

A CLASS OF SIMULTANEOUS METHODS FOR THE ZEROS OF ANALYTIC FUNCTIONS

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Abstract—A family of iterative methods for simultaneously approximating simple zeros of analytic functions (inside a simple smooth closed contour in the complex plane) is presented. The order of convergence of the considered methods is $m + 2$ ($m = 1, 2, \dots$), where m is the order of the highest derivative of analytic function appearing in the iterative formula. A special attention is paid to the total-step and single-step methods with Newton's and Halley's corrections because of their high computational efficiency. Numerical examples are also included.

1. INTRODUCTION

In the recent papers [1] and [2] the two classes of iterative methods for the simultaneous determination of simple zeros of analytic functions have been proposed. These methods can be regarded as generalizations of the methods for polynomial zeros presented in [3] and [4, Section 5.5]. Recently, Sakurai, Torii and Sugiura have derived in [5] a class of algorithms for simultaneously approximating simple complex zeros of a polynomial using the Padé approximation. In this paper we will show that these algorithms can be adapted for the determination of the zeros of analytic functions. To construct a family of zero-finding methods with arbitrary high order of convergence we use a quite different and, at the same time, much simpler procedure based a specific fixed-point relations (Section 2). The order of convergence of the presented methods is $m + 2$ ($m = 1, 2, \dots$), where m is the order of the highest derivative of analytic function appearing in the iterative formula.

The price to be paid in order to attain very high convergence consists of the requirement for great number of numerical operations. Practically, the iterative methods from the considered class with the order of convergence greater than four need too much computational effort. For this reason, in Section 3 we concentrate our attention to the fourth order method obtained for $m = 2$. We also construct several modified methods in parallel and serial fashion with the R -order of convergence between 4 and 7. The acceleration of convergence is attained by only a few additional operations in reference to the basic fourth order method, which points to the great computational efficiency of the modified methods.

2. FAMILY OF ITERATIVE METHODS

Let $z \mapsto \Phi(z)$ be an analytic function inside and on the simple smooth closed contour Γ , without zeros on Γ and with a known number n of simple zeros inside Γ . Then following Smirnov [6] Φ will be of the form

$$\Phi(z) = \Psi(z) \prod_{j=1}^n (z - \xi_j) \quad (1)$$

inside Γ , where ξ_1, \dots, ξ_n are the zeros of Φ (inside Γ) and Ψ is an analytic function such that $\Psi(z) \neq 0$ for $z \in \text{int } \Gamma$. The number of zeros n of Φ inside Γ is determined by the argument principle [7]

$$n = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Phi'(w)}{\Phi(w)} dw = \frac{1}{2\pi} [\arg \Phi(w)]_{\Gamma}. \quad (2)$$

Let ζ be an arbitrary point inside Γ such that $\Phi(\zeta) \neq 0$. The analytic function Ψ can be expressed as

$$\Psi(z) = \exp(Y(z)) \quad (3)$$

inside Γ , where Y is also an analytic function inside Γ given by

$$Y(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\log[(w - \zeta)^{-n} \Phi(w)]}{w - z} dw \quad (4)$$

(see [1]).

Let $\Delta_m f$ denote the m^{th} derivative of a function f , that is

$$\Delta_m f = \frac{d^m f}{dz^m} \quad \text{with} \quad \Delta_0 f \equiv f(z).$$

It is understood that $\Delta_r f(z_i)$ is the r^{th} derivative of a considered function $f(z)$ at z_i .

By virtue of the factorization (1) we find

$$\frac{1}{z - \xi_i} = \frac{1}{\Phi(z)} \exp(Y(z)) \prod_{\substack{j=1 \\ j \neq i}}^n (z - \xi_j) = G_i(z). \quad (5)$$

From the obvious identity

$$\xi_i = z + m \frac{\Delta_{m-1} \left(\frac{1}{z - \xi_i} \right)}{\Delta_m \left(\frac{1}{z - \xi_i} \right)} \quad (m = 1, 2, \dots)$$

and (5) it follows

$$\xi_i = z + m \frac{\Delta_{m-1}(G_i(z))}{\Delta_m(G_i(z))} \quad (m = 1, 2, \dots). \quad (6)$$

Applying the differentiation operators to the function $G_i(z)$ (given by (5)), we obtain from (6) the whole set of fixed-point relations depending on m . For illustration, let us consider the cases $m = 1$ and $m = 2$. We will use the notation

$$q_{k,i}(z) = \sum_{\substack{j=1 \\ j \neq i}}^n (z - \xi_j)^{-k}, \quad \sigma_k(z) = \frac{\Phi^{(k)}(z)}{\Phi(z)} \quad (k = 1, 2).$$

Using the logarithmic derivative of $G_i(z)$ we get from (5)

$$\frac{d}{dz} \log G_i(z) = \frac{G'_i(z)}{G_i(z)} = Y'(z) + \sum_{\substack{j=1 \\ j \neq i}}^n (z - \xi_j)^{-1} - \frac{\Phi'(z)}{\Phi(z)}. \quad (7)$$

Then relation (6) for $m = 1$ yields

$$\xi_i = z - \frac{1}{\sigma_1(z) - Y'(z) - q_{1,i}(z)} \quad (i = 1, \dots, n). \quad (8)$$

Formula (8) provides a construction of iterative methods for finding the zeros of analytic functions, as presented in [1].

From (6) it follows

$$G''_i(z) = G'_i(z) \left[q_{1,i}(z) + Y'(z) - \sigma_1(z) \right] - G_i(z) \left[q_{2,i}(z) - Y''(z) + \sigma_2(z) - \sigma_1^2(z) \right]. \quad (9)$$

Using (7) and (9) we obtain from (6) for $m = 2$

$$\begin{aligned} \xi_i &= z + \frac{2G'_i(z)}{G''_i(z)} \\ &= z - \frac{2[q_{1,i}(z) + Y'(z) - \sigma_1(z)]}{\sigma_2(z) - \sigma_1^2(z) - Y''(z) + q_{2,i}(z) - [q_{1,i}(z) + Y'(z) - \sigma_1(z)]^2} \quad (i = 1, \dots, n). \end{aligned} \tag{10}$$

Assume that z_1, \dots, z_n are reasonably good approximations to the zeros ξ_1, \dots, ξ_n of Φ and consider the function

$$g_i(z) = \frac{1}{\Phi(z)} \exp(Y(z)) \prod_{\substack{j=1 \\ j \neq i}}^n (z - z_j), \tag{11}$$

which is obtained from (5) substituting the zeros ξ_j by their approximations z_j ($j \neq i$). Let $z = z_i$ be sufficiently close to the zero ξ_i . Then, obviously, $g_i(z_i)$ is an approximation to $G_i(z_i)$ ($= (z_i - \xi_i)^{-1}$) so that we can expect that the complex number \hat{z}_i given by

$$\hat{z}_i = z_i + m \left[\frac{\Delta_{m-1}(g_i(z))}{\Delta_m(g_i(z))} \right]_{z=z_i} \tag{12}$$

is an approximation to the zero ξ_i (according to (6)). This assumption will be proved in the sequel. In passing, we note that for $m = 1$ we obtain from (11) and (12) the iterative formula

$$\hat{z}_i = z_i - \frac{1}{\frac{\Phi'(z_i)}{\Phi(z_i)} - Y'(z_i) - \sum_{\substack{j=1 \\ j \neq i}}^n (z_i - z_j)^{-1}} \quad (i = 1, \dots, n)$$

of the third order considered in [1].

Let us introduce the abbreviations

$$\varepsilon_i = \xi_i - z_i, \quad |\varepsilon_{M,i}| = \max_{j \neq i} |\varepsilon_j| \quad (i = 1, \dots, n), \quad |\varepsilon| = \max_{1 \leq i \leq n} |\varepsilon_i|.$$

THEOREM 1. *Assume that approximations z_1, \dots, z_n are sufficiently close to the zeros ξ_1, \dots, ξ_n of Φ . Then for the approximations \hat{z}_i ($i = 1, \dots, n$) given by (12) we have*

$$|\hat{z}_i - \xi_i| = O(|\varepsilon_{M,i}| |\varepsilon_i|^{m+1}).$$

The proof of Theorem 1 follows according to Lemma 3 and Theorem 1 from [5] and will be omitted.

Suppose that all the errors $|\varepsilon_i|$ are of the same order of magnitude, that is

$$|\varepsilon_i| = O(|\varepsilon|) \quad (i = 1, \dots, n).$$

Then Theorem 1 implies that the order of convergence of the iterative method (12) is $m + 2$ ($m = 1, 2, \dots$).

3. ON SOME EFFICIENT ALGORITHMS

As mentioned in the Introduction, the iterative methods of the form (12) have the high computational cost for $m > 2$. For this reason, in this section we will consider some efficient algorithms which follow from the fixed-point relation (10) taking $m = 2$.

Let $z \in \mathbb{C}$ and let

$$\sum_{k,i} z_i(r, s) = \sum_{j=1}^{i-1} (z_i - r_j)^{-k} + \sum_{j=i+1}^n (z_i - s_j)^{-k} \quad (k = 1, 2),$$

where $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{C}^n$ and $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ are some vectors. By means of $\sigma_1(z)$ and $\sigma_2(z)$, we define

$$N(z) = \frac{1}{\sigma_1(z)} \quad (\text{Newton's correction})$$

and

$$H(z) = \left[\sigma_1(z) - \frac{\sigma_2(z)}{2\sigma_1(z)} \right]^{-1} \quad (\text{Halley's correction}).$$

We recall that the correction terms $N(z)$ and $H(z)$ appear in the iterative formulas

$$\hat{z} = z - N(z)$$

and

$$\hat{z} = z - H(z),$$

with the order of convergence *two* and *three*, respectively.

Let $\mathbf{z}^{(k)} = (z_1^{(k)}, \dots, z_n^{(k)})$ ($k = 1, 2, 3$) be the vectors of approximations whose components are given by

$$\begin{aligned} z_i^{(1)} &= z_i && (\text{the current approximations}), \\ z_i^{(2)} &= z_i - N(z_i) && (\text{the Newton approximations}), \\ z_i^{(3)} &= z_i - H(z_i) && (\text{the Halley approximations}). \end{aligned}$$

We emphasize that the superscript index indicates the type of approximation and it should be strongly distinguished from the iteration index. The vector of new approximations will be denoted by $\hat{\mathbf{z}} = (\hat{z}_1, \dots, \hat{z}_n)$.

Suppose that we have found sufficiently good approximations z_1, \dots, z_n of the zeros ξ_1, \dots, ξ_n of Φ . Setting $z = z_i$ and $\xi_i := \hat{z}_i$ in (10), and taking certain approximations of ξ_j in the sums $q_{1,i}$ and $q_{2,i}$ on the right-hand side of (10), we can construct total-step (shorter TS) and single-step (SS) methods for the simultaneous determination of the zeros of an analytic function.

Let $(TS)_1$, $(TS)_2$ and $(TS)_3$ denote the total-step methods obtained by the substitution of the zeros ξ_j ($j \neq i$) in (10) by $z_j^{(1)}$, $z_j^{(2)}$ and $z_j^{(3)}$, respectively. The notations $(SS)_1$, $(SS)_2$ and $(SS)_3$ have the same meaning in the case of the single-step methods where we take $\xi_j := \hat{z}_j$ ($j < i$) and $\xi_j := z_j^{(k)}$ ($j > i$) ($k \in \{1, 2, 3\}$). Therefore, the subscript indices $k = 2$ and $k = 3$ point to the use of Newton's and Halley's correction, respectively. The above notation makes it possible for us to represent the new methods by the unique formulas as follows:

Total-step methods $(TS)_k$:

$$\hat{z}_i = z_i - \frac{2 \left[\sum_{1,i}(\mathbf{z}^{(k)}, \mathbf{z}^{(k)}) + Y'(z_i) - \sigma_1(z_i) \right]}{\sigma_2(z_i) - \sigma_1^2(z_i) - Y''(z_i) + \sum_{2,i}(\mathbf{z}^{(k)}, \mathbf{z}^{(k)}) - \left[\sum_{1,i}(\mathbf{z}^{(k)}, \mathbf{z}^{(k)}) + Y'(z_i) - \sigma_1(z_i) \right]^2} \quad (i = 1, \dots, n; k \in \{1, 2, 3\}). \quad (13)$$

Single-step methods $(SS)_k$:

$$\hat{z}_i = z_i - \frac{2 \left[\sum_{1,i}(\hat{\mathbf{z}}, \mathbf{z}^{(k)}) + Y'(z_i) - \sigma_1(z_i) \right]}{\sigma_2(z_i) - \sigma_1^2(z_i) - Y''(z_i) + \sum_{2,i}(\hat{\mathbf{z}}, \mathbf{z}^{(k)}) - \left[\sum_{1,i}(\hat{\mathbf{z}}, \mathbf{z}^{(k)}) + Y'(z_i) - \sigma_1(z_i) \right]^2} \quad (i = 1, \dots, n; k \in \{1, 2, 3\}). \quad (14)$$

The order of convergence of the total-step methods (13) is given in the following theorem:

THEOREM 2. Assume that z_1, \dots, z_n are reasonably good approximations to the zeros ξ_1, \dots, ξ_n . If $|\varepsilon_i| = O(|\varepsilon|)$ for each $i = 1, \dots, n$, then the iterative method $(TS)_k$, defined by (13), has the order of convergence equal to $k + 2$, $k \in \{1, 2, 3\}$.

The proof of Theorem 2 is very similar to the proof of Theorem 1 presented in [2] and will be omitted.

The convergence analysis of single-step methods is most frequently based on the notion of the R -order of convergence (see the book [8]). For the R -order of the single-step method $(SS)_k$ with the limit point $\xi = (\xi_1, \dots, \xi_n)$ (the vector of exact zeros) we will use the notation $O_R((SS)_k, \xi)$. Using the procedure applied in the papers [9] and [10] (see, also [11,12]) and the result from [13] it is easy to prove the following assertion:

THEOREM 3. *The R -order of convergence of the single-step method $(SS)_k$, defined by (14), is given by*

$$O_R((SS)_k, \xi) \geq 3 + \tau_n(k),$$

where $\tau_n(k)$ is the unique positive root of the equation

$$\tau^n - \tau k^{n-1} - 3k^{n-1} = 0 \quad (k \in \{1, 2, 3\}).$$

The values of the lower bounds of the R -order of convergence of the considered single-step methods are shown in Table 1.

Table 1. The lower bounds of the R -order of convergence.

method	n	3	4	5	6	7	8	9	10	15
$(SS)_1$		4.672	4.453	4.341	4.274	4.229	4.196	4.172	4.153	4.098
$(SS)_2$		5.862	5.585	5.443	5.357	5.299	5.257	5.225	5.200	5.130
$(SS)_3$		6.974	6.662	6.502	6.404	6.338	6.291	6.255	6.227	6.147

The improved iterative methods $(SS)_2$, $(SS)_3$, $(TS)_2$ and $(TS)_3$ with Newton's and Halley's corrections require only few additional operations compared to the basic methods $(TS)_1$ and $(SS)_1$ because these corrections are calculated using already found values $\sigma_1(z_j)$ and $\sigma_2(z_j)$. For this reason and the fact that the increase of the order of convergence is considerable we conclude that the computational efficiency of the methods mentioned above is remarkably high.

4. NUMERICAL RESULTS

The iterative formulas (13) and (14) require the calculation of the derivatives $Y'(z)$ and $Y''(z)$ at the points z_i ($i = 1, \dots, n$). From (4) it follows that

$$Y'(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\log[(w - \zeta)^{-n} \Phi(w)]}{(w - z)^2} dw, \tag{15}$$

and then

$$Y''(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{\log[(w - \zeta)^{-n} \Phi(w)]}{(w - z)^3} dw. \tag{16}$$

In (15) and (16) it can be taken that $\zeta = 0$. Applying an integration by parts we find that, from (15),

$$Y'(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Phi'(w)}{\Phi(w)} \frac{dw}{w - z} \tag{17}$$

(see [6]). Then the second derivative is given by

$$Y''(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Phi'(w)}{\Phi(w)} \frac{dw}{(w - z)^2}. \tag{18}$$

As it was advised in [1], the number of zeros n , given by (2), and $Y'(z)$ and $Y''(z)$, given by (17) and (18), should be computed by applying a suitable sufficiently accurate quadrature rule for contours of the form

$$\frac{1}{2\pi i} \int_{\Gamma} f(w) dw \cong \sum_{k=1}^m a_{km} f(w_{km}),$$

where a_{km} are the *weights* and w_{km} the corresponding *nodes* of the quadrature rule. As recommended in [1], it is convenient to apply the trapezoidal quadrature rule along the circumference $\Gamma = \{w : |w| = R\}$ with nodes

$$w_{km} = R \exp(i\theta_{km}), \quad \theta_{km} = (2k - 1)\pi/m \quad (k = 1, \dots, m)$$

(see, also, the book [14]). We observe that the same values $\Phi(w_{km})$ and $\Phi'(w_{km})$ may be used in the quadrature formulas for the computation of n , $Y'(z_i)$ and $Y''(z_i)$ because of the similar structure of the formulas (2), (17) and (18).

Furthermore, to avoid poor results of the numerical integration it is desirable to eliminate the singularities of the integrands. Formulas (17) and (18) are slightly simpler compared to (15) and (16) but the singularities in (17) and (18) (except $z = z_i$) are just the sought zeros of Φ ; on the other side, such a problem does not appear in dealing with (15) and (16). Finally, the result of the numerical integration applied to the contour integral (2) could be verified using some computational procedure for the argument principle

$$n = n(\Phi(\Gamma), 0) = \frac{1}{2\pi} [\arg \Phi(w)]_{\Gamma} \quad (19)$$

(see, for instance, the book [7] and [15]). More details about the mentioned problems may be found in [14], [16] and, particularly, in [1, Sections 3 and 4] and the references cited therein.

The influence of the error of numerical integration on the accuracy of the produced approximations to the zeros of Φ was discussed extensively in [1]. It was remarked there that this influence is very small for increasing values of the number of iterative steps. Numerical results from [1] have shown that even very rough approximations to $Y'(z_i)$ and $Y''(z_i)$ can provide significantly good approximations to the wanted zeros of Φ . The same conclusion may be drawn for the iterative methods presented in the previous section. For illustration, let us consider the basic fourth order method $(TS)_1$. With the abbreviations

$$h_i = \frac{\Phi(z_i)}{\Phi'(z_i)}, \quad S_{k,i} = \sum_{\substack{j=1 \\ j \neq i}}^n (z_i - z_j)^{-k} \quad (k = 1, 2),$$

$$R_i = S_{1,i} + Y'(z_i), \quad A_i = \frac{\Phi''(z_i)}{2\Phi'(z_i)} + R_i,$$

$$B_i = \frac{S_{2,i} - Y''(z_i) - R_i}{2},$$

formula (13) for $k = 1$ may be rewritten in the form

$$\hat{z}_i = z_i - \frac{h_i - R_i h_i^2}{1 - A_i h_i - B_i h_i^2}. \quad (20)$$

In the sequel, for two complex numbers z and w we will write $z = O_M(w)$ if $|z| = O(|w|)$ (the same order of their moduli).

Let us assume that $|h_i|$ is sufficiently small (which is the case if z_i is a considerably good approximation to the zero ξ_i). Then, taking advantage of Taylor's expansion

$$\frac{1}{1 - A_i h_i - B_i h_i^2} = 1 + A_i h_i + (A_i^2 + B_i) h_i^2 + (A_i^3 + 2A_i B_i) h_i^3 + \dots,$$

we obtain from (20)

$$\hat{z}_i = z_i - h_i + h_i^2(R_i - A_i) + h_i^3(A_i R_i - A_i^2 - B_i) + O_M(h_i^4),$$

that is,

$$\hat{z}_i = z_i - h_i - h_i^2 \frac{\Phi''(z_i)}{2\Phi'(z_i)} + \frac{1}{2} h_i^3 \left\{ \left[1 - \frac{\Phi''(z_i)}{\Phi'(z_i)} \right] \left[S_{1,i} + Y'(z_i) \right] - \frac{1}{2} \left[\frac{\Phi''(z_i)}{\Phi'(z_i)} \right]^2 + Y''(z_i) - S_{2,i} \right\} + O_M(h_i^4). \quad (21)$$

Table 2. The improved approximations obtained by the total-step methods (13) and the single-step methods (14). The underlined digit indicates the first incorrect digit, the symbol * denotes that all digits are correct.

	<i>i</i>	$z_i^{(2)}$	$z_i^{(3)}$
$(TS)_1$	1	-1.22970 <u>9</u> 152729003615292560624452258	-1.229708718114713729321759293128968
	2	-0.82193 <u>1</u> 317332681611134196811240404	-0.821932206573811152411099805830177
	3	0.5640643677 <u>4</u> 3854859159865289348227	0.564064367739056317926859206067799 *
$(SS)_1$	1	-1.229708 <u>8</u> 78965880515906631548819407	-1.229708718114713729321758216508564
	2	-0.821932206573 <u>2</u> 80813170468816427490	-0.821932206573811152411104312884845 *
	3	0.564064367739056317926859 <u>1</u> 95058748	0.564064367739056317926859206067799 *
$(TS)_2$	1	-1.22970871811 <u>1</u> 6704389262850741788402	-1.229708718114713729321758216508531 *
	2	-0.82193220657 <u>0</u> 888236976792723780922	-0.821932206573811152411104312884845 *
	3	0.56406436773905631 <u>8</u> 300745354670170	0.564064367739056317926859206067799 *
$(SS)_2$	1	-1.229708718114 <u>8</u> 84201880036356908862	-1.229708718114713729321758216508531 *
	2	-0.82193220657381115241 <u>2</u> 278954502608	-0.821932206573811152411104312884845 *
	3	0.5640643677390563179268592060 <u>7</u> 3319	0.564064367739056317926859206067799 *
$(TS)_3$	1	-1.2297087181147 <u>6</u> 8972454194610221481	-1.229708718114713729321758216508531 *
	2	-0.821932206573 <u>7</u> 53211172970980131917	-0.821932206573811152411104312884845 *
	3	0.5640643677390563179 <u>7</u> 4770549691611	0.564064367739056317926859206067799 *
$(SS)_3$	1	-1.22970871811471 <u>4</u> 476995170273322641	-1.229708718114713729321758216508531 *
	2	-0.82193220657381115241110 <u>0</u> 313833764	-0.821932206573811152411104312884845 *
	3	0.56406436773905631792685920606 <u>7</u> 801	0.564064367739056317926859206067799 *

From the last relation we note that the quantities $Y'(z_i)$ and $Y''(z_i)$ are multiplied by h_i^3 . Therefore, the influence of the quadrature errors in the iterative formula (13) (for $k = 1$) is neutralized due to the very small (in magnitude) factor h_i^3 . The same is valid for the remaining methods given by (13) and (14).

By the way, we observe that the main part

$$\hat{z}_i = z_i - h_i - h_i^2 \frac{\Phi''(z_i)}{\Phi'(z_i)} = z_i - \frac{\Phi(z_i)}{\Phi'(z_i)} \left[1 - \frac{\Phi''(z_i)\Phi(z_i)}{2\Phi'(z_i)^2} \right]$$

in (21) is the well-known Chebyshev's method of the third order.

To demonstrate the efficiency of the iterative methods presented in the previous section, we give numerical results for the zeros of the analytic function

$$\Phi(z) = \exp(z) - 2 \cos(3z) - 2 \tag{22}$$

inside the circle $S = \{z : |z| < 1.5\}$. The number of zeros of Φ in the circle S was computed by the argument principle. As mentioned in [15], to make formula (19) of any computational interest, the contour $\Gamma : |z| = 1.5$ of the circle S was first replaced by a polygon of vertices V_1, V_2, \dots, V_M , belonging to Γ . Then the variation of the argument was stepwise calculated from V_k to V_{k+1} (where $k = 1, \dots, M$ and $V_{M+1} = V_1$) and it was found that the number of zeros in S is $n = 3$.

For the iterative improvement of the initial approximations $z_1^{(0)} = -1.5$, $z_2^{(0)} = -0.5$ and $z_3^{(0)} = 0.8$ (found by a search algorithm including a proximity test for the detection of the presence of a zero) we have applied the total-step and single-step methods (13) and (14) (for $k = 1, 2, 3$) as well as the Newton method (for comparison purpose). Because of very fast convergence of these methods, quad precision arithmetic (about 34 significant decimal digits) was employed on the Micro VAX II computer. The improved approximations $z_i^{(m)}$ ($i = 1, 2, 3$) obtained by (13) and (14), where m is the iteration index, are shown in Table 2. We emphasize that the implemented algorithms possess a great power, so that they could be produced in the third iterative step to even more than 33 correct decimal digits, if a computer with arithmetic of higher precision were used.

For the sake of comparison, the Newton method

$$z_i^{(m+1)} = z_i^{(m)} - \frac{\Phi(z^{(m)})}{\Phi'(z^{(m)})} \quad (m = 0, 1, \dots)$$

was also applied for the determination of zeros of the analytic function Φ given by (22). The same initial approximations as in the case of the simultaneous methods (13) and (14) were used. The obtained numerical results are given in Table 3.

Table 3. The improved approximations obtained by the Newton method. The underlined digit indicates the first incorrect digit.

m	$z_1^{(m)}$	$z_2^{(m)}$	$z_3^{(m)}$
1	-1.277	-0.785	0.529
2	-1.234	-0.819	0.564113
3	-1.229742	-0.821922	0.56406436763
4	-1.229708721	-0.82193220636	0.56406436773905631792655
5	-1.229708718114713742	-0.82193220657381115232	33 correct digits
6	33 correct digits	33 correct digits	

Let $\varepsilon^{(m)} = \max_{1 \leq i \leq 3} |z_i^{(m)} - \xi_i|$ be the maximal error corresponding the produced approximations. These errors are displayed in Table 4 for all the applied methods.

Table 4. The maximal errors of approximations. $A(-h)$ means $A \times 10^{-h}$.

	Newton's method	$(TS)_1$	$(SS)_1$	$(TS)_2$	$(SS)_2$	$(TS)_3$	$(SS)_3$
$\varepsilon^{(1)}$	4.77(-2)	1.79(-2)	1.23(-2)	2.8(-3)	2.05(-3)	3.01(-3)	2.89(-3)
$\varepsilon^{(2)}$	3.91(-3)	8.89(-7)	1.61(-7)	2.92(-12)	1.7(-13)	5.79(-14)	7.48(-16)
$\varepsilon^{(3)}$	3.31(-5)	4.51(-24)	3.25(-32)	1.(-34)	1.(-34)	1.(-34)	1.(-34)

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