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Invariant metrics and Laplacians on Siegel–Jacobi space ☆

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Abstract

In this paper, we compute Riemannian metrics on the Siegel–Jacobi space which are invariant under the natural action of the Jacobi group explicitly and also provide the Laplacians of these invariant metrics. These are expressed in terms of the trace form.

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1. Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \left\{ Z \in \mathbb{C}^{(n,n)} \,\middle|\, Z = {}^tZ,\ \operatorname{Im} Z > 0 \right\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \left\{ M \in \mathbb{R}^{(2n,2n)} \,\middle|\, {}^tM J_n M = J_n \right\}$$

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be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

We see that $Sp(n, \mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot Z = (AZ + B)(CZ + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $Z \in \mathbb{H}_n$.

For two positive integers n and m, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{(\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = \big(\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\,{}^t\mu' - \mu\,{}^t\lambda'\big).$$

We define the semidirect product of $Sp(n, \mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J := Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\big(M, (\lambda, \mu; \kappa)\big) \cdot \big(M', (\lambda', \mu'; \kappa')\big) = \big(MM', \big(\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}\,{}^t\mu' - \tilde{\mu}\,{}^t\lambda'\big)\big)$$

with $M, M' \in Sp(n, \mathbb{R})$, $(\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. We call this group G^J the *Jacobi group* of degree n and index m. We have the *natural action* of G^J on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ defined by

$$\big(M, (\lambda, \mu; \kappa)\big) \cdot (Z, W) = \big(M \cdot Z, (W + \lambda Z + \mu)(CZ + D)^{-1}\big), \tag{1.2}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}),\ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z, W) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. The homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is called the *Siegel–Jacobi space* of degree n and index m. We refer to [2,3,6,7,11,14–21] for more details on materials related to the Siegel–Jacobi space.

For brevity, we write $\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. For a coordinate $(Z, W) \in \mathbb{H}_{n,m}$ with $Z = (z_{\mu\nu}) \in \mathbb{H}_n$ and $W = (w_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\begin{aligned}
&Z = X + iY, \quad X = (x_{\mu\nu}), \quad Y = (y_{\mu\nu}) \text{ real}, \\
&W = U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \text{ real}, \\
&dZ = (dz_{\mu\nu}), \quad d\overline{Z} = (d\bar{z}_{\mu\nu}), \quad dY = (dy_{\mu\nu}), \\
&dW = (dw_{kl}), \quad d\overline{W} = (d\overline{w}_{kl}), \quad dV = (dv_{kl}), \\
&\frac{\partial}{\partial Z} = \left(\frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial z_{\mu\nu}}\right), \qquad \frac{\partial}{\partial \overline{Z}} = \left(\frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial \bar{z}_{\mu\nu}}\right),
\end{aligned}$$

$$\frac{\partial}{\partial X}=\left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial x_{\mu\nu}}\right),\qquad \frac{\partial}{\partial Y}=\left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial y_{\mu\nu}}\right),$$

$$\frac{\partial}{\partial W}=\begin{pmatrix}\frac{\partial}{\partial w_{11}} & \cdots & \frac{\partial}{\partial w_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial w_{1n}} & \cdots & \frac{\partial}{\partial w_{mn}}\end{pmatrix},\qquad \frac{\partial}{\partial \overline{W}}=\begin{pmatrix}\frac{\partial}{\partial \overline{w}_{11}} & \cdots & \frac{\partial}{\partial \overline{w}_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial \overline{w}_{1n}} & \cdots & \frac{\partial}{\partial \overline{w}_{mn}}\end{pmatrix},$$

$$\frac{\partial}{\partial U}=\begin{pmatrix}\frac{\partial}{\partial u_{11}} & \cdots & \frac{\partial}{\partial u_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial u_{1n}} & \cdots & \frac{\partial}{\partial u_{mn}}\end{pmatrix},\qquad \frac{\partial}{\partial V}=\begin{pmatrix}\frac{\partial}{\partial v_{11}} & \cdots & \frac{\partial}{\partial v_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial v_{1n}} & \cdots & \frac{\partial}{\partial v_{mn}}\end{pmatrix},$$

where δ_{ij} denotes the Kronecker delta symbol.

C.L. Siegel [12] introduced the symplectic metric ds_n^2 on $\mathbb{H}_n$ invariant under the action (1.1) of $Sp(n,\mathbb{R})$ given by

$$ds_n^2=\sigma\big(Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big) \tag{1.3}$$

and H. Maass [8] proved that the differential operator

$$\Delta_n=\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right) \tag{1.4}$$

is the Laplacian of $\mathbb{H}_n$ for the symplectic metric ds_n^2. Here $\sigma(A)$ denotes the trace of a square matrix A.

In this paper, for arbitrary positive integers n and m, we express the G^J-invariant metrics on $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ and their Laplacians explicitly.

In fact, we prove the following theorems.

Theorem 1.1. *For any two positive real numbers A and B, the following metric*

$$\begin{aligned} ds^2_{n,m;A,B}&=A\sigma\big(Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big)\\ &\quad+B\big\{\sigma\big(Y^{-1}\,{}^tV\,VY^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big)+\sigma\big(Y^{-1}\,{}^t(dW)\,d\overline{W}\big)\\ &\quad-\sigma\big(VY^{-1}\,dZ\,Y^{-1}\,{}^t(d\overline{W})\big)-\sigma\big(VY^{-1}\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big)\big\}\end{aligned}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) *of the Jacobi group G^J.*

Theorem 1.2. *For any two positive real numbers A and B, the Laplacian $\Delta_{n,m;A,B}$ of $(\mathbb{H}_{n,m},ds^2_{n,m;A,B})$ is given by*

$$\begin{aligned}\Delta_{n,m;A,B}&=\frac{4}{A}\left\{\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right)+\sigma\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right)\right.\\ &\quad\left.+\sigma\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial W}\right)+\sigma\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial Z}\right)\right\}\\ &\quad+\frac{4}{B}\sigma\left(Y\frac{\partial}{\partial W}\,{}^t\!\left(\frac{\partial}{\partial\overline{W}}\right)\right).\end{aligned}$$

The following differential form

$$dv = (\det Y)^{-(n+m+1)}[dX] \wedge [dY] \wedge [dU] \wedge [dV]$$

is a G^J-invariant volume element on $\mathbb{H}_{n,m}$, where

$$[dX] = \bigwedge_{\mu \leqslant \nu} dx_{\mu\nu}, \qquad [dY] = \bigwedge_{\mu \leqslant \nu} dy_{\mu\nu}, \qquad [dU] = \bigwedge_{k,l} du_{kl} \quad \text{and}$$
$$[dV] = \bigwedge_{k,l} dv_{kl}.$$

The point is that the invariant metric $ds^2_{n,m;A,B}$ and its Laplacian $\Delta_{n,m;A,B}$ are expressed in terms of the trace form.

For the case $n = m = 1$ and $A = B = 1$, Berndt proved in [1] (cf. [19]) that the metric $ds^2_{1,1}$ on $\mathbb{H} \times \mathbb{C}$ defined by

$$ds^2_{1,1} := ds_{1,1;1,1} = \frac{y + v^2}{y^3}\left(dx^2 + dy^2\right) + \frac{1}{y}\left(du^2 + dv^2\right)$$
$$- \frac{2v}{y^2}(dx\,du + dy\,dv)$$

is a Riemannian metric on $\mathbb{H} \times \mathbb{C}$ invariant under the action (1.2) of the Jacobi group and its Laplacian $\Delta_{1,1}$ is given by

$$\Delta_{1,1} := \Delta_{1,1;1,1} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \left(y + v^2\right)\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)$$
$$+ 2yv\left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v}\right).$$

Notations. We denote by $\mathbb{R}$ and $\mathbb{C}$ the field of real numbers, and the field of complex numbers, respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. E_n denotes the identity matrix of degree n. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For $A \in \mathbb{C}^{(k,l)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = {}^t\overline{A}BA$.

2. Proof of Theorem 1.1

Let $g = (M, (\lambda, \mu; \kappa))$ be an element of G^J with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(n, \mathbb{R})$ and $(Z, W) \in \mathbb{H}_{n,m}$ with $Z \in \mathbb{H}_n$ and $W \in \mathbb{C}^{(m,n)}$. If we put $(Z_*, W_*) := g \cdot (Z, W)$, then we have

$$Z_* = M \cdot Z = (AZ + B)(CZ + D)^{-1},$$
$$W_* = (W + \lambda Z + \mu)(CZ + D)^{-1}.$$

Thus we obtain

$$dZ_* = dZ\big[(CZ+D)^{-1}\big] = {}^t(CZ+D)^{-1}\,dZ(CZ+D)^{-1} \tag{2.1}$$

and

$$dW_* = dW(CZ+D)^{-1} + \big\{\lambda - (W+\lambda Z+\mu)(CZ+D)^{-1}C\big\}\,dZ(CZ+D)^{-1}. \tag{2.2}$$

Here we used the following facts that

$$d(CZ+D)^{-1} = -(CZ+D)^{-1}C\,dZ(CZ+D)^{-1}$$

and that $(CZ+D)^{-1}C$ is symmetric.

We put

$$Z_* = X_* + iY_*, \quad W_* = U_* + iV_*, \quad X_*, Y_*, U_*, V_* \text{ real}.$$

From [9, p. 33] or [13, p. 128], we know that

$$Y_* = Y\big\{(CZ+D)^{-1}\big\} = {}^t(C\overline{Z}+D)^{-1}Y(CZ+D)^{-1}. \tag{2.3}$$

First of all, we recall that the following matrices

$$t(b) = \begin{pmatrix} E_n & b \\ 0 & E_n \end{pmatrix}, \quad b = {}^tb \text{ real},$$
$$g_0(h) = \begin{pmatrix} {}^th & 0 \\ 0 & h^{-1} \end{pmatrix}, \quad h \in GL(n,\mathbb{R}),$$
$$J_n = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

generate the symplectic group $Sp(n,\mathbb{R})$ (cf. [4,5]). Therefore the following elements $t(b;\lambda,\mu,\kappa)$, $g(h)$ and σ_n of G^J defined by

$$t(b;\lambda,\mu,\kappa) = \left(\begin{pmatrix} E_n & b \\ 0 & E_n \end{pmatrix},\ (\lambda,\mu;\kappa)\right), \quad b = {}^tb \text{ real},\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)},$$
$$g(h) = \left(\begin{pmatrix} {}^th & 0 \\ 0 & h^{-1} \end{pmatrix}, (0,0;0)\right), \quad h \in GL(n,\mathbb{R}),$$
$$\sigma_n = \left(\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}, (0,0;0)\right)$$

generate the Jacobi group G^J. So it suffices to prove the invariance of the metric $ds^2_{n,m;A,B}$ under the action of the generators $t(b;\lambda,\mu,\kappa)$, $g(h)$ and σ_n. For brevity, we write

$$
\begin{aligned}
(a) &= \sigma\big(Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big),\\
(b) &= \sigma\big(Y^{-1}\,{}^tV\,V\,Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big),\\
(c) &= \sigma\big(Y^{-1}\,{}^t(dW)\,d\overline{W}\big),\\
(d) &= -\sigma\big(V\,Y^{-1}\,dZ\,Y^{-1}\,{}^t(d\overline{W}) + V\,Y^{-1}\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
(a)_* &= \sigma\big(Y_*^{-1}\,dZ_*\,Y_*^{-1}\,d\overline{Z}_*\big),\\
(b)_* &= \sigma\big(Y_*^{-1}\,{}^tV_*V_*Y_*^{-1}\,dZ_*\,Y_*^{-1}\,d\overline{Z}_*\big),\\
(c)_* &= \sigma\big(Y_*^{-1}\,{}^t(dW_*)\,d\overline{W}_*\big),\\
(d)_* &= -\sigma\big(V_*Y_*^{-1}\,dZ_*\,Y_*^{-1}\,{}^t(d\overline{W}_*) + V_*Y_*^{-1}\,d\overline{Z}_*\,Y_*^{-1}\,{}^t(dW_*)\big).
\end{aligned}
$$

Case I. $g = t(b;\lambda,\mu,\kappa)$ with $b = {}^tb$ real and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$.

In this case, we have

$$Z_* = Z + b, \quad Y_* = Y, \quad W_* = W + \lambda Z + \mu, \quad V_* = V + \lambda Y$$

and

$$dZ_* = dZ, \quad dW_* = dW + \lambda\,dZ.$$

Therefore

$$
\begin{aligned}
(a)_* &= \sigma\big(Y_*^{-1}\,dZ_*\,Y_*^{-1}\,d\overline{Z_*}\big) = \sigma\big(Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big) = (a),\\
(b)_* &= \sigma\big(Y^{-1}\,{}^tV\;V\,Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big) + \sigma\big(Y^{-1}\,{}^tV\lambda\,dZ\,Y^{-1}\,d\overline{Z}\big)\\
&\quad + \sigma\big({}^t\lambda\;V\,Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big) + \sigma\big({}^t\lambda\,\lambda\,dZ\,Y^{-1}\,d\overline{Z}\big),\\
(c)_* &= \sigma\big(Y^{-1}\,{}^t(dW)\,d\overline{W}\big) + \sigma(Y^{-1}\,{}^t(dW)\lambda\,d\overline{Z})\\
&\quad + \sigma\big(Y^{-1}\,dZ\;{}^t\lambda\,d\overline{W}\big) + \sigma\big(Y^{-1}\,dZ\;{}^t\lambda\,\lambda\,d\overline{Z}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
(d)_* = &-\sigma\big(V\,Y^{-1}\,dZ\,Y^{-1}\,{}^t(d\overline{W})\big) - \sigma\big(\lambda\,dZ\,Y^{-1}\,{}^t(d\overline{W})\big)\\
&- \sigma\big(V\,Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\;{}^t\lambda\big) - \sigma\big(\lambda\,dZ\,Y^{-1}\,d\overline{Z}\,{}^t\lambda\big)\\
&- \sigma\big(V\,Y^{-1}\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big) - \sigma\big(\lambda\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big)\\
&- \sigma\big(V\,Y^{-1}\,d\overline{Z}\,Y^{-1}\,dZ\;{}^t\lambda\big) - \sigma\big(\lambda\,d\overline{Z}\,Y^{-1}\,dZ\;{}^t\lambda\big).
\end{aligned}
$$

Thus we see that

$$(a) = (a)_* \quad \text{and} \quad (b) + (c) + (d) = (b)_* + (c)_* + (d)_*.$$

Hence

$$ds^2_{n,m;A,B} = A(a) + B\big\{(b) + (c) + (d)\big\}$$

is invariant under the action of $t(B; \lambda, \mu, \kappa)$.

Case II. $g = g(h)$ with $h \in GL(n, \mathbb{R})$.

In this case, we have

$$Z_* = {}^t h\, Z h, \quad Y_* = {}^t h\, Y h, \quad W_* = W h, \quad V_* = V h$$

and

$$dZ_* = {}^t h\, dZ\, h, \quad dW_* = dW\, h.$$

Therefore by an easy computation, we see that each of (a), (b), (c) and (d) is invariant under the action of all $g(h)$ with $h \in GL(n, \mathbb{R})$. Hence the metric $ds^2_{n,m;A,B}$ is invariant under the action of all $g(h)$ with $h \in GL(n, \mathbb{R})$.

Case III. $g = \sigma_n = \big(\big(\begin{smallmatrix} 0 & -E_n \\ E_n & 0 \end{smallmatrix}\big), (0, 0; 0)\big)$.

In this case, we have

$$Z_* = -Z^{-1} \quad \text{and} \quad W_* = W Z^{-1}. \tag{2.4}$$

We set

$$\theta_1 := \operatorname{Re} Z^{-1} \quad \text{and} \quad \theta_2 := \operatorname{Im} Z^{-1}.$$

Then θ_1 and θ_2 are symmetric matrices and we have

$$Y_* = -\theta_2 \quad \text{and} \quad V_* := \operatorname{Im} W_* = V\theta_1 + U\theta_2. \tag{2.5}$$

It is easy to see that

$$Y = -Z\theta_2 \overline{Z} = -\overline{Z}\theta_2 Z, \tag{2.6}$$

$$\theta_1 Y + \theta_2 X = 0 \tag{2.7}$$

and

$$\theta_1 X - \theta_2 Y = E_n. \tag{2.8}$$

According to (2.6) and (2.7), we obtain

$$X = (-\theta_2)^{-1}\theta_1 Y \quad \text{and} \quad Y^{-1} = \theta_1(-\theta_2)^{-1}\theta_1 - \theta_2. \tag{2.9}$$

From (2.1) and (2.2), we have

$$dZ_* = Z^{-1}\, dZ\, Z^{-1} \tag{2.10}$$

and

$$dW_* = dW\, Z^{-1} - WZ^{-1}\, dZ\, Z^{-1} = \big(dW - WZ^{-1}\, dZ\big)Z^{-1}. \tag{2.11}$$

Therefore we have, according to (2.6) and (2.10),

$$\begin{aligned}(a)_* &= \sigma\big((-\theta_2)^{-1} Z^{-1}\, dZ\, Z^{-1}(-\theta_2)^{-1}\overline{Z}^{-1}\, d\overline{Z}\,\overline{Z}^{-1}\big)\\ &= \sigma\big(Y^{-1}\, dZ\, Y^{-1}\, d\overline{Z}\big) = (a).\end{aligned}$$

According to (2.5)–(2.10), we have

$$\begin{aligned}(b)_* &= \sigma\big((-\theta_2)^{-1}\big(\theta_1\, {}^tV + \theta_2\, {}^tU\big)(V\theta_1 + U\theta_2)(-\theta_2)^{-1} Z^{-1}\, dZ\, Y^{-1} d\overline{Z}\,\overline{Z}^{-1}\big)\\ &= \sigma\big(\big\{{}^tU - (-\theta_2)^{-1}\theta_1\, {}^tV\big\}\big\{U - V\theta_1(-\theta_2)^{-1}\big\}Z^{-1}\, dZ\, Y^{-1}\, d\overline{Z}\,\overline{Z}^{-1}\big)\\ &= \sigma\big(\big\{{}^t\overline{W} + \big(iE_n - (-\theta_2)^{-1}\theta_1\big)\, {}^tV\big\}\big\{W - V\big(iE_n + \theta_1(-\theta_2)^{-1}\big)\big\}Z^{-1} dZ\, Y^{-1}\, d\overline{Z}\,\overline{Z}^{-1}\big),\\ (c)_* &= \sigma\big((-\theta_2)^{-1}\big(Z^{-1}\, {}^t(dW) - Z^{-1}\, dZ\, Z^{-1}\, {}^tW\big)\big(d\overline{W}\,\overline{Z}^{-1} - \overline{W}\,\overline{Z}^{-1}\, d\overline{Z}\,\overline{Z}^{-1}\big)\big)\\ &= \sigma\big((-\theta_2)^{-1} Z^{-1}\, {}^t(dW)\, d\overline{W}\ \overline{Z}^{-1} - (-\theta_2)^{-1} Z^{-1}\, {}^t(dW)\overline{W}\,\overline{Z}^{-1}\, d\overline{Z}\,\overline{Z}^{-1}\\ &\quad - (-\theta_2)^{-1} Z^{-1}\, dZ\, Z^{-1}\, {}^tW d\overline{W}\,\overline{Z}^{-1}\\ &\quad + (-\theta_2)^{-1} Z^{-1}\, dZ\, Z^{-1}\, {}^tW\ \overline{W}\,\overline{Z}^{-1}\, d\overline{Z}\,\overline{Z}^{-1}\big)\\ &= \sigma\big(Y^{-1}\, {}^t(dW)\, d\overline{W}\big) - \sigma\big(Y^{-1}\, {}^t(dW)\overline{W}\,\overline{Z}^{-1}\, d\overline{Z}\big)\\ &\quad - \sigma\big(Y^{-1}\, dZ\, Z^{-1}\, {}^tW\, d\overline{W}\big) + \sigma\big(Y^{-1}\, dZ\, Z^{-1}\, {}^tW\ \overline{W}\,\overline{Z}^{-1}\, d\overline{Z}\big)\end{aligned}$$

and

$$\begin{aligned}(d)_* &= -\sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1} Z^{-1}\, dZ\, Z^{-1}(-\theta_2)^{-1}\big\{\overline{Z}^{-1}\, {}^t(d\overline{W}) - \overline{Z}^{-1}\, d\overline{Z}\,\overline{Z}^{-1}\, {}^t\overline{W}\big\}\big)\\ &\quad - \sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1}\overline{Z}^{-1}\, d\overline{Z}\,\overline{Z}^{-1}(-\theta_2)^{-1}\big\{Z^{-1}\, {}^t(dW) - Z^{-1}\, dZ\, Z^{-1}\, {}^tW\big\}\big)\\ &= -\sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1} Z^{-1}\, dZ\, Y^{-1}\, {}^t(d\overline{W})\big)\\ &\quad + \sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1} Z^{-1}\, dZ\, Y^{-1}\, d\overline{Z}\,\overline{Z}^{-1}\, {}^t\overline{W}\big)\\ &\quad - \sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1}\overline{Z}^{-1}\, d\overline{Z}\, Y^{-1}\, {}^t(dW)\big)\\ &\quad + \sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1}\overline{Z}^{-1}\, d\overline{Z}\, Y^{-1}\, dZ\, Z^{-1}\, {}^tW\big).\end{aligned}$$

Taking the $(dZ, d\overline{W})$-part $\square(Z, \overline{W})$ in $(b)_* + (c)_* + (d)_*$, we have

$$\begin{aligned}\square(Z, \overline{W}) &= -\sigma\big(VY^{-1}\, dZ\, Y^{-1}\, {}^t(d\overline{W})\big) + \sigma\big(Y^{-1}\, dZ\big({}^tW_* - Z^{-1}\, {}^tW\big)\, d\overline{W}\big)\\ &= -\sigma\big(VY^{-1}\, dZ\, Y^{-1}\, {}^t(d\overline{W})\big) \quad \text{because } W_* = WZ^{-1}\ \big(\text{cf. (2.4)}\big).\end{aligned}$$

Similarly, if we take the $(d\overline{Z}, dW)$-part $\square(\overline{Z}, W)$ in $(b)_* + (c)_* + (d)_*$, we have

$$\begin{aligned}\square(\overline{Z}, W) &= -\sigma\big(VY^{-1}\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big) + \sigma\big(d\overline{Z}\,Y^{-1}\,{}^t(dW)\big(\overline{W_*} - \overline{W}\,\overline{Z}^{-1}\big)\big)\\ &= -\sigma\big(VY^{-1}\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big) \quad \text{because } W_* = WZ^{-1}.\end{aligned}$$

If we take the $(dW, d\overline{W})$-part $\square(W, \overline{W})$ in $(b)_* + (c)_* + (d)_*$, we have

$$\square(W, \overline{W}) = \sigma\big(Y^{-1}\,{}^t(dW)\,d\overline{W}\big).$$

Finally, if we take the $(dZ, d\overline{Z})$-part $\square(Z, \overline{Z})$ in $(b)_* + (c)_* + (d)_*$, we have

$$\begin{aligned}\square(Z, \overline{Z}) = {}&\sigma\big(\big\{{}^t\overline{W} + \big(iE_n - (-\theta_2)^{-1}\theta_1\big)\,{}^tV\big\}\big\{W - V\big(iE_n + \theta_1(-\theta_2)^{-1}\big)\big\}Z^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\big)\\ &+ \sigma\big(Z^{-1}\,{}^tW\,\overline{W}\,\overline{Z}^{-1}\,d\overline{Z}\,Y^{-1}\,dZ\big)\\ &+ \sigma\big({}^t\overline{W}(V\theta_1 + U\theta_2)(-\theta_2)^{-1}Z^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\big)\\ &+ \sigma\big({}^tW(V\theta_1 + U\theta_2)(-\theta_2)^{-1}\overline{Z}^{-1}\,d\overline{Z}\,Y^{-1}\,dZ\,Z^{-1}\big).\end{aligned}$$

Since

$$\begin{aligned}(V\theta_1 + U\theta_2)(-\theta_2)^{-1} &= -U + V\theta_1(-\theta_2)^{-1}\\ &= -W + V\big\{iE_n + \theta_1(-\theta_2)^{-1}\big\}\\ &= -\overline{W} - V\big\{iE_n - \theta_1(-\theta_2)^{-1}\big\},\end{aligned}$$

we have

$$\begin{aligned}\square(Z, \overline{Z}) &= \sigma\big(\overline{Z}^{-1}\big\{iE_n - (-\theta_2)^{-1}\theta_1\big\}\,{}^tV\big\{W - V\big(iE_n + \theta_1(-\theta_2)^{-1}\big)\big\}Z^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big)\\ &\quad - \sigma\big(\overline{Z}^{-1}\big\{iE_n - (-\theta_2)^{-1}\theta_1\big\}\,{}^tV\,WZ^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big)\\ &= -\sigma\big(\overline{Z}^{-1}\big\{iE_n - (-\theta_2)^{-1}\theta_1\big\}\,{}^tV\,V\big\{iE_n + \theta_1(-\theta_2)^{-1}\big\}Z^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big).\end{aligned}$$

By the way, according to (2.9), we obtain

$$\begin{aligned}\overline{Z}^{-1}\big\{iE_n - (-\theta_2)^{-1}\theta_1\big\} &= (\theta_1 - i\theta_2)\big\{iE_n - (-\theta_2)^{-1}\theta_1\big\}\\ &= \theta_2 - \theta_1(-\theta_2)^{-1}\theta_1 = -Y^{-1}\end{aligned}$$

and

$$\big\{iE_n + \theta_1(-\theta_2)^{-1}\big\}Z^{-1} = \theta_1(-\theta_2)^{-1}\theta_1 - \theta_2 = Y^{-1}.$$

Therefore

$$\square(Z, \overline{Z}) = \sigma\big(Y^{-1}\,{}^tV\,VY^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big).$$

Hence $(a) = (a)_*$ and

$$(b)_* + (c)_* + (d)_* = \square(Z, \overline{W}) + \square(\overline{Z}, W) + \square(W, \overline{W}) + \square(Z, \overline{Z})$$
$$= (b) + (c) + (d).$$

This implies that the metric

$$ds^2_{n,m;A,B} = A(a) + B\big\{(b) + (c) + (d)\big\}$$

is invariant under the action (1.2) of σ_n.

Consequently $ds^2_{n,m;A,B}$ is invariant under the action (1.2) of the Jacobi group G^J. In particular, for $(Z, W) = (iE_n, 0)$, we have

$$\begin{aligned} ds^2_{n,m;A,B} &= A \cdot \sigma(dZ\, d\overline{Z}) + B \cdot \sigma\big({}^t(dW)\, d\overline{W}\big) \\ &= A\left\{ \sum_{\mu=1}^{n} \big(dx^2_{\mu\mu} + dy^2_{\mu\mu}\big) + 2 \sum_{1\leqslant \mu<\nu\leqslant n} \big(dx^2_{\mu\nu} + dy^2_{\mu\nu}\big)\right\} \\ &\quad + B\left\{ \sum_{1\leqslant k\leqslant m, 1\leqslant l\leqslant n} \big(du^2_{kl} + dv^2_{kl}\big)\right\}, \end{aligned}$$

which is clearly positive definite. Since G^J acts on $\mathbb{H}_{n,m}$ transitively, $ds^2_{n,m;A,B}$ is positive definite everywhere in $\mathbb{H}_{n,m}$. This completes the proof of Theorem 1.1. $\square$

Remark 2.1. The scalar curvature of the Siegel–Jacobi space $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$ is constant because of the transitive group action of G^J on $\mathbb{H}_{n,m}$. In the special case $n = m = 1$ and $A = B = 1$, by a direct computation, we see that the scalar curvature of $(\mathbb{H}_{1,1}, ds^2_{1,1;1,1})$ is -3.

3. Proof of Theorem 1.2

If $(Z_*, W_*) = g \cdot (Z, W)$ with $g = \big(\big(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\big), (\lambda, \mu; \kappa)\big) \in G^J$, we can easily see that

$$\begin{aligned} \frac{\partial}{\partial Z_*} &= (CZ + D)\,{}^t\!\left\{(CZ + D)\frac{\partial}{\partial Z}\right\} \\ &\quad + (CZ + D)\,{}^t\!\left\{\big(C\,{}^tW + C\,{}^t\mu - D\,{}^t\lambda\big)\,{}^t\!\left(\frac{\partial}{\partial W}\right)\right\} \end{aligned} \tag{3.1}$$

and

$$\frac{\partial}{\partial W_*} = (CZ + D)\frac{\partial}{\partial W}. \tag{3.2}$$

For brevity, we put

$$(\alpha) := 4\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial \overline{Z}}\right)\frac{\partial}{\partial Z}\right),$$
$$(\beta) := 4\sigma\left(Y\frac{\partial}{\partial W}\,{}^t\!\left(\frac{\partial}{\partial \overline{W}}\right)\right),$$

$$(\gamma) := 4\sigma\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right),$$
$$(\delta) := 4\sigma\left(V\,{}^t\!\left(Y\frac{\partial}{\partial \overline{Z}}\right)\frac{\partial}{\partial W}\right)$$

and

$$(\epsilon) := 4\sigma\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial Z}\right).$$

We also set

$$(\alpha)_* := 4\sigma\left(Y_*\,{}^t\!\left(Y_*\frac{\partial}{\partial \overline{Z}_*}\right)\frac{\partial}{\partial Z_*}\right),$$
$$(\beta)_* := 4\sigma\left(Y_*\frac{\partial}{\partial W_*}\,{}^t\!\left(\frac{\partial}{\partial \overline{W}_*}\right)\right),$$
$$(\gamma)_* := 4\sigma\left(V_*Y_*^{-1}\,{}^tV_*\,{}^t\!\left(Y_*\frac{\partial}{\partial \overline{W}_*}\right)\frac{\partial}{\partial W_*}\right),$$
$$(\delta)_* := 4\sigma\left(V_*\,{}^t\!\left(Y_*\frac{\partial}{\partial \overline{Z}_*}\right)\frac{\partial}{\partial W_*}\right)$$

and

$$(\epsilon)_* := 4\sigma\left({}^tV_*\,{}^t\!\left(Y_*\frac{\partial}{\partial \overline{W}_*}\right)\frac{\partial}{\partial Z_*}\right).$$

We need the following lemma for the proof of Theorem 1.2. H. Maass [8] observed the following useful fact.

Lemma 3.1.

(a) *Let A be an $n\times k$ matrix and let B be a $k\times n$ matrix. Assume that the entries of A commute with the entries of B. Then $\sigma(AB)=\sigma(BA)$.*
(b) *Let A be an $m\times n$ matrix and B an $n\times l$ matrix. Assume that the entries of A commute with the entries of B. Then ${}^t(AB)={}^tB\,{}^tA$.*
(c) *Let A, B and C be a $k\times l$, an $n\times m$ and an $m\times l$ matrix, respectively. Assume that the entries of A commute with the entries of B. Then*

$${}^t\big(A\,{}^t(BC)\big) = B\,{}^t\big(A\,{}^tC\big).$$

Proof. The proof follows immediately from the direct computation. □

Now we are ready to prove Theorem 1.2. First of all, we shall prove that $\Delta_{n,m;A,B}$ is invariant under the action of the generators $t(b;\lambda,\mu,\kappa)$, $g(h)$ and σ_n.

Case I. $g=t(b;\lambda,\mu,\kappa)=\left(\begin{pmatrix} E_n & b\\ 0 & E_n\end{pmatrix},(\lambda,\mu;\kappa)\right)$ with $b={}^tb$ real.

In this case, we have

$$Y_* = Y, \qquad V_* = V + \lambda Y$$

and

$$\frac{\partial}{\partial Z_*} = \frac{\partial}{\partial Z} - {}^t\!\left({}^t\lambda\, {}^t\!\left(\frac{\partial}{\partial W}\right)\right) \quad \text{and} \quad \frac{\partial}{\partial W_*} = \frac{\partial}{\partial W}.$$

Using Lemma 3.1, we obtain

$$\begin{aligned}
(\alpha)_* &= (\alpha) - \sigma\left(\lambda Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{Z}}\right) \frac{\partial}{\partial W}\right) \\
&\quad - \sigma\left(Y\, {}^t\lambda\, {}^t\!\left(Y \frac{\partial}{\partial \overline{W}}\right) \frac{\partial}{\partial Z}\right) + \sigma\left(\lambda Y\, {}^t\lambda\, {}^t\!\left(Y \frac{\partial}{\partial \overline{W}}\right) \frac{\partial}{\partial W}\right) \\
(\beta)_* &= \sigma\left(Y_* \frac{\partial}{\partial W_*}\, {}^t\!\left(\frac{\partial}{\partial \overline{W}_*}\right)\right) = (\beta), \\
(\gamma)_* &= (\gamma) + \sigma\left(\lambda\, {}^tV\, {}^t\!\left(Y \frac{\partial}{\partial \overline{W}}\right) \frac{\partial}{\partial W}\right) \\
&\quad + \sigma\left(V\, {}^t\lambda\, {}^t\!\left(Y \frac{\partial}{\partial \overline{W}}\right) \frac{\partial}{\partial W}\right) + \sigma\left(\lambda Y\, {}^t\lambda\, {}^t\!\left(Y \frac{\partial}{\partial \overline{W}}\right) \frac{\partial}{\partial W}\right), \\
(\delta)_* &= (\delta) + \sigma\left(\lambda Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{Z}}\right) \frac{\partial}{\partial W}\right) \\
&\quad - \sigma\left(V\, {}^t\lambda\, {}^t\!\left(Y \frac{\partial}{\partial \overline{W}}\right) \frac{\partial}{\partial W}\right) + \sigma\left(\lambda Y\, {}^t\lambda\, {}^t\!\left(Y \frac{\partial}{\partial \overline{W}}\right) \frac{\partial}{\partial W}\right)
\end{aligned}$$

and

$$\begin{aligned}
(\epsilon)_* &= (\epsilon) + \sigma\left(Y\, {}^t\lambda\, {}^t\!\left(Y \frac{\partial}{\partial \overline{W}}\right) \frac{\partial}{\partial Z}\right) \\
&\quad - \sigma\left(\lambda\, {}^tV\, {}^t\!\left(Y \frac{\partial}{\partial \overline{W}}\right) \frac{\partial}{\partial W}\right) - \sigma\left(\lambda Y\, {}^t\lambda\, {}^t\!\left(Y \frac{\partial}{\partial \overline{W}}\right) \frac{\partial}{\partial W}\right).
\end{aligned}$$

Thus $(\beta) = (\beta)_*$ and

$$(\alpha) + (\gamma) + (\delta) + (\epsilon) = (\alpha)_* + (\gamma)_* + (\delta)_* + (\epsilon)_*.$$

Hence

$$\Delta_{n,m;A,B} = \frac{4}{B}(\beta) + \frac{4}{A}\big\{(\alpha) + (\gamma) + (\delta) + (\epsilon)\big\}$$

is invariant under the action of all $t(b; \lambda, \mu, \kappa)$.

Case II. $g = g(h) = \left(\begin{pmatrix} {}^th & 0 \\ 0 & h^{-1} \end{pmatrix}, (0, 0; 0)\right)$ with $h \in GL(n, \mathbb{R})$.

In this case, we have

$$Y_* = {}^t h\, Y h, \qquad V_* = V h$$

and

$$\frac{\partial}{\partial Z_*} = h^{-1}\, {}^t\!\left(h^{-1}\frac{\partial}{\partial Z}\right), \qquad \frac{\partial}{\partial W_*} = h^{-1}\frac{\partial}{\partial W}.$$

According to Lemma 3.1, we see that each of $(\alpha), (\beta), (\gamma), (\delta)$ and (ϵ) is invariant under the action of all $g(h)$ with $h \in GL(n,\mathbb{R})$. Therefore $\Delta_{n,m;A,B}$ is invariant under the action of all $g(h)$ with $h \in GL(n,\mathbb{R})$.

Case III. $g = \sigma_n = \left(\left(\begin{smallmatrix} 0 & -E_n \\ E_n & 0 \end{smallmatrix}\right), (0,0;0)\right)$.

In this case, we have

$$Z_* = -Z^{-1} \quad \text{and} \quad W_* = W Z^{-1}.$$

We set

$$\theta_1 := \operatorname{Re} Z^{-1} \quad \text{and} \quad \theta_2 := \operatorname{Im} Z^{-1}.$$

Then we obtain the relations (2.5)–(2.9). From (2.6), we have the relation

$$\theta_2 \overline{Z} = -Z^{-1} Y. \tag{3.3}$$

It follows from the relation (2.3) that

$$Y_* = \overline{Z}^{-1} Y Z^{-1} = Z^{-1} Y \overline{Z}^{-1} = -\theta_2. \tag{3.4}$$

From (2.9), we obtain

$$\theta_1 \theta_2^{-1} \theta_1 = -Y^{-1} - \theta_2. \tag{3.5}$$

According to (3.1) and (3.2), we have

$$\frac{\partial}{\partial Z_*} = Z\, {}^t\!\left(Z \frac{\partial}{\partial Z}\right) + Z\, {}^t\!\left({}^t W\, {}^t\!\left(\frac{\partial}{\partial W}\right)\right) \tag{3.6}$$

and

$$\frac{\partial}{\partial W_*} = Z \frac{\partial}{\partial W}. \tag{3.7}$$

From (2.6), (3.3) and Lemma 3.1, we obtain

$$
\begin{aligned}
(\alpha)_* = (\alpha) &- \sigma\left(U\theta_2\overline{Z}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial W}\right) - i\sigma\left(V\theta_2\overline{Z}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial W}\right)\\
&- \sigma\left(Z\theta_2\,{}^tU\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial Z}\right) + i\sigma\left(Z\theta_2\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial Z}\right)\\
&- \sigma\left(W\theta_2\,{}^tU\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right) + \sigma\left(W\theta_2\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right).
\end{aligned}
$$

From the relation (3.4), we see $(\beta)_* = (\beta)$. According to (3.3), (3.5) and Lemma 3.1, we obtain

$$
(\gamma)_* = (\gamma) + \sigma\left(\left(V\theta_2\,{}^tV - V\theta_1\,{}^tU - U\theta_1\,{}^tV - U\theta_2\,{}^tU\right){}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right).
$$

Using the relation (3.3) and Lemma 3.1, we finally obtain

$$
\begin{aligned}
(\delta)_* = &\ \sigma\left(V\theta_1\overline{Z}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial W}\right) + \sigma\left(U\theta_2\overline{Z}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial W}\right)\\
&+ \sigma\left(V\theta_1\,{}^t\overline{W}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right) + \sigma\left(U\theta_2\,{}^t\overline{W}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(\epsilon)_* = &\ \sigma\left(Z\theta_1\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial Z}\right) + \sigma\left(Z\theta_2\,{}^tU\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial Z}\right)\\
&+ \sigma\left(W\theta_1\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right) + \sigma\left(W\theta_2\,{}^tU\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right).
\end{aligned}
$$

Using the fact $Z^{-1} = \theta_1 + i\theta_2$, we can show that

$$
(\alpha) + (\gamma) + (\delta) + (\epsilon) = (\alpha)_* + (\gamma)_* + (\delta)_* + (\epsilon)_*.
$$

Hence

$$
\Delta_{n,m;A,B} = \frac{4}{B}(\beta) + \frac{4}{A}\big\{(\alpha) + (\gamma) + (\delta) + (\epsilon)\big\}
$$

is invariant under the action of σ_n.

Consequently $\Delta_{n,m;A,B}$ is invariant under the action (1.2) of G^J. In particular, for $(Z, W) = (iE_n, 0)$, the differential operator $\Delta_{n,m;A,B}$ coincides with the Laplacian for the metric $ds^2_{n,m;A,B}$. It follows from the invariance of $\Delta_{n,m;A,B}$ under the action (1.2) and the transitivity of the action of G^J on $\mathbb{H}_{n,m}$ that $\Delta_{n,m;A,B}$ is the Laplacian of $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$. The invariance of the differential form dv follows from the fact that the following differential form

$$
(\det Y)^{-(n+1)}[dX] \wedge [dY]
$$

is invariant under the action (1.1) of $Sp(n,\mathbb{R})$ (cf. [13, p. 130]). $\square$

4. Remark on spectral theory of $\Delta_{n,m;A,B}$ on Siegel–Jacobi space

Before we describe a fundamental domain for the Siegel–Jacobi space, we review the Siegel's fundamental domain for the Siegel upper half plane.

We let

$$\mathcal{P}_n = \left\{ Y \in \mathbb{R}^{(n,n)} \;\middle|\; Y = {}^tY > 0 \right\}$$

be an open cone in $\mathbb{R}^{n(n+1)/2}$. The general linear group $GL(n,\mathbb{R})$ acts on $\mathcal{P}_n$ transitively by

$$h \circ Y := hY\,{}^th, \quad h \in GL(n,\mathbb{R}),\ Y \in \mathcal{P}_n.$$

Thus $\mathcal{P}_n$ is a symmetric space diffeomorphic to $GL(n,\mathbb{R})/O(n)$. We let

$$GL(n,\mathbb{Z}) = \left\{ h \in GL(n,\mathbb{R}) \;\middle|\; h \text{ is integral} \right\}$$

be the discrete subgroup of $GL(n,\mathbb{R})$.

The fundamental domain $\mathcal{R}_n$ for $GL(n,\mathbb{Z}) \backslash \mathcal{P}_n$ which was found by H. Minkowski [10] is defined as a subset of $\mathcal{P}_n$ consisting of $Y = (y_{ij}) \in \mathcal{P}_n$ satisfying the following conditions (M.1)–(M.2) (cf. [9, p. 123]):

(M.1) $aY\,{}^ta \geqslant y_{kk}$ for every $a = (a_i) \in \mathbb{Z}^n$ in which $a_k, \ldots, a_n$ are relatively prime for $k = 1, 2, \ldots, n$.
(M.2) $y_{k,k+1} \geqslant 0$ for $k = 1, \ldots, n-1$.

We say that a point of $\mathcal{R}_n$ is *Minkowski reduced* or simply *M-reduced*.

Siegel [12] determined a fundamental domain $\mathcal{F}_n$ for $\Gamma_n \backslash \mathbb{H}_n$, where $\Gamma_n = Sp(n,\mathbb{Z})$ is the Siegel modular group of degree n. We say that $\Omega = X + iY \in \mathbb{H}_n$ with X, Y real is *Siegel reduced* or *S-reduced* if it has the following three properties:

(S.1) $\det(\operatorname{Im}(\gamma \cdot \Omega)) \leqslant \det(\operatorname{Im}(\Omega))$ for all $\gamma \in \Gamma_n$;
(S.2) $Y = \operatorname{Im} \Omega$ is M-reduced, that is, $Y \in \mathcal{R}_n$;
(S.3) $|x_{ij}| \leqslant \frac{1}{2}$ for $1 \leqslant i, j \leqslant n$, where $X = (x_{ij})$.

$\mathcal{F}_n$ is defined as the set of all Siegel reduced points in $\mathbb{H}_n$. Using the highest point method, Siegel [12] proved the following (F1)–(F3) (cf. [9, p. 169]):

(F1) $\Gamma_n \cdot \mathcal{F}_n = \mathbb{H}_n$, i.e., $\mathbb{H}_n = \bigcup_{\gamma \in \Gamma_n} \gamma \cdot \mathcal{F}_n$.
(F2) $\mathcal{F}_n$ is closed in $\mathbb{H}_n$.
(F3) $\mathcal{F}_n$ is connected and the boundary of $\mathcal{F}_n$ consists of a finite number of hyperplanes.

The metric ds_n^2 given by (1.3) induces a metric $ds_{\mathcal{F}_n}^2$ on $\mathcal{F}_n$. Siegel [12] computed the volume of $\mathcal{F}_n$

$$\operatorname{vol}(\mathcal{F}_n) = 2 \prod_{k=1}^{n} \pi^{-k} \Gamma(k) \zeta(2k),$$

where $\Gamma(s)$ denotes the Gamma function and $\zeta(s)$ denotes the Riemann zeta function. For instance,

$$\mathrm{vol}(\mathcal{F}_1)=\frac{\pi}{3},\quad \mathrm{vol}(\mathcal{F}_2)=\frac{\pi^3}{270},\quad \mathrm{vol}(\mathcal{F}_3)=\frac{\pi^6}{127575},\quad \mathrm{vol}(\mathcal{F}_4)=\frac{\pi^{10}}{200930625}.$$

Let f_{kl} $(1\leqslant k\leqslant m,\ 1\leqslant l\leqslant n)$ be the $m\times n$ matrix with entry 1 where the kth row and the lth column meet, and all other entries 0. For an element $\Omega\in\mathbb{H}_n$, we set for brevity

$$h_{kl}(\Omega):=f_{kl}\Omega,\quad 1\leqslant k\leqslant m,\ 1\leqslant l\leqslant n.$$

For each $\Omega\in\mathcal{F}_n$, we define a subset P_Ω of $\mathbb{C}^{(m,n)}$ by

$$P_\Omega=\left\{\sum_{k=1}^{m}\sum_{j=1}^{n}\lambda_{kl}f_{kl}+\sum_{k=1}^{m}\sum_{j=1}^{n}\mu_{kl}h_{kl}(\Omega)\ \Big|\ 0\leqslant\lambda_{kl},\mu_{kl}\leqslant 1\right\}.$$

For each $\Omega\in\mathcal{F}_n$, we define the subset D_Ω of $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ by

$$D_\Omega:=\left\{(\Omega,Z)\in\mathbb{H}_n\times\mathbb{C}^{(m,n)}\ \big|\ Z\in P_\Omega\right\}.$$

We define

$$\mathcal{F}_{n,m}:=\bigcup_{\Omega\in\mathcal{F}_n}D_\Omega.$$

Theorem 4.1. *Let*

$$\Gamma_{n,m}:=Sp(n,\mathbb{Z})\ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)}=\left\{(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}\ \big|\ \lambda,\mu,\kappa\ \textit{are integral}\right\}.$$

Then $\mathcal{F}_{n,m}$ is a fundamental domain for $\Gamma_{n,m}\backslash\mathbb{H}_{n,m}$.

Proof. The proof can be found in [20]. □

In the case $n=m=1$, R. Berndt [2] introduced the notion of Maass–Jacobi forms. Now we generalize this notion to the general case.

Definition 4.1. For brevity, we set $\Delta_{n,m}:=\Delta_{n,m;1,1}$ (cf. Theorem 1.2). Let

$$\Gamma_{n,m}:=Sp(n,\mathbb{Z})\ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)}=\left\{(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}\ \big|\ \lambda,\mu,\kappa\ \text{are integral}\right\}.$$

A smooth function $f : \mathbb{H}_{n,m} \to \mathbb{C}$ is called a *Maass–Jacobi form* on $\mathbb{H}_{n,m}$ if f satisfies the following conditions (MJ1)–(MJ3):

(MJ1) f is invariant under $\Gamma_{n,m}$.
(MJ2) f is an eigenfunction of the Laplacian $\Delta_{n,m}$.
(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$\left|f(X+iY,Z)\right| \leqslant C\left|p(Y)\right|^N \quad \text{as } \det Y \to \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

It is natural to propose the following problems.

Problem A. Construct Maass–Jacobi forms.

Problem B. Find all the eigenfunctions of $\Delta_{n,m}$.

We consider the simple case $n = m = 1$. A metric $ds^2_{1,1}$ on $\mathbf{H}_1 \times \mathbb{C}$ given by

$$ds^2_{1,1} = \frac{y+v^2}{y^3}\left(dx^2+dy^2\right) + \frac{1}{y}\left(du^2+dv^2\right) - \frac{2v}{y^2}(dx\,du + dy\,dv)$$

is a G^J-invariant Kähler metric on $\mathbf{H}_1 \times \mathbb{C}$. Its Laplacian $\Delta_{1,1}$ is given by

$$\Delta_{1,1} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \left(y+v^2\right)\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + 2yv\left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v}\right).$$

We provide some examples of eigenfunctions of $\Delta_{1,1}$.

(1) $h(x,y) = y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|a|y)e^{2\pi iax}$ ($s \in \mathbb{C}$, $a \neq 0$) with eigenvalue $s(s-1)$. Here

$$K_s(z) := \frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}\left(t+t^{-1}\right)\right\} t^{s-1}\,dt,$$

where $\operatorname{Re} z > 0$.
(2) y^s, $y^s x$, $y^s u$ ($s \in \mathbb{C}$) with eigenvalue $s(s-1)$.
(3) $y^s v$, $y^s uv$, $y^s xv$ with eigenvalue $s(s+1)$.
(4) x, y, u, v, xv, uv with eigenvalue 0.
(5) All Maass wave forms.

We fix two positive integers m and n throughout this section.
For an element $\Omega \in \mathbb{H}_n$, we set

$$L_\Omega := \mathbb{Z}^{(m,n)} + \mathbb{Z}^{(m,n)}\Omega.$$

It follows from the positivity of $\operatorname{Im}\Omega$ that the elements $f_{kl}, h_{kl}(\Omega)$ $(1 \leqslant k \leqslant m,\ 1 \leqslant l \leqslant n)$ of L_Ω are linearly independent over $\mathbb{R}$. Therefore L_Ω is a lattice in $\mathbb{C}^{(m,n)}$ and the set $\{f_{kl}, h_{kl}(\Omega) \mid 1 \leqslant k \leqslant m,\ 1 \leqslant l \leqslant n\}$ forms an integral basis of L_Ω. We see easily that if Ω is an element of $\mathbb{H}_n$, the period matrix $\Omega_* := (I_n, \Omega)$ satisfies the Riemann conditions (RC.1) and (RC.2):

(RC.1) $\Omega_* J_n\, {}^t\Omega_* = 0$;
(RC.2) $-\frac{1}{i}\Omega_* J_n\, {}^t\overline{\Omega}_* > 0$.

Thus the complex torus $A_\Omega := \mathbb{C}^{(m,n)}/L_\Omega$ is an abelian variety.

It might be interesting to investigate the spectral theory of the Laplacian $\Delta_{n,m}$ on a fundamental domain $\mathcal{F}_{n,m}$. But this work is very complicated and difficult at this moment. It may be that the first step is to develop the spectral theory of the Laplacian Δ_Ω on the abelian variety A_Ω. The second step will be to study the spectral theory of the Laplacian Δ_n (see (1.4)) on the moduli space $\Gamma_n \setminus \mathbb{H}_n$ of principally polarized abelian varieties of dimension n. The final step would be to combine the above steps and more works to develop the spectral theory of the Laplacian $\Delta_{n,m}$ on $\mathcal{F}_{n,m}$. *Maass–Jacobi forms* play an important role in the spectral theory of $\Delta_{n,m}$ on $\mathcal{F}_{n,m}$. Here we deal only with the spectral theory Δ_Ω on $L^2(A_\Omega)$.

We fix an element $\Omega = X + iY$ of $\mathbb{H}_n$ with $X = \operatorname{Re}\Omega$ and $Y = \operatorname{Im}\Omega$. For a pair (A, B) with $A, B \in \mathbb{Z}^{(m,n)}$, we define the function $E_{\Omega;A,B} : \mathbb{C}^{(m,n)} \to \mathbb{C}$ by

$$E_{\Omega;A,B}(Z) = e^{2\pi i(\sigma({}^tAU) + \sigma((B - AX)Y^{-1}\,{}^tV))},$$

where $Z = U + iV$ is a variable in $\mathbb{C}^{(m,n)}$ with real U, V.

Lemma 4.1. *For any $A, B \in \mathbb{Z}^{(m,n)}$, the function $E_{\Omega;A,B}$ satisfies the following functional equation*

$$E_{\Omega;A,B}(Z + \lambda\Omega + \mu) = E_{\Omega;A,B}(Z), \quad Z \in \mathbb{C}^{(m,n)},$$

for all $\lambda, \mu \in \mathbb{Z}^{(m,n)}$. Thus $E_{\Omega;A,B}$ can be regarded as a function on A_Ω.

Proof. We write $\Omega = X + iY$ with real X, Y. For any $\lambda, \mu \in \mathbb{Z}^{(m,n)}$, we have

$$\begin{aligned} E_{\Omega;A,B}(Z + \lambda\Omega + \mu) &= E_{\Omega;A,B}\big((U + \lambda X + \mu) + i(V + \lambda Y)\big) \\ &= e^{2\pi i\{\sigma({}^tA(U + \lambda X + \mu)) + \sigma((B - AX)Y^{-1}\,{}^t(V + \lambda Y))\}} \\ &= e^{2\pi i\{\sigma({}^tAU + {}^tA\lambda X + {}^tA\mu) + \sigma((B - AX)Y^{-1}\,{}^tV + B\,{}^t\lambda - AX\,{}^t\lambda)\}} \\ &= e^{2\pi i\{\sigma({}^tAU) + \sigma((B - AX)Y^{-1}\,{}^tV)\}} \\ &= E_{\Omega;A,B}(Z). \end{aligned}$$

Here we used the fact that ${}^tA\mu$ and $B\,{}^t\lambda$ are integral. $\square$

Lemma 4.2. *The metric*

$$ds_\Omega^2 = \sigma\big((\operatorname{Im}\Omega)^{-1}\,{}^t(dZ)\,d\overline{Z}\big)$$

is a Kähler metric on A_Ω *invariant under the action* (1.2) *of* $\Gamma^J = Sp(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(m,n)}$ *on* (Ω, Z) *with* Ω *fixed. Its Laplacian* Δ_Ω *of* ds_Ω^2 *is given by*

$$\Delta_\Omega = \sigma\left((\operatorname{Im}\Omega) \frac{\partial}{\partial Z}\, {}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right)\right).$$

Proof. The proof can be found [20]. □

We let $L^2(A_\Omega)$ be the space of all functions $f : A_\Omega \to \mathbb{C}$ such that

$$\|f\|_\Omega := \int_{A_\Omega} \left|f(Z)\right|^2 dv_\Omega,$$

where dv_Ω is the volume element on A_Ω normalized so that $\int_{A_\Omega} dv_\Omega = 1$. The inner product $(\,,\,)_\Omega$ on the Hilbert space $L^2(A_\Omega)$ is given by

$$(f,g)_\Omega := \int_{A_\Omega} f(Z)\overline{g(Z)}\, dv_\Omega, \quad f, g \in L^2(A_\Omega).$$

Theorem 4.2. *The set* $\{E_{\Omega;A,B} \mid A, B \in \mathbb{Z}^{(m,n)}\}$ *is a complete orthonormal basis for* $L^2(A_\Omega)$. *Moreover we have the following spectral decomposition of* Δ_Ω:

$$L^2(A_\Omega) = \bigoplus_{A,B\in\mathbb{Z}^{(m,n)}} \mathbb{C}\cdot E_{\Omega;A,B}.$$

Proof. The complete proof can be found in [20]. □

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References

[1] R. Berndt, Sur l'arithmétique du corps des fonctions élliptiques de niveau N, in: Séminaire de Théorie de nombres de Paris 1982–1983, Birkhäuser, Boston, 1984, pp. 21–32.
[2] R. Berndt, R. Schmidt, Elements of the Representation Theory of the Jacobi Group, Birkhäuser, 1998.
[3] M. Eichler, D. Zagier, The Theory of Jacobi Forms, Progr. Math., vol. 55, Birkhäuser, Boston, 1985.
[4] E. Freitag, Siegelsche Modulfunktionen, Grundlehren Math. Wiss., Springer-Verlag, 1983.
[5] M. Kashiwara, M. Vergne, On the Segal–Shale–Weil representations and harmonic polynomials, Invent. Math. 44 (1978) 1–47.
[6] J. Kramer, A geometrical approach to the theory of Jacobi forms, Compos. Math. 79 (1991) 1–19.
[7] J. Kramer, An arithmetical theory of Jacobi forms in higher dimensions, J. Reine Angew. Math. 458 (1995) 157–182.
[8] H. Maass, Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen, Math. Ann. 126 (1953) 44–68.
[9] H. Maass, Siegel's Modular Forms and Dirichlet Series, Lecture Notes in Math., vol. 216, Springer-Verlag, Berlin, 1971.

[10] H. Minkowski, Gesammelte Abhandlungen, Chelsea, New York, 1967.
[11] B. Runge, Theta functions and Siegel–Jacobi functions, Acta Math. 175 (1995) 165–196.
[12] C.L. Siegel, Symplectic geometry, in: Gesammelte Abhandlungen, Band II, Springer-Verlag, 1966, pp. 274–359.
[13] C.L. Siegel, Topics in Complex Function Theory: Abelian Functions and Modular Functions of Several Variables, vol. III, Wiley–Interscience, 1973.
[14] J.-H. Yang, The Siegel–Jacobi operator, Abh. Math. Sem. Univ. Hamburg 63 (1993) 135–146.
[15] J.-H. Yang, Remarks on Jacobi forms of higher degree, in: Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, Pyungsan Institute for Mathematical Sciences, Seoul, 1993, pp. 33–58.
[16] J.-H. Yang, Singular Jacobi forms, Trans. Amer. Math. Soc. 347 (6) (1995) 2041–2049.
[17] J.-H. Yang, Construction of modular forms from Jacobi forms, Canad. J. Math. 47 (1995) 1329–1339.
[18] J.-H. Yang, A geometrical theory of Jacobi forms of higher degree, in: Tadao Oda (Ed.), Proceedings of Symposium on Hodge Theory and Algebraic Geometry, Sendai, Japan, 1996, pp. 125–147, or Kyungpook Math. J. 40 (2) (2000) 209–237, or arXiv: math.NT/0602267.
[19] J.-H. Yang, A note on Maass–Jacobi forms, Kyungpook Math. J. 43 (2003) 547–566.
[20] J.-H. Yang, A note on a fundamental domain for Siegel–Jacobi space, Houston J. Math. 32 (3) (2006) 701–712.
[21] C. Ziegler, Jacobi forms of higher degree, Abh. Math. Sem. Hamburg 59 (1989) 191–224.