Some Remarks on Filtrations and Plethysms

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INTRODUCTION

Plethysm, introduced by Littlewood, is an operation on symmetric functions that corresponds to composition of Schur and/or Weyl functors. Let $R$ be a commutative ring with identity and $F$ a free $R$-module of finite rank. We denote by $A F$, $S F$, and $D F$ the exterior, symmetric, and divided power algebra of $F$, respectively. Over the last decade or so, a number of authors have studied plethysms of type $A_k(B_2 F)$, where $A, B \in \{ A, S, D \}$, mostly in connection with invariant theory, resolutions of determinantal ideals, and the characteristic-free representation theory of the general linear group (see [A, ADF, B, K1, K2, RS1, RS2]). In a recent article [B], Boffi shows that of the nine plethysms $A(B_2 F)$ with $A, B \in \{ A, S, D \}$, only four have universal filtrations and he gives explicit descriptions of these. (A “universal” filtration for us is one defined over any ring whose composition factors are all Schur or all Weyl modules.)

In this note we are interested in filtrations associated to the plethysms $A_2(B_k F)$, $A, B \in \{ A, S, D \}$. If $R$ is a field of characteristic zero, the irreducible decomposition of each $A_2(B_k F)$ is known [L, Part II]. However, from [AB1] and [B] it follows that no $A_2(B_k F)$, with $A, B \in \{ A, S, D \}$, has a universal filtration. All the counterexamples contained in [AB1] and [B] involve rings where 2 is not invertible. Hence we assume from now on that 2 is invertible in $R$. In this case, the second divided power functor is naturally isomorphic to the second symmetric power, and therefore there are only six distinct plethysms of type $A_2(B_k F)$, $A, B \in \{ A, S, D \}$. The purpose of this article is to show (by providing explicit filtrations) that each of

$$D_2(D_k F), \ A^2(D_k F), \ S_2(S_k F), \ A^2(S_k F), \ S_2(A_k F), \ A^2(A_k F),$$

where $k \geq 1$, admits a natural filtration (over any ring in which 2 is invertible) so that the associated graded object is the one predicted by the characteristic zero theory. Informally speaking, we see that 2 is the only bad characteristic for these plethysms.

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We treat the cases of $D_2(D_k F)$, $A^2(D_k F)$, $S_2(A^k F)$, and $A^2(A^k F)$ in detail by providing explicit filtrations, while the other two cases follow from contravariant duality. (We prefer to write $D_2(D_k F)$ instead of $S_2(D_k F)$ in order to emphasize that the filtration is in terms of Weyl modules rather than Schur modules. In the opposite direction, we write $S_2(S_k F)$ instead of $D_2(S_k F)$.) In summary, we obtain the following statements:

\begin{align*}
D_2(D_k F) &\approx K_{(2k)} F + K_{(2k-2, 2)} F + K_{(2k-4, 4)} F + \cdots, \\
A^2(D_k F) &\approx K_{(2k-1, 1)} F + K_{(2k-3, 3)} F + K_{(2k-5, 5)} F + \cdots, \\
S_2(S_k F) &\approx L_{(1, 2k)} F + L_{(2, 12k-1)} F + L_{(2, 12k-3)} F + \cdots, \\
A^2(S_k F) &\approx L_{(2, 12k-1)} F + L_{(2, 12k-3)} F + L_{(2, 12k-5)} F + \cdots, \\
S_2(A^k F) &\approx L_{(k,k)} F + L_{(k+2, k-2)} F + L_{(k+4, k-4)} F + \cdots, \\
A^2(A^k F) &\approx L_{(k+1, k-1)} F + L_{(k+3, k-3)} F + L_{(k+5, k-5)} F + \cdots,
\end{align*}

where $\approx$ means "...admits a natural filtration whose associated graded object is ..." and $K_\lambda F$ (respectively, $L_\lambda F$) denotes the Weyl (respectively, Schur) module associated to $\lambda$. We remark that the assumption on the invertibility of 2 is used at two key points: to show that our filtrations are exhaustive and that certain maps on Weyl modules are well-defined.

I express my thanks to Professor J. Duncan for showing me how to use Derive and for some useful discussions concerning the important matrices that appear in Lemma 1.5.

\section*{1. $D_2(D_k F)$}

Unless specified otherwise, $R$ will denote a commutative ring with identity in which 2 is invertible. Our notation will follow [ABW]. In particular, let $F$ be a free $R$-module of finite rank. We fix once and for all an ordered basis $\{1, 2, \ldots, r\}$ of $F$.

Fix a positive integer $k$ and let $E$ be the set of all partitions, $\lambda$, of the form $\lambda = (2\lambda_1, 2\lambda_2)$ with $|\lambda| = 2\lambda_1 + 2\lambda_2 = 2k$.

The purpose of this section is to show that the divided power $D_2(D_k F)$ admits a natural filtration whose associated graded object is $\bigoplus_{\lambda \in E} K_\lambda F$.

**Definition 1.1.** For each $\lambda = (2\lambda_1, 2\lambda_2) \in E$, we define a $GL(F)$-map $\varphi_\lambda : D_{2\lambda_1} F \otimes D_{2\lambda_2} F \to D_2(D_k F)$ as the composition
\[ D_{2k} F \otimes D_{2k} F \xrightarrow{\Delta \otimes 1} D_k F \otimes D_{2k-1} F \]
\[ \otimes D_{2k} F \xrightarrow{1 \otimes m^r} D_k F \otimes D_k F \xrightarrow{m'} D_2(D_k F), \]

where \( \Delta \) is the diagonal map of \( DF \), \( m \) is the multiplication map of \( DF \), and \( m' \) is multiplication in \( D_2(D_k F) \).

**Remark 1.2.** In the special case where \( \lambda = (2k) \), we note that \( \varphi_{(2k)}(1^{2k}) = 2(1^{2k})^{(2)} \) and hence \( \varphi_{(2k)} \) introduces a coefficient of 2. We want to point out that the assumption on the invertibility of 2 is not needed here. One can replace \( \varphi_{(2k)} \) by the generator, \( \varphi \), of \( \text{Hom}_{\text{GL}(F)}(D_{2k} F, D_2(D_k F)) \). (That this \( R \)-module has rank 1 follows from [AB, Sect. 2].) We have \( \varphi_{(2k)} = 2\varphi \).

We did not replace \( \varphi_{(2k)} \) by \( \varphi \) in Definition 1.1 for the sake of having a uniform description of the various maps. Later (Remark 1.10), we will indicate precisely where the assumption on 2 is necessary.

**Definition 1.3.** For each \( \lambda \in E \), we define \( \text{GL}(F) \)-submodules, \( M_{\lambda} \) and \( \tilde{M}_{\lambda} \), of \( D_2(D_k F) \) by

\[ M_{\lambda} = \sum_{\mu \in \mathbb{E}, \mu \geq \lambda} \text{Im} \varphi_{\mu} \quad \text{and} \quad \tilde{M}_{\lambda} = \sum_{\mu \in \mathbb{E}, \mu > \lambda} \text{Im} \varphi_{\mu}, \]

where \( \geq \) denotes the lexicographic ordering of partitions.

**Remark 1.4.** If \( k \) is even, the smallest partition in \( E \) is \((k, k)\). Since 2 is invertible in \( R \), we see that the map \( \varphi_{(k, k)} : D_k F \otimes D_k F \to D_2(D_k F) \) is surjective. These remarks show that

\[ 0 \subseteq M_{(2k)} \subseteq M_{(2k-2, 2)} \subseteq \cdots \subseteq M_{(k, k)} = D_2(D_k F) \]

is indeed a filtration of \( D_2(D_k F) \), \( k \) even. The case where \( k \) is odd is more involved because \((k, k) \notin E \) (see Lemma 1.7).

First, we will need the following elementary lemma which states that certain matrices with entries from \( R \) are invertible.

**Lemma 1.5.** Let \( m \) and \( n \) be integers satisfying \( m \geq 2n-1 \geq 1 \), and let \( A(m, n) \) be the \( n \times n \) matrix whose \((i, j)\) entry is \((m-n-j+1)+(m+j-2n)\). Then

\[ \det A(m, n) = 2^n. \]

**Proof.** We use induction on \( n \), the case \( n = 1 \) being trivial. Assume now that \( \det A(m, n-1) = 2^{n-1} \). We perform the following column operations
on $A(m,n)$: substitute the second column from the first, the third from the second, ..., the $n$th from the $(n-1)$st. The binomial identity \((\binom{n}{s} - \binom{n}{s-1}) = \binom{n-1}{s-1}\) shows that the new matrix is

\[
\begin{bmatrix}
(\frac{m-1}{n-1}) & (\frac{m-1}{n-1} - 2) & \cdots & (\frac{m-1}{n-1} - n+1) \\
(\frac{m-1}{n-1} - 2) & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
(\frac{m-1}{n-1} - n+1) & \cdots & \cdots & \frac{m-1}{n-1}
\end{bmatrix}
\]

Let $B(m-1, n-1)$ denote the $(n-1) \times (n-1)$ matrix obtained from the above matrix by deleting the last row and column. Then $\det A(m,n) = 2 \det B(m-1, n-1)$. We apply now to $B(m-1, n-1)$ the same column operations that we applied to $A(m,n)$. Via the binomial identity \((\binom{n}{s} - \binom{n}{s-1}) = \binom{n-1}{s-1}\), one recognizes that the new matrix thus obtained is $A(m-2, n-1)$. Hence $\det A(m,n) = 2 \det A(m-2, n-1) = 2^n$, as desired.

**Corollary 1.6.** Let $s \geq 0$. Then $\det[\binom{2s+1}{2s-2j-1} + \binom{2s+1}{2s-2j}]_{0 \leq i,j \leq s} = 2^{s+1}$ and $\det[\binom{2s+1}{2s-2j} - \binom{2s+1}{2s-2j-1}]_{0 \leq i,j \leq s} = 2^s$.

**Proof.** Let $m = 2s+1$ and $n = s+1$. The first matrix in the statement of the corollary is the transpose of $A(m,n)$, while the second matrix is the transpose of $B(m-1, n-1)$.

**Lemma 1.7.** \(\{M_s\}, \ k \in E\), is a filtration of $D_2(D_k F)$.

**Proof.** By Remark 1.4 we may assume that $k$ is odd, say $k = 2s + 1$. Let $x \cdot y \in D_2(D_k F)$ with $x, y \in D_k F$. We will show that $x \cdot y \in M_{(k+1,k-1)}$.

For each $j = 0, 1, \ldots, s$, note that $(k+2j+1, k-2j-1)$ is a partition with $(k+2j+1, k-2j-1) \in E$. Consider the element $\sum_a y_a (2j+1) \otimes y_a (k-2j-1) \in D_{k+2j+1} F \otimes D_{k-2j-1} F$, where $\sum_a y_a (2j+1) \otimes y_a (k-2j-1)'$ is the image of $y$ under the diagonal $D_k F \to D_{2j+1} F \otimes D_{k-2j-1} F$. A quick computation yields

\[
\varphi_{(k+2j+1, k-2j-1)} \left( \sum_a x_a (2j+1) \otimes y_a (k-2j-1)' \right) = \sum_{i=0}^k \left( \begin{array}{c} k-i \\ 2j-1 \end{array} \right) \sum_{a,b} y_{a,b} (k-i) \cdot x_a (i) \cdot y_b (k-i)',
\]

where the binomial coefficient comes from multiplication in $D_k F$. Since multiplication in $D_2(D_k F)$ is commutative, we have
\[ \sum_{i,b} x_b(k-i) y_a(i) \cdot x_b(i') y_a(k-i') = \sum_{i,b} x_b(k-i) y_a(k-i) \cdot x_b(k-i') y_a(i'). \]

Hence, we rewrite (1) as follows:

\[
\varphi_{(k+2j+1,k-2j-1)} \left( \sum_a x_a(2j+1) \otimes y_a(k-2j-1) \right) = \sum_{j=0}^s \left( \begin{array}{c} k-i \\ k-2j-1 \end{array} \right) + \left( \begin{array}{c} k-i \\ k-2j-1 \end{array} \right) \times \sum_{i,b} x_b(k-i) y_a(i) \cdot x_b(i') y_a(k-i').
\]

(2)

\( j = 0, 1, ..., s \). We regard (2) as a system of linear equations \((j = 0, 1, ..., s)\) in the \( \sum_{i,b} x_b(k-i) y_a(i) \cdot x_b(i') y_a(k-i') \). Since \( k = 2s + 1 \), the determinant of this system is \( 2^{s+1} \) by the first statement of Corollary 1.6. Since 2 is invertible in \( R \), system (2) is solvable, and therefore each \( \varphi_{(k+2j+1,k-2j-1)} \) is an \( R \)-linear combination of the \( \varphi_{(k+2j+1,k-2j-1)} \) for \( j = 0, 1, ..., s \). In other words, \( \sum_{i,b} x_b(k-i) y_a(i) \cdot x_b(i') y_a(k-i') \in \sum_{j=0}^s M_{(k+2j+1,k-2j-1)} \subset M_{(k+1,k-1)} \), because \( (k+2j+1, k-2j-1) \geq (k+1, k-1) \), \( j = 0, 1, ..., s \). In particular (for \( i = 0 \)), \( x \cdot y \in M_{(k+1,k-1)} \) and the proof is complete.

**Theorem 1.8.** The associated graded object of the filtration \( \{ M_j \} \) is \( \sum_{j \in \mathbb{N}} K_j F \).

For the proof of the previous theorem we need to show:

**Proposition 1.9.** Let \( \lambda \in E \). Then the map \( \varphi_\lambda \) induces a surjective map \( K_\lambda F \to M_\lambda / M_\lambda \).

**Proof.** Let \( \lambda = (2\lambda_1, 2\lambda_2) \) and \( 0 \leq t < 2\lambda_2 \). It suffices to show \cite[ABW, Sect. II.3]{5} that \( \text{Im} ( \varphi_\lambda \circ \square ) \subset M_\lambda \), where \( \square \lambda \) is the composition\n
\[
D_{2\lambda_1 + 2\lambda_2 - t} F \otimes D_{2\lambda_2 - t} F \overset{\Delta \otimes 1}{\longrightarrow} D_{2\lambda_1} F \otimes D_{2\lambda_2 - t} F \otimes D_{2\lambda_1} F \overset{1 \otimes m}{\longrightarrow} D_{2\lambda_1} F \otimes D_{2\lambda_2} F.
\]

In order to simplify the notation, we restrict our attention to the element \( 1^{2\lambda_1 + 2\lambda_2 - t} \otimes 2^{(t)} \in D_{2\lambda_1 + 2\lambda_2 - t} F \otimes D_{2\lambda_1} F \) (as was done in \cite[B, Proof of Lemma 1.7]{5}). A quick computation yields

\[
\square \lambda (1^{2\lambda_1 + 2\lambda_2 - t} \otimes 2^{(t)}) = 1^{2\lambda_1} (2^{(k)}) \otimes 1^{2\lambda_2 - t} (2^{(t)}),
\]

and

\[
\varphi_\lambda \circ \square \lambda (1^{2\lambda_1 + 2\lambda_2 - t} \otimes 2^{(t)}) = \left( \frac{k-t}{2\lambda_1 - k} \right) 1^{(k)} \cdot 1^{(k-t)} (2^{(t)}).
\]
where the binomial coefficient \( \binom{k-l}{t-1} \) comes from the product \( (2k-l-k) \binom{k-l}{t-1} 2^{(t)} = (2k-l-k) \binom{1}{t-1} 2^{(t)} \in D_2 F \) (remember that \( k = \lambda_1 + \lambda_2 \)), and \( \cdot \) indicates multiplication in \( D_2(F) \).

We will show that \( 1^{(k)}, 1^{(k-t-1) 2^{(t)}} \in M_j \) by distinguishing two cases.

**Case 1.** Suppose that \( t \) is even. Then \( (2k-t, t) \) is a partition in \( E \) and \( (2k-t, t) > \lambda \) because \( t < 2\lambda_2 \). Hence \( M_{(2k-t, t)} \subseteq M_j \). Finally, we have \( 1^{(k)}, 1^{(k-t-1) 2^{(t)}} = \sigma_{(2k-t, t)}(1^{(2k-t)} \otimes 2^{(t)}) \in M_{(2k-t, t)} \subseteq M_j \).

**Case 2.** Suppose \( t \) is odd, say \( t = 2s + 1 \). For each \( l = 0, 1, \ldots, s \), \( (2k-2l, 2l) \) is a partition in \( E \) with \( (2k-2l, 2l) > \lambda \). Hence, \( \sigma_{(2k-2l, 2l)}(1^{(2k-2l)} \otimes 2^{(2l)}) \in M_j \) and, in particular,

\[
\sigma_{(2k-2l, 2l)}(1^{(2k-2l)} 2^{(t-2l)} 2^{(2l)}) \in M_j, \tag{3}
\]

for \( l = 0, 1, \ldots, s \). We compute the left-hand side of (3) to obtain

\[
\sigma_{(2k-2l, 2l)}(1^{(2k-2l)} 2^{(t-2l)} 2^{(2l)}) = \sum_{i=0}^{t} \binom{t-i}{2l} 1^{(k-t-1)} 2^{(i)}, \tag{4}
\]

where it is understood that \( \binom{t-i}{2l} = 0 \) if \( t-i < 2l \). Note that \( 1^{(k-t-1)} 2^{(i)}, 1^{(k-t-i)} 2^{(t-i)} \in M_j \) if \( i = j \) or \( i = t-j \), because multiplication in \( D_2(F) \) is commutative. Now (4) can be written as follows:

\[
\sigma_{(2k-2l, 2l)}(1^{(2k-2l)} 2^{(t-2l)} 2^{(2l)}) = \sum_{j=0}^{t-2l} \binom{t-i}{2l} 1^{(k-t-1)} 2^{(i)}, \tag{5}
\]

\( l = 0, 1, \ldots, s \). One of the elements of \( D_2(F) \) in the right-hand side of (5) is \( 1^{(k)}, 1^{(k-t-1) 2^{(t)}} \). In light of (1), we will have \( 1^{(k)}, 1^{(k-t-1) 2^{(t)}} \in M_j \) once we show that the system (5) (consisting of \( (s+1) \) equations and the \( (s+1) \) “unknowns” \( 1^{(k-t-1) 2^{(t)}}, 1^{(k-t-i) 2^{(t-i)}}, 0 \leq i \leq s \)) is solvable. By permuting columns, substituting \( t = 2s + 1 \), and applying Corollary 1.6 we have

\[
\det \begin{bmatrix} \binom{t-i}{2l} + \binom{1}{2l} \\ \binom{i}{2l} \end{bmatrix}_{0 \leq i, j \leq s} = \pm \det \begin{bmatrix} 2s+1-i \\ 2s-2j \end{bmatrix}_{0 \leq i, j \leq s} = \pm 2^{s+1}.
\]

Since \( 2 \) is invertible in \( R \), we see that our system is solvable. The proof is complete.
Proof of Theorem 1.8. Over the rationals, we have (see, for example, [FH, Exercise 6.16])

\[ \text{rk}(D_2(D_k F)) = \sum_{\bar{\lambda} \in E} \text{rk}(K_{\bar{\lambda}} F). \tag{6} \]

Since the ranks of the modules involved are independent of the ground ring [ABW], (6) is also valid over \( R \). It follows then that the surjective maps \( K_{\bar{\lambda}} F \to M_{\bar{\lambda}}/M_{\bar{\lambda}}, \bar{\lambda} \in E, \) of Proposition 1.9 are isomorphisms.

Remark 1.10. We remark that the assumption on the invertibility of 2 was used twice: once to show that \( \{ M_{\bar{\lambda}} \} \) is a filtration of \( D_2(D_k F) \) (Lemma 1.7), and once to show that our maps \( \varphi_{\bar{\lambda}}, \bar{\lambda} \in E, \) induce maps \( K_{\bar{\lambda}} F \to M_{\bar{\lambda}}/M_{\bar{\lambda}} \) (Proposition 1.9). In fact, the proofs show that our assumption on 2 is necessary in order to obtain our result via the maps \( \varphi_{\bar{\lambda}} \).

Remark 1.11. We conclude this section by observing that contravariant duality implies the third statement in \((*)\) of the Introduction.

2. \( \Lambda^2(D_k F) \)

Fix a positive integer \( k \) and let \( O \) denote the set of all partitions \( \bar{\lambda} = (2\lambda_1 + 1, 2\lambda_2 - 1) \) with \( |\bar{\lambda}| = \lambda_1 + \lambda_2 = k \); i.e., \( O \) consists of the partitions of \( 2k \) into two odd parts. The purpose of this section is to filter \( \Lambda^2(D_k F) \) so that the associated graded object is \( \sum_{\bar{\lambda} \in O} K_{\bar{\lambda}} F. \) As usual, we assume that 2 is invertible in \( R. \)

Definition 2.1. For each \( \bar{\lambda} = (2\lambda_1 + 1, 2\lambda_2 - 1) \in O \), we define a \( GL(F) \)-map \( \varphi_{\bar{\lambda}} : D_{2\lambda_1 + 1} F \otimes D_{2\lambda_2 - 1} F \to \Lambda^2(D_k F) \) as the composition

\[
D_{2\lambda_1 + 1} F \otimes D_{2\lambda_2 - 1} F \xrightarrow{A \otimes 1} D_k F \otimes D_{2\lambda_1 + 1 - k} F \otimes D_{2\lambda_2 - 1} F \xrightarrow{1 \otimes m} D_k F \otimes D_k F \xrightarrow{m'} \Lambda^2(D_k F),
\]

where \( A \) is the diagonal map of \( DF \), \( m \) is the multiplication map of \( DF \), and \( m' \) is multiplication in \( \Lambda^2(D_k F) \).

Definition 2.2. For each \( \bar{\lambda} \in O \), we define \( GL(F) \)-submodules \( M_{\bar{\lambda}} \) and \( \tilde{M}_{\bar{\lambda}} \) of \( \Lambda^2(D_k F) \) by

\[
M_{\bar{\lambda}} = \sum_{\mu < \bar{\lambda} \atop \mu \in O} \text{Im} \varphi_\mu \quad \text{and} \quad \tilde{M}_{\bar{\lambda}} = \sum_{\mu < \bar{\lambda} \atop \mu \in O} \text{Im} \varphi_\mu,
\]

where \( \geq \) denotes the lexicographic ordering of partitions.
Remark 2.3. If $k$ is odd our filtration looks like
\[ 0 \leq M_{(2k-1, 1)} \leq \cdots \leq M_{(k, k)} = A^2(D_k F), \]
while if $k$ is even we have
\[ 0 \leq M_{(2k-1, 1)} \leq \cdots \leq M_{(k+1, k-1)}, \]
and it remains to be shown that $M_{(k+1, k-1)} = A^2(D_k F)$.

Lemma 2.4. Let $k \in \mathbb{Z}$, then
\[ [M^*, *, O, *] \text{ is a filtration of } A^2(D_k F). \]

Proof. By Remark 2.3 we may assume that $k$ is even, say $k = 2s$. Let $x, y \in A^2(D_k F)$ with $x, y \in M_{(k+1, k-1)}$. For each $j = 0, 1, \ldots, s-1$, note that $(k+2j+1, k-2j-1)$ is a partition in 0 and
\[ (k+2j+1, k-2j-1) \gg (k+1, k-1). \]

Consider the element
\[ \sum_a x_a(2j+1) \otimes y_a(k-2j-1)' \in D_{k+2j+1} F \otimes D_{k-2j-1} F. \]
A quick computation yields (cf. Eq. (1))
\begin{align*}
\varphi_{(k+2j+1, k-2j-1)} \left( \sum_a x_a(2j+1) \otimes y_a(k-2j-1)' \right) & = \sum_{i=0}^{k-2j-1} \binom{k-i}{k-2j-1} \sum_{a, b} x_{a}(k-i) \cdot x_{b}(i) \cdot y_{a}(k-i)' \\
& = -\sum_{a, b} x_{a}(i) \cdot x_{b}(k-i) \cdot y_{a}(k-i)', \\
& = \sum_{a, b} x_{a}(k-i) \cdot x_{b}(i) \cdot y_{a}(k-i)' \\
& = \sum_{a, b} x_{a}(k-i) \cdot x_{b}(i) \cdot y_{a}(k-i)', \tag{8}
\end{align*}
In particular (for $i = s$), we have $\sum_{a, b} x_{a}(s) \cdot x_{b}(s) \cdot y_{a}(s)' = 0$. These remarks allow us to rewrite (8) as follows:
\begin{align*}
\varphi_{(k+2j+1, k-2j-1)} \left( \sum_a x_a(2j+1) \otimes y_a(k-2j-1)' \right) & = \sum_{i=0}^{k-2j-1} \binom{k-i}{k-2j-1} - \binom{i}{k-2j-1} \\
& \times \sum_{a, b} x_{a}(k-i) \cdot x_{b}(i) \cdot y_{a}(k-i)', \tag{9}
\end{align*}
Since \( k = 2s \), the matrix of system (9) is 
\[
\begin{pmatrix}
(2s - 2j - 1) & \cdots & (2s - 2j - 1) \\
\vdots & \ddots & \vdots \\
(2s - 2j - 1) & \cdots & (2s - 2j - 1)
\end{pmatrix}_{0 \leq i, j \leq s - 1},
\]
which is invertible by the second statement in Corollary 1.6. This and (7) imply that
\[
\sum_{a, b} x_{a}(k - i) y_{a}(i) \cdot X_{b}(i)' y_{b}(k - i)' \in M_{(k + 1, k - 1)},
\]
for each \( i = 0, 1, \ldots, s - 1 \). For \( i = 0 \) we obtain \( x \cdot y \in M_{(k + 1, k - 1)} \), and the proof is complete.

**Theorem 2.5.** The associated graded object of the filtration \( \{ M_{a} \} \) is
\[
\sum_{i \in \mathbb{O}} K_{i} F.
\]

The above theorem will follow once we show

**Proposition 2.6.** Let \( \lambda \in \mathbb{O} \). Then the map \( \varphi_{\lambda} \) induces a surjective map \( K_{i} F \to M_{\lambda} / \lambda \).

**Proof.** Let \( \lambda = (2\lambda_{1} + 1, 2\lambda_{2} - 1) \) and \( 0 \leq t < 2\lambda_{2} - 1 \). It suffices to show (ABW, Sect. II.3) that \( \text{Im}(\varphi_{\lambda} \circ \Box_{i}) \subseteq M_{\lambda_{i}} \), where \( \Box_{i} \) is the composition
\[
D_{2\lambda_{1} + 2\lambda_{2} - t} F \otimes D_{1} F \xrightarrow{\Delta \otimes 1} D_{2\lambda_{1} + 1} F
\]
\[
\otimes D_{2\lambda_{1} - t - 1} F \otimes D_{1} F \xrightarrow{1 \otimes m} D_{2\lambda_{1} + 1} F \otimes D_{2\lambda_{1} - 1} F.
\]

As in the Proof of Proposition 1.9, we restrict our attention to
\[
1(2\lambda_{1} + 2\lambda_{2} - t) \otimes 2^{(t)} \in D_{2\lambda_{1} + 2\lambda_{2} - t} F \otimes D_{1} F.
\]
A quick computation yields
\[
\varphi_{\lambda} \circ \Box_{i}(1(2\lambda_{1} + 2\lambda_{2} - t) \otimes 2^{(t)}) = \begin{pmatrix} k - t \\ 2\lambda_{1} + 1 - k \end{pmatrix} 1^{(k)}, 1^{(k - t)} 2^{(t)}.
\]

We will show that \( 1^{(k)} 1^{(k - t)} 2^{(t)} \in M_{\lambda_{i}} \) by distinguishing two cases.

1. Suppose that \( t \) is odd. Then \( (2k - t, t) \) is a partition in \( \mathbb{O} \) and \( (2k - t, t) > \lambda \), because \( t < 2\lambda_{2} - 1 \). Hence \( M_{(2k - t, t)} \subseteq M_{\lambda} \). Finally, we have
\[
1^{(k)} 1^{(k - t)} 2^{(t)} = \varphi_{(2k - t, t)}(1^{(2k - t)} \otimes 2^{(t)}) \in M_{(2k - t, t)} \subseteq M_{\lambda}.
\]

2. Suppose that \( t \) is even, say \( t = 2s \). For each \( l = 0, 1, \ldots, s - 1 \), \( (2k - 2l - 1, 2l + 1) \) is a partition in \( \mathbb{O} \) with \( (2k - 2l - 1, 2l + 1) > \lambda \), because \( t < 2\lambda_{2} - 1 \). Hence \( \text{Im} \varphi_{(2k - 2l - 1, 2l + 1)} \subseteq M_{\lambda} \), and, in particular,
\[
\varphi_{(2k - 2l - 1, 2l + 1)}(1^{(2k - t)} 2^{(t - 2l - 1)} \otimes 2^{(2l + 1)}) \in M_{\lambda}, \quad (10)
\]
where it is understood that \((t \leq i) = 0\) if \(t < 2l + 1\). Since multiplication in \(A^2(D_k F)\) is skew commutative, we have

\[
1 \cdot 1 \cdot 1 - 1 \cdot 1 \cdot 1 = 1 \cdot 1 \cdot 1 - 1 \cdot 1 \cdot 1 \cdot 1 = 0.
\]

Therefore, system (12) is solvable and we obtain \(1 \cdot 1 \cdot 1 = M_{4*}\) because of (10). For \(i = 0\) we have \(1 \cdot 1 \cdot 1 = M_{4*}\) and the proof is complete.

The proof of Theorem 2.5 now follows trivially once we recall that over the rationals, \(rk(4^2(D_k F)) = \sum_{i=0}^{4*} \text{rk}(K_i F)\) (see, for example, [FH, Exercise 6.16]).

Remark 2.7. As in the case of \(D_2(D_k F)\), we remark that the assumption on \(2\) is necessary in order to obtain our result via the map \(\varphi_{\lambda}\).

Remark 2.8. Contravariant duality implies the fourth statement in (*) of the Introduction.
3. $S_2(A^k F)$

We continue to assume that 2 is invertible in our ground ring $R$. Our goal is to construct in an explicit fashion on natural filtration of $S_2(A^k F)$, so that the composition factors are Schur modules.

Fix a positive integer $k$ and let $P$ denote the set of all partitions of the form $\lambda = (k + 2m, k - 2m)$.

**Definition 3.1.** For each $\lambda = (k + 2m, k - 2m) \in P$, we define a $GL(F)$-map $\psi_\lambda : A^{k + 2m} F \otimes A^{k - 2m} F \to S_2(A^k F)$ as the composition

$$A^{k + 2m} F \otimes A^{k - 2m} F \xrightarrow{A \otimes 1} A^k F \otimes A^{2m} F \xrightarrow{1 \otimes m} A^k F \otimes A^k F \xrightarrow{m'} S_2(A^k F),$$

where $A$ is the diagonal map of $AF$, $m$ is the multiplication map of $AF$, and $m'$ is multiplication in $S_2(A^k F)$.

**Definition 3.2.** For each $\lambda \in P$, we define $GL(F)$-submodules $N_\lambda$ and $\tilde{N}_\lambda$ of $S_2(A^k F)$ by

$$N_\lambda = \sum_{\mu \in P} \text{Im} \psi_\mu \quad \text{and} \quad \tilde{N}_\lambda = \sum_{\mu \in P} \text{Im} \psi_\mu,$$

where $\geq$ denotes the lexicographic ordering of partitions.

**Remark 3.3.** If $k$ is even, our filtration looks like

$$0 \subseteq N_{(2k)} \subseteq N_{(2k - 2, 2)} \subseteq \cdots \subseteq N_{(k, k)} = S_2(A^k F),$$

while for $k$ odd we have

$$0 \subseteq N_{(2k - 1, 1)} \subseteq N_{(2k - 3, 3)} \subseteq \cdots \subseteq N_{(k, k)} = S_2(A^k F).$$

Unlike Remark 1.4, the assumption on 2 is not needed here.

**Theorem 3.4.** The associated graded object of $\{N_\lambda\}$ is $\sum_{\lambda \in P} L_\lambda F$.

As usual, the previous theorem will follow once we prove

**Proposition 3.5.** Let $\lambda \in P$. Then the map $\psi_\lambda$ induces a map $L_\lambda F \to N_\lambda / \tilde{N}_\lambda$. 


Proof. Let \( \lambda = (k + 2m, k - 2m) \) and \( 0 \leq t < k - 2m \). It suffices to show \([ABW, \text{Sect. II.2}]\) that \( \text{Im}(\psi_\lambda \circ \square_\lambda) \subseteq \mathcal{N}_\lambda \), where \( \square_\lambda \) is the composition

\[
A^{k-t}F \otimes A^tF \xrightarrow{\text{A} \otimes 1} A^{k+2m}F \otimes A^{k+2m-t}F \xrightarrow{1 \otimes \text{m}} A^{k+2m}F \otimes A^{k-2m}F.
\]

Let \( x \otimes y \in A^{2k-t}F \otimes A^tF \). A quick computation yields

\[
\psi_\lambda \circ \square_\lambda(x \otimes y) = \left( \frac{k-t}{2m} \right)^a x_a(k) \cdot x_a(k-t)' y,
\]

where the binomial coefficient comes from the composition of diagonalization and multiplication in \( AF \), \( \sum_a x_a(k) \otimes x_a(k-t)' \) is the image of \( x \) under the diagonal \( A^{2k-t}F \to A^kF \otimes A^{k-t}F \), multiplication in \( AF \) is denoted by juxtaposition, and multiplication in \( S_2(A^tF) \) is denoted by \( \cdot \).

We will show that \( \sum_a x_a(k) \cdot x_a(k-t)' y \in \mathcal{N}_\lambda \) by distinguishing two cases.

Case 1. Suppose that \( k \) and \( t \) have the same parity. Then \((2k-t, t)\) is a partition in \( P \) and \((2k-t, t) > \lambda \). Hence \( N_{(2k-t, t)} \subseteq \mathcal{N}_\lambda \). Finally, we have

\[
\sum_a x_a(k) \cdot x_a(k-t)' y = \psi_{(2k-t, t)}(x \otimes y) \in N_{(2k-t, t)} \subseteq \mathcal{N}_\lambda.
\]

Case 2. Suppose that \( k \) and \( t \) have distinct parities. For the sake of clarity, we consider two subcases.

Case 2(a). Suppose \( k \) is even and \( t \) is odd. Let \( t = 2s + 1 \). For each \( l = 0, 1, \ldots, s \), \((2k-2l, 2l)\) is a partition in \( P \) satisfying \((2k-2l, 2l) > \lambda \). Hence, \( \text{Im} \psi_{(2k-2l, 2l)} \subseteq \mathcal{N}_\lambda \) and, in particular,

\[
\psi_{(2k-2l, 2l)} \left( \sum_a x_a(t-2l) \otimes y_a(2l)' \right) \in \mathcal{N}_\lambda,
\]

for \( l = 0, 1, \ldots, s \), where \( x \otimes y \in A^{2k-t}F \otimes A^tF \) and \( \sum_a y_a(t-2l) \otimes y_a(2l)' \) is the image of \( y \) under the diagonal \( A^tF \to A^{t-2l}F \). A quick computation yields

\[
\psi_{(2k-2l, 2l)} \left( \sum_a x_a(t-2l) \otimes y_a(2l)' \right) = \sum_{l=0}^t (-1)^{(k-t+l)} \left( \frac{t-l}{2l} \right) \times \sum_{a,b} x_{a}(k-i) \cdot y_{a}(i) \cdot x_{b}(k-t+i)' \cdot y_{b}(t-i)'.
\]

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We now use the identity

\[ \sum_{a,b} x_a(k-i) y_a(i) \cdot x_a(k-t+i)' y_a(t-i)' = \begin{cases} (-1)^{k-i(k-t+i) + i(t-i)} \\ \times \sum_{a,b} x_a(k-t+i) y_a(t-i) \cdot x_a(k-i)' y_a'(i)' \end{cases} \]

(15)

(which holds because multiplication in \( S_2(A^kF) \) is commutative and multiplication in \( AF \) is skew commutative) and the fact that

\[ (-1)^{k-t+i} = 1 = (-1)^{k-i(k-t+i) + i(t-i)} \]

(which holds because \( k \) is even and \( t \) is odd) in order to collect terms in the right-hand side of (14). Thus,

\[ \psi_{(2k-2l,2l)} \left( \sum_a x_a(t-2l) \otimes y_a(2l)' \right) = \sum_{i=0}^{s} \left( \begin{pmatrix} t-i \\ 2l \end{pmatrix} + \begin{pmatrix} i \\ 2l \end{pmatrix} \right) \times \sum_{a,b} x_a(k-i) y_a(i) \cdot x_a(k-t+i)' y_a(t-i), \]

(16)

for \( l = 0, 1, ..., s \). The matrix of this system is identical to the matrix appearing in (5) (see Case 2 of the Proof of Proposition 1.9) which we saw is invertible. From this, (16), and (13), we have

\[ \sum_{a,b} x_a(k-i) y_a(i) \cdot x_a(k-t+i)' y_a(t-i)' \in \hat{N}_k \]

for each \( i = 0, 1, ..., s \). In particular, for \( i = 0 \) we obtain \( \sum_{b} x_b(k) \cdot x_b(k-t)' y \in \hat{N}_k \), which is the desired result.

**Case 2(b).** Suppose \( k \) is odd and \( t \) is even. Let \( t = 2s \). For each \( l = 0, 1, ..., s-1 \), \((2k-2l-1, 2l+1)\) is a partition in \( P \) satisfying \((2k-2l-1, 2l+1) > \lambda\). Hence, \( \text{Im} \psi_{(2k-2l-1,2l+1)} \subseteq \hat{N}_k \) and, in particular,

\[ \psi_{2k-2l-1,2l+1} \left( \sum_a x_a(t-2l-1) \otimes y_a(2l+1)' \right) \in \hat{N}_k, \]

(17)
for \( l = 0, 1, \ldots, s - 1 \), where \( x \otimes y \in A^{2k-1}F \otimes A^tF \). A quick computation gives (cf. Eq. (14))

\[
\psi_{(2k-2l-1, 2l+1)} \left( \sum_{a} xy_a (t-2l-1) \otimes y_d (2l+1) \right)
\]

\[
= \sum_{i=0}^{t} \left( -1 \right)^{i-t+i} \left( \frac{t-i}{2l+1} \right) \times \sum_{a,b} x_a (k-i) y_a (i) \cdot x_b (k-t+i) \cdot y_d (t-i),
\]

(18)

for \( l = 0, 1, \ldots, s - 1 \). Under the assumption on \( k \) and \( t \) we have

\[
(-1)^{i-t+i} = 1 \quad \text{and} \quad (-1)^{k-i(k-t+i) + t(t-i)} = -1.
\]

Also note that the summand of the right-hand side of (18) corresponding to \( i = s \) is a multiple of \( \sum_{a,b} x_a (k-s) y_a (s) \cdot x_b (k-s) \cdot y_d (s) \cdot y_a (t) \), which is zero because \( k \) is odd. (This follows also from Eq. (15)). These remarks, Eq. (15), and the fact that \( t = 2s \) allow us to collect terms in the right-hand side of (18). Thus,

\[
\psi_{(2k-2l-1, 2l+1)} \left( \sum_{a} xy_a (t-2l-1) \otimes y_d (2l+1) \right)
\]

\[
= \sum_{i=0}^{t-1} \left[ \left( \frac{t-i}{2l+1} \right) - \left( \frac{t-1}{2l+1} \right) \right] \sum_{a,b} x_a (k-i) y_a (i) \cdot x_b (k-t+i) \cdot y_d (t-i),
\]

(19)

\( l = 0, 1, \ldots, s - 1 \). The matrix of this system is identical to the matrix of (12) in the Proof of Proposition 2.6, which we saw is invertible. From this, (19), and (17) we obtain \( \sum_{a,b} x_a (k-i) y_a (k-t+i) \cdot y_d (t-i) \in N_{2s} \), which is the desired result. This completes the proof.

The proof of Theorem 3.4 follows now trivially once we recall that over the rationals we have (see, for example, [FH, Exercise 6.16])

\[
\text{rk}(S_2(A^kF)) = \sum_{x \in P} \text{rk}(L_x F).
\]

(20)

Remark 3.6. Similarly to [B, Remark 2.8] one could, instead of invoking [FH], prove directly Eq. (20) over the rationals by showing that \( L_{x,F} \rightarrow N_{2s}/N_{2s} \) is nonzero (consider the image of the canonical tableau) and hence is an isomorphism (by the irreducibility of the \( GL(F) \)-module \( L_{x,F} \) over the rationals).
4. $A^2(A^k F)$

Fix a positive odd integer $k$ and let $Q$ denote the set of all partitions of the form $\lambda = (k + 2m + 1, k - 2m - 1)$. As usual, we assume that $2$ is invertible in $R$. The purpose of this section is to show that the exterior power $A^2(A^k F)$ has a natural filtration whose associated graded object is $\sum_{\lambda \in Q} L_\lambda F$.

Our approach is along the lines followed in the previous sections. Hence, we will only indicate the appropriate (minor) modifications that are needed for $A^2(A^k F)$.

For each $\lambda = (k + 2m + 1, k - 2m - 1) \in Q$, define a $GL(F)$-map $\psi_\lambda : A^k + 2m + 1 F \otimes A^k - 2m - 1 F \to A^2(A^k F)$ as the composition

$$A^k + 2m + 1 F \otimes A^k - 2m - 1 F \to A^k F \otimes A^2 2m + 1 F \otimes A^k F \to A^2(A^k F).$$

Now let $N_\lambda = \sum_{\mu \in Q, \mu \geq \lambda} \text{Im } \psi_\mu$ and $\tilde{N}_\lambda = \sum_{\mu \in Q, \mu \geq \lambda} \text{Im } \psi_\mu$. Our filtration looks like

$$0 \subseteq \cdots \subseteq N_{(k+3,k-3)} \subseteq N_{(k+1,k-1)},$$

and we must show that $N_{(k+1,k-1)} = A^2(A^k F)$. The proof of this fact is very similar to the Proof of Lemma 1.7 or that of Lemma 2.4 (depending on whether $k$ is even or odd) and is thus omitted.

Let $\lambda \in Q$. One then proves that $\psi_\lambda$ induces a surjective map $L_\lambda F \to N_\lambda / \tilde{N}_\lambda$ in a way very similar to the proof of Proposition 3.5. Note that Eq. (15) must be replaced by

$$\sum_{a,b} x_a(k-i) y_a(i) x_a(k-t+i) y_a(t-i) = (-1)^{k-i}(k-t+i) a(t-i)+1 \sum_{a,b} x_a(k-t+i) y_a(t-i) x_a(k-i) y_a(i)$$

in order to account for the skew commutativity of multiplication in $A^2(A^k F)$.

It follows now that the associated graded object of the filtration $\{N_\lambda\}$ is $\sum_{\lambda \in Q} L_\lambda F$ as desired.

Note Added in Proof. At the time this article was accepted for publication, S. Donkin informed me that the existence of our good filtrations follows also from his work. However, his methods do not give these filtrations explicitly. To be more precise, consider the exact sequence of $GL(F)$-modules

$$0 \to A^2(A^k F) \to A^k F \otimes A^k F \to S_2(A^k F) \to 0$$
which is split (for the GL \((F)\) action) if \(2\) is invertible in \(R\). Since \(A^kF \otimes A^lF\) has a good filtration (for example, see S. Donkin, “Rational Representations of Algebraic Groups: Tensor Products and Filtrations,” Lecture Notes in Math, Vol. 1140, Springer-Verlag, New York/Berlin, or [AB1, p. 168]), it follows from S. Donkin, loc. cit., Prop. 3.2.5, that \(A^k(A^lF)\) and \(S_2(A^lF)\) have good filtrations if \(2\) is invertible in \(R\). Similar remarks apply to the other plethysms considered in this paper. Finally, to the counterexamples of existence of good filtrations in characteristic \(2\) mentioned in the second paragraph of the Introduction one should add the example on p. 71 of S. Donkin, Good filtrations of rational modules for reductive groups, in “Arcata Conference on Representations of Finite Groups,” Proc. of Symposia in Pure Mathematics, Vol. 47, Part 1, pp. 69–80, Amer. Math. Soc., Providence, RI. I thank Stephen Donkin for pointing out these matters to me.

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