Regular Matrices in the Semigroup of Hall Matrices

Han H. Cho*

Department of Mathematics
College of Education
Seoul National University
Seoul, Korea

Submitted by Richard A. Brualdi

ABSTRACT

We study the regular matrices in the semigroup $H_n$ of Hall matrices (Boolean matrices with positive permanent). We study some necessary and sufficient conditions for a Hall matrix to be regular in $H_n$ in terms of idempotent matrices, adjoint matrices, and identifying permutation matrices.

1. INTRODUCTION

Let $\beta = \{0, 1\}$ be the Boolean algebra of order two with operations $(\cdot, \cdot): 1 + 0 = 0 + 1 = 1 + 1 = 1 \cdot 1 = 1 & 0 + 0 = 0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$ and an order $<: 0 < 1$. Then under these Boolean operations, the set $B_n$ of all $n \times n$ matrices over $\beta$ (Boolean matrices) and the set $H_n$ of all $n \times n$ matrices over $\beta$ with positive permanent (Hall matrices) form multiplicative semigroups. There have been many researches on such semigroup properties as primeness, regularity, and indices in $B_n$ and $H_n$. In this paper we study the regular matrices in the semigroup $H_n(s)$ ($= \{A \in B_n \mid \text{per} A \geq s\}$) for each positive integer $s$.

DEFINITION 1.1. Let $S$ be a multiplicative semigroup, and let $A$ be an element of $S$. $A$ is regular in $S$ if $AGA = A$ for some $G \in S$ ($G$ is called a
generalized inverse of $A$). $A$ is semi-invertible in $S$ if $AGA = A$ and $GAG = G$ for some $G \in S$ ($G$ is called a semi-inverse of $A$). For a Boolean matrix $A \in B_n$, the Boolean rank of $A$ is the smallest integer $r$ for which there exist $n \times r$ and $r \times n$ Boolean matrices $B$ and $C$ providing the factorization $A = B \cdot C$. The Boolean rank of a zero matrix is 0, and if the Boolean rank of $A \in B_n$ is $n$, then $A$ is called a rank-$n$ matrix.

Throughout this paper, we will regard the regular elements of $H_n(s)$ as a generalization of the invertible elements in the semigroup $R_n$ of $n \times n$ real matrices. With this point of view in mind, we derive some properties of the regular matrices of $H_n(s)$ by examining the well-known properties of the invertible matrices in $R_n$. Also in this paper we show that many properties that hold for the rank-$n$ regular matrices of $B_n$ also hold for the regular matrices of $H_n(s)$.

DEFINITION 1.2. Let $A \in B_n$ be an $n \times n$ Boolean matrix. Then for each pair of integers $i$ and $j$, $A_{ij}$ and $A_{ji}$ denote respectively the $i$th row and the $j$th column of $A$, and $A_{ij}$ denotes the $(i, j)$ entry of $A$. As usual, $A'$ denotes the transpose of $A$. For $A$ and $R$ in $B_n$, $R \leq A$ if and only if $R_{ij} \leq A_{ij}$ for all $i$ and $j$. If $R \leq A$, then $R$ is called a spanning submatrix of $A$ (we also say that $R$ is contained in $A$), and $A - R$ denotes a Boolean matrix such that $(A - R)_{ij} = 1$ if and only if $A_{ij} = 1$ and $R_{ij} = 0$. The permanent per $A$ of $A$ is the number of elements in $S_A$, where $S_A = \{P \in S_n | P \leq A\}$ and $S_n$ is the set of all $n \times n$ permutation matrices ($S_n$ also denotes the set of all permutations on $\{1, \ldots, n\}$). Finally, both $|A|$ and $\sigma(A)$ denote the number of ones of $A \in B_n$, and $A$ is called a $J$-matrix and denoted by $J_n$ if $\sigma(A) = n^2$.

Consider the following commutative diagram:

Here, $N_n$ denotes the monoid of $n \times n$ nonnegative matrices, $\Omega_n$ denotes the monoid of $n \times n$ doubly stochastic matrices, and $T_n$ denotes the monoid of $n \times n$ Boolean matrices with total support (Boolean matrices that can be expressed as a sum of permutation matrices). Finally $\iota$ denotes the canonical inclusion map, and $\pi$ denotes the support map that sends a nonnegative
matrix to a Boolean matrix in such a way that for each \( \alpha \in \mathbb{N}_n \), the \((i, j)\) entry \( \pi(\alpha)_{ij} \) of \( \pi(\alpha) \) is 1 if and only if \( \alpha_{ij} > 0 \) for each \( i \) and \( j \). Since \( 1 \cdot 1 = 1 \) in \( \beta \) and the product of any two positive numbers is positive in the real number system, \( \pi(\alpha) = \pi(\beta)\pi(\gamma) \) when \( \alpha = \beta\gamma \). Therefore \( \pi \) is a semigroup homomorphism, and using this semigroup homomorphism \( \pi \) we can compare the regular elements of each semigroup in the diagram as follows.

**Theorem 1.3.** Let \( \alpha \) be an element of \( \mathbb{N}_n \) (respectively \( \Omega_n \)) and let \( A \) be \( \pi(a) \). Then

1. If \( \alpha \) is regular in \( \mathbb{N}_n \) (respectively \( \Omega_n \)), then \( A \) is regular in \( B_n \) (respectively \( T_n \)).
2. \( A \) is regular in \( B_n \) if and only if \( (A'A'A')^c \) \( (A^c \text{ is } J_n - A) \) is the largest generalized inverse of \( A \) in \( B_n \).
3. \( \alpha \in \Omega_n \) is regular in \( \Omega_n \) if and only if \( \alpha\alpha'\alpha = \alpha \).


Consider the matrices

\[
\gamma = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]

Then \( \gamma \) is not regular in \( \mathbb{N}_2 \), but \( C \) is regular in \( B_2 \). So the converse of (1) in Theorem 1.3 does not hold. By (2) and Corollary 4.4 we can see that for \( A \in H_n \), \( A \) is regular in \( H_n \) iff \( (A'A'A')^c = P'AP' \) for some \( P \in S_A \) iff \( (A'A'A')^c \in H_n \). Consider the matrices

\[
A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

Then \( A \in H_4 \), and \( A \) is regular in \( B_4 \) since \( AGA = A \) and \( GAG = G \) hold. But \( A \) is not regular in \( H_4 \), by Theorem 4.3, since the first row \( A_{11} \) of \( A \) contains only one row of \( A \) even though \( |A_{14}| = 2 \). Therefore there is a Hall matrix \( A \in H_n \) of rank less than \( n \) such that \( A \) is regular in \( B_n \) but not in \( H_n \). Note that for a rank-\( n \) Boolean matrix or a prime Boolean matrix \( A \in B_n \), \( A \) is regular in \( B_n \) if and only if \( A \) is regular in \( H_n \) (cf. [2]).
2. REGULAR MATRICES AND IDEMPOTENT MATRICES

In this section we characterize the regularity of a Hall matrix \( A \in H_n \) in terms of idempotent matrices. Then for each positive integer \( s \), we compare the regular matrices in \( H_n(s) \) and \( T_n \) with the regular matrices in \( H_n \).

**Definition 2.1.** For \( A \in H_n \), \( A \) is called an idempotent matrix if \( AA = A \). For any Boolean matrices \( A \) and \( B \) in \( B_n \), we say \( A \) is (permutationally) equivalent to \( B \) if \( B = PAQ \) for some permutation matrices \( P \) and \( Q \). Also, if \( B = PAP^t \) for some permutation matrix \( P \), then we say \( A \) is (permutationally) similar to \( B \).

**Lemma 2.2.** Let \( A \) and \( G \) be Hall matrices such that \( AGA = A \). Then for each \( P \in S_G \), \( APA = A \), and \( P^t \in S_A \).

**Proof.** For each \( P \in S_G \), \( AGA = A \) means \( APA \leq A \) and \( |APA| \leq |A| \). Since \( AP \) contains a permutation matrix, \( |(AP)A| \geq |A| \) holds. Thus we have \( APA = A \) and \( PAPA = PA \) for any \( P \in S_G \). Since \( PAPA = PA \) and \( (P \cdot S_A) \cdot (P \cdot S_A) \subseteq P \cdot S_A \), \( P \cdot S_A \) is a subsemigroup of the symmetric group \( S_n \) (\( P \cdot S_A \) is \( \{P \cdot Q \mid Q \in S_A\} \)). So \( P \cdot S_A \) is a group. Thus \( P^tQ = I \) for some \( Q \in S_A \), and \( P^t \) is in \( S_A \).

**Theorem 2.3.** Let \( A \) be an \( n \times n \) Hall matrix. Then the following statements are all equivalent:

1. \( A \) is regular in \( H_n \).
2. \( A \) has a unique semiinverse in \( H_n \).
3. \( A \) is permutationally equivalent to an idempotent matrix.
4. \( A \) is regular in \( H_n(s) \) if \( A \in H_n(s) \).

**Proof.** (1) \( \rightarrow \) (2): Let \( S = \{X \in H_n \mid AXA = A\} \), and let \( G \) be the Boolean sum \( \sum_{X \in S} X \). Then \( AGA = A(\sum_{X \in S} X)A = \sum_{X \in S} (AXA) = A \). Therefore \( G \) is the largest generalized inverse of \( A \) in \( H_n \). Since \( A(GAG)A = (AGA)GA = AGA = A \), we obtain \( GAG \leq G \). Note that \( AGA = A \) and \( GAG \leq G \) imply \( |G| \leq |A| \) and \( |A| \leq |G| \) respectively. Thus \( |G| = |A| \) and \( GAG = G \). Hence \( G \) is also the largest semiinverse of \( A \) in \( H_n \). Now consider any semiinverse \( H \) of \( A \) in \( H_n \). Since \( A(G + H)A = A \) and \( G \) is the largest generalized inverse of \( A \), we obtain \( H \leq G \). Since \( |A| = |G| \) and \( HAH = H \), we have \( |H| \leq |A| \) and \( |A| \leq |H| \). Thus \( |A| = |H| = |G| \) and \( H = G \). Therefore there exists a unique semiinverse of \( A \) in \( H_n \).
(2) \( \rightarrow \) (3): By Lemma 2.2, \( AGA = A \) implies \( APA = A \) for any \( P \in S_G \). So \( (APA)P = AP \) and \( AP \) is an idempotent matrix of \( H_n \). Thus \( A \) is permutationally equivalent to an idempotent matrix.

(3) \( \rightarrow \) (4): Suppose \( (PAQ)(PAQ) = PAQ \) for some permutation matrices \( P \) and \( Q \). Then \( A(QP) = A \), and \( A \) is regular in \( H_n \). Hence for the largest generalized inverse \( G \) of \( A \) in \( H_n \), \( AGA = A \) and \( GAG = G \), and this means per \( A = \) per \( G \). Thus \( A \) is regular in \( H_n(s) \) for any positive integer \( s \) less than or equal to per \( A \).

(4) \( \rightarrow \) (1): Obvious because \( H_n(1) = H_n \).

**Corollary 2.4.** Let \( A \) be an \( n \times n \) Hall matrix with total support. Then the following statements are all equivalent:

1. \( A \) is regular in \( T_n \).
2. \( A = (\sum_{Q \in H} Q)P \) for some subgroup \( H \subseteq S_n \) and for some \( P \in S_n \).
3. \( A' \) is the unique semiinverse of \( A \) in \( T_n \).

**Proof.** (1) \( \rightarrow \) (2): Let \( A \in T_n \) be a regular matrix. Note that \( A \in T_n \) means \( A = \sum_{Q \in S_n} Q \). Hence if \( AA = A \) then \( S_A \cdot S_A \subseteq S_A \) and \( S_A \) is a subgroup of \( S_n \). Hence \( A \in T_n \) is an idempotent matrix if and only if \( A = \sum_{Q \in K} Q \) for some subgroup \( K \) of \( S_n \). If \( AGA = A \), then \( APA = A \) and \( APAP = AP \) for any \( P \in S_G \). Thus \( AP \) is an idempotent matrix for some \( P \in S_n \). Then \( S_A \cdot P \) is a subgroup \( H \) of \( S_n \), and \( S_A = H \cdot P' \). So \( A = \sum_{Q \in H} (QP') = (\sum_{Q \in H} Q)P' \) for some subgroup \( H \subseteq S_n \) and \( P \in S_n \).

(2) \( \rightarrow \) (3): Let \( A = (\sum_{Q \in H} Q)P \) for some subgroup \( H \subseteq S_n \) and for some \( P \in S_n \). Then \( A' = P'\sum_{Q \in H} Q \), and we obtain \( AA'A = A \) and \( A'AA' = A' \), since \( \sum_{Q \in H} Q \cdot \sum_{Q \in H} Q = \sum_{Q \in H} Q \). Thus \( A' \) is a semiinverse of \( A \). Now let \( B \) be another semiinverse of \( A \) in \( T_n \). Then by Theorem 2.3, \( A' \) is \( B \), and this means \( A' \) is the unique semiinverse of \( A \).

(3) \( \rightarrow \) (1): Obvious.

From Corollary 2.4 we conclude that if \( \alpha \) is regular in \( \Omega_n \), then \( A \) [\( = \pi(\alpha) \)] is regular in \( T_n \) and \( AA'A = A \). In fact, by (2) of Theorem 1.3 the semiinverse of \( \pi(\alpha) \) is \( \pi(\alpha') \). But in general the semiinverse of \( A \) may not be \( A' \) for a regular matrix \( A \) of \( H_n \).

3. **REGULAR MATRICES AND ADJOINT MATRICES**

In this section we characterize the regular matrices of \( H_n(s) \) in terms of their adjoint matrices. By Theorem 2.3, if \( A \) is regular in \( H_n \), then \( A \) is regular in \( H_n(s) \). Therefore we will only consider the regularity in \( H_n \).
DEFINITION 3.1. Let \( A \in B_n \) be an \( n \times n \) Boolean matrix. Then \( A \) is fully indecomposable if \( A \) is not equivalent to a matrix of the form
\[
\begin{bmatrix}
B_1 & * \\
0 & B_2
\end{bmatrix},
\]
where the \( B_i \)'s are square matrices. \( A \) is partly decomposable if \( A \) is not fully indecomposable. For each pair of positive integers \( i \) and \( j \), \( E_n(i, j) \) denotes an \( n \times n \) Boolean matrix whose \((i, j)\) entry is the only nonzero entry. For \( A \in B_n \), the adjoint matrix \( \text{adj} \ A \) of \( A \) is an \( n \times n \) Boolean matrix whose \((i, j)\) entry \((\text{adj} \ A)_{ij}\) is 1 if \( \text{per} \ A(j|i) > 0 \), and 0 if \( \text{per} \ A(j|i) = 0 \). Here \( A(j|i) \) denotes an \((n-1)\)-by-\((n-1)\) Boolean matrix obtained from \( A \) by deleting its \( j \)th row and the \( i \)th column.

LEMMA 3.2. Let \( A \) be an \( n \times n \) Hall matrix. Then:

1. If \( A \) is an idempotent matrix, then \( \text{adj} \ A = A \).
2. \( A \) is fully indecomposable if and only if \( \text{adj} \ A = I_n \).
3. \( \text{adj}(PAQ) = Q^t(\text{adj} A)P^t \) for any \( P \) and \( Q \) in \( S_n \).
4. \( \text{adj} \ A \) is a regular matrix of \( H_n \).

Proof. (1): If \( A \in H_n \) is an idempotent matrix, then \( S_A \cdot S_A = S_A \) and \( S_A \) is a group. Thus there is an identity permutation matrix in \( A \). Hence \( A_{ii} = (\text{adj} A)_{ii} = 1 \), and \( A_{ij} = 1 \) implies \((\text{adj} A)_{ij} = 1 \) for any \( i \) and \( j \) (\( i \neq j \)). Now suppose that \((\text{adj} A)_{ij} = 1 \) for some \( i \) and \( j \) (\( i \neq j \)). Then there is a permutation \( \sigma \) on \( \{1, \ldots, n\} \) such that \( \sigma(j) = i \) and \( A_{t \sigma(t)} = 1 \) if \( t \neq j \). Consider \( \sigma(t) \) and \( \sigma^2(t) \) (\( = \sigma(\sigma(t)) \)), and \( \sigma^m(t) \) in general. Note that there is a smallest integer \( d (> 2) \) such that \( \sigma^d(i) = i \) and \( \sigma^{d-1}(i) = j \), since \( \sigma \) is a permutation. Also note that \( A_{i \sigma^{-1}(i)} = 1 \) for any positive integer \( m < d \), since \( A_{i \sigma^{-1}(i)} = 1 \) and \( A_{i \sigma^{m-1}(i) \sigma^{-1}(i)} = 1 \) and \( A \) is an idempotent matrix. Thus \((\text{adj} A)_{ij} = 1 \) if and only if \( A_{ij} = 1 \).

(2): It is well known that \( A \in H_n \) is fully indecomposable if and only if \( A(i|j) \) is a Hall matrix for each \( i \) and \( j \) (cf. [5]).

(3): If \( P \) is in \( S_n \), then \( P \) can be expressed as a Boolean sum \( \sum_{i=1}^n E_n(i, \sigma(i)) \) for some permutation \( \sigma \) on the set \( \{1, \ldots, n\} \). Now we claim that \( \text{adj}(PA) = (\text{adj} A)P^t \). Choose any \( i \) and \( j \) from \( \{1, \ldots, n\} \), and let \( \sigma(i) = \alpha \) and \( \sigma(j) = \beta \). Then \((\text{adj} A)P^t)_{ij} = (\text{adj} A)_{i \beta} \) and \( \text{per}(A(\beta|i)) = \text{per}(PA(j|i)) \). Thus \((\text{adj}(PA))_{ij} = ((\text{adj} A)P^t)_{ij} \) and \( \text{adj}(PA) = (\text{adj} A)P^t \).
By the same method, we have \( \text{adj}(AQ) = Q' \text{adj} A \). Thus we conclude that 
\[
\text{adj}(PAQ) = [\text{adj}(AQ)]P' = Q'(\text{adj} A)P'.
\]

(4): If \( A \) is fully indecomposable, then \( \text{adj} A = J_n \) and \( \text{adj} A \) is regular in \( H_n \). Now let \( A \) be partly decomposable. Then \( A \) is permutationally equivalent to a canonical form \( N \) of \( A \), where

\[
N = \begin{bmatrix}
B_1 & B_{12} & \cdots & B_{1\lambda} \\
B_{21} & B_2 & \cdots & B_{2\lambda} \\
\vdots & \vdots & \ddots & \vdots \\
B_{\lambda 1} & B_{\lambda 2} & \cdots & B_\lambda
\end{bmatrix}
\]

Here, the \( B_s \)'s are the nonzero fully indecomposable components of \( A \), and each block matrix \( B_{st} \) is a zero matrix if \( s > t \). In addition we may assume that each \( B_s \) contains an identity permutation matrix, since \( B_s \) is a Hall matrix. We now claim that \( \text{adj} N \) is in fact an idempotent matrix. Note that for any \((i, j)\) entry of \( N \) located inside of any \( B_s \), \( \text{per} N(ij) \) is positive by (2). Hence if \( \text{adj} N \) is partitioned in the same way as \( N \), then the diagonal blocks of \( \text{adj} N \) are \( J \)-matrices and \( \text{adj} N \) contains an identity permutation matrix. We now show that \( \text{adj} N \) has the transitive property. Suppose \((\text{adj} N)_{ij} = 1 \) and \((\text{adj} N)_{jk} = 1 \) \((i < j < k)\). Now let the \((j, j)\) entry of \( N \) be located in \( B_s \), and let the \((\alpha + 1, \alpha + 1)\) entry of \( N \) be the first entry of \( B_s \) and the \((\beta - 1, \beta - 1)\) entry of \( N \) be the last entry of \( B_s \). Since \( \text{per} N(ij) > 0 \) and \( \text{per} N(kj) > 0 \), there are permutations \( \sigma \) and \( \tau \) in \( S_n \) such that \( \sigma(j) = i \), \( \tau(k) = j \), \( N_{i\sigma(x)} = 1 \) \((x \neq j)\), and \( N_{x\tau(x)} = 1 \) \((x \neq k)\). Then, without loss of generality, we may assume that \( \sigma = (i, \ldots, \sigma_0, \sigma_1, \ldots, \sigma_n) \) and \( \tau = (\tau_1, \ldots, \tau_b, \tau_{b+1}, \ldots, k) \) \((\sigma_b = \tau_j = i)\), where \( \sigma_1, \ldots, \sigma_n \) \((\tau_1, \ldots, \tau_b)\) are the components of \( \sigma \) \((\tau)\) that are greater than \( \alpha \) and less than \( \beta \). If \( \sigma_1 = \tau_b \), then construct a permutation \((\ldots, \sigma_0, \sigma_1, \tau_{b+1}, \ldots, k)\) by joining \( \sigma \) and \( \tau \) after deleting \( \tau_1, \ldots, \tau_b \) from \( \tau \). Thus we have \((\text{adj} N)_{ik} = 1 \) in this case. Now let \( \sigma_1 \neq \tau_b \). If \( \sigma_1 = \alpha + f \) and \( \tau_b = \alpha + g \), then \( \text{per} B_c(gf) \) is positive, since \( B_c \) is fully indecomposable. Consider a permutation \( \rho = (i, \ldots, \sigma_0, \sigma_1, \rho_1, \ldots, \rho_c, \tau_b, \tau_{b+1}, \ldots, k) \), where \( N_{x\rho(x)} = 1 \) \((x \neq k)\), \( \rho(k) = i \), and \((\sigma_1, \rho_1, \ldots, \rho_c, \tau_b) \) is a permutation obtained from the condition \( \text{per} B_c(gf) > 0 \). Hence \((\text{adj} N)_{ik} = 1 \) in this case. Using similar arguments, we can show that \((\text{adj} N)_{ik} = 1 \) for general \( i, j, \) and \( k \). Thus \( \text{adj} N \) is an idempotent matrix, since it contains an identity permutation matrix and satisfies transitivity. Therefore we now conclude that \( \text{adj} A \) is a regular matrix in \( H_n \), since \( \text{adj} A \) is permutationally equivalent to an idempotent matrix \( \text{adj} N \).
THEOREM 3.3. For an \( n \times n \) Hall matrix \( A \in H_n \), the following statements are all equivalent:

(1) \( A \) is regular in \( H_n \).

(2) \( \text{adj} \, A \) is the unique semiinverse of \( A \).

(3) \( \text{adj} \, \text{adj} \, A = A \).

Proof. (1) \( \rightarrow \) (2): Let \( A \) be regular in \( H_n \). Then \( A \) is permutationally equivalent to an idempotent matrix \( D (= PAQ \) for some \( P \) and \( Q \) in \( S_n \)) by Theorem 2.3. Note that \( \text{adj} \, D = D \) by Lemma 3.2. Hence \( \lambda(\text{adj} \, A) \lambda = (P^tDQ^t)(Q(\text{adj} \, D)P)(P^tDQ^t) = P^tDQ^t = A \). Thus \( \text{adj} \, A \) is a generalized inverse of \( A \). Note that \( |\text{adj} \, A| = |A| \), since \( |A| = |D| = |\text{adj} \, D| = |\text{adj} \, A| \). Thus \( \text{adj} \, A \) is the largest generalized inverse of \( A \). Therefore \( \text{adj} \, A \) is the unique semiinverse of \( A \) by Theorem 2.3.

(2) \( \rightarrow \) (3): From the assumption, \( A \) is regular in \( H_n \). Now let \( D (= PAQ \) for some \( P \) and \( Q \) in \( S_n \)) be an idempotent matrix. Then \( A = P^tDQ^t \) and \( \text{adj} \, \text{adj}(PAQ) = \text{adj}[Q'((\text{adj} \, A)^tP') = P(\text{adj} \, \text{adj} \, A)Q \) by Lemma 3.2. Note that \( P(\text{adj} \, \text{adj} \, A)Q = D \), since \( \text{adj}(PAQ) = D \). Therefore we obtain \( \text{adj} \, \text{adj} \, A = A \).

(3) \( \rightarrow \) (1): \( A \in H_n \) implies \( \text{adj} \, A \in H_n \), and \( \text{adj} \, A \in H_n \) implies \( \text{adj} \, \text{adj} \, A \) is a regular matrix in \( H_n \) by Lemma 3.2(4). Thus if \( A = \text{adj} \, \text{adj} \, A \), then \( A \) is regular in \( H_n \).

4. REGULAR MATRICES AND IDENTIFYING PERMUTATION MATRICES

In this section we characterize the regularity of a Hall matrix \( A \in H_n(s) \) in terms of its row (and column) sums and its identifying permutation matrices. We also give simple criteria for any Boolean matrix \( A \) to be regular in \( H_n \) in terms of \( S_A \).
DEFINITION 4.1. For any Hall matrix $A$, a permutation matrix $P$ in $A$ is called an identifying permutation matrix of $A$ if $P_i \preceq A_j$ implies $A_i \preceq A_j$ for each $i$ and $j$. Let $B$ be a submatrix of $A$. We say a row of $R$ of $A$ passes $B$ if there is an $(i, j)$ entry place of $A$ such that $R$ and $B$ meet at that $(i, j)$ entry place.

LEMMA 4.2. Let $A \in H_n$ be a Hall matrix such that each row $A_{i*}$ of $A$ contains exactly $|A_{i*}|$-many rows of $A$. Then:

1. $A$ is a $J$-matrix if $A$ is fully indecomposable.
2. $A$ is regular in $H_n$.

Proof. (1): Choose a row $A_{i*}$ whose row sum $\sigma(A_{i*})$ is the minimum among $\sigma(A_{i*})$'s. Then by the assumption, there are $|A_{i*}|$-many rows of $A$ that are contained in $A_{i*}$. So $A$ can have a $|A_{i*}| \times (n - |A_{i*}|)$ zero submatrix of $A$ if $A_{i*}$ is not an all-one vector. Therefore each $i$th row $A_{i*}$ must be an all-one vector, and $A$ is a $J$-matrix, since $A$ is a fully indecomposable matrix.

(2): If $A$ is fully indecomposable, then $A$ is $J_n$ and $A$ is regular in $H_n$ by (1). Now let $A$ be partly decomposable, and let $N (= PAQ$ for some $P$ and $Q$ in $S_n$) be a canonical form of $A$, where

$$N = \begin{bmatrix}
B_1 & B_{12} & \cdots & B_{1\lambda} \\
B_{21} & B_2 & \cdots & B_{2\lambda} \\
\vdots & \vdots & \ddots & \vdots \\
B_{\lambda 1} & B_{\lambda 2} & \cdots & B_{\lambda \lambda}
\end{bmatrix}.$$ 

Here, all the $B_j$'s are nonzero fully indecomposable components of $A$, and $B_{st}$ is a zero matrix if $s > t$. By Theorem 2.3, we know $A$ is regular in $H_n$ if $A$ is permutationally equivalent to an idempotent matrix. We will show that this $N$ is in fact an idempotent matrix by induction. For each positive integer $k$ with $k < \lambda$, let $N(k)$ be

$$N(k) = \begin{bmatrix}
B_{\lambda-k} & B_{\lambda-k,b} & \cdots & B_{\lambda-k,\lambda} \\
B_{\lambda-k+1,\lambda-k} & B_{\lambda-k+1} & \cdots & B_{\lambda-k+1,\lambda} \\
\vdots & \vdots & \ddots & \vdots \\
B_{\lambda,\lambda-k} & B_{\lambda,\lambda-k+1} & \cdots & B_{\lambda}
\end{bmatrix},$$

where $b = \lambda - k + 1$. Note that each $B_j$ of $N$ is a Hall matrix, and each
$N(k)$ satisfies the condition that each row $R$ of $N(k)$ contains exactly $\sigma(R)$ many rows passing $N(k)$. Now we do induction on this $k$. First let $k = 1$. We know that $B_\lambda$ in $N(1)$ is a $J$-matrix by (1). Now let $\lfloor (B_{\lambda-1})_r \rfloor$ be the minimum value among $(B_{\lambda-1})_r$'s $((B_{\lambda-1})_r)$ denotes the $r$th row of $B_{\lambda-1}$, and let $R$ be $(B_{\lambda-1})_r$. If $R$ does not contain $\sigma(R)$ many rows of $B_{\lambda-1}$, then $N(1)_r$ must contain at least one row of $N(1)$ passing $B_\lambda$. Thus any entry of $N(1)_r$ located inside of $B_{\lambda-1}$ must be one, and $N(1)_r$ must contain $\sigma(R)$ many rows of $B_{\lambda-1}$, since $N(1)_r$ contains $|N(1)_r|$ many rows of $N(1)$. Thus $B_{\lambda-1}$ is a $J$-matrix too by (1). Hence $N(1)$ is an idempotent matrix, since $B_{\lambda-1}$ is a zero matrix, or a $J$-matrix if there is a positive entry inside of $B_{\lambda-1}$. By the induction hypothesis, we assume that $\lambda \geq 3$ and $N(k)$ is an idempotent matrix for any $k \leq \lambda - 2$. Now let $k = \lambda - 1$ and $N(k) = N$. Let $R$ be the $i$th row of $B_1$ such that $\sigma(R)$ is the minimum value among $(B_1)_r$'s. If $R$ contains less than $\sigma(R)$ many rows of $B_1$, then $N_i$ should contain more than $|N_i| - \sigma(R)$ many rows passing $N(k-1)$. Note that the number of nonzero entries in the Boolean sum of more than $|N_i| - \sigma(R)$ many rows of $N(k-1)$ is greater than $|N_i| - \sigma(R)$, since $N(k-1)$ is a Hall matrix. Thus $B_1$ is a $J$-matrix, and every block $B_{1b}$ in $N$ is a zero matrix or a $J$-matrix (if there is a positive entry inside of the block $B_{1b}$). Since the right-hand side of the row $N_i$, not contained in $B_1$ is the Boolean sum of $|N_i| - \sigma(R)$ many rows of $N(k-1)$. Thus if there is a positive $(i, j)$ entry of $N$ in the block $B_{1b}$, then $N_i$ contains a row $S$ passing $N(k-1)$ such that $j$th entry of $S$ is one. Note that there exists a row $T$ of $N$ passing $B_{1b}$ such that $T \leq S$, since $N(k-1)$ has the transitivity property by the induction hypothesis. Thus $T \leq S \leq N_i$, and $N$ also has the transitivity property. Hence $N$ is an idempotent matrix, and $A$ is regular in $H_n$.

**Theorem 4.3** Let $A \in H_n$ be an $n \times n$ Hall matrix. Then the following statements are all equivalent:

1. $A$ is regular in $H_n$.
2. Each row $A_i$ contains exactly $|A_i|$ many rows of $A$.
3. $A$ has an identifying permutation matrix.

**Proof.** (1) $\rightarrow$ (2): If $A$ is regular in $H_n$, then by Theorem 2.3, $A$ is permutationally equivalent to an idempotent matrix. Thus each row $A_i$ contains exactly $|A_i|$ many rows of $A$.

(2) $\rightarrow$ (3): By Lemma 4.2, $A$ is regular in $H_n$, and $N (= PAQ$ for some $P$ and $Q$ in $S_n$) is an idempotent matrix in $H_n$. Note that $P'Q'$ is in $A$ and an identifying permutation matrix of $A$, since the main diagonal of $N$ is an identifying permutation matrix of $N$. 


(3) \( \rightarrow \) (1): For each row \( A_{i*} \) of \( A \), \( A_{i*} \) contains \( |A_{i*}| \) many rows of \( P \) if there is an identifying permutation matrix \( P \) of \( A \). Thus \( A_{i*} \) contain at least \( |A_{i*}| \) many rows of \( A \). Since \( A \) is a Hall matrix, \( \sum_{j \in I} A_{j*} \geq |I| \) for each subset \( I \) of \( \{1, \ldots, n\} \), and this means \( A_{i*} \) contains exactly \( |A_{i*}| \) many rows of \( A \). Thus by Lemma 4.2, \( A \) is regular in \( H_n \).

Consider the following four \( 4 \times 4 \) Hall matrices:

\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\end{bmatrix}
\]

\[
\text{adj } A = \begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
\end{bmatrix}, \quad \text{adj } B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\end{bmatrix}
\]

Then by Theorem 3.3, \( A \) is not regular in \( H_n \), but \( B \) is. Also, by Theorem 4.3, \( A \) is not regular, since \( A_{1*} \) contains \( A_{1*} \) only, even though \( |A_{1*}| = 2 \), but \( B \) is regular, since the main diagonal of \( B \) is an identifying permutation matrix of \( B \). If we interpret \( C \in H_n \) as the incidence matrix of the subsets \( S_1, \ldots, S_n \) of \( \{1, \ldots, n\} \), this incidence matrix \( C \) is regular if and only if \( C \) has a special kind of system of distinct representatives of the \( S_1, \ldots, S_n \) (and there occurs an identifying permutation), by Theorem 4.3. The following corollary presents simple criteria for \( A \) to be regular in the semigroup of Hall matrices. The regularity of \( A \) in the semigroup \( H_n \) can be determined from \( S_A \) as follows.

**Corollary 4.4.** Let \( A \) be an \( n \times n \) Hall matrix. Then the following statements are all equivalent:

1. \( A \) is regular in \( H_n \).
2. \( S_A = \{ P \in S_n \mid AP^tA = A \} \).
3. \( S_A \subseteq \{ P \in S_n \mid P'AP' = \text{adj } A \} \).
4. \( S_A = S_A \cap \{ P \in S_n \mid P \text{ is an identifying permutation matrix of } A \} \).
5. For some \( P \in S_A \), \( AP^tA \leq A \).
6. For some \( P \in S_A \), \( P_{i*} \leq A_{j*} \) if and only if \( A_{i*} \leq A_{j*} \).

**Proof.** (1) \( \rightarrow \) (2): From Theorem 2.3, \( AGA = A \) and \( GAG = G \), where \( G \) is the largest semiinverse (generalized inverse) of \( A \) in \( H_n \). Then for any \( P^t \in S_G \), we have \( AP^tA = A \) and \( P \in S_A \) by Lemma 2.2. Similarly, for any \( P \in S_A \) we have \( GPG = G \) and \( P^t \in S_G \). Thus for any \( P \) we have \( P \in S_A \) iff \( P^t \in S_G \).
(2) $\rightarrow$ (3): $AP' = A$ means $A$ is regular in $H_n$. Let $G$ be the unique semiinverse of $A$. Then for $P \in S_A$, $P^tAP' < CAC$ ($= C$) and $P^tAP'$ is a semiinverse of $A$, because $(P^tAP')A(P^tAP') = P^tAP'$ and $A(P^tAP')A = A$. Thus by Theorem 3.3, $adj A = P^tAP'$ for any $P \in S_A$.

(3) $\rightarrow$ (4): $adj A$ is regular, and $adj A$ is permutationally equivalent to an idempotent matrix by Theorem 2.3. Thus if $P^tAP' = adj A$ for any $P \in S_A$, then $A$ is also permutationally equivalent to an idempotent matrix. Thus $A$ is regular in $H_n$, and its canonical form $N$ in Lemma 3.2 becomes an idempotent matrix. Note that any permutation matrix contained in $A$ is an identifying permutation matrix of $A$, since any permutation matrix of $N$ is an identifying permutation matrix of $N$.

(4) $\rightarrow$ (5): Since $A$ has an identifying permutation, $A$ is regular in $H_n$. Therefore for any $Q \in S_{adj A}$, $AQA < A$. Thus $AP' < A$ for some $P \in S_A$.

(5) $\rightarrow$ (6): Since $|AP' A| > |A|$ for any $P \in S_A$, $AP' A < A$ implies $A = AP' A$. Thus $A$ is regular in $H_n$, and any $P \in S_A$ is an identifying permutation matrix of $A$.

(6) $\rightarrow$ (1): Obvious. 

The author would like to thank the referee for suggesting improvements in the original version of this paper.

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