. So set $J_0 = J_7$ and again apply Lemma 2.12 to give triviality of $K_0(\mathscr{A}_6(H))$.

3. FREDHOLM MODULES AND ALGEBRAS ALMOST COMMUTING WITH CLIFFORD ALGEBRAS

In this section we introduce a new approach to Kasparov's theory [Ka]. We will concentrate on the complex case, the results for the real case are similar with the usual modifications of the Bott periodicity results. We begin with a C^*-algebra $\mathscr{B}$ and the (complex) right Hilbert $\mathscr{B}$-module $E = \mathscr{H}_{\mathscr{B}}$ (cf. [Ka, Sect. 4]). Let $\mathscr{K}(\mathscr{B})$ denote the $\mathscr{B}$-compact operators on E and $\mathscr{L}(E)$ the operators on E which, along with their adjoints commute with the $\mathscr{B}$-action [B1, p. 127]. Let $\varepsilon \colon \mathscr{C}_k \to \mathscr{L}(E)$ denote an action of $\mathscr{C}_k$ on E satisfying the conditions of Section 2. We refer to E as a $(\mathscr{C}_k, \mathscr{B})$-bimodule.

DEFINITION 3.1. Let $\mathscr{A}_{k,\mathscr{B}}$ denote the C^*-algebra of operators A in $\mathscr{L}(E)$ with

$$[A, \varepsilon(J_i)] = 0, \qquad i = 1, 2, ..., k - 1; \tag{3.1}$$

$$[A, \varepsilon(J_k)] \in \mathcal{K}(\mathcal{B}). \tag{3.2}$$

Now let $\mathcal{A}$ be a C^*-algebra and let ρ be a *-homomorphism into $\mathcal{A}_{k,\mathcal{B}}$.

We define ρ to be trivial if $[\rho(a), J_k] = 0$ for all $a \in \mathcal{A}$. Two *-homo-morphisms ρ_0 and ρ_1 are defined to be homotopic if there is a family (ρ_t, ε_t) with $t \in [0, 1]$ consisting of *-homomorphisms

$$\rho_t : \mathcal{A} \to \mathcal{A}^t_{k,\mathcal{B}}, \qquad \varepsilon_t : \mathcal{C}_k \to \mathcal{L}(E)$$

(where $\mathcal{A}^t_{k,\mathcal{B}}$ is given by Definition 3.1 with ε replaced by ε_t) which define in the obvious way a pair $(\hat{\rho}, \hat{\varepsilon})$ of homomorphisms into $\mathcal{A}_{k,\mathcal{B}[0,1]}$ acting on the $\mathcal{B}[0, 1]$ module $\mathcal{H}_{\mathcal{B}[0,1]}$ (with $\mathcal{B}[0, 1]$ being continuous functions from $[0, 1]$ to $\mathcal{B}$).

We now define $\mathcal{G}_k(\mathcal{A}, \mathcal{B})$ to be the set of homotopy classes of *-homo-morphisms from $\mathcal{A}$ into $\mathcal{A}_{k,\mathcal{B}}$ modulo trivial homomorphisms. Then $\mathcal{G}_k(\mathcal{A}, \mathcal{B})$ is a semigroup under the operation of forming direct sums. Given a pair (ρ, ε) we now define inverses by taking $\varepsilon' : \mathcal{C}_k \to \mathcal{L}(E)$ to be defined by $\varepsilon'(J_i) = -\varepsilon(J_i)$, and we now claim that $(\rho \oplus \rho, \varepsilon \oplus \varepsilon')$ is homotopic to the trivial *-homomorphism. To see this introduce the operator

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which commutes with $\{\rho(a) \oplus \rho(a) | a \in \mathcal{A}\}$. Now let

$$\varepsilon_t(J_k) = (\varepsilon \oplus \varepsilon')(J_k) \cos(\pi t/2) + J \sin(\pi t/2)$$

$$\varepsilon_t(J_i) = (\varepsilon \oplus \varepsilon')(J_i) \qquad (i < k),$$

where $t \in [0, 1]$. Then $(\rho \oplus \rho, \varepsilon_t)$ is a homotopy connecting $(\rho \oplus \rho, \varepsilon \oplus \varepsilon')$ to $(\rho \oplus \rho, \varepsilon_1)$, where $\varepsilon_1(J_k) = J$.

Thus as $(\rho \oplus \rho, \varepsilon'_1)$ is a trivial *-homomorphism we conclude that we have made $\mathcal{G}_k(\mathcal{A}, \mathcal{B})$ into a group. Theorem 3.2 below identifies $G_k(\mathcal{A}, \mathcal{B})$ with Kasparov's group $KK^k(\mathcal{A}, \mathcal{B})$ for $k = 0, 1$. For the reader's con-venience we recall Kasparov's definition of the latter. He begins with the set of triples (η, ϕ, F), where $E = \mathcal{H}_{\mathcal{B}}$ is now a $\mathbb{Z}_2$-graded right Hilbert $\mathcal{B}$-module with η the grading $((\mathcal{A}, \mathcal{B})$ have $\mathbb{Z}_2$-gradings as well), ϕ is a *-homomorphism from $\mathcal{A}$ into $\mathcal{L}(E)$, and F is an operator on E of degree 1 such that $[F, \phi(a)]$, $(F^2 - 1)\phi(a)$, $(F - F^*)\phi(a)$ are all in $K(\mathcal{B})$ for all $a \in A$. Then $KK^0(\mathcal{A}, \mathcal{B})$ is the set of equivalence classes of such triples (modulo trivial ones) under a homotopy equivalence which implies that for $\mathcal{G}_k(\mathcal{A}, \mathcal{B})$ above as in our picture the roles of the $\mathbb{Z}_2$-grading and F are played by the generators of $\mathcal{C}_2$. Kasparov defines $KK^1(\mathcal{A}, \mathcal{B})$ as

$KK^0(\mathscr{A}, \mathscr{B} \hat{\otimes} \mathscr{C}_1)$ which, because our picture is trivially graded, does not obviously correspond to $G_1(\mathscr{A}, \mathscr{B})$. The following is the main result of this section.

THEOREM 3.2.   (i) $\mathscr{G}_k(\mathscr{A}, \mathscr{B}) \approx \mathscr{G}_{k-2}(\mathscr{A}, \mathscr{B})$.

(ii) $\mathscr{G}_k(\mathscr{A}, \mathscr{B}) \approx KK^0(\mathscr{A}, \mathscr{B})$, k even.

(iii) $\mathscr{G}_k(\mathscr{A}, \mathscr{B}) \approx KK^1(\mathscr{A}, \mathscr{B})$, k odd.

Proof.   (i) Given a pair (ρ, ε) defining an element of $\mathscr{G}_k(\mathscr{A}, \mathscr{B})$ we use the fact that $\mathscr{C}_k \approx \mathscr{C}_{k-2} \hat{\otimes} \mathscr{C}_2$ ($\hat{\otimes}$ denotes graded tensor product) to write $E = E' \otimes \mathbb{C}^2$ and correspondingly $\varepsilon = \varepsilon' \otimes \varepsilon_2$, where ε_2 is the usual representation of $\mathscr{C}_2$ on $\mathbb{C}^2$.

Let J_1 and J_2 generate this copy of $\mathscr{C}_2$ and $J_3, ..., J_k$ the copy of $\mathscr{C}_{k-2}$. Then since $\rho(\mathscr{A})$ commutes with the $\mathscr{C}_2$ action we may define ρ' by letting $\rho(a) = \rho'(a) \otimes 1$, where 1 denotes the identity operator on $\mathbb{C}^2$. Then (ρ', ε') defines an element of $\mathscr{G}_{k-2}(\mathscr{A}, \mathscr{B})$. It is not difficult to check that the map so defined from $\mathscr{G}_k(\mathscr{A}, \mathscr{B})$ to $\mathscr{G}_{k-2}(\mathscr{A}, \mathscr{B})$ preserves homotopies. Moreover the procedure is completely reversible proving the isomorphism.

(ii)   Following Rosenberg [R] (see also [B1, Sect. 17.5]) an element of $KK^0(\mathscr{A}, \mathscr{B})$ is specified by the data (E, ϕ, F) where

(a) $E = E^+ \oplus E^-$ with $E^\pm$ Hilbert $\mathscr{B}$-modules;

(b) $\phi = \phi^+ \oplus \phi^-$ with $\phi^\pm$ *-homomorphisms from $\mathscr{A}$ into $\mathscr{L}(E^\pm)$;

(c) $F = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ with $T: E^+ \to E^-$ and $F^2 = 1$ $(F \in \mathscr{L}(E))$

such that for all a in $\mathscr{A}$, $[\phi(a), F] \in \mathscr{K}(\mathscr{B})$.

To realise this data in our case it is sufficient by (i) to demonstrate how each pair (ρ, ε) in $\mathscr{G}_2(\mathscr{A}, \mathscr{B})$ gives rise to it. So we now define:

(a)' $E^\pm$ = spectral subspaces of $\varepsilon(J_1)$;

(b)' $\rho = \rho^+ \oplus \rho^-$, the corresponding decomposition of ρ into a direct sum of *-homomorphisms which are now identified with $\phi^\pm$;

(c') $F = i\varepsilon(J_2)$.

Then clearly for all a in $\mathscr{A}$, $[\phi(a), F] \in \mathscr{K}(\mathscr{B})$ and $F^2 = 1$. (Note that T can be defined by observing that as J_1 and J_2 anticommute F maps E^+ to E^- and E^- to E^+.) It is now clear that homotopy classes of pairs (ρ, ε) coincide with homotopy classes of triples $(\eta(J_1), \phi, F)$ which completes the proof.

(iii) This result follows as in Blackadar [B1, Sect. 17.6.4], from Cuntz' quasi-homomorphism picture of KK.

4. Other Algebras

The algebras introduced in the preceding sections seem the most canonical if one is interested in K-theory. However, other possibilities exist and it is the purpose of this section to explore one of them. We will concentrate on the case of a complex Hilbert space H with a $\mathscr{C}_k$ action and introduce the algebras

$$\mathscr{B}_k = \{ A \in B(E) | [A, J_i] \in \mathfrak{S}, \ i = 1, ..., k \}.$$

Then $\mathscr{B}_k$ is also a Banach algebra in the norm given by (2.1). Clearly $\mathscr{A}_k(H)$ is a subalgebra of $\mathscr{B}_k$ and it seems an interesting question whether $\mathscr{B}_k$ shares any of the properties of the former algebra. The main result of this section is the following.

THEOREM 4.1. (i) *For k odd, $K_1(\mathscr{B}_k) \approx \mathbb{Z}$.*

(ii) *For k even, $K_1(\mathscr{B}_k) \approx 0$.*

(iii) *$K_0(\mathscr{B}_k)$ is a non-trivial torsion group.*

Proof. (i) We let H_1 carry a $\mathscr{C}_{k-1}$ action and define on $H = H_1 \oplus H_1$ the action of J_k by setting it equal to $-J_1 J_2 \cdots J_{k-1}$ on the first copy of H_1 and $J_1 J_2 \cdots J_{k-1}$ on the second copy. Now if U is a unitary in $\mathscr{B}_k$ acting on H we can write it as

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

with respect to the decomposition $H = H_1 \oplus H_1$. Now we have

$$U_{ij} J_1 - J_1 U_{ij} \in \mathfrak{S} \tag{4.1}$$

$$J_1 \cdots J_{k-1} U_{ij} + U_{ij} J_1 \cdots J_{k-1} \in \mathfrak{S}, \qquad i \neq j \tag{4.2}$$

$$J_1 \cdots J_{k-1} U_{ii} + U_{ii} J_i \cdots J_{k-1} \in \mathfrak{S}. \tag{4.3}$$

From (4.1) and (4.2) we have $U_{ij} \in \mathfrak{S}$ for $i \neq j$. Conversely, if U is unitary with $U_{ij} \in \mathfrak{S}$ for $i \neq j$ and U_{ii} satisfying (4.3) then $U \in \mathscr{B}_k$. Now introduce the map j from the unitary group $\mathscr{U}_k$ of $\mathscr{B}_k$ to $\mathbb{Z}$ given by

$$j(U) = \text{Fredholm index of } U_{11} = -(\text{Fredholm index of } U_{22}).$$

To show that j labels the connected components of $\mathscr{U}_k$ it is sufficient to show that any U with $j(U) = 0$ is connected to the identity. We may use Lemma 2.2 to write

$$U = \begin{pmatrix} W_{11} & 0 \\ 0 & W_{22} \end{pmatrix} (1 + X), \tag{4.4}$$

where X lies in $\mathfrak{S}$ and W_{ii} are the isometries in the polar decomposition of U_{ii} extended to be unitaries on H_1 by setting them to equal, on ker U_{ii}, isometries mapping ker U_{ii} onto ker U_{ii}^*. We may assume by induction that $\mathcal{U}_{k-1}$ is connected so that, as W_{ii} is in $\mathcal{U}_{k-1}$, we can connect U to $1+X$. But the group $\mathcal{U}_{\mathfrak{S}}$ is connected so U is connected to the identity. From the fact that $\mathcal{B}_k \otimes M_n$ is isomorphic to $\mathcal{B}_k$ we conclude that $K_1(\mathcal{B}_k)$ is $\mathbb{Z}$.

(ii) Choose the representation of $\mathscr{C}_k$ to have the form (on $H = H_1 \oplus H_1$),

$$J_k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad J_{k-1} = \begin{pmatrix} -J_1 \cdots J_{k-2} & 0 \\ 0 & J_1 \cdots J_{k-2} \end{pmatrix}.$$

Any U in $\mathcal{U}_k$ also lies in $\mathcal{U}_{k-1}$. So writing as in part (i)

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

as before we conclude that U_{12}, U_{21} are in $\mathfrak{S}$. But from $[U, J_k] \in \mathfrak{S}$ we also have

$$U_{11} - U_{22} \in \mathfrak{S}. \tag{4.5}$$

This forces $j(U) = 0$ as $j(U_{11}) = j(U_{22})$ by (4.5) but $j(U_{11}) = -j(U_{22})$ by general arguments. Hence we have $\mathcal{U}_k$ contained in the connected component of the identity of $\mathcal{U}_{k-1}$. Now write, as in (4.4),

$$U = \begin{pmatrix} W_{11} & 0 \\ 0 & W_{22} \end{pmatrix} (1 + X).$$

Then W_{ii} satisfies $W_{ii} J_j - J_j W_{ii} \in \mathfrak{S}$ and so $W_{ii} \in \mathcal{U}_{k-2}$ (regarded as acting on H_1). Induction on k gives that $\mathcal{U}_{k-2}$ is connected so that U is connected to $1 + X$ and hence to the identity proving that $\mathcal{U}_k$ is connected. Again it follows that $K_1(\mathcal{B}_k) = 0$. Note that the induction starts since we know that $\pi_0(\mathcal{U}_1)$ is $\mathbb{Z}$ and that the group of unitaries on a Hilbert space is contractible.

(iii) As $\mathcal{B}_k$ is stable under the holomorphic functional calculus following Connes [Co] we can assert that $K_0(\mathcal{B}_k)$ is independent of $\mathfrak{S}$ and take $\mathfrak{S}$ to be the compact operators. We let π denote the quotient map onto the Calkin algebra. Then we have the exact sequence

$$0 \to \mathscr{K} \to \mathcal{B}_k \to \pi(\mathcal{B}_k) \to 0,$$

where $\mathscr{K}$ denotes the compact operators on H. Now Paschke [P] has shown that

$$K_0(\pi(\mathscr{C}_k)^c) = \text{Ext}(\mathscr{C}_k)$$

$$K_1(\pi(\mathscr{C}_k)^c) = \text{Ext}(\Omega\mathscr{C}_k),$$

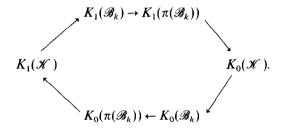where the superscripted c denotes the commutant in the Calkin algebra.

But $\pi(\mathscr{C}_k)^c = \pi(\mathscr{B}_k)$. Now $C_k \cong M_k$ for k even and $M_k \oplus M_k$ when k is odd and hence both $K_0(\pi(\mathscr{B}_k))$ and $K_1(\pi(\mathscr{B}_k))$ are known:

$$K_1(\pi(\mathscr{B}_k)) = \begin{cases} \mathbb{Z}, & k \text{ even} \\ \mathbb{Z}^2, & k \text{ odd} \end{cases}$$

(see Brown [B]) while

$$K_0(\pi(\mathscr{B}_k)) = \begin{cases} \mathbb{Z}_k, & k \text{ even} \\ \mathbb{Z}_k \oplus \mathbb{Z}_k, & k \text{ odd}. \end{cases}$$

From the K-theory exact sequence:

$$
\begin{array}{ccc}
 & K_1(\mathscr{B}_k) \to K_1(\pi(\mathscr{B}_k)) & \\
\nearrow & & \searrow \\
K_1(\mathscr{K}) & & K_0(\mathscr{K}). \\
\nwarrow & & \swarrow \\
 & K_0(\pi(\mathscr{B}_k)) \leftarrow K_0(\mathscr{B}_k) &
\end{array}
$$

We therefore obtain the following exact sequences (using parts (i) and (ii) of the theorem)

$$0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to K_0(\mathscr{B}_k) \to \mathbb{Z}_k \to 0 \qquad (k \text{ even})$$

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to K_0(\mathscr{B}_k) \to \mathbb{Z}_k \oplus \mathbb{Z}_k \to 0 \qquad (k \text{ odd}).$$

Neither of these sequences is enough to determine $K_0(\mathscr{B}_k)$ completely, however, since the only injections of $\mathbb{Z}$ into itself are of the form $\mathbb{Z} \to l\mathbb{Z}$, for $l \in \mathbb{Z}$ we conclude for k even that $K_0(\mathscr{B}_k)$ is torsion and a similar argument gives the same conclusion for k-odd proving the result.

5. THE EQUIVARIANT CASE

In this section we discuss some of the modifications necessary to prove results analogous to the above when the Hilbert space admits a compact group action which commutes with the $\mathscr{C}_k$ or $\mathscr{C}_k \otimes \mathbb{C}$ action. So let G be the compact group acting on E or H and note that G acts by conjugation as automorphisms of $\mathscr{A}_k(E)$ and $\mathscr{A}_k(H)$. Hence let $\mathscr{A}_k^G(E)(\mathscr{A}_k^G(H))$ denote the fixed point algebra of this G action. Let $\mathcal{O}_k^G$ and $\mathcal{U}_k^G$ denote the unitary groups of these two algebras. If O lies in one of these groups then $\ker \mu_k(O)$ is an element of the representation ring $R(G)$ of G and also a $\mathscr{C}_k$ module. Hence we introduce the group $R(G)_k$ (resp. $R(G)_k^C$) of

$(\mathscr{C}_k, G)$-bimodules (resp. $(\mathscr{C}_k \otimes \mathbb{C}, G)$-bimodules) modulo those extendable to $(\mathscr{C}_{k+1}, G)$-bimodules (resp. $(\mathscr{C}_{k+1} \otimes \mathbb{C}, G)$-bimodules). Now define the index map

$$\operatorname{ind}_k^G(O) = [\ker \mu_k(O)],$$

where the RHS denotes an element of $R(G)_k$ or $R(G)_k^{\mathbb{C}}$ depending on whether O is in $\mathcal{O}_k^G$ or $\mathcal{U}_k^G$.

Now we reconsider the results of Section 2 as they apply to ind_k^G.

Firstly the crucial Lemma 2.2 goes through for elements of $\mathcal{O}_k^G$ and $\mathcal{U}_k^G$ provided we restrict to elements of index zero when the Clifford algebra is not simple. In the latter case it is not difficult to see that one may interpret the index as the element of $R(G)_k$ defined by taking the difference in $R(G)_k$ of the elements defined by the kernels $\ker P_+ O P_+$ and $\ker P_- O^* P_-$ which is the usual version of the Fredholm index in the presence of a G action.

Secondly, Proposition 2.5 goes through with no modification proving continuity of the index. A corollary of this fact is a proof of continuity of Matsui's "determinant index" [M].

Finally Proposition 2.6 also follows without change proving that ind_k labels the connected components of the relevant group. Thus we have the following.

THEOREM 5.1.   *The maps*

$$\operatorname{ind}_k^G : \mathcal{O}_k \to R(G)_k, \qquad \operatorname{ind}_k^G : \mathcal{U}_k \to R(G)_k^{\mathbb{C}}$$

are homomorphisms which separate the connected components of the respective groups.

To go beyond these simple observations to use this as an approach to equivariant KK theory seems less fruitful since it apparently requires special assumptions on the nature of the G-action. We will not go into this here.

REFERENCES

[A-E]   H. ARAKI AND D. E. EVANS, On a C^*-algebra approach to phase transition in the dimensional Ising model, *Comm. Math. Phys.* **91** (1983), 489–503.

[A-B-S]   M. F. ATIYAH, R. BOTT, AND A. SHAPIRO, Clifford modules, *Topology* **3**, Supp. 1 (1964), 3–38.

[A-S]   M. F. ATIYAH AND I. M. SINGER, Index theory for skew-adjoint Fredholm operators, *Publ. I.H.E.S.* **37** (1969), 305–326.

[Bl]   B. BLACKADAR, K-theory for operator algebras, to appear.

[B]   L. BROWN, Ext of certain free product C^*-algebras, *J. Operator Theory* **6** (1981), 135–141.

[C-O]   A. L. CAREY AND D. M. O'BRIEN, Automorphisms of the infinite dimensional Clifford algebra and the Atiyah–Singer mod 2 index, *Topology* **22** (1983), 437–448.

[C1]   A. L. CAREY, Some homogeneous spaces and representations of the Hilbert Lie group $U(H)_2$, *Rev. Roumaine Math. Pures Appl.* **30** (1985), 505–520.

[C2]   A. L. CAREY, Some infinite dimensional groups and bundles, *Publ. Res. Inst. Math. Sci.* **20** (1984), 1103–1117.

[C-E]   A. L. CAREY AND D. E. EVANS, The operator algebras of the two-dimensional Ising model, Contemp. Math. Amer. Math. Soc., Providence, RI, 1988.

[C-K]   A. CONNES AND M. KAROUBI, Caractere multiplicatif d'un module de Fredholm, preprint, 1988.

[Co]   A. CONNES, Non-commutative differential geometry, *Publ. I.H.E.S.* **62** (1985), 257–360.

[D]   R. G. DOUGLAS, C^*-algebra extensions and K-homology, *in* "Ann. Math. Studies," Princeton Univ. Press, Princeton, NJ, 1980.

[H]   P. DE LA HARPE, Classical Banach–Lie algebras and Banach–Lie groups of operators on Hilbert space, *in* "Springer Lecture Notes in Mathematics," Vol. 285, Springer-Verlag, New York/Berlin, 1972.

[Ka]   G. G. KASPAROV, The operator K-functor and extensions of C^*-algebras, *Math. USSR-Izv.* **16** (1981), 513–572.

[K]   N. KUIPER, Contractibility of the unitary group in Hilbert space, *Topology* **3** (1964), 19–30.

[L-S]   J. T. LEWIS AND P. N. M. SISSON, A C^*-algebra of the two-dimensional Ising model, *Comm. Math. Phys.* **44** (1975), 279–282.

[M]   T. MATSUI, On quasifree states of gauge invariant CAR algebras, RIMS preprint, 1985.

[P]   W. PASCHKE, K-theory for commutants in the Calkin algebra, *Pacific J. Math.* **95** (1981), 427–434.

[Mi]   J. MILNOR, "Morse Theory," Princeton Univ. Press, Princeton, NJ, 1963.

[R]   J. ROSENBERG, K-theory for group C^*-algebras, foliation C^*-algebras and crossed products, *in* "Operator Algebras, Foliations, and Index Theory" (J. Kaminker, K. Mallett, and C. Schochet, Eds.), Contemp. Math., Amer. Math. Soc., Providence, RI, in press.

[Si]   B. SIMON, "Trace Ideals and Their Applications," Cambridge University Press, Cambridge, 1979.

[T]   J. T. TAYLOR, Banach algebras and topology, *in* "Algebras and Analysis" (J. H. Williamson, Ed.), Academic Press, London/New York, 1975.