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Finite projective planes admitting a projective linear group PSL(2, q)[☆]

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Abstract

Let $\mathscr{S}$ be a projective plane, and let $G \leqslant \mathrm{Aut}(\mathscr{S})$ and $\mathrm{PSL}(2, q) \leqslant G \leqslant \mathrm{P\Gamma L}(2, q)$ with $q > 3$. If G acts point-transitively on $\mathscr{S}$, then $q = 7$ and $\mathscr{S}$ is of order 2.

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1. Introduction

A linear space $\mathscr{S}$ is a set $\mathscr{P}$ of points, together with a set $\mathscr{L}$ of distinguished subsets called lines such that any two points lie on exactly one line. If a linear space with an automorphism group which acts transitive on the lines, then its every line has the same number of points and we shall call such a linear space a regular linear space. Moreover, we shall also assume that $\mathscr{P}$ is finite and that $|\mathscr{L}| > 1$.

Let G be a line-transitive automorphism group of a linear space $\mathscr{S} = (\mathscr{P}, \mathscr{L})$. Let the parameters of $\mathscr{S}$ be (b, v, r, k), where b is the number of lines, v is the number

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of points, r is the number of lines through a point and k is the number of points on a line with $k > 2$. By Block [1], transitivity of G on lines implies transitivity of G on points. The groups of automorphisms of linear spaces which are line-transitive have greatly been considered by Camina, Preager, Neumann, Spiezia, Li, Liu and others (see [3–7,11–20,22]).

Recently, Camina and Spiezia have proved the following theorem.

Theorem 1.1 [7]. *Let G be a simple group acting line-transitively, point-primitively, but not flag-transitively on a linear space, then G is not* $\mathrm{PSL}(n, q)$ *with q odd and* $n \geqslant 13$.

Therefore, it is necessary to consider the case where n is small. Weijun Liu has proved the following theorem.

Theorem 1.2 [13]. *Let* $G = \mathrm{PSL}(2, q)$ *with* $q > 3$ *act line-transitively on a finite linear space* $\mathscr{S}$, *then* $\mathscr{S}$ *is one of the following cases*:

(i) *A projective plane*;
(ii) *A regular linear space with parameters* $(b, v, r, k) = (32\,760, 2080, 189, 12)$, *in this case* $q = 2^6$;
(iii) *A regular linear space with parameters* $(b, v, r, k) = (q^2 - 1, q(q - 1)/2, q + 1, q/2)$, *where q is a power of* 2, *it is called a witt-bose-shrikhande space*.

In this short article we considered the case (i) of the Theorem 1.2. and proved the following:

Theorem 1.3. *Let* $G \leqslant \mathrm{Aut}(\mathscr{S})$, *where* $\mathscr{S}$ *is a projective plane, and* $\mathrm{PSL}(2, q) \leqslant G \leqslant \mathrm{P\Gamma L}(2, q)$ *with* $q > 3$. *If G acts point-transitively on* $\mathscr{S}$, *then* $q = 7$ *and* $\mathscr{S}$ *is of order* 2.

Section 2 introduce notions and preliminary results about $\mathrm{PSL}(2, q)$ and the projective planes. In Section 3, we shall prove Theorem 1.3.

The original version of this article discussed only the case where $G = \mathrm{PSL}(2, q)$. The present version is done according to the referee's suggestions. The authors are grateful to the referee for his helpful suggestions.

2. Some preliminary results

We begin by recalling some fundamental properties of $\mathrm{PSL}(2, q)$ with $q = p^a > 3$, where p is a prime number and a is a positive integer.

Lemma 2.1 (Theorem 8.27, chapter II of [8]). *Every subgroup of* $G = \mathrm{PSL}(2, q)$ *is isomorphic to one of the following groups*:

(1) *An elementary Abelian p-group of order at most* p^a;
(2) *A cyclic group of order z, where z divides* $(p^a \pm 1)/d$ *and* $d = (2, q - 1)$;
(3) *A dihedral group of order* $2z$, *where z is as above*;
(4) *The alternating group* A_4, *in this case*, $p > 2$ *or* $p = 2$ *and* $2|a$;
(5) *The symmetric group* S_4, *in this case*, $p^{2a} - 1 \equiv 0 \pmod{16}$;
(6) *The alternating group* A_5, *in this case*, $p = 5$ *or* $p^{2a} - 1 \equiv 0 \pmod 5$;
(7) $Z_p^m : Z_t$, *where t divides* $(p^m - 1)/d$ *and* $q - 1$ *and* $m \leqslant a$;
(8) $\mathrm{PSL}(2, p^m)$, *where* $m|a$, *and* $\mathrm{PGL}(2, p^m)$, *where* $2m|a$.

Now, we suppose that G is a line-transitive automorphism group of a linear space with parameters (b, v, r, k) and $k > 2$. Recall the basic inequalities for linear space.

$$vr = bk;$$
$$v = r(k - 1) + 1;$$
$$b \geqslant v \text{ (Fisher's inequality)}.$$

It is well-known that a linear space is a projective plane if and only if $b = v$. As the projective plane is concerned, we can get the following equations by $b = v$:

$$r = k; \qquad v = k^2 - k + 1.$$

Thus both b and v are odd.

We denote G_L and G_α as setwise stabilizer of L in G and point stabilizer of α in G respectively.

Lemma 2.2. *Let G be a transitive group on* Ω, *and K a conjugacy class of an element x of G. Let* $\mathrm{Fix}_\Omega(\langle x\rangle)$ *denote the set of fixed points of* $\langle x\rangle$. *Then* $|\mathrm{Fix}_\Omega(\langle x\rangle)| = |G_\alpha \cap K| \cdot |\Omega|/|K|$, *where* $\alpha \in \Omega$. *Especially, if G has only one conjugacy class of the involutions, then*

$$|\mathrm{Fix}_\Omega(\langle i\rangle)| = \frac{e(G_\alpha) \cdot |\Omega|}{e(G)},$$

where $e(G)$ *is the number of involutions in G.*

Proof. Count the number of the ordered pairs (α, x), where $\alpha \in \mathrm{Fix}_\Omega(\langle x\rangle)$, we can get

$$|\mathrm{Fix}_\Omega(\langle x\rangle)| = \frac{|G_\alpha \cap K| \cdot |\Omega|}{|K|}.$$

If G has only one conjugacy class of the involution, then $G_\alpha \cap K$ denotes the set of involution of G_α, that is, $|G_\alpha \cap K| = e(G_\alpha)$, and $|K|$ is the length of the conjugacy class of the involution, that is, $|K| = e(G)$. Thus we get

$$|\mathrm{Fix}_\Omega(\langle i\rangle)| = \frac{e(G_\alpha) \cdot |\Omega|}{e(G)}. \qquad \square$$

Lemma 2.3 (Theorem 13.3 of [10]). *A collineation of a finite projective plane has an equal number of fixed points and lines.*

Lemma 2.4 (Theorem 13.4 of [10]). *Let Π be a collineation group of a finite projective plane. Then Π has an equal number of points and lines orbits.*

Lemma 2.5 (Theorem 20.9.7 of [9, 21]). *Let i be an involution in a finite projective plane of order n. Then either*

(1) *$n = m^2$ and fixed points and lines of i form a subplane of order m, or*
(2) *i is a central collineation. In particular, if G acts transitively on points of the projective plane, and $i \in G$, then $G \geqslant \mathrm{PSL}(3, n)$.*

Lemma 2.6. *Let $\mathscr{S}$ be a projective plane.*

Let $G \leqslant \mathrm{Aut}(\mathscr{S})$, and let G be point-transitive. If G has the unique conjugacy class of involutions and $|\mathrm{Fix}(\langle i\rangle)| = n + \sqrt{n} + 1$, where i is an involution of G and n is order of $\mathscr{S}$, then

(1) *$n - \sqrt{n} + 1 = \frac{e(G)}{e(G_\alpha)}$, and so $e(G_\alpha)$ divides $e(G)$;*
(2) *the number v of points of $\mathscr{S}$ is grester than $\left(\frac{e(G)}{e(G_\alpha)}\right)^2$.*

Proof

(1) By Lemma 2.2,

$$|\mathrm{Fix}(\langle i\rangle)| = \frac{|G_\alpha \cap K|v}{|K|} = \frac{e(G_\alpha)v}{e(G)}.$$

Since

$$v = n^2 + n + 1 = (n + \sqrt{n} + 1)(n - \sqrt{n} + 1),$$

we have $n - \sqrt{n} + 1 = \frac{e(G)}{e(G_\alpha)}$, an integer.

(2) Since $n - \sqrt{n} + 1 = \frac{e(G)}{e(G_\alpha)}$, we have $v > (n - \sqrt{n} + 1)^2 = \left(\frac{e(G)}{e(G_\alpha)}\right)^2$. □

Lemma 2.7. *Let $G = T : \langle\delta\rangle$. Suppose that $G \leqslant \mathrm{Aut}(\mathscr{S})$, where $\mathscr{S}$ is a projective plane. If any proper subgroup of G is not point-transitive, then the order of δ is odd.*

Proof. If $\mathrm{o}(\delta)$ is even, then there exists a subgroup H of G, such that $|G/H| = 2$ (for example $H = T : \langle\delta^2\rangle$).

Let P denote a Sylow 2-subgroup of G. Then $G = HP$. Since v is odd, $P \leqslant G_\alpha$ for some point α of $\mathscr{S}$.

Thus H acts point-transitively on $\mathscr{S}$, a contradiction. □

The following lemma, suggested by the referee, is very useful.

Lemma 2.8 (Lemma 5.1 of [4]). *If p is a prime congruent to* 2 *modulo* 3 *then the point-stabiliser X contains some Sylow p-subgroup of G.*

Moreover, X contains a subgroup of index 3 *in a Sylow* 3*-subgroup of G.*

Lemma 2.9. $\mathrm{PSL}(2,q)$ *has exactly one conjugacy class of involutions.*

Proof. We know that the involutions in $\mathrm{PSL}(2,q)$ shall satisfy the following equalitiy:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} Z \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} Z = Z,$$

where $Z = Z(\mathrm{SL}(2,q))$, the centre of $\mathrm{SL}(2,q)$.

(1) When $q = 2^f$, $Z = E$, we have $a = d$ and $a^2 + bc = 1$. This deduces $\mathrm{PSL}(2, 2^f)$ has exactly $q^2 - 1$ involutions.
On the other hand, $|C_G(i)| = q$, $|G| = q(q^2 - 1)$, then $|G : C_G(i)| = q(q^2 - 1)/q = q^2 - 1$. Thus $\mathrm{PSL}(2, 2^f)$ has exactly one conjugacy class of involutions.
(2) When $q = p^f$, p is an odd prime number, $Z = \{E, -E\}$, then $a = -d$ and $a^2 + bc = \pm 1$. Thus $G = \mathrm{PSL}(2, p^f)$ has exactly $q(q + \epsilon)/2$ involutions, where $4 | \frac{q-\epsilon}{2}$, and $\epsilon = 1$ or -1. Note that $|C_G(i)| = q - \epsilon$, $|G| = q(q^2 - 1)/2$, then $|G : C_G(i)| = q(q + \epsilon)/2$. Thus we know that $\mathrm{PSL}(2, p^f)$ has exactly one conjugacy class of involutions. □

Lemma 2.10. *If $T = \mathrm{PSL}(2,q) \leqslant G \leqslant \mathrm{P\Gamma L}(2,q)$ with $q = p^a$, and G acts point-transitively on the projective plane, then*

(i) *if $p = 2$, then $G_\alpha \cap T$ is conjugate to one of the subgroups of* (1) *or* (7) *in Lemma* 2.1;
(ii) *if p is odd, then $G_\alpha \cap T$ is conjugate to one of the subgroups of* (3), (4), (5), (6) *or* (8) *in Lemma* 2.1.

Proof

(i) Because $v = |G|/|G_\alpha|$ is odd and greater than 3, this means that G_α contains a Sylow 2-subgroup of G, and hence $G_\alpha \cap T$ contains a Sylow 2-subgroup of T. Note that $q = 2^a$, check the groups containing a Sylow 2-subgroup of $\mathrm{PSL}(2,q)$ in Lemma 2.1, we will get the conclusion.
(ii) Proved as above. □

Lemma 2.11. $\mathrm{PGL}(2,q)$ *contains exactly q^2 involutions, where q is odd.*

Proof. We know that the involutions in $\mathrm{PGL}(2, q)$ shall satisfy the following equality:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} Z \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} Z = Z,$$

where $Z = Z(\mathrm{GL}(2, q))$, the centre of $\mathrm{GL}(2, q)$. Thus $a = -d$ and $a^2 + bc \neq 0$. This implies that the number of involutions of $\mathrm{PGL}(2, q)$ equal to $\frac{q^3-q^2}{q-1} = q^2$. □

Lemma 2.12 [2, p. 65]. *The equation $x^2 + 3 = 4p^a$ has exactly solutions $p^a = 7$ and 7^3.*

3. The proof of Theorem 1.3

Let $\mathscr{S}$ be a projective plane of order n, and let G act point-transitively on $\mathscr{S}$.

If $G = \mathrm{PGL}(2, q)$, then $\mathrm{PSL}(2, q)$ is point-transitive by Lemma 2.2 of [6]. Let $G = T : \langle\delta\rangle$, where $T = \mathrm{PSL}(2, q)$ with $q = p^a > 3$. Then by Lemma 2.7, we can suppose that $\mathrm{o}(\delta)$ is odd, and so $e(G) = e(T)$ and $e(G_\alpha) = e(T_\alpha)$.

Thus by Lemma 2.9, G has the unique conjugacy class of involutions. Moreover, T_α contains a Sylow 2-subgroup of T. Since G is point-transitive, we have $v = |G|/|G_\alpha| = (|T|/|T_\alpha|) \cdot t$ for some odd divisor t of a.

Let i be an involution of G. Since v is odd, we have i must fix at least one point of $\mathscr{P}$. By Lemmas 2.3 and 2.4, we know that $|\mathrm{Fix}_{\mathscr{P}}(\langle i\rangle)| = |\mathrm{Fix}_{\mathscr{L}}(\langle i\rangle)|$ and G also acts line-transitively on $\mathscr{S}$. If i is a central collineation, then by Lemma 2.5, we have $G \geqslant \mathrm{PSL}(3, n)$. Since $\mathrm{PSL}(3, n) \ncong \mathrm{PSL}(2, q)$ unless $n = 2$ and $q = 7$ (as required), we may assume that $|\mathrm{Fix}_{\mathscr{P}}(\langle i\rangle)| = n + \sqrt{n} + 1$, and so

$$n + \sqrt{n} + 1 = \frac{e(T_\alpha) \cdot v}{e(T)} \tag{1}$$

by Lemma 2.2.

Since $\mathrm{PSL}(2, 4) \cong \mathrm{PSL}(2, 5) \cong A_5$ and $\mathrm{PSL}(2, 9) \cong A_6$, by [4], we can let $q \geqslant 7$, and if $q = 3^a$ and 5^a, then $a \geqslant 3$ and $a \geqslant 2$, respectively.

If $q = 3^a$ with $a \geqslant 3$, then by Lemma 2.8, G_α contains a subgroup of index 3 in a Sylow 3-subgroup of G. Thus T_α must lie in a parabolic subgroup of T, and so T_α cannot contain any Sylow 2-subgroup of T. Hence $p \neq 3$.

If $p = 2$, then $|\mathrm{PSL}(2, q)| = q(q^2 - 1)$.

By Lemma 2.1, we get

$$T_\alpha \cong Z_p^a \rtimes Z_t \cong Z_2^a \rtimes Z_t, \quad \text{where } t|2^f - 1.$$

Thus $e(T_\alpha) = q - 1$. By Lemma 2.6(1), $n - \sqrt{n} + 1 = (q^2 - 1)/(q - 1) = q + 1$, which is impossible.

Now we can assume that $p \geqslant 5$ and $q \geqslant 7$. Since T_α does not contain any Sylow p-subgroup of T, we have p divides v, together with Lemma 2.8, we have $p \equiv 1 \pmod 3$, and so $q \equiv 1 \pmod 3$.

From now on, we assume that $4|(q-\epsilon)$, where $\epsilon = \pm 1$. Thus $e(T) = \frac{q(q+\epsilon)}{2}$.

By Lemma 2.10, we only consider the cases of T_α as (3), (4), (5), (6) or (8) in Lemma 2.1.

When $T_\alpha \cong A_4$, we have $e(T_\alpha) = 3$, and so 3 divides $q + \epsilon$ by Lemma 2.6(1). It follows that $\epsilon = -1$.

Since $q + 1 \equiv 2 \pmod 3$ and $4|(q+1)$, we have $q + 1 = 6l + 2$ for some integer l. Moreover, $(q+1)/2$ is even, and so $(q+1)/2 = 6l' + 4$, where $l = 2l' + 1$. This deduces that $\frac{q+1}{4} \equiv 2 \pmod 3$.

Since $v = \frac{|T|}{|T_\alpha|} \cdot t = \frac{q(q^2-1)}{24} \cdot t$, we know that $(q+1)/4$ divides v. Thus there exists a prime s congruent to 2 modulo 3, such that $s|v$, which conflicts with Lemma 2.8.

When $T_\alpha \cong A_5$, we may get a contradiction as in the case where $T_\alpha \cong A_4$.

When $T_\alpha \cong S_4$, by Lemma 2.1(5), $8|(q-\epsilon)$, and $e(T_\alpha) = 9$. By Lemma 2.6(2),

$$v > \frac{q^2(q+\epsilon)^2}{18^2} \geqslant \frac{q^2(q-1)^2}{18^2}.$$

Namely,

$$t \cdot \frac{q(q^2-1)}{48} > \frac{q^2(q-1)^2}{18^2}.$$

It follows that

$$t > \frac{4q}{27} \cdot \left(1 - \frac{2}{q+1}\right) \geqslant \frac{q}{9}$$

(note that here $q \geqslant 7$), and so

$$a > \frac{p^a}{9}. \tag{2}$$

Note that $p \geqslant 5$ and $p^a \geqslant 7$, and we get $p^a = 5^2$ or 7 from (2) which conflicts with $e(T_\alpha)|e(T)$.

When $T_\alpha \cong D_{2z}$, where z divides $(q-\epsilon)/2$ and $\frac{q-\epsilon}{2z}$ is odd, we get $e(T_\alpha) = z + 1$. Since

$$(n + \sqrt{n} + 1)(n - \sqrt{n} + 1) = n^2 + n + 1 = v = \frac{q(q^2-1)}{4z} \cdot t,$$

we have q divides v. Together with $n - \sqrt{n} + 1 = \frac{e(T)}{e(T_\alpha)} = \frac{q(q+\epsilon)}{2(z+1)}$ is divided by p, and so we have q divides $\frac{q(q+\epsilon)}{2(z+1)}$.

This implies that $z + 1$ divides $(q+\epsilon)/2$. Suppose that $q - \epsilon = 2zl_1$ and $q + \epsilon = 2(z+1)l_2$, where both l_1 and l_2 are positive integers. Then $2\epsilon = 2(z+1)l_2 - 2zl_1 = 2z(l_2 - l_1) + 2l_2$, and so $l_2 \leqslant l_1$. If $l_2 = l_1$, then $l_1 = l_2 = \epsilon = 1$, and so $z = (q-1)/2$. It follows that $n - \sqrt{n} + 1 = q$, and so $1 - 4(1-q)$ is a square, that is, the equation $x^2 + 3 = 4p^a$ has positive integer solutions for x, p and a. By Lemma 2.12, we have $p^a = 7$ or 7^3. The two cases contradict z is even. Thus $l_2 < l_1$. Moreover, $l_2 + 1 \leqslant l_1$, that is,

$$\frac{q+\epsilon}{2(z+1)} + 1 \leqslant \frac{q-\epsilon}{2z}.$$

It follows that

$$2z^2 + 2z + 2\epsilon z \leqslant q - \epsilon.$$

This deduces that $z < \sqrt{q}$.

On the other hand, by Lemma 2.6(2), we have

$$v > \left(\frac{q(q+\epsilon)}{2(z+1)}\right)^2,$$

that is,

$$\frac{q(q^2-1)}{4z} \cdot t > \left(\frac{q(q+\epsilon)}{2(z+1)}\right)^2 \geqslant \frac{q^2(q-1)^2}{4(z+1)^2}.$$

It follows that

$$t > \frac{q(q-1)}{q+1} \cdot \frac{z}{z+1} \cdot \frac{1}{z+1} \geqslant \frac{3q}{4} \cdot \frac{2}{3} \cdot \frac{1}{z+1} = \frac{q}{2(z+1)}. \tag{3}$$

If $2a > p^{a/2} - 1$, then $p^a = 7$ (note that $p \geqslant 5$ and $p^a \geqslant 7$ again). In this case, $e(T) = 21$ and $e(T_\alpha) = 5$, contrary to $e(T_\alpha)|e(T)$.

Thus we can suppose that $2a \leqslant p^{a/2} - 1$. By (3), we have $z + 1 > q/(2a) \geqslant q/\left(\sqrt{q} - 1\right) > \sqrt{q} + 1$. It follows that $z > \sqrt{q}$, a contradiction.

When $T_\alpha \cong \mathrm{PSL}(2, p^m)$, then $e(T_\alpha) = \frac{q'(q'+\epsilon)}{2}$, where $\frac{a}{m} > 1$ and $q' = p^m$.

By Lemma 2.6(2), we have

$$v > \frac{q^2(q+\epsilon)^2}{q'^2(q'+\epsilon)^2} > \frac{q^2(q-1)^2}{q'^2(q'+1)^2},$$

that is,

$$\frac{q(q^2-1)}{q'(q'^2-1)} \cdot t > \frac{q^2(q-1)^2}{q'^2(q'+1)^2}.$$

This deduces that

$$q't > q \cdot \left(1 - \frac{2}{q+1}\right) \cdot \left(1 - \frac{2}{q'+1}\right) > \frac{3q}{4} \cdot \frac{2}{3} = q/2$$

(note that here $q \geqslant 7$ and $q' \geqslant 5$).

It follows that

$$2a \geqslant 2t > p^{a-m} \geqslant 5^{a/2},$$

since $a/m \geqslant 2$, which is impossible.

When $T_\alpha \cong \mathrm{PGL}(2, p^m)$, then $e(T_\alpha) = q'^2$ by Lemma 2.11, where $2m|a$ and $q' = p^m$. Hence by Lemma 2.6(2), we have

$$v > \frac{q^2(q+\epsilon)^2}{4q'^4} \geqslant \frac{q^2(q-1)^2}{4q'^4},$$

that is,

$$\frac{q(q^2-1)}{2q'(q'^2-1)} \cdot t > \frac{q^2(q-1)^2}{4q'^4}.$$

This deduces that

$$2q't > q \cdot \frac{q-1}{q+1} \cdot \left(1 - \frac{1}{q'^2}\right) > \frac{3q}{4} \cdot \left(1 - \frac{1}{25}\right) = \frac{18q}{25}.$$

Therefore,

$$a \geqslant t > \frac{9}{25} \cdot p^{a-m} \geqslant 9 \cdot 5^{a/2-2},$$

which forces that $p^a = 5^2$ since $a \geqslant 2$. In this case, we have $n - \sqrt{n} + 1 = 13$, and so $n = 16$. Thus by (1), we get $21 = 5t$, a contradiction.

Now we finished the proof of Theorem 1.3. □

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