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Vector measures and Mackey topologies

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Abstract

Let Σ be a σ-algebra of subsets of a non-empty set Ω. Let $B(\Sigma)$ be the Banach lattice of all bounded Σ-measurable real-valued functions defined on Ω, equipped with the natural Mackey topology $\tau(B(\Sigma), ca(\Sigma))$. We study $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous linear operators from $B(\Sigma)$ to a quasicomplete locally convex space (E, ξ). A generalized Nikodym convergence theorem and a Vitali–Hahn–Saks type theorem for operators on $B(\Sigma)$ are obtained. It is shown that the space $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ has the strict Dunford–Pettis property. Moreover, a Yosida–Hewitt type decomposition for weakly compact operators on $B(\Sigma)$ is given.

Keywords: Bounded measurable functions; σ-smooth operators; Mackey topologies; Strongly Mackey space; Weakly compact operators; Countably additive measures; Strict Dunford–Pettis property

1. Introduction and terminology

Properties of bounded linear operators from the space $B(\Sigma)$ to a Banach space E can be expressed in terms of the properties of their representing vector measures (see [6, Theorem 2.2], [7, Theorem 1, p. 148], [11, Corollary 12], [16, Theorem 10], [18, Theorem 2.1]). In this paper we study linear operators from $B(\Sigma)$ to a quasicomplete locally convex space (E, ξ). In particular, we obtain a Vitali–Hahn–Saks type theorem and a Nikodym convergence type theorem for $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous linear operators $T : B(\Sigma) \to E$. It is shown that the space

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$(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ has the strict Dunford–Pettis property. Moreover, a Yosida–Hewitt type decomposition for weakly compact operators on $B(\Sigma)$ is given.

For terminology concerning vector lattices we refer the reader to [2,3,1]. We denote by $\sigma(L, K)$, $\tau(L, K)$ and $\beta(L, K)$ the weak topology, the Mackey topology and the strong topology on L with respect to the dual pair $\langle L, K\rangle$. We assume that (E, ξ) is a locally convex Hausdorff space (for short, lcHs). By $(E, \xi)'$ or E'_ξ we denote the topological dual of (E, ξ). Recall that (E, ξ) is a strongly Mackey space if every relatively countably $\sigma(E'_\xi, E)$-compact subset of E'_ξ is ξ-equicontinuous. By E''_ξ we denote the bidual of (E, ξ), i.e., $E''_\xi = (E'_\xi, \beta(E'_\xi, E))'$.

Let Σ be a σ-algebra of subsets of a non-empty set Ω. By $ca(\Sigma, E)$ we denote the space of all ξ-countably additive vector measures $m : \Omega \to E$, and we will write $ca(\Sigma)$ if $E = \mathbb{R}$. By $\mathcal{S}(\Sigma)$ we denote the space of all real-valued Σ-simple functions defined on Ω. Then $\mathcal{S}(\Sigma)$ can be endowed with the (locally convex) universal measure topology τ of Graves [10], that is, τ is the coarsest locally convex topology on $\mathcal{S}(\Sigma)$ such that the integration map $T_m : \mathcal{S}(\Sigma) \ni s \mapsto \int_\Omega s\,dm \in E$ is continuous for every locally convex space (E, ξ) and every $m \in ca(\Sigma, E)$ (see [10, p. 5]). Let $(L(\Sigma), \hat{\tau})$ stand for the completion of $(\mathcal{S}(\Sigma), \tau)$. It is known that both $(\mathcal{S}(\Sigma), \tau)$ and $(L(\Sigma), \hat{\tau})$ are strongly Mackey spaces (see [10, Corollaries 11.7 and 11.8]). It follows that $\tau = \tau(\mathcal{S}(\Sigma), ca(\Sigma))$ and $\hat{\tau} = \tau(L(\Sigma), ca(\Sigma))$ (see [10,11]). Moreover, if (E, ξ) is complete in its Mackey topology, then for each $m \in ca(\Sigma, E)$, the integration map T_m can be uniquely extended to a $(\hat{\tau}, \xi)$-continuous map $\widetilde{T}_m : L(\Sigma) \to E$ (see [11]).

Let $B(\Sigma)$ denote the Dedekind σ-complete Banach lattice of all bounded Σ-measurable functions $u : \Omega \to \mathbb{R}$, provided with the uniform norm $\|\cdot\|$. Then $B(\Sigma)$ is the $\|\cdot\|$-closure of $\mathcal{S}(\Sigma)$, so $\mathcal{S}(\Sigma) \subset B(\Sigma) \subset L(\Sigma)$ and the restriction $\hat{\tau}$ from $L(\Sigma)$ to $B(\Sigma)$ coincides with the Mackey topology $\tau(B(\Sigma), ca(\Sigma))$ (see [11, p. 199]). Moreover, $(B(\Sigma), \hat{\tau}|_{B(\Sigma)})$ is a strongly Mackey space (see [10, Corollary 11.8]). Note that the topology γ_1 on $B(\Sigma)$ studied by Khurana [12], coincides with the topology $\hat{\tau}|_{B(\Sigma)} (= \tau(B(\Sigma), ca(\Sigma)))$ (see [12, Theorem 2, Corollary 6]).

Denote by $ba(\Sigma)$ the Banach lattice of all bounded finitely additive measures $\nu : \Sigma \to \mathbb{R}$ with the norm $\|\nu\| = |\nu|(\Omega)$, where $|\nu|(A)$ denotes the variation of ν on $A \in \Sigma$. It is well known that the Banach dual $B(\Sigma)^*$ of $B(\Sigma)$ can be identified with $ba(\Sigma)$ through the mapping $ba(\Sigma) \ni \nu \mapsto \Phi_\nu \in B(\Sigma)^*$, where

$$\Phi_\nu(u) = \int_\Omega u\,d\nu \quad \text{for } u \in B(\Sigma).$$

Then $\|\Phi_\nu\| = |\nu|(\Omega)$ (see [1, Theorem 13.4]). The σ-order continuous dual $B(\Sigma)^*_c$ of $B(\Sigma)$ is a band of $B(\Sigma)^*$ (separating the points of $B(\Sigma)$) and $B(\Sigma)^*_c$ can be identified with $ca(\Sigma)$ (see [1, Theorem 13.5]). Hence

$$(B(\Sigma), \hat{\tau}|_{B(\Sigma)})' = (B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))' = B(\Sigma)^*_c.$$

Moreover, it is well known that $\tau(B(\Sigma), ca(\Sigma))$ is a locally solid σ-Lebesgue topology on $B(\Sigma)$ (see [3, Ex. 18, p. 178], [12, Theorem 3]).

2. σ-smooth operators on $B(\Sigma)$

In this section we study $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous linear operators from $B(\Sigma)$ to a locally convex Hausdorff space (E, ξ). Recall that a sequence (u_n) in $B(\Sigma)$ is order convergent to $u \in B(\Sigma)$ (in symbols, $u_n \xrightarrow{(o)} u$) if there is a sequence (v_n) in $B(\Sigma)$ such that $|u_n - u| \le v_n \downarrow 0$ in $B(\Sigma)$ (see [3]).

Definition 2.1. A linear operator $T : B(\Sigma) \to E$ is said to be σ-*smooth* if $T(u_n) \to 0$ in ξ whenever $u_n \xrightarrow{(o)} 0$ in $B(\Sigma)$.

Proposition 2.1. *For a linear operator $T : B(\Sigma) \to E$ the following statements are equivalent:*

(i) *$e' \circ T \in B(\Sigma)^*_c$ for each $e' \in E'_\xi$.*
(ii) *T is $(\sigma(B(\Sigma), ca(\Sigma)), \sigma(E, E'_\xi))$-continuous.*
(iii) *T is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous.*
(iv) *T is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-sequentially continuous.*
(v) *T is σ-smooth.*

Proof. (i) $\Longleftrightarrow$ (ii) See [2, Theorem 9.26].

(ii) $\Longrightarrow$ (iii) Assume that T is $(\sigma(B(\Sigma), ca(\Sigma)), \sigma(E, E'_\xi))$-continuous. Then T is $(\tau(B(\Sigma), ca(\Sigma)), \tau(E, E'_\xi))$-continuous (see [2, Ex. 11, p. 149]). It follows that T is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous because $\xi \subset \tau(E, E'_\xi)$.

(iii) $\Longrightarrow$ (iv) It is obvious.

(iv) $\Longrightarrow$ (v) Assume that T is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-sequentially continuous, and let $u_n \xrightarrow{(o)} 0$ in $B(\Sigma)$. Then $u_n \to 0$ for $\tau(B(\Sigma), ca(\Sigma))$ because $\tau(B(\Sigma), ca(\Sigma))$ is a σ-Lebesgue topology. Hence $T(u_n) \to 0$ for ξ, i.e., T is σ-smooth.

(v) $\Longrightarrow$ (i) It is obvious. $\square$

Note that every σ-smooth operator $T : B(\Sigma) \to E$ is $(\|\cdot\|, \xi)$-continuous because $\tau(B(\Sigma), ca(\Sigma)) \subset \mathcal{T}_{\|\cdot\|}$.

Let $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$ stand for the space of all $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous linear operators from $B(\Sigma)$ to E, equipped with the topology $\mathcal{T}_s$ of simple convergence. Then $T_\alpha \to T$ for $\mathcal{T}_s$ in $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$ if and only if $T_\alpha(u) \to T(u)$ in ξ for all $u \in B(\Sigma)$.

The following result will be of importance (see [16, Theorem 2]).

Theorem 2.2. *Let $\mathcal{K}$ be a $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$. If C is a $\sigma(E'_\xi, E)$-closed and ξ-equicontinuous subset of E'_ξ, then $\{e' \circ T : T \in \mathcal{K}, e' \in C\}$ is a $\sigma(B(\Sigma)^*_c, B(\Sigma))$-compact subset of $B(\Sigma)^*_c$.*

Now using Theorem 2.2 and the property that $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ is a strongly Mackey space, we are ready to prove our main result.

Theorem 2.3. *Let $\mathcal{K}$ be a subset of $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$. Then the following statements are equivalent:*

(i) *$\mathcal{K}$ is relatively $\mathcal{T}_s$-compact.*
(ii) *$\mathcal{K}$ is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-equicontinuous and for each $u \in B(\Sigma)$, the set $\{T(u) : T \in \mathcal{K}\}$ is relatively ξ-compact in E.*

Proof. (i) $\Longrightarrow$ (ii) Assume that $\mathcal{K}$ is relatively $\mathcal{T}_s$-compact. Let W be an absolutely convex and ξ-closed neighbourhood of 0 for ξ in E. Then the polar W^0 of W, with respect to the dual pair $\langle E, E'_\xi\rangle$, is a $\sigma(E'_\xi, E)$-closed and ξ-equicontinuous subset of E'_ξ (see [2, Theorem 9.21]). Hence in view of Theorem 2.2 the set $H = \{e' \circ T : T \in \mathcal{K}, e' \in W^0\}$ in $B(\Sigma)^*_c$ is relatively $\sigma(B(\Sigma)^*_c, B(\Sigma))$-compact. Since $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ is a strongly Mackey space, the set H is $\tau(B(\Sigma), ca(\Sigma))$-equicontinuous. It follows that there exists a $\tau(B(\Sigma), ca(\Sigma))$-neighbourhood V of 0 in $B(\Sigma)$ such that $H \subset V^0$, where V^0 denotes the polar of V with respect to the dual pair $\langle B(\Sigma), B(\Sigma)_c^*\rangle$. It follows that for each $T \in \mathcal{K}$ we have that $\{e' \circ T : e' \in W^0\} \subset V^0$, i.e., if $e' \in W^0$, then $|e'(T(u))| \leq 1$ for all $u \in V$. This means that for each $T \in \mathcal{K}$ we get $W^0 \subset T(V)^0$. Hence $T(V) \subset T(V)^{00} \subset W^{00} = W$ for each $T \in \mathcal{K}$, i.e., $\mathcal{K}$ is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-equicontinuous. Clearly, for each $u \in B(\Sigma)$, the set $\{T(u) : T \in \mathcal{K}\}$ is relatively ξ-compact in E.

(ii) $\Longrightarrow$ (i) It follows from [5, Chap. 3, Section 3.4, Corollary 1]. □

Corollary 2.4. *Assume that $\mathcal{K}$ is a relatively $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$. Then $\mathcal{K}$ is uniformly σ-smooth, i.e., for each ξ-continuous seminorm p on E we have that $\sup_{T\in\mathcal{K}} p(T(u_n)) \to 0$ whenever $u_n \xrightarrow{(o)} 0$ in $B(\Sigma)$.*

Proof. In view of Theorem 2.3 $\mathcal{K}$ is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-equicontinuous. Let p be a ξ-continuous seminorm on E, and let $\varepsilon > 0$ be given. Then there exists a $\tau(B(\Sigma), ca(\Sigma))$-neighbourhood V of 0 in $B(\Sigma)$ such that for each $T \in \mathcal{K}$ we have $p(T(u)) \leq \varepsilon$ for all $u \in V$. Assume that (u_n) is a sequence in $B(\Sigma)$ such that $u_n \xrightarrow{(o)} 0$ in $B(\Sigma)$. Then $u_n \longrightarrow 0$ for $\tau(B(\Sigma), ca(\Sigma))$ because $\tau(B(\Sigma), ca(\Sigma))$ is a σ-Lebesgue topology, and hence there exists $n_\varepsilon \in \mathbb{N}$ such that $u_n \in V$ for $n \geq n_\varepsilon$. Hence $\sup_{T\in\mathcal{K}} p(T(u_n)) \leq \varepsilon$ for $n \geq n_\varepsilon$. □

3. Integration operators on $B(\Sigma)$

For terminology and basic results concerning the integration with respect to vector measures we refer the reader to [14,15,13]. In this section we study integration operators on $B(\Sigma)$ in terms of their representing vector measures.

Let (E, ξ) be a quasicomplete lcHs and $m : \Sigma \to E$ be a ξ-bounded additive vector measure (i.e., the range of m is ξ-bounded in E). Given $u \in B(\Sigma)$, let (s_n) be a sequence of Σ-simple functions that converges uniformly to u on Ω. Following [14, Definition 1] we say that u is m-integrable and define

$$\int_\Omega u\, dm := \xi - \lim \int_\Omega s_n\, dm.$$

The $\int_\Omega u\, dm$ is well defined (see [14, Lemma 5]) and the map $T_m : B(\Sigma) \to E$ given by $T_m(u) = \int_\Omega u\, dm$ is $(\|\cdot\|, \xi)$-continuous and linear, and for each $e' \in E'_\xi$

$$e'\left(\int_\Omega u\, dm\right) = \int_\Omega u\, d(e' \circ m) \quad \text{for } u \in B(\Sigma)$$

(see [14, Lemma 5]). Conversely, let $T : B(\Sigma) \to E$ be a $(\|\cdot\|, \xi)$-continuous linear operator, and let $m(A) = T(\mathbb{1}_A)$ for $A \in \Sigma$. Then $m : \Sigma \to E$ is a ξ-bounded vector measure, called the *representing measure* of T and $T_m(u) = T(u)$ for $u \in B(\Sigma)$ (see [14, Definition 2]).

An important example of a quasicomplete locally convex Hausdorff space is the space $\mathcal{L}(X, Y)$ of all bounded linear operators between Banach spaces X and Y, provided with the strong operator topology.

Now we present a characterization of $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous linear operators $T : B(\Sigma) \to E$ in terms of their representing measures.

Proposition 3.1. *Assume that (E, ξ) is a quasicomplete lcHs. Let $T : B(\Sigma) \to E$ be a $(\|\cdot\|, \xi)$-continuous linear operator and $m : \Sigma \to E$ be its representing measure. Then the following statements are equivalent:*

(i) $e' \circ m \in ca(\Sigma)$ *for each* $e' \in E'_\xi$.
(ii) $e' \circ T \in B(\Sigma)^*_c$ *for each* $e' \in E'_\xi$.
(iii) T *is* $(\tau(B(\Sigma), ca(\Sigma)), \xi)$*-continuous.*
(iv) T *is* σ*-smooth.*
(v) $T(u_n) \longrightarrow 0$ *for* ξ *whenever* $u_n(\omega) \longrightarrow 0$ *for each* $\omega \in \Omega$ *and* $\sup_n \|u_n\| < \infty$.
(vi) m *is* ξ*-countably additive.*
In particular, if $(E, \|\cdot\|_E)$ *is a Banach space, then each of the statements* (i)–(vi) *is equivalent to the following:*
(vii) T *is* $(\tau(B(\Sigma), ca(\Sigma)), \|\cdot\|_E)$*-weakly compact, i.e.,* $T(V)$ *is relatively weakly compact in* E *for some* $\tau(B(\Sigma), ca(\Sigma))$*-neighbourhood* V *of* 0 *in* $B(\Sigma)$.

Proof. (i) $\Longleftrightarrow$ (ii) For each $e' \in E'_\xi$ we have

$$(e' \circ T)(u) = \int_\Omega u\, d(e' \circ m) \quad \text{for all } u \in B(\Sigma).$$

Hence, $e' \circ m \in ca(\Sigma)$ if and only if $e' \circ T \in B(\Sigma)^*_c$ (see [1, Theorem 13.5]).

(ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) See Proposition 2.1.

(iv) $\Longrightarrow$ (v) Assume that (iv) holds and let (u_n) be a sequence in $B(\Sigma)$ such that $u_n(\omega) \longrightarrow 0$ for each $\omega \in \Omega$ and $\sup \|u_n\| < \infty$. Let $v_n(\omega) = \sup_{m \ge n} |u_m(\omega)|$ for $\omega \in \Omega$, $n \in \mathbb{N}$. Then $v_n \in B(\Sigma)$ and $|u_n(\omega)| \le v_n(\omega) \downarrow 0$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$. It follows that $u_n \xrightarrow{(o)} 0$ in $B(\Sigma)$ and by (iv) $T(u_n) \to 0$ for ξ.

(v) $\Longrightarrow$ (vi) Assume that (v) holds, and let $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. Then $\mathbb{1}_{A_n}(\omega) \downarrow 0$ for $\omega \in \Omega$ and $\sup_n \|\mathbb{1}_{A_n}\| \le 1$. It follows that $m(A_n) = T(\mathbb{1}_{A_n}) \to 0$ for ξ, i.e., m is ξ-countably additive.

(vi) $\Longrightarrow$ (i) It is obvious.

Assume that $(E, \|\cdot\|_E)$ is a Banach space. Then by [11, Corollary 12] we have (vi) $\Leftrightarrow$ (vii). $\square$

Graves and Ruess [11, Theorem 7] characterized relative compactness in $ca(\Sigma, E)$ in the topology $\mathcal{T}_s$ of simple convergence (convergence on each $A \in \Sigma$) in terms of the properties of the integration operators from $\mathcal{S}(\Sigma)$ to E or from $L(\Sigma)$ to E.

For a subset $\mathcal{M}$ of $ca(\Sigma, E)$ let $\mathcal{K}_\mathcal{M} = \{T_m \in \mathcal{L}_{\tau,\xi}(B(\Sigma), E) : m \in \mathcal{M}\}$. Now using [11, Theorem 7] and Theorem 2.3 we are ready to state the following generalized Vitali–Hahn–Saks theorem for operators from $B(\Sigma)$ to E (see [16, Theorem 10]).

Theorem 3.2. *Assume that* (E, ξ) *is a quasicomplete lcHs that is complete in its Mackey topology (in particular,* E *is a Banach space). Then for a set* $\mathcal{M}$ *in* $ca(\Sigma, E)$ *the following statements are equivalent:*

(i) $\mathcal{K}_\mathcal{M}$ *is a relatively* $\mathcal{T}_s$*-compact set in* $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$.
(ii) $\mathcal{K}_\mathcal{M}$ *is* $(\tau(B(\Sigma), ca(\Sigma)), \xi)$*-equicontinuous and for each* $u \in B(\Sigma)$*, the set* $\{T_m(u) : m \in \mathcal{M}\}$ *is relatively* ξ*-compact in* E.
(iii) $\mathcal{M}$ *is uniformly* ξ*-countably additive and for each* $A \in \Sigma$*, the set* $\{m(A) : m \in \mathcal{M}\}$ *is relatively* ξ*-compact in* E.
(iv) $\mathcal{M}$ *is a relatively* $\mathcal{T}_s$*-compact set in* $ca(\Sigma, E)$.

Proof. (i) $\Longleftrightarrow$ (ii) See Theorem 2.3.

(ii) $\Longrightarrow$ (iii) Assume that (ii) holds and let $A_n \downarrow \emptyset$, $(A_n) \subset \Sigma$. Then $\mathbb{1}_{A_n} \downarrow \emptyset$ in $B(\Sigma)$, and we can apply Proposition 3.1 and Corollary 2.4.

(iii) $\Longrightarrow$ (ii) Assume that (iii) holds. Then in view of [11, Theorem 7] the set $\widetilde{\mathcal{K}}_{\mathcal{M}} = \{\widetilde{T}_m : m \in \mathcal{M}\}$ is $(\hat{\tau}, \xi)$-equicontinuous and $\widetilde{\mathcal{K}}_{\mathcal{M}}$ is a relatively $\mathcal{T}_s$-compact set in $\mathcal{L}_{\hat{\tau},\xi}(L(\Sigma), E)$ (=the space of all $(\hat{\tau}, \xi)$-continuous linear operators from $L(\Sigma)$ to E). It follows that $\mathcal{K}_{\mathcal{M}}$ is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-equicontinuous and for each $u \in B(\Sigma)$, the set $\{T_m(u) : m \in \mathcal{M}\}$ is relatively ξ-compact in E, i.e., (ii) holds.

(iii) $\Longleftrightarrow$ (iv) See [11, Theorem 7]. $\square$

Recall that the Nikodym convergence theorem says that if (m_k) is a sequence of measures on a σ-algebra Σ taking values in a locally convex space (E, ξ) and $m(A) := \xi - \lim m_k(A)$ for each $A \in \Sigma$, then $m : \Sigma \to E$ is ξ-countably additive and the family $\{m_k : k \in \mathbb{N}\}$ is uniformly ξ-countably additive (see [8, Theorem 8.6], [11, Theorem 9], [16, Corollary 9]). Now we shall prove a generalized Nikodym convergence type theorem for operators from $B(\Sigma)$ to a quasicomplete lcHs (E, ξ).

For this purpose we first establish some terminology. For each ξ-continuous seminorm p on E, let $E_p = (E, p)$ be the associated seminormed space. Denote by $(\widetilde{E}_p, \|\cdot\|_p^{\sim})$ the completion of the quotient normed space $E/p^{-1}(0)$. Let $\Pi_p : E_p \to E/p^{-1}(0) \subset \widetilde{E}_p$ be the canonical quotient map (see [14, p. 92]).

Given a vector measure $m : \Sigma \to E$, let $m_p : \Sigma \to \widetilde{E}_p$ be given by

$$m_p(A) := (\Pi_p \circ m)(A) \quad \text{for } A \in \Sigma.$$

Then m_p is a Banach space-valued measure on Σ. We define the p-semivariation $\|m\|_p$ of m by

$$\|m\|_p(A) := \|m_p\|(A) \quad \text{for } A \in \Sigma,$$

where $\|m_p\|$ denotes the semivariation of $m_p : \Sigma \to \widetilde{E}_p$. Note that m is ξ-bounded if and only if $\|m\|_p(\Omega) < \infty$ for each ξ-continuous seminorm p on E. Moreover, we have (see [14, Lemma 7])

$$\|m\|_p(\Omega) = \|T_m\|_p = \sup\left\{p\left(\int_\Omega u\,dm\right) : u \in B(\Sigma), \|u\| \le 1\right\}. \tag{3.1}$$

Now we can prove our desired theorem.

Theorem 3.3. *Assume that (E, ξ) is a quasicomplete lcHs. Let $m_k : \Sigma \to E$ be ξ-countably additive vector measures for $k \in \mathbb{N}$ and assume that $m(A) = \xi - \lim m_k(A)$ exists for each $A \in \Sigma$. Then the following statements hold:*

(i) *$m : \Sigma \to E$ is a ξ-countably additive vector measure, and the integration operator $T_m : B(\Sigma) \to E$ is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous.*
(ii) *$T_m(u) = \xi - \lim_k T_{m_k}(u)$ for all $u \in B(\Sigma)$.*
(iii) *The family $\{T_{m_k} : k \in \mathbb{N}\}$ is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-equicontinuous.*

Proof. In view of the Nikodym convergence theorem (see [8, Theorem 8.6]) the vector measure $m : \Sigma \to E$ is ξ-countably additive, and by Proposition 3.1 $T_m : B(\Sigma) \to E$ is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous.

Let p be a ξ-continuous seminorm on E. We show that $p(T_{m_k}(u) - T_m(u)) \to 0$ for each $u \in B(\Sigma)$. Indeed, since $p(m_k(A) - m(A)) \to 0$ for all $A \in \Sigma$, we have $\|\Pi_p(m_k(A) - m(A))\|_p^{\sim} \to 0$, i.e., $\|(m_k)_p(A) - m_p(A)\|_p^{\sim} \to 0$ for all $A \in \Sigma$. It follows that $\sup_k \|(m_k)_p(A)\|_p^{\sim} < \infty$ for all $A \in \Sigma$, and in view of the Nikodym boundedness theorem (see [7, Theorem 1, p. 14]) and (3.1) we get

$$c = \sup_k \|T_{m_k}\|_p = \sup_k \|m_k\|_p(\Omega) < \infty.$$

Let $u \in B(\Sigma)$ and $\varepsilon > 0$ be given. Choose $s_0 \in \mathcal{S}(\Sigma)$ such that $\|u - s_0\| \leq \frac{\varepsilon}{3a}$, where $a = \max(c, \|T_m\|_p)$. Then there is $k_0 \in \mathbb{N}$ such that $p(T_{m_k}(s_0) - T_m(s_0)) \leq \frac{\varepsilon}{3}$ for $k \geq k_0$. Hence for $k \geq k_0$ we have

$$\begin{aligned} p(T_{m_k}(u) - T_m(u)) &\leq p(T_m(u - s_0)) + p(T_m(s_0) - T_{m_k}(s_0)) + p(T_{m_k}(s_0) - T_{m_k}(u)) \\ &\leq \|T_m\|_p \cdot \|u - s_0\| + p(T_m(s_0) - T_{m_k}(s_0)) + \|T_{m_k}\|_p \cdot \|s_0 - u\| \\ &\leq a \cdot \frac{\varepsilon}{3a} + \frac{\varepsilon}{3} + a \cdot \frac{\varepsilon}{3a} = \varepsilon. \end{aligned}$$

It follows that $T_{m_k} \to T_m$ in $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$ for $\mathcal{T}_s$. Since $\{T_{m_k} : k \in \mathbb{N}\} \cup \{T_m\}$ is a $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\tau,\xi}(B(\Sigma), E)$, by Theorem 2.3 the set $\{T_{m_k} : k \in \mathbb{N}\}$ is $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-equicontinuous. □

Finally, we shall show that every $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous linear operator $T : B(\Sigma) \to E$ is a strict Dunford–Pettis operator.

Theorem 3.4. *Assume that (E, ξ) is a quasicomplete lcHs. If $T : B(\Sigma) \to E$ is a $(\tau(B(\Sigma), ca(\Sigma)), \xi)$-continuous linear operator, then T maps $\sigma(B(\Sigma), B(\Sigma)^*_c)$-Cauchy sequences in $B(\Sigma)$ onto ξ-convergent sequences in E.*

Proof. For each $\omega \in \Omega$ let $\Phi_\omega(u) = u(\omega)$ for all $u \in B(\Sigma)$. Note that $\Phi_\omega \in B(\Sigma)^*_c$ (see Proposition 3.1). Let (u_n) be a $\sigma(B(\Sigma), B(\Sigma)^*_c)$-Cauchy sequence in $B(\Sigma)$. Then the set $\{u_n : n \in \mathbb{N}\}$ is $\tau(B(\Sigma), ca(\Sigma))$-bounded, and it follows that $\sup_n \|u_n\| < \infty$ (see [12, Theorem 2]). It follows that for each $\omega \in \Omega$, $\lim u_n(\omega) = \lim \Phi_\omega(u_n) = u_0(\omega)$ exists in $\mathbb{R}$. It follows that $u_0 : \Omega \to \mathbb{R}$ is Σ-measurable, and since $\sup_n \|u_n\| < \infty$, we obtain that u_0 is bounded, i.e., $u_0 \in B(\Sigma)$.

Let $m(A) = T(\mathbb{1}_A)$ for $A \in \Sigma$. Then by Proposition 3.1, m is ξ-countably additive. Hence making use of the Lebesgue type convergence theorem (see [15, Proposition 7, p. 4854]), u_0 is m-integrable and we have

$$T(u_n) = \int_\Omega u_n \, dm \xrightarrow{\xi} \int_\Omega u_0 \, dm \in E.$$

Thus the proof is complete. □

As a consequence of Theorem 3.4 we have the following result (see [9, Section 9.4]).

Corollary 3.5. *The space $(B(\Sigma), \tau(B(\Sigma), ca(\Sigma)))$ has the strict Dunford–Pettis property.*

4. Yosida–Hewitt decomposition for weakly compact operators on $B(\Sigma)$

In this section we derive a Yosida–Hewitt type decomposition theorem for weakly compact operators on $B(\Sigma)$. In view of the Yosida–Hewitt decomposition theorem (see [17]) we have

$$ba(\Sigma) = ca(\Sigma) \oplus \text{pfa}(\Sigma),$$

where pfa(Σ) ($=ca(\Sigma)^d$—the disjoint complement of $ca(\Sigma)$ in $ba(\Sigma)$) stands for the band of purely finitely additive members of $ba(\Sigma)$. On the other hand, we have (see [2, Theorem 3.8])

$$B(\Sigma)^* = B(\Sigma)^*_c \oplus (B(\Sigma)^*_c)^d,$$

where $(B(\Sigma)^*_c)^d$ stands for the disjoint complement of $B(\Sigma)^*_c$ in $B(\Sigma)^*$. It follows that

$$(B(\Sigma)^*_c)^d = \{\Phi_\nu : \nu \in \mathrm{pfa}(\Sigma)\}.$$

Since $B(\Sigma)$ is an AM-space, $ba(\Sigma)$ is an AL-space (see [1, Theorem 13.2]). This means that $\|\nu\| = \|\nu_c\| + \|\nu_p\|$ and $\|\Phi_\nu\| = \|\Phi_{\nu_c}\| + \|\Phi_{\nu_p}\|$, where $\nu = \nu_c + \nu_p$ with $\nu_c \in ca(\Sigma)$ and $\nu_p \in \mathrm{pfa}(\Sigma)$.

Definition 4.1. A $(\|\cdot\|, \xi)$-continuous linear operator $T : B(\Sigma) \to E$ is said to be *purely non-σ-smooth* if $e' \circ T \in (B(\Sigma)^*_c)^d$ for each $e' \in E'_\xi$.

Moreover, since $(B(\Sigma)^*_c)^d = \{\Phi_\nu : \nu \in \mathrm{pfa}(\Sigma)\}$ we have the following result.

Proposition 4.1. *Assume that (E, ξ) is a quasicomplete lcHs. Let $T : B(\Sigma) \to E$ be a $(\|\cdot\|, \xi)$-continuous linear operator and $m : \Sigma \to E$ be its representing measure. Then the following statements are equivalent:*

(i) *T is purely non-σ-smooth.*
(ii) *$e' \circ m \in \mathrm{pfa}(\Sigma)$ for each $e' \in E'_\xi$.*

A vector measure $m : \Sigma \to E$ is said to be ξ-strongly bounded (ξ-exhausting) if $m(A_n) \to 0$ in ξ for each pairwise disjoint sequence (A_n) in Σ. It is well known that each ξ-countably additive measure $m : \Sigma \to E$ is ξ-strongly bounded. The following theorem is of importance (see [14, Lemma 3 and Theorem 1]).

Theorem 4.2. *Assume that (E, ξ) is a quasicomplete lcHs. Let $T : B(\Sigma) \to E$ be a $(\|\cdot\|, \xi)$-continuous linear operator and $m : \Sigma \to E$ be its representing measure. Then the following statements are equivalent:*

(i) *T is weakly compact, i.e., T maps bounded sets in $B(\Sigma)$ onto relatively $\sigma(E, E'_\xi)$-compact sets in E.*
(ii) *m is ξ-strongly bounded.*

From Theorem 4.2 and Proposition 3.1 it follows that every σ-smooth operator from $B(\Sigma)$ to a quasicomplete lcHs is weakly compact.

Now we are ready to prove the following Yosida–Hewitt type decomposition for weakly compact operators from $B(\Sigma)$ to a quasicomplete lcHs.

Theorem 4.3. *Assume that (E, ξ) is a quasicomplete lcHs. Let $T : B(\Sigma) \to E$ be a weakly compact operator and $m : \Sigma \to E$ be its representing measure. Then:*

(i) *m can be uniquely decomposed as $m = m_c + m_p$, where $m_c : \Sigma \to E$ is ξ-countably additive, $m_p : \Sigma \to E$ is ξ-strongly bounded and $e' \circ m_p \in \mathrm{pfa}(\Sigma)$ for each $e' \in E'_\xi$.*
(ii) *T can be uniquely decomposed as $T = T_1 + T_2$, where T_1 is σ-smooth and T_2 is weakly compact and purely non σ-smooth, and*

$$T_1(u) = \int_\Omega u\, dm_c \quad \text{and} \quad T_2(u) = \int_\Omega u\, dm_p \quad \text{for all } u \in B(\Sigma).$$

Proof. For $\Phi \in B(\Sigma)^*$ we have $\Phi = \Phi_1 + \Phi_2$, where $\Phi_1 \in B(\Sigma)^*_c$ and $\Phi_2 \in (B(\Sigma)^*_c)^d$. Then we have two natural projections

$$P_k : B(\Sigma)^* \longrightarrow B(\Sigma)^*,$$

where $P_k(\Phi) = \Phi_k$ and $\|P_k\| \leq 1$ for $k = 1, 2$. Now consider the conjugate mappings

$$P_k' : B(\Sigma)^{**} \longrightarrow B(\Sigma)^{**}$$

defined by $P_k'(V)(\Phi) = V(P_k(\Phi))$ for $V \in B(\Sigma)^{**}$ (=the Banach bidual of $B(\Sigma)$).

Since T is $(\sigma(B(\Sigma), B(\Sigma)^*), \sigma(E, E_\xi'))$-continuous (see [9, Corollary 8.6.5]), we can define the conjugate mapping

$$T' : E_\xi' \longrightarrow B(\Sigma)^*$$

by putting $T'(e') = e' \circ T$ for $e' \in E_\xi'$. Then T' is $(\beta(E_\xi', E), \beta(B(\Sigma)^*, B(\Sigma)))$-continuous (see [9, Proposition 8.7.1]), and hence T' is $(\sigma(E_\xi', E_\xi''), \sigma(B(\Sigma)^*, B(\Sigma)^{**}))$-continuous. It follows that one can define the conjugate mapping (see [2, Theorem 9.2])

$$T'' : B(\Sigma)^{**} \longrightarrow E_\xi''$$

by putting $T''(V)(e') = V(T'(e'))$ for $V \in B(\Sigma)^{**}$ and $e' \in E_\xi'$. Then T'' is $(\sigma(B(\Sigma)^{**}, B(\Sigma)^*), \sigma(E_\xi'', E_\xi'))$-continuous.

We have a natural isometric embedding $\pi : B(\Sigma) \to B(\Sigma)^{**}$, where $\pi(u)(\Phi) = \Phi(u)$ for all $\Phi \in B(\Sigma)^*$, $u \in B(\Sigma)$. Then $(P_k' \circ \pi)(u) = \pi(u) \circ P_k$ for $u \in B(\Sigma)$.

Let $i : E \to E_\xi''$ stand for the canonical mapping, where $i(e)(e') = e'(e)$ for $e' \in E_\xi'$ and $e \in E$. Moreover, let $j : i(E) \longrightarrow E$ denote the left inverse of i, i.e., $j \circ i = id_E$. Then $T'' \circ \pi = i \circ T$.

By the Gantmacher type theorem (see [9, Theorem 9.3.2]) we have

$$T''(B(\Sigma)^{**}) \subset i(E).$$

Define linear mappings ($k = 1, 2$)

$$T_k := j \circ T'' \circ P_k' \circ \pi : B(\Sigma) \longrightarrow E.$$

To show that $T_k : B(\Sigma) \to E$ are weakly compact, note that for $u \in B_{B(\Sigma)}$ (=the closed unit ball in $B(\Sigma)$) we have

$$\|P_k'(\pi(u))\|_{B(\Sigma)^{**}} = \|\pi(u) \circ P_k\|_{B(\Sigma)^{**}} \leq 1.$$

Hence by the Banach–Alaoglou theorem the set $\{P_k'(\pi(u)) : u \in B_{B(\Sigma)}\}$ is relatively $\sigma(B(\Sigma)^{**}, B(\Sigma)^*)$-compact in $B(\Sigma)^{**}$. Since T'' is $(\sigma(B(\Sigma)^{**}, B(\Sigma)^*), \sigma(E_\xi'', E_\xi'))$-continuous and $T''(B(\Sigma)^{**}) \subset i(E)$, the set $\{T''(P_k'(\pi(u))) : u \in B_{B(\Sigma)}\}$ is relatively $\sigma(i(E), E_\xi')$-compact in $i(E)$. But j is $(\sigma(i(E), E_\xi'), \sigma(E, E_\xi'))$-continuous, so the set $T_k(B_{B(\Sigma)}) = \{j(T''(P_k'(\pi(u)))) : u \in B_{B(\Sigma)}\}$ is relatively $\sigma(E, E_\xi')$-compact in E, and this means that the T_k are weakly compact.

For $A \in \Sigma$ let us put

$$m_c(A) := T_1(\mathbb{1}_A) \quad \text{and} \quad m_p(A) := T_2(\mathbb{1}_A).$$

In view of Theorem 4.2 the measures $m_c : \Sigma \to E$ and $m_p : \Sigma \to E$ are ξ-strongly bounded, and we have

$$T_1(u) = \int_\Omega u\, dm_c \quad \text{and} \quad T_2(u) = \int_\Omega u\, dm_p \quad \text{for all } u \in B(\Sigma).$$

Note that for each $e' \in E'_\xi$ and $u \in B(\Sigma)$ we have

$$\begin{aligned}(e' \circ T_k)(u) &= e'(j \circ (T'' \circ P'_k \circ \pi)(u)) \\ &= ((T'' \circ P'_k \circ \pi)(u))(e') \\ &= (T''(\pi(u) \circ P_k))(e') \\ &= (\pi(u) \circ P_k)(T'(e')) \\ &= \pi(u)(P_k(e' \circ T)) = P_k(e' \circ T)(u).\end{aligned}$$

Hence $e' \circ T_1 = P_1(e' \circ T) \in B(\Sigma)^*_c$, and by Proposition 3.1, T_1 is σ-smooth. Moreover, $e' \circ T_2 = P_2(e' \circ T) \in (B(\Sigma)^*_c)^d$, i.e., T_2 is purely non-σ-smooth. For every $e' \in E'_\xi$ and $u \in B(\Sigma)$ we have

$$e'(T_1(u) + T_2(u)) = P_1(e' \circ T)(u) + P_2(e' \circ T)(u) = e'(T(u)),$$

so $T_1(u) + T_2(u) = T(u)$. The uniqueness of the decomposition $T = T_1 + T_2$ follows from the uniqueness of the decomposition $e' \circ T = e' \circ T_1 + e' \circ T_2$ for each $e' \in E'_\xi$.

Moreover, for each $e' \in E'_\xi$ we have

$$(e' \circ T_1)(u) = \int_\Omega u\, d(e' \circ m_c) \quad \text{and} \quad (e' \circ T_2)(u) = \int_\Omega u\, d(e' \circ m_p) \quad \text{for } u \in B(\Sigma).$$

Hence by Proposition 3.1, m_c is ξ-countably additive, and by Proposition 4.1, $e' \circ m_p \in \mathrm{pfa}(\Sigma)$. Clearly, $m = m_c + m_p$. □

Remark. A Yosida–Hewitt type decomposition theorem for order-weakly compact operators acting from a vector lattice to a Banach space was derived in [4, Theorem 4].

References

[1] C.D. Aliprantis, K.C. Border, Infinite Dimensional Analysis, second ed., Springer-Verlag, 1999.
[2] C.D. Aliprantis, O. Burkinshaw, Positive Operators, Academic Press, New York, 1985.
[3] C.D. Aliprantis, O. Burkinshaw, Locally Solid Riesz Spaces with Applications to Economics, second ed., in: Math. Surveys and Monographs, vol. 105, 2003.
[4] A. Basile, A.V. Bukhvalov, M.Ya. Yakubson, The generalized Yosida–Hewitt theorem, Math. Proc. Cambridge Philos. Soc. 116 (1994) 527–533.
[5] N. Bourbaki, Elements of Mathematics, Topological Vector Spaces, Springer-Verlag, Berlin, 1987 (Chapter 1–5).
[6] J. Diestel, Applications of weak compactness and bases to vector measures and vectorial applications, Rev. Roumaine Math. Pures Appl. 28 (2) (1973) 211–224.
[7] J. Diestel, J.J. Uhl, Vector Measures, in: Math. Surveys, vol. 15, Amer. Math. Soc., Providence, RI, 1977.
[8] L. Drewnowski, Topological rings of sets, continuous set functions, integration, III, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. 20 (6) (1972) 439–445.
[9] R.E. Edwards, Functional Analysis, Theory and Applications, Holt, Rinehart and Winston, New York, 1965.
[10] W.H. Graves, On the Theory of Vector Measures, in: Memoirs, vol. 195, Amer. Math. Soc., Providence, RI, 1977.
[11] W.H. Graves, W. Ruess, Compactness in spaces of vector-valued measures and a natural Mackey topology for spaces of bounded measurable functions, Contemp. Math. 2 (1980) 189–203.
[12] S.S. Khurana, A topology associated with vector measures, J. Indian Math. Soc. 45 (1981) 167–179.
[13] D.R. Lewis, Integration with respect to vector measures, Pacific J. Math. 33 (1) (1970) 157–165.
[14] T.V. Panchapagesan, Applications of a theorem of Grothendieck to vector measures, J. Math. Anal. Appl. 214 (1997) 89–101.
[15] T.V. Panchapagesan, Characterizations of weakly compact operators on $C_0(T)$, Trans. Amer. Math. Soc. 350 (12) (1998) 4849–4867.
[16] H. Schaefer, Xiao-Dong Zhang, On the Vitali–Hahn–Saks Theorem, in: Operator Theory: Advances and Applications, vol. 75, Birkhäuser, Basel, 1995, pp. 289–297.
[17] K. Yosida, E. Hewitt, Finitely additive measures, Trans. Amer. Math. Soc. 72 (1952) 46–66.
[18] Xiao-Dong Zhang, On weak compactness in spaces of measures, J. Funct. Anal. 143 (1997) 1–9.