An International Journal
computers & mathematics with applications
PERGAMON Computers and Mathematics with Applications 44 (2002) 1157-1173
www.elsevier.com/locate/camwa

Errors of Linear Multistep Methods and Runge-Kutta Methods for Singular Perturbation Problems with Delays

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(Received and accepted July 2001)

Abstract—This paper is concerned with the error analysis of linear multistep methods and Runge-Kutta methods applied to some classes of one-parameter stiff singularly perturbed problems with delays. We derive the global error estimates of $A(\alpha)$-stable linear multistep methods and algebraically and diagonally stable Runge-Kutta methods with Lagrange interpolation procedure. Numerical experiments confirm our theoretical analysis.

Keywords—Singular perturbation problems, Delays, Interpolation, Linear multistep methods, Runge-Kutta methods, Error analysis.

1. INTRODUCTION

Let $\langle\ ,\ \rangle$ be the standard inner product on R^N and $\|\cdot\|$ the corresponding norm. Consider the singular perturbation problems (SPPs) with delays

$$\begin{aligned} x'(t) &= f(x(t), x(t-\tau), y(t), y(t-\tau)), \qquad t \in [0, T], \\ \epsilon y'(t) &= g(x(t), x(t-\tau), y(t), y(t-\tau)), \qquad 0 < \epsilon \ll 1, \end{aligned} \tag{1.1}$$

$$x(t) = \varphi(t), \quad y(t) = \psi(t), \qquad t \le 0,$$

where τ and ϵ are constants, and $\tau > 0$. φ and ψ are given continuous functions. $f : R^M \times R^M \times R^N \times R^N \to R^M$ and $g : R^M \times R^M \times R^N \times R^N \to R^N$ are given mappings, which are sufficiently

This work is supported by the National Natural Science Foundation of China (Nos. 19871086 and 10101027).
The authors would like to thank the anonymous referees for their many valuable suggestions. Moreover, the authors are indebted to Dr. Chengming Huang and Dr. Aiguo Xiao for useful discussions.

PII: S0898-1221(02)00223-7

smooth. In order to make the error analysis feasible, we always assume that problem (1.1) has a unique solution $(x(t), y(t))$ which is sufficiently differentiable and satisfies

$$\left\|\frac{d^i x(t)}{dt^i}\right\| \le M_i, \qquad \left\|\frac{d^i y(t)}{dt^i}\right\| \le N_i,$$

where M_i and N_i are constants which are independent of the stiffness of the problem.

The numerical solution of delay differential equations (DDEs) has been the object of interesting research in recent years. Many papers investigated the local and global error behaviour of DDE solvers (cf. [1–6]). However, they are only suitable for nonstiff DDEs. In 1997, the concept of D-convergence for stiff DDEs was introduced (cf. [7]). Subsequently, D-convergence theory was further developed [8,9].

Now we briefly recall the concept of D-convergence (cf. [7–9]). Consider the following nonlinear problem:

$$\begin{aligned} x'(t) &= f(t, x(t), x(t-\tau)), \qquad & t \ge 0, \\ x(t) &= \varphi(t), & t \le 0, \end{aligned} \tag{1.2}$$

where $f : [0, +\infty) \times C^M \times C^M \to C^M$ is a given mapping which satisfies the following conditions:

$$\operatorname{Re}\langle f(t, x_1, z) - f(t, x_2, z), x_1 - x_2\rangle \le \beta \|x_1 - x_2\|^2, \qquad t \ge 0, \tag{1.3a}$$

$$\|f(t, x, z_1) - f(t, x, z_2)\| \le \gamma \|z_1 - z_2\|, \qquad t \ge 0, \tag{1.3b}$$

with moderately-sized constants β and γ, here x, x_1, x_2, z, z_1, and $z_2 \in C^M$, here $\langle\ ,\ \rangle$ is an inner product on C^M, and $\|\cdot\|$ the corresponding norm.

Let (A, b, c) denote a given Runge-Kutta method with $s \times s$ matrix $A = (a_{ij})$ and vectors $b = (b_1, \dots, b_s)^\top$, $c = (c_1, \dots, c_s)^\top$. A Runge-Kutta method applied to (1.2) gives

$$X_i^{(n)} = x_n + h \sum_{j=1}^{s} a_{ij} f\left(t_n + c_j h, X_j^{(n)}, \bar{X}_j^{(n)}\right), \qquad i = 1, 2, \dots, s,$$

$$x_{n+1} = x_n + h \sum_{j=1}^{s} b_j f\left(t_n + c_j h, X_j^{(n)}, \bar{X}_j^{(n)}\right).$$

The argument $\bar{X}_j^{(n)}$ denotes an approximation to $x(t_n + c_j h - \tau)$, which is obtained by a specific interpolation procedure at the point $t = t_n + c_j h - \tau$.

DEFINITION 1.1. *A Runge-Kutta method (A, b, c) with an interpolation procedure is called D-convergent of order p for problem (1.2) satisfying (1.3), if the global error admits an estimate*

$$\|x(t_n) - x_n\| \le C(t_n) h^p, \qquad n \ge 1, \quad h \in (0, h_0],$$

where the function $C(t)$ and the maximum stepsize h_0 depend only on the method, the parameters β, γ, and τ, and bounds for certain derivatives of the exact solution.

Zhang and Zhou [7] gave a sufficient condition which guarantees D-convergence of the Runge-Kutta method. Huang *et al.* [8,9] further discussed D-convergence of Runge-Kutta methods, one-leg methods, and general linear methods.

Convergence of numerical methods for SPPs is also an important issue. Many papers analyzed the error behaviour of numerical methods for single and multiple stiff SPPs (cf. [10–17]).

But up to now, there existed no results of numerical methods for SPPs with delays. Although stiff SPPs with delays are considered as a special class of stiff initial value problems of delay differential equations, they cannot be covered by D-theory because their parameters β and γ corresponding to (1.3) are in general $\mathcal{O}(\epsilon^{-1})$. Therefore, it is meaningful to investigate convergence of numerical methods for SPPs with delays. This paper is concerned with the error analysis of linear multistep methods and Runge-Kutta methods applied to some classes of one-parameter stiff SPPs with delays.

This paper is organized as follows. In Section 2, for some classes of stiff singularly perturbed problems with delays, we derive the global error estimate of $A(\alpha)$-stable multistep method with Lagrange interpolation procedure. In fact, the result (Theorem 2.1) can be considered as an extension of that obtained by Lubich (cf. [13]) for the case of singular perturbation problems without delay. In Section 3, for some classes of multiple stiff singularly perturbed problems with delays, we obtain the global error estimate of algebraically and diagonally stable Runge-Kutta methods with Lagrange interpolation procedure. The result (Theorem 3.3) can be considered an extension of that obtained by Xiao (cf. [16]) for the case of singular perturbation problems without delay. In Section 4, we illustrate our main results by numerical experiments.

2. ERROR OF LINEAR MULTISTEP METHODS FOR SPPS WITH DELAYS

In this section, we assume that (cf. [12])

$$\text{the eigenvalues } \lambda \text{ of } g_y(x,u,y,v) \text{ lie in } |\arg\lambda - \pi| < \alpha, \tag{2.1}$$

for (x,u,y,v) in a neighbourhood of the considered solution. A linear multistep method applied to system (1.1) gives

$$\sum_{i=0}^{k} \alpha_i x_{n+i} = h \sum_{i=0}^{k} \beta_i f\left(x_{n+i}, \bar{x}_{n+i}, y_{n+i}, \bar{y}_{n+i}\right), \tag{2.2a}$$

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \frac{h}{\epsilon} \sum_{i=0}^{k} \beta_i g\left(x_{n+i}, \bar{x}_{n+i}, y_{n+i}, \bar{y}_{n+i}\right), \tag{2.2b}$$

where $h > 0$ is the stepsize, $t_n = nh$, $n = 0,1,\dots,I$, $(I+k)h \le T$, and x_n and y_n are an approximation to the exact solution $x(t_n)$ and $y(t_n)$, respectively. α_i, β_i $(i = 0,1,\dots,k)$ are given constants, $\alpha_k\beta_k \neq 0$. The arguments $\bar{x}_n$ and $\bar{y}_n$ denote an approximation to $x(t_n - \tau)$ and $y(t_n - \tau)$, respectively, which are obtained by a specific interpolation procedure at the point $t = t_n - \tau$ using x_l and y_l, respectively, with $l \le n-1$.

Process (2.2) is defined completely by the linear multistep method and the interpolation procedure for $\bar{x}_n$ and $\bar{y}_n$.

Let $\mu, \nu \ge 0$ be integers, $\tau = (m-\delta)h$ with integer $m \ge k+\nu+1$ and $\delta \in [0,1)$. We consider the following interpolation procedure:

$$\bar{x}_n = \begin{cases} \sum\limits_{i=-\mu}^{\nu} L_i(\delta) x_{n-m+i}, & t_n - \tau > 0, \\ \varphi(t_n - \tau), & t_n - \tau \le 0, \end{cases} \tag{2.3a}$$

$$\bar{y}_n = \begin{cases} \sum\limits_{i=-\mu}^{\nu} L_i(\delta) y_{n-m+i}, & t_n - \tau > 0, \\ \psi(t_n - \tau), & t_n - \tau \le 0, \end{cases} \tag{2.3b}$$

where $x_j = \varphi(t_j)$ and $y_j = \psi(t_j)$ for $j \le 0$, and

$$L_i(\theta) = \prod_{\substack{j=-\mu \\ j \neq i}}^{\nu} \frac{\theta - j}{i - j}, \qquad \theta \in [0,1). \tag{2.4}$$

Here we assume $m \ge k+\nu+1$ not only so as to guarantee that, in the interpolation procedure, no unknown values x_l and y_l with $l > n+k-1$ are used, but also for simplicity in the discussion of Part (c) in this section. In this section, the constants h_i, C, C_i, $\tilde{C}$, $\tilde{C}_i$, and κ used later are independent of stiffness of the considered problem.

THEOREM 2.1. *Suppose that a multistep method is of order p, $A(\alpha)$-stable, and strictly stable at infinity. If problem (1.1) satisfies (2.1), then the global error is bounded for $h \geq \epsilon$ and $nh \leq T$ by*

$$\|x_n - x(t_n)\| + \|y_n - y(t_n)\| \leq C \left(\max_{0 \leq j < k} \|x_j - x(t_j)\| + h^p \int_0^{t_n} \left\| x^{(p+1)}(t) \right\| dt \right.$$
$$\left. + \max_{0 \leq j < k} \|y_j - y(t_j)\| + \epsilon h^p \max_{0 \leq t \leq t_n} \left\| y^{(p+1)}(t) \right\| + h^{\mu+\nu+1} \right).$$

This estimate holds for $h \leq h_0$ (h_0 sufficiently small, but independent of ϵ), and provided that the starting values are in a sufficiently small, h- and ϵ-independent neighbourhood of the exact solution.

PROOF. The basic idea of the following proof comes from that of Theorem 1.3 in [12, p. 412].

(a) First we derive recursive estimates for the global error. We insert the exact solution of (1.1) into method (2.2) and so obtain

$$\sum_{i=0}^{k} \alpha_i x(t_{n+i}) = h \sum_{i=0}^{k} \beta_i f\left(x(t_{n+i}), x(t_{n+i} - \tau), y(t_{n+i}), y(t_{n+i} - \tau)\right) + d_{n+k}, \tag{2.5a}$$

$$\sum_{i=0}^{k} \alpha_i y(t_{n+i}) = \frac{h}{\epsilon} \sum_{i=0}^{k} \beta_i g\left(x(t_{n+i}), x(t_{n+i} - \tau), y(t_{n+i}), y(t_{n+i} - \tau)\right) + e_{n+k}, \tag{2.5b}$$

where the perturbations d_{n+k}, e_{n+k} can be estimated (for $n \geq 0$) as

$$\|d_{n+k}\| \leq C_1 h^p \int_{t_n}^{t_{n+k}} \left\| x^{(p+1)}(t) \right\| dt, \tag{2.6a}$$

$$\|e_{n+k}\| \leq C_2 h^{p+1} \max_{t_n \leq t \leq t_{n+k}} \left\| y^{(p+1)}(t) \right\|. \tag{2.6b}$$

We then denote the global errors by $\Delta x_n = x_n - x(t_n)$, $\Delta y_n = y_n - y(t_n)$, and introduce the differences

$$\begin{aligned}
\Delta f_{n+k} = \sum_{i=0}^{k} \beta_i \big(& f\left(x_{n+i}, \bar{x}_{n+i}, y_{n+i}, \bar{y}_{n+i}\right) \\
& - f\left(x(t_{n+i}), x(t_{n+i} - \tau), y(t_{n+i}), y(t_{n+i} - \tau)\right)\big), \qquad n \geq 0, \\
\Delta g_{n+k} = \sum_{i=0}^{k} \beta_i \big(& g(x_{n+i}, \bar{x}_{n+i}, y_{n+i}, \bar{y}_{n+i}) \\
& - g\left(x(t_{n+i}), x(t_{n+i} - \tau), y(t_{n+i}), y(t_{n+i} - \tau)\right) - J \Delta y_{n+i}\big), \qquad n \geq 0,
\end{aligned}$$

where $\Delta f_j = 0$ and $\Delta g_j = 0$ for $j < k$, $J = g_y(x(0), x(-\tau), y(0), y(-\tau))$. Subtraction of (2.5a) from (2.2a) yields, for $n \geq 0$,

$$\sum_{i=0}^{k} \alpha_i \Delta x_{n+i} = h \Delta f_{n+k} - d_{n+k}. \tag{2.7}$$

We take the difference of (2.2b) and (2.5b) and then subtract from both sides the quantity $(h/\epsilon) \sum_{i=0}^{k} \beta_i J \Delta y_{n+i}$. This yields, for $n \geq 0$,

$$\sum_{i=0}^{k} \left(\alpha_i I - \beta_i \frac{h}{\epsilon} J \right) \Delta y_{n+i} = \frac{h}{\epsilon} \Delta g_{n+k} - e_{n+k}. \tag{2.8}$$

We define $d_0, \dots, d_{k-1}$, $e_0, \dots, e_{k-1}$ such that (2.7) and (2.8) also hold for negative n. Using equations (2.7), (2.8), and a similar technique in [12, p. 413], we obtain

$$\|\Delta x_n\| \le h \sum_{j=0}^{n} \left(M_x \|\Delta x_j\| + M_u \|\Delta \bar{x}_j\| + M_y \|\Delta y_j\| + M_v \|\Delta \bar{y}_j\| \right) + C_3 \sum_{j=0}^{n} \|d_j\|, \tag{2.9}$$

$$\|\Delta y_n\| \le \sum_{j=0}^{n} \kappa^{n-j} \left(L \|\Delta x_j\| + \bar{L} \|\Delta \bar{x}_j\| + l \|\Delta y_j\| + \bar{l} \|\Delta \bar{y}_j\| \right) + C_4 \frac{\epsilon}{h} \sum_{j=0}^{n} \kappa^{n-j} \|e_j\|, \tag{2.10}$$

where the constants M_x, M_u, M_y, M_v, L, $\bar{L}$, l, and $\bar{l}$ are independent of ϵ and h, and

$$\Delta \bar{x}_j = \bar{x}_j - x(t_j - \tau), \qquad \Delta \bar{y}_j = \bar{y}_j - y(t_j - \tau).$$

On the other hand, it follows from (2.3) that

$$\|\bar{x}_j - x(t_j - \tau)\| \le \left\| \sum_{i=-\mu}^{\nu} L_i(\delta)(x_{j-m+i} - x(t_{j-m+i})) \right\| + \left\| \sum_{i=-\mu}^{\nu} L_i(\delta) x(t_{j-m+i}) - x(t_j - \tau) \right\|.$$

From the remainder estimate of the Lagrange interpolation formula, we have

$$\left\| \sum_{i=-\mu}^{\nu} L_i(\delta) x\left(t_{j-m+i}\right) - x\left(t_j - \tau\right) \right\| \le \frac{M_{\mu+\nu+1}}{(\mu+\nu+1)!} h^{\mu+\nu+1} \prod_{i=-\mu}^{\nu} |\delta - i| \le M_{\mu+\nu+1} h^{\mu+\nu+1}.$$

Let $\tilde{L}_0 = \max_{-\mu \le i \le \nu} \sup_{\theta \in [0,1)} |L_i(\theta)|$. Therefore, from the Cauchy inequality, we further obtain

$$\|\bar{x}_j - x\left(t_j - \tau\right)\|^2 \le 2 \left((\mu+\nu+1) \tilde{L}_0^2 \sum_{i=-\mu}^{\nu} \|\Delta x_{j-m+i}\|^2 + M_{\mu+\nu+1}^2 h^{2(\mu+\nu+1)} \right),$$

which gives

$$\|\bar{x}_j - x(t_j - \tau)\| \le C_5 \left(\sum_{i=-\mu}^{\nu} \|\Delta x_{j-m+i}\| + h^{\mu+\nu+1} \right), \tag{2.11a}$$

where $C_5 = \sqrt{2} \max(\sqrt{\mu+\nu+1}\, \tilde{L}_0, M_{\mu+\nu+1}, N_{\mu+\nu+1})$. Similarly,

$$\|\bar{y}_j - y(t_j - \tau)\| \le C_5 \left(\sum_{i=-\mu}^{\nu} \|\Delta y_{j-m+i}\| + h^{\mu+\nu+1} \right). \tag{2.11b}$$

A combination of (2.9)–(2.11) leads to

$$\|\Delta x_n\| \le h \sum_{j=0}^{n} \left(\tilde{M} \|\Delta x_j\| + \tilde{N} \|\Delta y_j\| \right) + C_6 \sum_{j=0}^{n} \left\| \tilde{d}_j \right\|, \qquad n \ge k, \tag{2.12a}$$

$$\|\Delta y_n\| \le \sum_{j=0}^{n} \kappa^{n-j} \left(L \|\Delta x_j\| + l \|\Delta y_j\| \right) + C_4 \frac{\epsilon}{h} \sum_{j=0}^{n} \kappa^{n-j} \|e_j\| + C_7 \sum_{j=0}^{n} \kappa^{n-j} w_j, \qquad n \ge k, \tag{2.12b}$$

where

$$\tilde{M} = M_x + (\mu+\nu+1) C_5 M_u, \qquad \tilde{N} = N_y + (\mu+\nu+1) C_5 M_v,$$

$$C_6 = \max(C_3, C_5 M_u + C_5 M_v), \qquad \left\| \tilde{d}_j \right\| \le \|d_j\| + h^{\mu+\nu+2},$$

$$C_7 = \frac{C_5(\mu+\nu+1) \max\left(\bar{L}, \bar{l}\right)}{\kappa^{\mu+\nu}},$$

$$w_j = \begin{cases} 0, & j < m - \nu, \\ \|\Delta x_{j-m+\nu}\| + \|\Delta y_{j-m+\nu}\| + 2h^{\mu+\nu+1}, & j \ge m - \nu. \end{cases}$$

(b) In order to solve inequalities (2.12a) and (2.12b), we define sequences $\{u_n\}$ and $\{v_n\}$ $(n \geq k)$ by

$$u_n = h\sum_{j=0}^{n}\left(\tilde{M}u_j + \tilde{N}v_j\right) + C_6\sum_{j=0}^{n}\left\|\tilde{d}_j\right\|, \tag{2.13a}$$

$$v_n = \sum_{j=0}^{n}\kappa^{n-j}\left(Lu_j + lv_j\right) + C_4\frac{\epsilon}{h}\sum_{j=0}^{n}\kappa^{n-j}\left\|e_j\right\| + C_7\sum_{j=0}^{n}\kappa^{n-j}w_j. \tag{2.13b}$$

Let $u_j = \|\Delta x_j\|$ and $v_j = \|\Delta y_j\|$ for $j < k$, an induction argument shows that, for $n \geq 0$,

$$\|\Delta x_n\| \leq u_n, \qquad \|\Delta y_n\| \leq v_n,$$

provided $l < 1$ and $h \leq h_1$. It is important to remark that the Lipschitz constant l can be made arbitrarily small by shrinking the considered interval, compact interval $[0, T]$ can be covered by repeated application of the below estimates (cf. [12]).

By a similar process of Part (b) in the proof of Theorem 1.3 in [12, p. 414], we easily show from (2.13) that there exists $h_0 > 0$ such that, for $\epsilon \leq h \leq h_0$,

$$u_n + v_n \leq C_8\left(\sum_{j=0}^{n}\hat{d}_j + \sum_{j=0}^{n}\left(h + Q^{n-j}\right)\hat{e}_j\right), \tag{2.14}$$

where $0 < Q = \kappa/(1-l) < 1$, and

$$\left|\hat{d}_n\right| \leq C_9\left(\left\|\tilde{d}_n\right\| + \epsilon\|e_n\| + hw_n\right), \qquad |\hat{e}_n| \leq C_{10}\left(\left\|\tilde{d}_n\right\| + \frac{\epsilon}{h}\|e_n\| + w_n\right),$$

which gives

$$\left|\hat{d}_n\right| \leq C_9\left(\|d_n\| + \epsilon\|e_n\| + hw_n + h^{\mu+\nu+2}\right),$$
$$|\hat{e}_n| \leq C_{10}\left(\|d_n\| + \frac{\epsilon}{h}\|e_n\| + w_n + h^{\mu+\nu+2}\right).$$

(c) Our next aim is to investigate the global error in successive subintervals.

For $n \in [k, m-\nu-1]$, $w_n = 0$. Since $d_0, \dots, d_{k-1}$ are a linear combination of the values Δx_j $(j < k)$, and $e_0, \dots, e_{k-1}$ are a linear combination of the Δy_j and $(h/\epsilon)\Delta y_j$ $(j < k)$, it follows from $\|\Delta x_n\| \leq u_n$, $\|\Delta y_n\| \leq v_n$, and (2.14) that

$$\begin{aligned}\|x_n - x(t_n)\| + \|y_n - y(t_n)\| \leq \tilde{C}_1 \Bigg(&\max_{0\leq j<k}\|x_j - x(t_j)\| + h^p\int_0^{t_n}\left\|x^{(p+1)}(t)\right\|dt \\ &+ (h + Q^n)\max_{0\leq j<k}\|y_j - y(t_j)\| + \epsilon h^p\max_{0\leq t\leq t_n}\left\|y^{(p+1)}(t)\right\| + h^{\mu+\nu+1}\Bigg).\end{aligned} \tag{2.15}$$

For $n \in [m-\nu, 2(m-\nu)-1]$, $w_n = \|\Delta x_{n-m+\nu}\| + \|\Delta y_{n-m+\nu}\| + 2h^{\mu+\nu+1}$. Using (2.14) and (2.15), we obtain

$$\begin{aligned}\|x_n - x(t_n)\| + \|y_n - y(t_n)\| \leq \tilde{C}_2 \Bigg(&\max_{0\leq j<k}\|x_j - x(t_j)\| + h^p\int_0^{t_n}\left\|x^{(p+1)}(t)\right\|dt \\ &+ \max_{0\leq j<k}\|y_j - y(t_j)\| + \epsilon h^p\max_{0\leq t\leq t_n}\left\|y^{(p+1)}(t)\right\| + h^{\mu+\nu+1}\Bigg).\end{aligned} \tag{2.16}$$

Generally, for $n \in [i(m-\nu), (i+1)(m-\nu)-1]$. by induction, inequality (2.16) is also valid with $\tilde{C}_2$ replaced by $\tilde{C}_{i+1}$.

Because of $mh \geq \tau$ and $m \geq k + \nu + 1$, we have

$$(m - \nu)h \geq \frac{k+1}{k+\nu+1}\tau.$$

Let

$$m_0 = \left[\frac{T}{((k+1)/(k+\nu+1))\tau}\right] + 1,$$

where $[a]$ is an integer with $a - 1 < [a] \leq a$. Repeating the above process n_0 ($n_0 \leq m_0$) times, we can obtain the global error estimate $\|x_n - x(t_n)\| + \|y_n - y(t_n)\|$ for all n ($nh \leq T$). Let

$$C = \max\left((1 + h_0)\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_{n_0}\right).$$

The proof is completed.

REMARK 2.2. It is well known that the k-step ($k \leq 6$) backward differentiation formulas (BDF) is of order k, $A(\alpha)$-stable, and strictly stable at infinity. Therefore, the methods satisfy the assumptions in Theorem 2.1.

REMARK 2.3. System (2.2a),(2.2b) constitutes a nonlinear system with respect to x_{n+k} and y_{n+k}. The Jacobian of the system is of the form

$$\begin{pmatrix} I_M + \mathcal{O}(h) & \mathcal{O}(h) \\ \mathcal{O}(1) & \dfrac{\epsilon}{h}\dfrac{\alpha_k}{\beta_k}I_N - g_y\left(x_{n+k}, \bar{x}_{n+k}, y_{n+k}, \bar{y}_{n+k}\right) \end{pmatrix}. \tag{2.17}$$

Since condition (2.1) and the fact that the method is $A(\alpha)$-stable and strictly stable at infinity, it follows from formula (VI.1.52) in [12] (there is a typing error in the formula, where $\sigma(\zeta^{-k})$ should be $\sigma(\zeta^{-1})$) that

$$\left\|\left(\frac{\epsilon}{h}\frac{\alpha_k}{\beta_k}I_N - g_y\left(x_{n+k}, \bar{x}_{n+k}, y_{n+k}, \bar{y}_{n+k}\right)\right)^{-1}\right\| \leq C_{11}.$$

Consequently, also the inverse of (2.17) is uniformly bounded for $\epsilon > 0$ and $h \leq h_0$. Hence, the nonlinear system (2.2a),(2.2b) possesses a locally unique solution.

REMARK 2.4. The result (Theorem 2.1) can be considered as an extension of that obtained by Lubich (cf. [13]) for the case of singular perturbation problems without delay.

3. ERROR OF RUNGE-KUTTA METHODS FOR MSPPS WITH DELAYS

In this section, we assume problem (1.1) satisfies the following conditions:

$$\langle f(x_1, u, y, v) - f(x_2, u, y, v), x_1 - x_2\rangle \leq \omega_1\|x_1 - x_2\|^2, \tag{3.1a}$$

$$\langle g(x, u, y_1, v) - g(x, u, y_2, v), y_1 - y_2\rangle \leq -\omega_2\|y_1 - y_2\|^2, \tag{3.1b}$$

with moderately-sized constant ω_1 and $-\omega_2$, where $x, x_1, x_2, u \in R^M$, $y, y_1, y_2, v \in R^N$, and $f(x,u,y,v)$ and $g(x,u,y,v)$ satisfy Lipschitz conditions with respect to other arguments. Without loss of generality, we assume $\omega_2 = 1$ (cf. [12]).

We note that the one-sided Lipschitz condition (3.1a) is weaker than the conventional Lipschitz condition

$$\|f(x_1, u, y, v) - f(x_2, u, y, v)\| \leq L\|x_1 - x_2\|, \tag{3.2}$$

since (3.2) implies (3.1a) with $\omega_1 = L$ for moderately-sized L. If problem (1.1) satisfies (3.2) with moderately-sized L, it is called a single stiff singularly perturbed problem (SSPP) with delays.

If $L \gg 1$, it is called a multiple stiff singularly perturbed problem (MSPP) with delays whose stiffness is caused by the small parameter ϵ and other factors. In 1988, Hairer *et al.* [10] obtained the sharp error bounds of Runge-Kutta methods for SPPs. However, it is restricted within the limits of SSPPs. In 1999, Xiao [16] investigated the error of Runge-Kutta methods for MSPPs. In this section, we extend the study of Xiao to MSPPs with delays.

A Runge-Kutta method (A, b, c) applied to system (1.1) gives

$$X_i^{(n)} = x_n + h\sum_{j=1}^{s} a_{ij} f\left(X_j^{(n)}, \bar{X}_j^{(n)}, Y_j^{(n)}, \bar{Y}_j^{(n)}\right), \qquad i = 1, 2, \ldots, s, \tag{3.3a}$$

$$\epsilon Y_i^{(n)} = \epsilon y_n + h\sum_{j=1}^{s} a_{ij} g\left(X_j^{(n)}, \bar{X}_j^{(n)}, Y_j^{(n)}, \bar{Y}_j^{(n)}\right), \qquad i = 1, 2, \ldots, s, \tag{3.3b}$$

$$x_{n+1} = x_n + h\sum_{i=1}^{s} b_i f\left(X_i^{(n)}, \bar{X}_i^{(n)}, Y_i^{(n)}, \bar{Y}_i^{(n)}\right), \tag{3.3c}$$

$$\epsilon y_{n+1} = \epsilon y_n + h\sum_{i=1}^{s} b_i g\left(X_i^{(n)}, \bar{X}_i^{(n)}, Y_i^{(n)}, \bar{Y}_i^{(n)}\right), \tag{3.3d}$$

where x_n and y_n are an approximation to the exact solutions $x(t_n)$ and $y(t_n)$, respectively. The arguments $\bar{X}_j^{(n)}$ and $\bar{Y}_j^{(n)}$ denote an approximation to $x(t_n + c_j h - \tau)$ and $y(t_n + c_j h - \tau)$, respectively, which are obtained by a specific interpolation procedure at the point $t = t_n + c_j h - \tau$ using values x_k and y_k, respectively, with $k \le n$.

We always assume that $0 \le c_i \le 1$ $(i = 1, \ldots, s)$.

Process (3.3) is defined completely by the Runge-Kutta method (A, b, c) and the interpolation procedure for $\bar{X}_j^{(n)}$ and $\bar{Y}_j^{(n)}$.

Let $\tau = (m - \delta)h$ with integer m and $\delta \in [0, 1)$, $c_j + \delta = l_j + \theta_j$ with integer l_j and $\theta_j \in [0, 1)$ for $1 \le j \le s$, then $0 \le l_j \le 1$. Let $\mu, \nu \ge 0$ be integers. We consider the following interpolation procedure:

$$\bar{X}_j^{(n)} = \begin{cases} \sum_{i=-\mu}^{\nu} L_i(\theta_j) x_{n-m+l_j+i}, & t_n + c_j h - \tau > 0, \quad \nu + 2 \le m, \\ \varphi(t_n + c_j h - \tau), & t_n + c_j h - \tau \le 0, \end{cases} \tag{3.4a}$$

$$\bar{Y}_j^{(n)} = \begin{cases} \sum_{i=-\mu}^{\nu} L_i(\theta_j) y_{n-m+l_j+i}, & t_n + c_j h - \tau > 0, \quad \nu + 2 \le m, \\ \psi(t_n + c_j h - \tau), & t_n + c_j h - \tau \le 0, \end{cases} \tag{3.4b}$$

where $x_k = \varphi(t_k)$ and $y_k = \psi(t_k)$ for $k \le 0$, $L_i(\theta)$ is defined by (2.4), and we assume $m \ge \nu + 2$ not only so as to guarantee that, in the interpolation procedure, no unknown values x_k and y_k with $k > n$ are used, but for simplicity in discussion of Part (c) in this section.

For any matrix H, let $\hat{H} = H \otimes I_M$, $\tilde{H} = H \otimes I_N$, where $\otimes$ denotes Kronecker product of two matrices, and I_l denotes an $l \times l$ unit matrix. Then process (3.3) can be written in the more compact form

$$X^{(n)} = e \otimes x_n + h\hat{A} F\left(X^{(n)}, \bar{X}^{(n)}, Y^{(n)}, \bar{Y}^{(n)}\right), \tag{3.5a}$$

$$\epsilon Y^{(n)} = \epsilon e \otimes y_n + h\tilde{A} G\left(X^{(n)}, \bar{X}^{(n)}, Y^{(n)}, \bar{Y}^{(n)}\right), \tag{3.5b}$$

$$x_{n+1} = x_n + h\hat{b}^{\top} F\left(X^{(n)}, \bar{X}^{(n)}, Y^{(n)}, \bar{Y}^{(n)}\right), \tag{3.5c}$$

$$\epsilon y_{n+1} = \epsilon y_n + h\tilde{b}^{\top} G\left(X^{(n)}, \bar{X}^{(n)}, Y^{(n)}, \bar{Y}^{(n)}\right), \tag{3.5d}$$

with the following notational conventions:

$$X^{(n)} = \begin{bmatrix} X_1^{(n)} \\ X_2^{(n)} \\ \vdots \\ X_s^{(n)} \end{bmatrix}, \quad \bar{X}^{(n)} = \begin{bmatrix} \bar{X}_1^{(n)} \\ \bar{X}_2^{(n)} \\ \vdots \\ \bar{X}_s^{(n)} \end{bmatrix}, \quad Y^{(n)} = \begin{bmatrix} Y_1^{(n)} \\ Y_2^{(n)} \\ \vdots \\ Y_s^{(n)} \end{bmatrix}, \quad \bar{Y}^{(n)} = \begin{bmatrix} \bar{Y}_1^{(n)} \\ \bar{Y}_2^{(n)} \\ \vdots \\ \bar{Y}_s^{(n)} \end{bmatrix},$$

$$F\left(X^{(n)}, \bar{X}^{(n)}, Y^{(n)}, \bar{Y}^{(n)}\right) = \begin{bmatrix} f\left(X_1^{(n)}, \bar{X}_1^{(n)}, Y_1^{(n)}, \bar{Y}_1^{(n)}\right) \\ f\left(X_2^{(n)}, \bar{X}_2^{(n)}, Y_2^{(n)}, \bar{Y}_2^{(n)}\right) \\ \vdots \\ f\left(X_s^{(n)}, \bar{X}_s^{(n)}, Y_s^{(n)}, \bar{Y}_s^{(n)}\right) \end{bmatrix},$$

$$G\left(X^{(n)}, \bar{X}^{(n)}, Y^{(n)}, \bar{Y}^{(n)}\right) = \begin{bmatrix} g\left(X_1^{(n)}, \bar{X}_1^{(n)}, Y_1^{(n)}, \bar{Y}_1^{(n)}\right) \\ g\left(X_2^{(n)}, \bar{X}_2^{(n)}, Y_2^{(n)}, \bar{Y}_2^{(n)}\right) \\ \vdots \\ g\left(X_s^{(n)}, \bar{X}_s^{(n)}, Y_s^{(n)}, \bar{Y}_s^{(n)}\right) \end{bmatrix},$$

and $e = [1, 1, \dots, 1]^\top \in R^s$.

It is well known that a method (A, b, c) is said to be algebraically stable if $B = \operatorname{diag}(b_1, b_2, \dots, b_s)$, $b_j \geq 0$, and the matrix

$$BA + A^\top B - bb^\top$$

is nonnegative definite (cf. [18]). A method is said to be diagonally stable if there exists an $s \times s$ diagonal matrix $Q > 0$ such that the matrix $QA + A^\top Q$ is positive definite (cf. [19]). A method is said to have stage order q if q is the largest integer such that the following simplifying conditions (cf. [20]) hold:

$$B(q): \; b^\top c^{j-1} = \frac{1}{j}, \qquad j = 1, 2, \dots, q,$$

$$C(q): \; Ac^{j-1} = \frac{c^j}{j}, \qquad j = 1, 2, \dots, q,$$

with $c^j = (c_1^j, c_2^j, \dots, c_s^j)^\top$.

In this section, the constants h_i, D, D_i, $\tilde{D}_i$, and D_{ij} used later are independent of the stiffness of the considered problem, and so are constants symbolized in the $\mathcal{O}(\cdots)$ terms.

In order to prove our results, we need the following lemmas [21], and suppose that ξ in the lemmas is a given real constant.

LEMMA 3.1. *Assume the method (A, b, c) is diagonally stable. Then there exist the positive constants γ_0, d_1, and d_2 such that for any given $h > 0$, $z \in E_\xi$, with $h\xi \leq \gamma_0$, the matrix $\hat{I}_s - h\hat{A}z$ is invertible and*

$$\left\| \left(\hat{I}_s - h\hat{A}z\right)^{-1} \right\| \leq d_1, \qquad \left\| h\hat{b}^\top z \left(\hat{I}_s - h\hat{A}z\right)^{-1} \right\| \leq d_2, \tag{3.6}$$

where $E_\xi = \{z : z = \operatorname{blockdiag}(z_1, z_2, \dots, z_s) \in R^{Ms \times Ms},\ z_i \in R^{M \times M},\ \mu(z_i) \leq \xi\}$, γ_0, d_1, and d_2 depend only on the method. Here $\mu(H)$ denotes the logarithmic norm of H.

LEMMA 3.2. *Assume the method (A, b, c) is algebraically and diagonally stable. Then there exist the positive constants γ_1, d_3 such that for any given $h > 0$, $z \in E_\xi$, with $h\xi \le \gamma_1$, the matrix $\hat{I}_s - h\hat{A}z$ is invertible and*

$$\left\| I_M + h\hat{b}^\top z \left(\hat{I}_s - h\hat{A}z\right)^{-1} \hat{e} \right\| \le 1 + d_3 h\xi\delta(\xi), \tag{3.7}$$

where $\delta(\xi) = 1$ for $\xi > 0$ and $\delta(\xi) = 0$ for $\xi \le 0$, γ_1 and d_3 depend only on the method.

THEOREM 3.3. *Suppose that an algebraically and diagonally stable Runge-Kutta method (A, b, c) is of stage order $q \ge 1$ and satisfies $|\eta| < 1$; the eigenvalues of A have positive real part. If problem (1.1) satisfies (3.1), then the global error of the method with interpolation procedure (3.4) satisfies, for $\epsilon \le D_0 h^2$, $h \le h_2$, and $nh \le T$,*

$$\|x_n - x(t_n)\| + \|y_n - y(t_n)\| \le D\left(\|x_0 - x(t_0)\| + \|y_0 - y(t_0)\| + h^q + h^{\mu+\nu+1}\right), \tag{3.8}$$

where

$$\eta = 1 - b^\top A^{-1} e.$$

PROOF.
(a) First we derive recursive estimates for the global error. Let $\Delta x_n = x(t_n) - x_n$, $\Delta y_n = y(t_n) - y_n$,

$$\begin{aligned}
X(t) &= \left(x(t+c_1h)^\top, x(t+c_2h)^\top, \dots, x(t+c_sh)^\top\right)^\top, \\
Y(t) &= \left(y(t+c_1h)^\top, y(t+c_2h)^\top, \dots, y(t+c_sh)^\top\right)^\top, \\
F(X(t), X(t-\tau), &Y(t), Y(t-\tau)) \\
&= \Big(f\left(x(t+c_1h), x(t+c_1h-\tau), y(t+c_1h), y(t+c_1h-\tau)\right)^\top, \dots, \\
&\qquad f\left(x(t+c_sh), x(t+c_sh-\tau), y(t+c_sh), y(t+c_sh-\tau)\right)^\top\Big)^\top, \\
G(X(t), X(t-\tau), &Y(t), Y(t-\tau)) \\
&= \Big(g\left(x(t+c_1h), x(t+c_1h-\tau), y(t+c_1h), y(t+c_1h-\tau)\right)^\top, \dots, \\
&\qquad g\left(x(t+c_sh), x(t+c_sh-\tau), y(t+c_sh), y(t+c_sh-\tau)\right)^\top\Big)^\top, \\
\Delta X^{(n)} &= X(t_n) - X^{(n)}, \qquad \Delta Y^{(n)} = Y(t_n) - Y^{(n)}, \\
\Delta \bar{X}^{(n)} &= X(t_n - \tau) - \bar{X}^{(n)}, \qquad \Delta \bar{Y}^{(n)} = Y(t_n - \tau) - \bar{Y}^{(n)}, \\
\Delta F^{(n)} &= F(X(t_n), X(t_n-\tau), Y(t_n), Y(t_n-\tau)) - F\left(X^{(n)}, \bar{X}^{(n)}, Y^{(n)}, \bar{Y}^{(n)}\right), \\
\Delta G^{(n)} &= G(X(t_n), X(t_n-\tau), Y(t_n), Y(t_n-\tau)) - G\left(X^{(n)}, \bar{X}^{(n)}, Y^{(n)}, \bar{Y}^{(n)}\right).
\end{aligned}$$

Conditions $B(q)$ and $C(q)$ imply

$$X(t_n) = e \otimes x(t_n) + h\hat{A}F\left(X(t_n), X(t_n-\tau), Y(t_n), Y(t_n-\tau)\right) + \mathcal{O}\left(h^{q+1}\right), \tag{3.9a}$$

$$Y(t_n) = e \otimes y(t_n) + \frac{h}{\epsilon}\tilde{A}G\left(X(t_n), X(t_n-\tau), Y(t_n), Y(t_n-\tau)\right) + \mathcal{O}\left(h^{q+1}\right), \tag{3.9b}$$

$$x(t_{n+1}) = x(t_n) + h\hat{b}^\top F\left(X(t_n), X(t_n-\tau), Y(t_n), Y(t_n-\tau)\right) + \mathcal{O}\left(h^{q+1}\right), \tag{3.9c}$$

$$y(t_{n+1}) = y(t_n) + \frac{h}{\epsilon}\tilde{b}^\top G\left(X(t_n), X(t_n-\tau), Y(t_n), Y(t_n-\tau)\right) + \mathcal{O}\left(h^{q+1}\right). \tag{3.9d}$$

Subtractions of (3.5a) from (3.9a), (3.5b) from (3.9b), (3.5c) from (3.9c), and (3.5d) from (3.9d) yield, for $n \geq 0$,

$$\Delta X^{(n)} = e \otimes \Delta x_n + h\hat{A}\Delta F^{(n)} + \mathcal{O}\left(h^{q+1}\right), \tag{3.10a}$$

$$\Delta Y^{(n)} = e \otimes \Delta y_n + \frac{h}{\epsilon}\tilde{A}\Delta G^{(n)} + \mathcal{O}\left(h^{q+1}\right), \tag{3.10b}$$

$$\Delta x_{n+1} = \Delta x_n + h\hat{b}^{\top}\Delta F^{(n)} + \mathcal{O}\left(h^{q+1}\right), \tag{3.10c}$$

$$\Delta y_{n+1} = \Delta y_n + \frac{h}{\epsilon}\tilde{b}^{\top}\Delta G^{(n)} + \mathcal{O}\left(h^{q+1}\right). \tag{3.10d}$$

Since diagonal stability of the method implies that A is invertible (cf. [19]), we can compute $\Delta F^{(n)}$ and $\Delta G^{(n)}$, from (3.10a) and (3.10b),

$$\Delta F^{(n)} = \frac{1}{h}\hat{A}^{-1}\left(\Delta X^{(n)} - e \otimes \Delta x_n + \mathcal{O}\left(h^{q+1}\right)\right), \tag{3.11a}$$

$$\Delta G^{(n)} = \frac{\epsilon}{h}\tilde{A}^{-1}\left(\Delta Y^{(n)} - e \otimes \Delta y_n + \mathcal{O}\left(h^{q+1}\right)\right). \tag{3.11b}$$

It follows from (3.10) and (3.11) that

$$\Delta x_{n+1} = \eta\Delta x_n + \hat{b}^{\top}\hat{A}^{-1}\Delta X^{(n)} + \mathcal{O}\left(h^{q+1}\right), \tag{3.12a}$$

$$\Delta y_{n+1} = \eta\Delta y_n + \tilde{b}^{\top}\tilde{A}^{-1}\Delta Y^{(n)} + \mathcal{O}\left(h^{q+1}\right). \tag{3.12b}$$

On the other hand,

$$\Delta F^{(n)} = F_X\Delta X^{(n)} + F_{\bar{X}}\Delta\bar{X}^{(n)} + F_Y\Delta Y^{(n)} + F_{\bar{Y}}\Delta\bar{Y}^{(n)}, \tag{3.13a}$$

$$\Delta G^{(n)} = G_X\Delta X^{(n)} + G_{\bar{X}}\Delta\bar{X}^{(n)} + G_Y\Delta Y^{(n)} + G_{\bar{Y}}\Delta\bar{Y}^{(n)}, \tag{3.13b}$$

where

$$\begin{aligned}
F_X = \operatorname{blockdiag}\Bigg(&\int_0^1 f_x\left(X_1^{(n)} + \theta\left(x(t_n + c_1h) - X_1^{(n)}\right),\right.\\
&\left. x(t_n + c_1h - \tau), y(t_n + c_1h), y(t_n + c_1h - \tau)\right) d\theta,\\
&\dots, \int_0^1 f_x\left(X_s^{(n)} + \theta\left(x(t_n + c_sh) - X_s^{(n)}\right),\right.\\
&\left. x(t_n + c_sh - \tau), y(t_n + c_sh), y(t_n + c_sh - \tau)\right) d\theta\Bigg),\\
F_{\bar{X}} = \operatorname{blockdiag}\Bigg(&\int_0^1 f_u\left(X_1^{(n)}, \bar{X}_1^{(n)} + \theta\left(x(t_n + c_1h - \tau) - \bar{X}_1^{(n)}\right),\right.\\
&\left. y(t_n + c_1h), y(t_n + c_1h - \tau)\right) d\theta,\\
&\dots, \int_0^1 f_u\left(X_s^{(n)}, \bar{X}_s^{(n)} + \theta\left(x(t_n + c_sh - \tau) - \bar{X}_s^{(n)}\right), y(t_n + c_sh), y(t_n + c_sh - \tau)\right) d\theta\Bigg),\\
F_Y = \operatorname{blockdiag}\Bigg(&\int_0^1 f_y\left(X_1^{(n)}, \bar{X}_1^{(n)}, Y_1^{(n)} + \theta\left(y(t_n + c_1h) - Y_1^{(n)}\right), y(t_n + c_1h - \tau)\right) d\theta,\\
&\dots, \int_0^1 f_y\left(X_s^{(n)}, \bar{X}_s^{(n)}, Y_s^{(n)} + \theta\left(y(t_n + c_sh) - Y_s^{(n)}\right), y(t_n + c_sh - \tau)\right) d\theta\Bigg),\\
F_{\bar{Y}} = \operatorname{blockdiag}\Bigg(&\int_0^1 f_v\left(X_1^{(n)}, \bar{X}_1^{(n)}, Y_1^{(n)}, \bar{Y}_1^{(n)} + \theta\left(y(t_n + c_1h - \tau) - \bar{Y}_1^{(n)}\right)\right) d\theta,\\
&\dots, \int_0^1 f_v\left(X_s^{(n)}, \bar{X}_s^{(n)}, Y_s^{(n)}, \bar{Y}_s^{(n)} + \theta\left(y(t_n + c_sh - \tau) - \bar{Y}_s^{(n)}\right)\right) d\theta\Bigg),
\end{aligned}$$

and likewise for G_X, $G_{\bar{X}}$, G_Y, and $G_{\bar{Y}}$, here

$$f_x = \frac{\partial f(x,u,y,v)}{\partial x}, \quad f_u = \frac{\partial f(x,u,y,v)}{\partial u}, \quad f_y = \frac{\partial f(x,u,y,v)}{\partial y}, \quad f_v = \frac{\partial f(x,u,y,v)}{\partial v},$$

and similarly for g_x, g_u, g_y, and g_v. From (3.10b) and (3.13b), we can obtain

$$\begin{aligned} \Delta Y^{(n)} &= \frac{h}{\epsilon}\left(\tilde{I}_s - \frac{h}{\epsilon}\tilde{A}G_Y\right)^{-1} \\ &\times \left(\frac{\epsilon}{h} e \otimes \Delta y_n + \tilde{A}G_X \Delta X^{(n)} + \tilde{A}G_{\bar{X}}\Delta\bar{X}^{(n)} + \tilde{A}G_{\bar{Y}}\Delta\bar{Y}^{(n)} + \mathcal{O}\left(\epsilon h^q\right)\right). \end{aligned} \tag{3.14}$$

Inserting (3.13a) and (3.14) into (3.10a) gives

$$\begin{aligned} &\left(\hat{I}_s - h\hat{A}F_X\right)\Delta X^{(n)} \\ &= h\hat{A}F_Y \frac{h}{\epsilon}\left(\tilde{I}_s - \frac{h}{\epsilon}\tilde{A}G_Y\right)^{-1} \tilde{A}G_X\Delta X^{(n)} + e\otimes \Delta x_n + h\hat{A}\left(F_{\bar{X}}\Delta\bar{X}^{(n)} + F_{\bar{Y}}\Delta\bar{Y}^{(n)}\right) \\ &+ h\hat{A}F_Y \frac{h}{\epsilon}\left(\tilde{I}_s - \frac{h}{\epsilon}\tilde{A}G_Y\right)^{-1}\left(\frac{\epsilon}{h}e\otimes\Delta y_n + \tilde{A}G_{\bar{X}}\Delta\bar{X}^{(n)} + \tilde{A}G_{\bar{Y}}\Delta\bar{Y}^{(n)} + \mathcal{O}\left(\epsilon h^q\right)\right) \\ &+ \mathcal{O}\left(h^{q+1}\right). \end{aligned} \tag{3.15}$$

Using (3.1b), diagonal stability, and the fact the eigenvalues of A have positive real part, by means of the technique in [21], we have, for any given $h > 0$,

$$\left\| \frac{h}{\epsilon}\left(\tilde{I}_s - \frac{h}{\epsilon}\tilde{A}G_Y\right)^{-1} \right\| \le D_2. \tag{3.16}$$

It follows from (3.14)–(3.16) and Lemma 3.1 that, for $h \le h_2$,

$$\left\|\Delta X^{(n)}\right\| \le D_3\left(\|\Delta x_n\| + \epsilon\|\Delta y_n\| + h\left\|\Delta\bar{X}^{(n)}\right\| + h\left\|\Delta\bar{Y}^{(n)}\right\| + \epsilon h^{q+1} + h^{q+1}\right), \tag{3.17a}$$

$$\left\|\Delta Y^{(n)}\right\| \le D_4\left(\|\Delta x_n\| + \tilde{\epsilon}\|\Delta y_n\| + \left\|\Delta\bar{X}^{(n)}\right\| + \left\|\Delta\bar{Y}^{(n)}\right\| + \epsilon h^{q} + h^{q+1}\right), \tag{3.17b}$$

where $\tilde{\epsilon} = \epsilon(1 + 1/h)$. By (3.10c) and (3.13a), we have

$$\Delta x_{n+1} = \Delta x_n + h\hat{b}^\top F_X \Delta X^{(n)} + \sigma_n, \tag{3.18}$$

where

$$\|\sigma_n\| \le D_5\left(h\left\|\Delta Y^{(n)}\right\| + h\left\|\Delta\bar{X}^{(n)}\right\| + h\left\|\Delta\bar{Y}^{(n)}\right\| + h^{q+1}\right).$$

From (3.15)–(3.18) and Lemmas 3.1 and 3.2, we easily obtain, for $h \le h_2$,

$$\begin{aligned} &\|\Delta x_{n+1}\| \le (1 + \mathcal{O}(h))\|\Delta x_n\| \\ &+ D_6\left(\epsilon\|\Delta y_n\| + h\left\|\Delta\bar{X}^{(n)}\right\| + h\left\|\Delta\bar{Y}^{(n)}\right\| + \epsilon h^{q+1} + h^{q+1}\right). \end{aligned} \tag{3.19a}$$

By (3.17b) and (3.12b), we estimate

$$\|\Delta y_{n+1}\| \le (\eta + \mathcal{O}(\tilde{\epsilon}))\|\Delta y_n\| + D_7\left(\|\Delta x_n\| + \left\|\Delta\bar{X}^{(n)}\right\| + \left\|\Delta\bar{Y}^{(n)}\right\| + \epsilon h^q + h^{q+1}\right). \tag{3.19b}$$

On the other hand, for the interpolation procedure (3.4), we have

$$\begin{aligned} \left\|\bar{X}_j^{(k)} - x\left(t_k + c_j h - \tau\right)\right\| &\le \left\| \sum_{i=-\mu}^{\nu} L_i\left(\theta_j\right)\left(x_{k-m+l_j+i} - x\left(t_{k-m+l_j+i}\right)\right)\right\| \\ &+ \left\| \sum_{i=-\mu}^{\nu} L_i\left(\theta_j\right) x\left(t_{k-m+l_j+i}\right) - x\left(t_k + c_j h - \tau\right)\right\|. \end{aligned}$$

In analogy to (2.11), we have the following estimate:

$$\left\|\Delta \bar{X}^{(k)}\right\| \leq D_8 \left(\sum_{i=-\mu}^{\nu} \|\Delta x_{k+l_j-m+i}\| + h^{\mu+\nu+1} \right), \tag{3.20a}$$

$$\left\|\Delta \bar{Y}^{(k)}\right\| \leq D_8 \left(\sum_{i=-\mu}^{\nu} \|\Delta y_{k+l_j-m+i}\| + h^{\mu+\nu+1} \right). \tag{3.20b}$$

It follows from (3.19) and (3.20) that

$$\|\Delta x_n\| \leq \|\Delta x_0\| + D_9 \sum_{i=0}^{n-1} (h\|\Delta x_i\| + \epsilon \|\Delta y_i\|) + D_9 \sum_{i=0}^{n-1} d_i + D_9 \sum_{i=0}^{n-1} \tilde{d}_i, \tag{3.21a}$$

$$\begin{aligned} \|\Delta y_n\| \leq \eta^n \|\Delta y_0\| &+ D_9 \sum_{i=0}^{n-1} \eta^{n-1-i} (\|\Delta x_i\| + \tilde{\epsilon} \|\Delta y_i\|) \\ &+ D_9 \sum_{i=0}^{n-1} \eta^{n-1-i} e_i + D_9 \sum_{i=0}^{n-1} \eta^{n-1-i} \tilde{e}_i, \end{aligned} \tag{3.21b}$$

where

$$\begin{aligned} d_i &= \mathcal{O}\left(h^{q+1} + \epsilon h^{q+1}\right), & e_i &= \mathcal{O}\left(h^{q+1} + \epsilon h^q\right), \\ \tilde{d}_i &= \begin{cases} 0, & i < m - \nu - 1, \\ h \|\Delta y_{i-m+\nu+1}\| + h^{\mu+\nu+2}, & i \geq m - \nu - 1, \end{cases} \\ \tilde{e}_i &= \begin{cases} 0, & i < m - \nu - 1, \\ \|\Delta x_{i-m+\nu+1}\| + \|\Delta y_{i-m+\nu+1}\| + h^{\mu+\nu+1}, & i \geq m - \nu - 1. \end{cases} \end{aligned}$$

(b) We define sequences $\{u_n\}$ and $\{v_n\}$ $(n \geq 1)$ by

$$\begin{aligned} u_n &= \|\Delta x_0\| + D_9 \sum_{i=0}^{n-1} (h u_i + \epsilon v_i) + D_9 \sum_{i=0}^{n-1} d_i + D_9 \sum_{i=0}^{n-1} \tilde{d}_i, \\ v_n &= \eta^n \|\Delta y_0\| + D_9 \sum_{i=0}^{n-1} \eta^{n-1-i} (u_i + \tilde{\epsilon} v_i) + D_9 \sum_{i=0}^{n-1} \eta^{n-1-i} e_i + D_9 \sum_{i=0}^{n-1} \eta^{n-1-i} \tilde{e}_i. \end{aligned} \tag{3.22}$$

By a similar process of Part (b) in the previous section, and noting the fact $(\eta + \mathcal{O}(\tilde{\epsilon}))^n = \mathcal{O}(\eta^n) + \mathcal{O}(\tilde{\epsilon})$ for $\epsilon \leq D_0 h^2$ and $nh \leq T$, we easily show from (3.22)

$$\begin{aligned} u_n &\leq D_{10} \left(u_0 + \epsilon v_0 + \sum_{i=0}^{n-1} \left(\hat{d}_i + \epsilon \hat{e}_i \right) \right), \\ v_n &\leq D_{10} \left(u_0 + (\eta^n + \tilde{\epsilon}) v_0 + \sum_{i=0}^{n-1} \left(\hat{d}_i + \left(\eta^{n-1-i} + \tilde{\epsilon}\right) \hat{e}_i \right) \right), \end{aligned} \tag{3.23}$$

for $n \geq 1$ and $h \leq h_2$, where $\hat{d}_i = \mathcal{O}(d_i + \tilde{d}_i)$, $\hat{e}_i = \mathcal{O}(e_i + \tilde{e}_i)$.

(c) Our next aim is to obtain the global error estimate in successive subintervals.

For $n \in [1, m-\nu-1]$, $\tilde{d}_{n-1} = 0$, $\tilde{e}_{n-1} = 0$, it follows from $\|\Delta x_n\| \leq u_n$, $\|\Delta y_n\| \leq v_n$, and (3.23) that

$$\begin{aligned} \|x_n - x(t_n)\| &\leq \tilde{D}_1 \left(\|x_0 - x(t_0)\| + \epsilon \|y_0 - y(t_0)\| + h^q + \epsilon h^q\right), \\ \|y_n - y(t_n)\| &\leq \tilde{D}_1 \left(\|x_0 - x(t_0)\| + (\eta^n + \tilde{\epsilon}) \|y_0 - y(t_0)\| + h^q + \epsilon h^q\right). \end{aligned} \tag{3.24}$$

For $n \in [m-\nu, 2m-2\nu-1]$, $\tilde{d}_{n-1} = h\|\Delta y_{n-m+\nu}\|+h^{\mu+\nu+2}$, $\tilde{e}_{n-1} = \|\Delta x_{n-m+\nu}\|+\|\Delta y_{n-m+\nu}\|+h^{\mu+\nu+1}$, by (3.23) and (3.24), we get

$$\begin{aligned} \|x_n - x(t_n)\| &\le \tilde{D}_2 \left(\|x_0 - x(t_0)\| + \|y_0 - y(t_0)\| + h^q + \epsilon h^q + h^{\mu+\nu+1}\right), \\ \|y_n - y(t_n)\| &\le \tilde{D}_2 \left(\|x_0 - x(t_0)\| + \|y_0 - y(t_0)\| + h^q + \epsilon h^q + h^{\mu+\nu+1}\right). \end{aligned} \tag{3.25}$$

Generally, for $n \in [i(m-\nu), (i+1)(m-\nu)-1]$, by induction, (3.25) is also valid with $\tilde{D}_2$ replaced by $\tilde{D}_{i+1}$.

Similar to the process of Part (c) in the previous section, repeating the above process n_1 times, where n_1 is independent of h, we can obtain the global error $\|x_n - x(t_n)\| + \|y_n - y(t_n)\|$ for all n ($nh \le T$). Let

$$D = 2\max\left(\left(1 + D_0 h_0 + D_0 h_0^2\right)\tilde{D}_1, \tilde{D}_2, \dots, \tilde{D}_{n_1}\right).$$

The proof is completed.

REMARK 3.4. It is well known that s-stage Radau IA and Radau IIA methods are all algebraically and diagonally stable and satisfy $1 - b^\top A^{-1} e = 0$ (cf. [12,19]). We have verified that the eigenvalues of A of the methods have positive real part for $s \le 5$. We note that s-stage Radau IA method is of stage order $p = s - 1$ and Radau IIA method $p = s$. Hence, Radau IA and Radau IIA methods all satisfy the assumptions in Theorem 3.3, and $p = s-1$, s ($s \le 5$), respectively.

We also can verify that the two-stage Lobatto IIIC method satisfies the assumptions in Theorem 3.3.

REMARK 3.5. System (3.5a),(3.5b) constitutes a nonlinear system with respect to $X^{(n)}$ and $Y^{(n)}$. The Jacobian of the system is of the form

$$\begin{pmatrix} \hat{I}_s - h\hat{A}Z_X & \mathcal{O}(h) \\ \mathcal{O}(1) & \frac{\epsilon}{h}\tilde{I}_s - \tilde{A}Z_Y \end{pmatrix}, \tag{3.26}$$

where

$$\begin{aligned} Z_X &= \operatorname{blockdiag}\left(f_x\left(X_1^{(n)}, \bar{X}_1^{(n)}, Y_1^{(n)}, \bar{Y}_1^{(n)}\right), \dots, f_x\left(X_s^{(n)}, \bar{X}_s^{(n)}, Y_s^{(n)}, \bar{Y}_s^{(n)}\right)\right), \\ Z_Y &= \operatorname{blockdiag}\left(g_y\left(X_1^{(n)}, \bar{X}_1^{(n)}, Y_1^{(n)}, \bar{Y}_1^{(n)}\right), \dots, g_y\left(X_s^{(n)}, \bar{X}_s^{(n)}, Y_s^{(n)}, \bar{Y}_s^{(n)}\right)\right). \end{aligned}$$

By Lemma 3.1 and condition (3.1a), we have, for $h \le h_2$,

$$\left\|\left(\hat{I}_s - h\hat{A}Z_X\right)^{-1}\right\| \le d_1.$$

We can show as (3.16), for any given $h > 0$,

$$\left\|\left(\frac{\epsilon}{h}\tilde{I}_s - \tilde{A}Z_Y\right)^{-1}\right\| \le d_4.$$

Hence, the nonlinear system (3.5a),(3.5b) possesses a locally unique solution.

REMARK 3.6. The result (Theorem 3.3) can be considered as an extension of that obtained by Xiao (cf. [16]) for the case of singular perturbation problems without delay.

4. NUMERICAL EXAMPLES

In order to illustrate the results obtained in Sections 2 and 3, we consider the following linear and nonlinear problems (4.1) and (4.2) whose exact solutions are given. Though (4.1) and (4.2) are all nonautonomous, we can transform them into autonomous form (1.1) by adding t to the variable x as

$$\begin{bmatrix} t \\ x \end{bmatrix}' = \begin{pmatrix} 1 \\ f(t, x(t), x(t-\tau), y(t), y(t-\tau)) \end{pmatrix}.$$

For the following given a_1 and a_2, we can easily verify conditions (2.1) and (3.1). We apply the two-step BDF (BDF2) and the two-stage Radau IIA method (RadauIIA2) to the problems, respectively. Noting that the BDF2 is of order $p = 2$ and the Radau IIA2 is of stage order $q = 2$, according to Theorems 2.1 and 3.3, we select linear interpolation procedure (i.e., $\mu = 0$, $\nu = 1$) for BDF2 and RadauIIA2. Moreover, in order to observe whether the order of convergence of the adapting RadauIIA2 increases when the order of the interpolation procedure increases, we also consider quadratic interpolation procedure (i.e., $\mu = -1$, $\nu = 1$ for $0 < \theta_j \leq 0.5$ or $\mu = 0$, $\nu = 2$ for $0.5 < \theta_j < 1$) for RadauIIA2. We denote BDF2 and RadauIIA2 with linear interpolation procedure by BDF2-1 and RadauIIA2-1, respectively, RadauIIA2 with quadratic interpolation procedure by RadauIIA2-2. Let err_x and err_y be the global errors of x- and y-components at $T = 10$, respectively, $\mathrm{err} = \mathrm{err}_x + \mathrm{err}_y$. Let $\epsilon = 10^{-6}$. The numerical results (i.e., err) are listed in Tables 1 and 2. For $a_1 = -5$ in problem (4.1), the result of RadauIIA2-2 is better than that of RadauIIA2-1, but for $a_1 = -1000$, the results are not improved apparently for RadauIIA2-2. For $a_2 = -1$ in problem (4.2), the result of RadauIIA2-2 is better than that of RadauIIA2-1, but for $a_2 = -1000$, no accuracy increase is observed for RadauIIA2-2. Therefore, for multiple stiff problems, it is sufficient to require that the order of the interpolation procedure matches the stage order of the method in Theorem 3.3; i.e., higher order of the interpolation is not necessary. It is clear that the results given by Tables 1 and 2 confirm Theorems 2.1 and 3.3.

Table 1. Numerical results for problem (4.1).

	BDF2-1	RadauIIA2-1		RadauIIA2-2	
a_1	−5	−5	−1000	−5	−1000
$h = 0.2$	1.7E−1	2.1E−1	3.0E−6	1.4E−2	2.5E−6
$h = 0.1$	4.5E−2	5.5E−2	8.3E−7	1.7E−3	6.0E−7
$h = 0.05$	1.2E−2	1.4E−2	2.5E−7	2.2E−4	1.4E−7

Table 2. Numerical results for problem (4.2).

	BDF2-1	RadauIIA2-1		RadauIIA2-2	
a_2	−1	−1	−1000	−1	−1000
$h = 0.2$	7.0E−4	2.5E−4	1.8E−8	1.6E−5	2.0E−8
$h = 0.1$	1.8E−4	6.5E−5	4.0E−9	2.0E−6	4.7E−9
$h = 0.05$	4.5E−5	1.7E−5	7.9E−10	2.6E−7	1.1E−9

EXAMPLE 4.1. Consider the linear problem

$$\begin{aligned} x'(t) &= 2x(t-1) + y(t-1) + a_1 x(t) + y(t) + r_x(t), & t > 0, \\ \epsilon y'(t) &= x(t-1) - y(t-1) + 3x(t) - y(t) + r_y(t), & t > 0, \\ x(t) &= 1 + 10e^{-(1/2)(t+1)} + 5e^{-(1/\epsilon)(t+1)}, & t \le 0, \\ y(t) &= -1 - 9e^{-(1/2)(t+1)} + 4e^{-(1/\epsilon)(t+1)}, & t \le 0, \end{aligned} \tag{4.1}$$

where a_1 is a parameter, and

$$r_x(t) = (4 - 10a_1)e^{-(1/2)(t+1)} - \left(\frac{5}{\epsilon} + 5a_1 + 4\right) e^{-(1/\epsilon)(t+1)} - 11e^{-(1/2)t} - 14e^{-(1/\epsilon)t} - a_1,$$
$$r_y(t) = \left(\frac{9}{2}\epsilon - 39\right) e^{-(1/2)(t+1)} - 15e^{-(1/\epsilon)(t+1)} - 19e^{-(1/2)t} - e^{-(1/\epsilon)t} - 6.$$

Problem (4.1) has the exact solution $x(t) = 1 + 10e^{-(1/2)(t+1)} + 5e^{-(1/\epsilon)(t+1)}$, $y(t) = -1 - 9e^{-(1/2)(t+1)} + 4e^{-(1/\epsilon)(t+1)}$, $t > 0$. $x(10) = 1.040867714384641$, $y(10) = -1.036780942946177$.

EXAMPLE 4.2. Consider the nonlinear problem

$$\begin{aligned} x'(t) &= x(t-1)y(t-1) + a_2 x(t) + 2y^2(t) + R_x(t), & t > 0, \\ \epsilon y'(t) &= x(t-1) - y(t-1) - (1 + x(t))y(t) + R_y(t), & t > 0, \\ x(t) &= e^{-0.5t} + e^{-0.2t}, & t \le 0, \\ y(t) &= -e^{-0.5t} + e^{-0.2t}, & t \le 0, \end{aligned} \tag{4.2}$$

where a_2 is a parameter, and

$$R_x(t) = -(0.5 + a_2)e^{-0.5t} - (0.2 + a_2)e^{-0.2t} + e^{-(t-1)} - e^{-0.4(t-1)} - 2e^{-t} - 2e^{-0.4t} + 4e^{-0.7t},$$
$$R_y(t) = (0.5\epsilon - 1)e^{-0.5t} + (1 - 0.2\epsilon)e^{-0.2t} - 2e^{-0.5(t-1)} - e^{-t} + e^{-0.4t}.$$

Problem (4.2) has the exact solution $x(t) = e^{-0.5t} + e^{-0.2t}$, $y(t) = -e^{-0.5t} + e^{-0.2t}$, $t > 0$. $x(10) = 0.1420732302356982$, $y(10) = 0.1285973362375272$.

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