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Note

On 3-colorable plane graphs without 5- and 7-cycles ☆

Baogang Xu

School of Mathematics and Computer Science, Nanjing Normal University, Ninghai Road 122, Nanjing 210097, PR China

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Abstract

In this note, it is proved that every plane graph without 5- and 7-cycles and without adjacent triangles is 3-colorable. This improves the result of [O.V. Borodin, A.N. Glebov, A. Raspaud, M.R. Salavatipour, Planar graphs without cycles of length from 4 to 7 are 3-colorable, J. Combin. Theory Ser. B 93 (2005) 303–311], and offers a partial solution for a conjecture of Borodin and Raspaud [O.V. Borodin, A. Raspaud, A sufficient condition for planar graphs to be 3-colorable, J. Combin. Theory Ser. B 88 (2003) 17–27].

Keywords: Plane graph; Cycle; Coloring

In [1], Borodin et al. proved that every plane graph G without cycles of length from 4 to 7 is 3-colorable that provides a new upper bound to Steinberg's conjecture (see [3, p. 229]). In [2], Borodin and Raspaud proved that every plane graph with neither 5-cycles nor triangles of distance less than four is 3-colorable, and they conjectured that every plane graph with neither 5-cycles nor adjacent triangles is 3-colorable, where the distance between triangles is the length of the shortest path between vertices of different triangles, and two triangles are said to be adjacent if they have an edge in common. In [4], Xu improved Borodin and Raspaud's result by showing that every plane graph with neither 5-cycles nor triangles of distance less than three is 3-colorable.

In this note, it is proved that every plane graph without 5- and 7-cycles and without adjacent triangles is 3-colorable. This improves the result of [1], and offers a partial solution for Borodin and Raspaud's conjecture [2].

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E-mail address: baogxu@njnu.edu.cn.

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Let $G = (V, E, F)$ be a plane graph, where V, E and F denote the sets of vertices, edges and faces of G, respectively. The neighbor set and degree of a vertex v are denoted by $N(v)$ and $d(v)$, respectively. Let f be a face of G. We use $b(f)$, $V(f)$ and $N(f)$ to denote the boundary of f, the set of vertices on $b(f)$, and the set of faces adjacent to f, respectively. The degree of f, denoted by $d(f)$, is the length of the facial walk of f. A k-vertex (k-face) is a vertex (face) of degree k.

Let C be a cycle of G. We use $\mathrm{int}(C)$ and $\mathrm{ext}(C)$ to denote the sets of vertices located inside and outside C, respectively. C is called a *separating* cycle if both $\mathrm{int}(C) \neq \emptyset$ and $\mathrm{ext}(C) \neq \emptyset$, and is called a *facial cycle* otherwise. For convenience, we still use C to denote the set of vertices of C.

An 11-face f of G is called a *special face* if the following hold:

(1) $b(f)$ is a cycle;
(2) f is adjacent to a triangle;
(3) every vertex $x \notin V(f)$ has at most two neighbors on $b(f)$; and
(4) for every edge uv of $G \setminus V(f)$, $|(N(u) \cap V(f)| + |N(v) \cap V(f)| \leqslant 3$.

A vertex in $G \setminus V(f)$ that violates (3) is called a *claw-center* of $b(f)$, and a pair of adjacent vertices in $G \setminus V(f)$ that violates (4) is called a *d-claw-center* of $b(f)$.

A separating 11-cycle C is called a *special cycle* if in $G \setminus \mathrm{ext}(C)$, C is the boundary of a special face. We use $\mathcal{G}$ to denote the set of plane graphs without 5- and 7-cycles and without adjacent triangles. Following is our main theorem.

Theorem 1. *Let G be a graph in $\mathcal{G}$ that contains cycles of length 4 or 6, f an arbitrary face that is a special face, or a 3-face, or a 9-face with $b(f)$ being a cycle. Then, any 3-coloring of f can be extended to G.*

As a corollary of Theorem 1, every plane graph in $\mathcal{G}$ is 3-colorable. To see this, let G be a plane graph in $\mathcal{G}$. By Grötzsch's theorem, we may assume that G contains triangles. If G contains neither 4-cycles nor 6-cycles, then by [1, Theorem 1.2], G is 3-colorable. Otherwise, for an arbitrary triangle T, any 3-coloring of T can be extended to $\mathrm{int}(T)$ and $\mathrm{ext}(T)$, that yields a 3-coloring of G.

Proof of Theorem 1. Assume that G is a counterexample to Theorem 1 with minimum $\sigma(G) = |V(G)| + |E(G)|$. Without loss of generality, assume that the unbounded face f_o is a special face, or a 3-face or a 9-face with $b(f)$ being a cycle, such that a 3-coloring ϕ of f_o cannot be extended to G. Let $C = b(f_o)$ and let $p = |C|$. Then, every vertex not in C has degree at least 3.

By our choice of G, neither 4-cycle nor 6-cycle is adjacent to triangles. Since $G \setminus \mathrm{int}(C')$ is still in $\mathcal{G}$ for any separating cycle C' of G.

Lemma 1. *G contains neither special cycles, nor separating k-cycles, $k = 3, 9$.*

Lemma 2. *G is 2-connected. That is, the boundary of every face of G is a cycle.*

Interested readers may find the proof of Lemma 2 in [1] (see that of Lemma 2.2).

Let C' be a cycle of G, and u and v two vertices on C'. We use $C'[u,v]$ to denote the path of C' clockwisely from u to v, and let $C'(u,v) = C'[u,v] \setminus \{u,v\}$. Unless specified particularly, we always write a cycle on its vertices sequence clockwisely.

Lemma 3. *C is chordless.*

Proof. Assume to the contrary that C has a chord uv. Let $S_1 = V(C(u,v))$, $S_2 = V(C(v,u))$, and assume that $|S_1| < |S_2|$. It is certain that $p = 9$ or 11, and $|S_1| \leqslant 4$. Since $|S_1| = 3$ provides $C[u,v] + uv$ is a 5-cycle, and $|S_1| = 4$ provides $C[v,u] + uv$ is a $(p-4)$-cycle, we assume that $|S_1| = 1$ or 2.

If $|S_1| = 1$, say $S_1 = \{w\}$, then $uvwu$ bounds a 3-face by Lemma 1. Let G' be the graph obtained from $G - w$ by inserting a new vertex into uv. Then, $G' \in \mathcal{G}$, $\sigma(G') = \sigma(G) - 1$. We can extend ϕ to a 3-coloring ϕ' of G'. This produces a contradiction because ϕ' and $\phi(w)$ yield a 3-coloring of G that extends ϕ.

Assume $|S_1| = 2$. Since $C[v,u] + uv$ is a $(p-2)$-cycle, $p = 11$ and $C[v,u] + uv$ bounds a 9-face by Lemma 1. Let G' be the graph obtained from $G \setminus S_2$ by inserting five vertices into uv. Then, $G' \in \mathcal{G}$, $\sigma(G') < \sigma(G)$. By assigning appropriate colors to the new added vertices, the restriction of ϕ on $C[u,v]$ can be extended to a 3-coloring ϕ' of G'. The restriction of ϕ' on $G \setminus C(v,u)$ and ϕ again yield a 3-coloring of G that extends ϕ, a contradiction. □

Lemma 4. *$N(u) \cap N(v) \cap \mathrm{int}(C_1) = \emptyset$ for separating 11-cycle C_1 and $uv \in E(C_1)$.*

Proof. Assume to the contrary that $x \in N(u) \cap N(v) \cap \mathrm{int}(C_1)$. By Lemma 1, $xuvx$ bounds a 3-face. We will show that C_1 has neither claw-center nor d-claw-center. Then, C_1 is a special cycle that contradicts Lemma 1.

If $xw \in E(G)$ for some $w \in C_1 \setminus \{u,v\}$, assume that u, v and w clockwisely lie on C_1, then $|V(C_1(v,w))| \geqslant 5$ and $|V(C_1(w,u))| \geqslant 5$ since $G \in \mathcal{G}$, and hence $|C_1| \geqslant 13$, a contradiction. If a vertex $y \in \mathrm{int}(C_1) \setminus \{x\}$ has three neighbors z_1, z_2 and z_3 on C_1, then by simply counting the number of vertices in $C_1 \setminus \{z_1, z_2, z_3\}$, G must contain a 9-cycle C_2 with $x \in \mathrm{int}(C_2)$, a contradiction to Lemma 1 because C_2 is a separating 9-cycle.

Assume that $\{a,b\}$ is a d-claw-center of C_1. Since G has no adjacent triangles, $|(N(a) \cup N(b)) \cap C_1| \geqslant 3$. If $(N(a) \cup N(b)) \cap C_1$ has exactly three vertices, say a_1, a_2 and a_3 clockwisely on C_1, we may assume that $a_1 \in N(a) \cap N(b)$, then $|V(C_1(a_1,a_2))| \geqslant 5$ and $|V(C_1(a_3,a_1))| \geqslant 5$ that provide $|C_1| \geqslant 13$. So, assume that a has two neighbors $a_1, a_2 \in C_1$, b has two neighbors $b_1, b_2 \in C_1 \setminus \{a_1, a_2\}$, and assume these four vertices clockwisely lie on C_1.

If $a_1a_2 \in E(C_1)$, then $|V(C_1(a_2,b_1))| \geqslant 4$ and $|V(C_1(b_2,a_1))| \geqslant 4$ providing $|C_1| \geqslant 12$, a contradiction. So, we may assume that $a_1a_2 \notin E(C_1)$ and $b_1b_2 \notin E(C_1)$, i.e., $|V(C_1(a_1,a_2))| \geqslant 1$ and $|V(C_1(b_1,b_2))| \geqslant 1$. By symmetry, we assume $x \in \mathrm{int}(C_1[a_1,b_1] \cup a_1abb_1)$. By simply counting the number of vertices in $C_1 \setminus \{a_1,a_2,b_1,b_2\}$, we get $|C_1| > 11$, a contradiction. □

Lemma 5. *For $u, v \in C$ and $x \notin C$, if $xu, xv \in E(G)$, then $uv \in E(C)$.*

Proof. Assume to the contrary that $uv \notin E(C)$. By Lemma 3, $uv \notin E(G)$. Let $|V(C[u,v])| = l < |V(C[v,u])|$. Then, $3 \leqslant l \leqslant \frac{p+1}{2} \leqslant 6$.

Since $C[u,v] \cup vxu$ is an $(l+1)$-cycle and $C[v,u] \cup uxv$ is a $(p-l+3)$-cycle, $l \notin \{4,6\}$, and $l \neq 5$ whenever $p = 9$. If $l = 5$ and $p = 11$, $C[v,u] \cup uxv$ must bound a 9-face by Lemma 1, then f_o has to be adjacent to a 3-face f_1 on $C[u,v]$, and hence $C[u,v] \cup vxu \cup b(f_1)$ yields a 7-cycle. So, $l = 3$. Let $C[u,v] = uwv$.

If $p = 11$, then f_o has to be adjacent to a 3-face on $C[v,u]$ that contradicts to Lemma 4 because $C[v,u] \cup vxu$ is a separating 11-cycle. Therefore, $p = 9$ and $C[v,u] \cup vxu$ bounds a 9-face by Lemma 1. Let G' be the graph obtained from $G \setminus V(C(v,u))$ by inserting 5 new vertices into ux. Then, $G' \in \mathcal{G}$, $\sigma(G') < \sigma(G)$, and the unbounded face of G' has degree 9. We can extend $\phi(u)$, $\phi(w)$ and $\phi(v)$ to a 3-coloring ϕ' of G' with $\phi'(u) \neq \phi'(x)$. But ϕ' and ϕ yield a 3-coloring of G that extends ϕ, a contradiction. □

Lemma 6. *G contains neither 4-cycles nor 6-cycles.*

Proof. First assume to the contrary that G contains a 4-cycle $C_1 = uvwxu$. If C_1 is a separating 4-cycle, let ψ be an extension of ϕ on $G \setminus \mathrm{int}(C_1)$, and let G_1 be the graph obtained from $G \setminus \mathrm{ext}(C_1)$ by inserting five new vertices into an edge of C_1. If $p \neq 3$ then $|C \setminus C_1| \geqslant 6$ since C is chordless, and hence $|\mathrm{ext}(C_1)| \geqslant 6$. If $p = 3$ then $|C \cap C_1| \leqslant 1$ and hence $E(C) \cap E(C_1) = \emptyset$, again $|\mathrm{ext}(C_1)| \geqslant 6$ because every face incident with some edge on C_1 is a 4^+-face. Therefore, $\sigma(G_1) < \sigma(G)$, and we can extend the restriction of ψ on C_1 to G_1, and thus get a 3-coloring of G that extends ϕ. So, we assume that G contains no separating 4-cycles, and let f be the face bounded by C_1. We proceed to show that one can identify a pair of diagonal vertices of C_1 such that ϕ can be extended to a 3-coloring of the resulting graph G'. Since any 3-coloring of G' offers a 3-coloring of G, this contradiction guarantees the nonexistence of 4-cycles in G.

If $f \notin N(f_o)$, C_1 contains a pair of diagonal vertices that are not on C. By symmetry, we assume that $u, w \in C_1 \setminus C$ whenever $f \notin N(f_o)$. Let $G_{u,w}$ be the graph obtained from G by identifying u and w, and let r_{uw} be the new vertex obtained by identifying u and w. It is clear that $G_{u,w}$ contains no adjacent triangles since no edge of C_1 is contained in triangles. If $f \notin N(f_o)$, it is certain that ϕ is still a proper coloring of C in $G_{u,w}$. If $f \in N(f_o)$, we may assume that $u \in C$, then $w \notin C$ and $N(w) \cap C \subset \{x, v\}$ by Lemmas 3 and 5, and thus ϕ is also a proper coloring of C in $G_{u,w}$ by letting $\phi(r_{u,w}) = \phi(u)$.

Since a cycle of length 5 or 7 in $G_{u,w}$ yields a 7-cycle or a separating 9-cycle in G, $G_{u,w} \in \mathcal{G}$. Now we need only to check that f_o is still a special face in $G_{u,w}$ in case of $p = 11$. Assume that $p = 11$. We first consider the case that $f \in N(f_o)$.

If C has a claw-center z with neighbors y_1, y_2 and y_3 clockwisely on C in $G_{u,w}$, then $y_i = r_{uw}$ for an i. Assume $y_1 = r_{uw}$. It is clear that $x \notin \{y_2, y_3\}$, and $y_2y_3 \in E(C)$ by Lemma 5. If $|V(C(x, y_2))| \leqslant 3$, then in G, $C(x, y_2) \cup xwzy_2 \cup zy_3$ contains a cycle of length 5 or 7. If $|V(C(y_3, u))| \leqslant 3$, then in G, $C(y_3, u) \cup C_1 \cup wzy_2 \cup zy_3$ contains a cycle of length 5 or 7, or a separating 9-cycle. Therefore, $|V(C(x, y_2))| \geqslant 4$, $|V(C(y_3, u))| \geqslant 4$, and hence $p \geqslant 12$, a contradiction.

Assume that C has a d-claw-center $\{z_1, z_2\}$ in $G_{u,w}$. Since C has no claw-center in $G_{u,w}$, $|N(z_i) \cap C| = 2$, $i = 1, 2$. Let $N(z_1) \cap C = \{y_1, y_2\}$ and $N(z_2) \cap C = \{y_3, y_4\}$. Since G contains no adjacent triangles, $\{y_1, y_2\} \cap \{y_3, y_4\} = \emptyset$ by Lemma 5. Since f_o is a special face in G, we may assume that $y_2 = r_{uw}$. Then, $y_3y_4 \in E(C)$ by Lemma 5. Using the similar argument as used in the last paragraph, we get $p \geqslant 12$ by counting the number of vertices in $C(x, y_3)$, $C(y_4, y_1)$ and $C(y_1, u)$, a contradiction.

In the case that $f \notin N(f_o)$, C has a claw-center z provides $z = r_{u,w}$, and C has a d-claw-center provides $r_{u,w}$ is in the d-claw-center. In either case, one may get a contradiction that $p \geqslant 12$ by almost the same arguments as above.

Now, assume that C' is a 6-cycle of G. Since G contains no 4-cycles as just proved above, every face incident with some edge on C' is a 6^+-face. If C' is a separating cycle, it is not difficult to verify that $|\mathrm{ext}(C')| \geqslant 4$, then by letting G'' be the graph obtained from $G \setminus \mathrm{int}(C')$ by inserting three vertices into an edge of C', we can first extend ϕ to $G \setminus \mathrm{int}(C')$, and then extend the restriction of ϕ on C' to G'', and thus get an extension of ϕ on G. So, we assume that C' bounds a face f'.

If $C' \cap C \neq \emptyset$, we choose u_0 to be a vertex in $C' \cap C$, and choose u_1 to be a vertex in $C' \setminus C$. If $C' \cap C = \emptyset$, since G contains no l-cycle for $l = 4, 5$ or 7, there must be a vertex on C' that has no neighbors on C, we choose such a vertex as u_1. Let $C' = u_0 u_1 \ldots u_5 u_0$, and let H be the graph obtained from G by identifying u_1 and u_5, u_2 and u_4, respectively. Since H contains no adjacent triangles, and any 5-cycle (7-cycle) of H yields a 7-cycle (separating 9-cycle) in G, $H \in \mathcal{G}$.

We will show that ϕ is still a coloring of f_o in H. It is trivial if $C' \cap C = \emptyset$, since the operation from G to H is independent of ϕ. Assume that $C' \cap C \neq \emptyset$. Then, $u_0 \in C$ and $u_1 \notin C$ by our choice, and $u_2 \notin C$ and $N(u_1) \cap C = \{u_0\}$ by Lemma 5. If either u_2 has no neighbors on C, or $u_4 \notin C$, then we are done. Otherwise, assume that $u_4 \in C$ and u_2 has a neighbor, say z, on C, and assume that u_0, z and u_4 lie on C clockwisely. Since G contains no 5-cycles, $u_0 u_4 \notin E(G)$, and hence $u_5 \in C$ by Lemma 5. Since G contains no 4-cycles and no separating 6-cycles, $|V(C(u_0, z))| \geqslant 4$, $|V(C(z, u_4))| \geqslant 4$, and hence $p \geqslant 12$, a contradiction.

Finally, we will prove that f_o is still a special face in H in case of $p = 11$. Then, a contradiction occurs again since ϕ can be extended to H that offers an extension of ϕ to G, this will end the proof of Lemma 6 and also the proof of our theorem.

The proof technique is again, as used repeatedly, to derive a contradiction by counting the number of vertices on the segments divided by the vertices adjacent to some claw-center or d-claw-center of C. We proceed only with the case $C' \cap C = \emptyset$. Assume to the contrary that $p = 11$ but f_o is not a special face in H. Let $r_{1,5}$ and $r_{2,4}$ be the vertices obtained by identifying u_1 and u_5, and u_2 and u_4, respectively.

Assume that C has a claw-center y with three neighbors y_1, y_2 and y_3, clockwisely on C in H. By symmetry, we may assume that $y = r_{1,5}$, and assume that $y_1 u_1 \in E(G)$ and $y_2 u_5, y_3 u_5 \in E(G)$. Then, $y_2 y_3 \in E(C)$ by Lemma 5. Since G contains no adjacent triangles, contains no cycles of length 4, 5 and 7, and contains no separating 9-cycles, $|V(C(y_1, y_2))| \geqslant 4$, $|V(C(y_3, y_1))| \geqslant 5$, and hence $p \geqslant 12$, a contradiction.

Assume that C has a d-claw-center $\{z_1, z_2\}$ in H. Then, each of z_1 and z_2 has two neighbors on C and these four vertices are all distinct. By symmetry, we may assume that $z_1 = r_{1,5}$. z_2 may be u_0, $r_{2,4}$ or a vertex not on $C \cup C'$. In each case, the same argument as above ensures that $p \geqslant 12$. This contradiction completes the proof of Lemma 6. □

Our proof is then completed because by the assumption in Theorem 1, G contains either 4-cycles or 6-cycles. □

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