JOURNAL OF COMBINATORIAL THEORY 2, 135-139 (1967)

Kronecker Products of Fully Indecomposable Matrices and of Ultrastrong Digraphs

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Communicated by F. Harary

ABSTRACT

The following theorem is proved: The Kronecker product of two fully indecomposable matrices is fully indecomposable. The theorem is then related to connectivity in directed graphs.

1. Let $A = [a_{ij}]$ and $B = [b_{kl}]$ be, respectively, m by m and n by n non-negative matrices. The *Kronecker product* [4, p. 8] of A with B is the mn by mn matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix} \tag{1}$$

The matrix A is *fully indecomposable* [4, p. 123–124] provided it does not contain an r by s block of 0's with $1 \leq r \leq m - 1$ and $r + s = m$; that is, A is fully indecomposable if there do not exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix} \tag{2}$$

where A_1 and A_2 are square non-vacuous matrices. The elements $a_{1\sigma(1)}$,

* The research of the author was partially supported by NSF grant GP-3993.

..., $a_{m\sigma(m)}$ form a *positive diagonal* of A provided σ is a permutation of 1, ..., m and $a_{1\sigma(1)}, \ldots, a_{m\sigma(m)} > 0$. In [1] it was shown by use of the Frobenius-König theorem [4, p. 123] that *the matrix A is fully indecomposable if and only if each of the matrices obtained from A by striking out a row and column has a positive diagonal.* The Frobenius-König theorem immediately implies that a fully indecomposable matrix has a positive diagonal.

Since the concept of full indecomposability is important combinatorially and since the operation of forming the Kronecker product is sometimes useful combinatorially and otherwise, the following theorem may be of some interest.

THEOREM. *The Kronecker product of two fully indecomposable matrices is a fully indecomposable matrix.*

PROOF. Suppose $A = [a_{ij}]$ and $B = [b_{kl}]$ are, respectively, m by m and n by n fully indecomposable non-negative matrices. Consider a typical element $a_{ij}b_{kl}$ of $A \otimes B$:

If $a_{ij} > 0$ and $b_{kl} > 0$, then from the above discussion a_{ij} belongs to a positive diagonal of A, say $a_{1\sigma(1)}, \ldots, a_{m\sigma(m)}$, and b_{kl} belongs to a positive diagonal of B, say $b_{1\tau(1)}, \ldots, b_{n\tau(n)}$. Then it is easy to verify that

$$a_{1\sigma(1)}b_{1\tau(1)}, \ldots, a_{1\sigma(1)}b_{n\tau(n)}, \ldots, a_{m\sigma(m)}b_{1\tau(1)}, \ldots, a_{m\sigma(m)}b_{n\tau(n)}$$

is a positive diagonal of $A \otimes B$ containing $a_{ij}b_{kl}$.

If $a_{ij} > 0$ and $b_{kl} = 0$, then a_{ij} belongs to a positive diagonal of A, say $a_{1\sigma(1)}, \ldots, a_{m\sigma(m)}$ with $\sigma(i) = j$, while the matrix obtained from B by striking out row k and column l has a positive diagonal, say $b_{1\tau(1)}, \ldots, b_{k-1\tau(k-1)}, b_{k+1\tau(k+1)}, \ldots, b_{n\tau(n)}$. Moreover, B itself has a positive diagonal $b_{1\varrho(1)}, \ldots, b_{n\varrho(n)}$. It is now easy to verify that

$$a_{1\sigma(1)}b_{1\varrho(1)}, \ldots, a_{1\sigma(1)}b_{n\varrho(n)}, \ldots, a_{i\sigma(i)}b_{1\tau(1)}, \ldots, a_{i\sigma(i)}b_{k-1\tau(k-1)},$$
$$a_{i\sigma(i)}b_{k+1\tau(k+1)}, \ldots, a_{i\sigma(i)}b_{n\tau(n)}, \ldots, a_{m\sigma(m)}b_{1\varrho(1)}, \ldots, a_{m\sigma(m)}b_{n\varrho(n)}$$

is a positive diagonal of the matrix obtained from $A \otimes B$ by striking out the row and column of $a_{ij}b_{kl}$.

If $a_{ij} = 0$ and $b_{kl} > 0$, then the situation is analogous to the previous case. For it follows by inspection that a permutation matrix P exists such that $A \otimes B = P^T(B \otimes A)P$. Since the matrix obtained from $B \otimes A$ by striking out the row and column of $b_{kl}a_{ij}$ has a positive diagonal, it follows that the matrix obtained from $A \otimes B$ by striking out the row and column of $a_{ij}b_{kl}$ has a positive diagonal.

We may now conclude that, if upon striking out the row and column of an element $a_{ij}b_{kl}$ of $A \otimes B$ we obtain a matrix with no positive diagonal, then both $a_{ij} = 0$ and $b_{kl} = 0$.

Suppose now $A \otimes B$ were not fully indecomposable, so that there exist permutation matrices P and Q with

$$P(A \otimes B)Q = \begin{bmatrix} C_1 & 0 \\ C_{21} & C_2 \end{bmatrix}, \tag{3}$$

where C_1 and C_1 are square non-vacuous matrices. The matrix C_{21} must be all zeros, for striking out the row and column of any element of C_{21} gives a matrix with no positive diagonal. But then it follows that striking out the row and column of any 0 in the zero block 0 in (3) gives a matrix with no positive diagonal. By what we have shown, each such 0 arises from a product $a_{ij}b_{kl}$ with $a_{ij} = 0$ and $b_{kl} = 0$. Hence the zero block 0 in (3) must arise by first choosing an r by s submatrix of 0's of A, say $a_{i_p j_q}$, $1 \leq p \leq r$, $1 \leq q \leq s$, and then choosing u_p by v_q submatrices of 0's of B, $1 \leq p \leq r$, $1 \leq q \leq s$. The zero block 0 in (3) is then formed by taking the corresponding u_p by v_q submatrices of 0's in $a_{i_p j_q}B$, $1 \leq p \leq r$, $1 \leq q \leq s$. This submatrix of 0's of $A \otimes B$ is then of size $\sum_{p=1}^{r} u_p$ by $\sum_{q=1}^{s} v_q$. Let

$$u_k = \max_{1 \leq p \leq r} \{u_p\} \qquad \text{and} \qquad v_l = \max_{1 \leq q \leq s} \{v_q\}.$$

Since B is fully indecomposable, $u_k + v_l < n$ and since, A is fully indecomposable, $r + s < m$. Hence

$$\begin{aligned} \sum_{p=1}^{r} u_p + \sum_{=1}^{s} v_q &\leq r u_k + s v_l \\ &\leq (r + s)(u_k + v_l) \\ &< mn. \end{aligned}$$

But this is a contradiction since the sum of the dimensions of the zero block 0 in (3) is mn. Hence $A \otimes B$ is fully indecomposable. This completes the proof.

2. We now relate the theorem to *directed graphs* or *digraphs* [2]. A digraph (we allow loops) is said to be *strongly connected* or *strong* provided every pair of distinct points of the digraph are mutually reach-

able by directed paths. Let D be a finite digraph with m points. Label the points of D with the integers 1, 2, ..., m. Then, in the usual way, we associate an $m \times m$ 0, 1 matrix $A = [a_{ij}]$ with D as follows: $a_{ij} = 1$ if and only if there is a directed line in D from point i to point j. Conversely, given an $m \times m$ 0, 1 matrix we can associate a digraph. It is well known [6] that the digraph D is strong if and only if the associated matrix A is irreducible. The matrix A is *irreducible* provided there does not exist a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix},$$

where A_1 and A_2 are square non-vacuous matrices. We say that the digraph D is *ultrastrong* provided each of the digraphs associated with the matrices PAQ for all permutation matrices P and Q is strong. (Note that it is enough to say that each of the digraphs associated with PA for all permutation matrices P is strong.) Thus the digraph D is ultrastrong if and only if the associated matrix A is fully indecomposable. In Figures 1 and 2 we give a digraph followed by all the digraphs (up to isomorphism) derived from it. The digraph for Figure 1 is ultrastrong while the digraph for Figure 2 is strong but not ultrastrong.

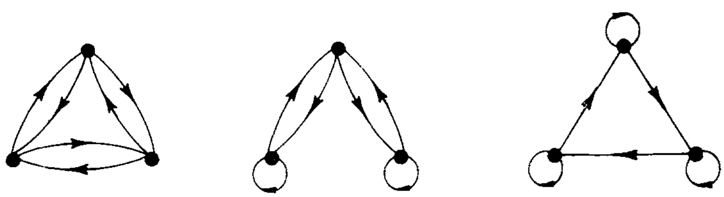

FIGURE 1

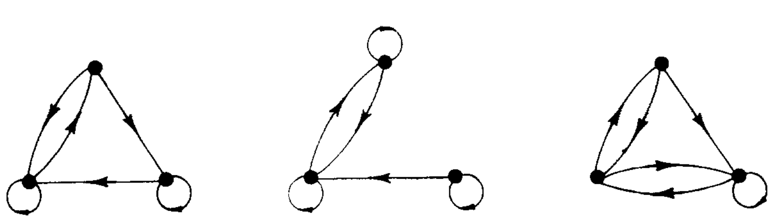

FIGURE 2

Now let D_1 and D_2 be two finite digraphs with associated matrices A_1 and A_2. The usual definition of the *Kronecker product* or *tensor product* $D_1 \otimes D_2$ of D_1 and D_2 is equivalent to: $D_1 \otimes D_2$ is the digraph associated with the matrix $A_1 \otimes A_2$. We may now restate our theorem as:

THEOREM. *The Kronecker product of two ultrastrong digraphs is also ultrastrong.*

Finally we remark that a discussion of Kronecker products of digraphs relative to other types of connectivity is given in [3].

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