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The total graph and regular graph of a commutative ring

S. Akbari [a,*], D. Kiani [b,c], F. Mohammadi [b], S. Moradi [b]

[a] Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran

[b] Department of Pure Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Iran

[c] School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran

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ABSTRACT

Let R be a commutative ring. The *total graph* of R, denoted by $T(\Gamma(R))$ is a graph with all elements of R as vertices, and two distinct vertices $x, y \in R$, are adjacent if and only if $x + y \in Z(R)$, where $Z(R)$ denotes the set of zero-divisors of R. Let *regular graph* of R, $Reg(\Gamma(R))$, be the induced subgraph of $T(\Gamma(R))$ on the regular elements of R. Let R be a commutative Noetherian ring and $Z(R)$ is not an ideal. In this paper we show that if $T(\Gamma(R))$ is a connected graph, then $\mathrm{diam}(Reg(\Gamma(R))) \leqslant \mathrm{diam}(T(\Gamma(R)))$. Also, we prove that if R is a finite ring, then $T(\Gamma(R))$ is a Hamiltonian graph. Finally, we show that if S is a commutative Noetherian ring and $Reg(S)$ is finite, then S is finite.

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1. Introduction

The study of algebraic structures using the properties of graphs has become an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring, see [1–8]. Throughout the paper R is a commutative ring with unity. We denote the set of zero-divisor elements and the set of *regular* elements of R, by $Z(R)$ and $Reg(R)$, respectively ($Reg(R) = R \setminus Z(R)$). Throughout the paper we assume that $0 \in Z(R)$. The *total graph* of R denoted by $T(\Gamma(R))$ was introduced in [8], as the graph with all elements of R as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$. Let the *regular graph* of R, $Reg(\Gamma(R))$, be the induced subgraph of $T(\Gamma(R))$ on the vertices $Reg(R)$. There are some rings R, for which $T(\Gamma(R))$ is a connected graph but $Reg(\Gamma(R))$ is not connected, see Example 3.2 of [8]. Also, the regular graph of $\mathbb{Z}_4$ is connected, but its total graph is not connected. It has been proved that if $Z(R)$ is not an ideal and $Reg(\Gamma(R))$ is connected, then $T(\Gamma(R))$ is connected, see Theorem 3.1 of [8]. The motivation of this paper is the study of interplay between the graph-theoretic properties of $T(\Gamma(R))$, $Reg(\Gamma(R))$ and the ring properties of R. In [8] it was proved that for every commutative ring R if $Z(R)$ is not an ideal of R, then $T(\Gamma(R))$ is connected if and only if the ideal generated by $Z(R)$ is R (i.e., $R = (z_1, \ldots, z_n)$ for some $z_1, \ldots, z_n \in Z(R)$). In particular, if R is a finite commutative ring and $Z(R)$ is not an ideal of R, then $T(\Gamma(R))$ is connected. Also we show that for every finite ring R, if $Z(R)$ is not an ideal, then $T(\Gamma(R))$ is a Hamiltonian graph. Among other results we prove that for every commutative Noetherian ring R, if $Reg(R)$ is finite, then R is finite.

Let G be a graph. A *path* of length n is an ordered list of distinct vertices $v_0, \ldots, v_n$ such that $v_{i-1}v_i$, for $i = 1, \ldots, n$ are edges. A (u, v)-*path* is a path with endpoints u and v. A *cycle* is a path $v_0, \ldots, v_n$ with an extra edge v_0v_n. For vertices x and y of G, let $d(x, y)$ be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path between x and y). The *diameter* of G is defined:

$$\mathrm{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}.$$

* Corresponding author.
E-mail addresses: s_akbari@sharif.edu (S. Akbari), dkiani@aut.ac.ir (D. Kiani), f_mohammadi@cic.aut.ac.ir (F. Mohammadi), s_moradi@cic.aut.ac.ir (S. Moradi).

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A graph G is *connected* if it has a (u, v)-path for each pair $u, v \in V(G)$. A *Hamilton cycle* is a spanning cycle in a graph. A graph G is called *Hamiltonian* if G has a Hamilton cycle. For a graph G, $\kappa(G)$, is the smallest number of vertex deletions sufficient to disconnect G. The *Cartesian product* of graphs G and H, $G \times H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$.
Anderson and Badawi showed that if R is a commutative ring, $Z(R)$ is not an ideal and $T(\Gamma(R))$ is connected, then $\mathrm{diam}(Reg(\Gamma(R))) \leqslant \mathrm{diam}(T(\Gamma(R)))$, see [8].

The main goal of this paper is to show that for every commutative Noetherian ring R, if $Z(R)$ is not an ideal, then $\mathrm{diam}(Reg(\Gamma(R))) \leqslant \mathrm{diam}(T(\Gamma(R)))$. In the last section we show that if R is a finite commutative ring and $Z(R)$ is not an ideal, then $T(\Gamma(R))$ is a Hamiltonian graph and $Reg(\Gamma(R))$ is a Hamiltonian graph if and only if R is isomorphic to none of the rings:

$$\mathbb{Z}_2{}^{n+1}, \mathbb{Z}_2{}^n \times \mathbb{Z}_3, \mathbb{Z}_2{}^n \times \mathbb{Z}_4, \mathbb{Z}_2{}^n \times \mathbb{Z}_2[x]/(x^2),$$

where n is a natural number.

2. The diameters of total graph and the regular graph

In this section we would like to study the relation between the diameter of the total graph and the diameter of the regular graph of a commutative Noetherian ring. We show that if R is a commutative Noetherian ring and $Z(R)$ is not an ideal, then $\mathrm{diam}(Reg(\Gamma(R))) \leqslant \mathrm{diam}(T(\Gamma(R)))$. First we start with the following theorem.

Theorem 1. *Let R be a commutative Noetherian ring and l be a natural number. If $a, b \in Reg(R)$ and $d_{T(\Gamma(R))}(a, b) = l$, then $d_{Reg(\Gamma(R))}(a, b) = l$.*

Proof. First assume that $l = 2$. If there exists a regular element c such that c is adjacent to both a and b, then we are done. Thus suppose that no regular element is adjacent to both a and b. Since R is a commutative Noetherian ring, then by Proposition 4.7, of [9, p.53], there are prime ideals $P_1, \ldots, P_n$ such that $Z(R) = \bigcup_{r=1}^{n} P_r$. Since a and b are not adjacent, so $a + b \in Reg(R)$. Since $d_{T(\Gamma(R))}(a, b) = 2$, there exists $z \in Z(R)$ such that $a + z, b + z \in Z(R)$. If $a - b \in Z(R)$, then $a, -b, b$ is a path in $Reg(\Gamma(R))$. Therefore we can assume that $a - b \in Reg(R)$. Set $a_1 = a + z$, $a_2 = b + z$, $a_3 = z$ and $a_4 = a_3 + a_1a_2$. If $a_4 \in Reg(R)$, then a, a_4, b is a path in $Reg(\Gamma(R))$, a contradiction. Now, assume that $a_4 \in Z(R)$. It is easy to see that for each i, $i = 1, \ldots, n$, we have $|P_i \cap \{a_1, a_2, a_3, a_4\}| \leqslant 1$. With no loss of generality assume that $a_i \in P_i$, for $i = 1, 2, 3, 4$. Clearly, $a + a_4 \in P_1$ and $b + a_4 \in P_2$. Now, we inductively construct a_i, $i > 4$ such that $a + a_i \in P_1$, $b + a_i \in P_2$, $a_i \notin \bigcup_{r=1}^{i-1} P_r$ and $a_i \in P_i$. Suppose that we constructed $a_1, \ldots, a_{i-1}$. Now, define $a_i = a_{i-1} + a_1a_2 \cdots a_{i-2}$. Since $a + a_{i-1} \in P_1$, $b + a_{i-1} \in P_2$, $a_1 \in P_1$ and $a_2 \in P_2$, we have $a + a_i \in P_1$ and $b + a_i \in P_2$. Since a_i is adjacent to a and b, so $a_i \in Z(R) = \bigcup_{r=1}^{n} P_r$. Thus there exists some j such that $a_i \in P_j$. We claim that $j \geqslant i$. By contradiction suppose that $j < i$. If $j \leqslant i - 2$, then $a_{i-1} \in P_j$, which is a contradiction. Thus $j = i - 1$, $(a_1 \cdots a_{i-3})a_{i-2} \in P_{i-1}$ and $a_{i-2} + a_1 \cdots a_{i-3} \in P_{i-1}$. So $a_1 \cdots a_{i-3}, a_{i-2} \in P_{i-1}$. By repeating this argument we have $a_1a_2a_3, a_4 \in P_{i-1}$. This implies that a_4 and a_t are contained in P_{i-1}, for some t, $1 \leqslant t \leqslant 3$, a contradiction. Therefore $a_i \notin \bigcup_{r=1}^{i-1} P_r$ and the claim is proved. With no loss of generality assume that $j = i$. Since the number of prime ideals which cover $Z(R)$ is n, we have $a_{n+1} \notin Z(R)$, a contradiction.

Next assume that $l = 3$. As we saw before there are prime ideals $P_1, \ldots, P_n$ such that $Z(R) = \bigcup_{i=1}^{n} P_i$. Because a and b are not adjacent, $a + b \in Reg(R)$. Since $d_{T(\Gamma(R))}(a, b) = 3$, there exist $z_1, z_2 \in R$ such that $a + z_1, z_1 + z_2, b + z_2 \in Z(R)$. If z_1 or z_2 is regular, then by the previous case we have nothing to prove. Thus let $z_1, z_2 \in Z(R)$. If $a - b \in Z(R)$, then $a, -b, b$ is a path in $T(\Gamma(R))$, a contradiction. Therefore we can assume that $a - b \in Reg(R)$. If $a + z_1 + z_2 \in Reg(R)$, then $a, -(a + z_1 + z_2), z_2, b$ is a path in $T(\Gamma(R))$ and by the case $l = 2$, there is a path of length 3 between a and b in $Reg(\Gamma(R))$. Thus we can suppose that $a + z_1 + z_2 \notin Reg(R)$. Similarly, one can assume that $b + z_1 + z_2 \notin Reg(R)$. Set $a_1 = a + z_1 + z_2$, $a_2 = b + z_1 + z_2$ and $a_3 = z_1 + z_2$. With no loss of generality assume that $a_1 \in P_1$. So $a_2, a_3 \notin P_1$. Also, a_2 and a_3 are not contained in the same prime ideal. With no loss of generality assume that $a_2 \in P_2$ and $a_3 \in P_3$. Set $a_4 = a_3 + a_1a_2$. Clearly, $a_4 \notin \bigcup_{i=1}^{3} P_i$, $a + a_4 \in P_1$ and $b + a_4 \in P_2$. Therefore a, a_4, b is a path in $T(\Gamma(R))$, a contradiction. Thus $d_{Reg(\Gamma(R))}(a, b) = 3$, as desired.

Now, by induction on r, we show that if $a, b \in Reg(R)$ and $a, z_1, \ldots, z_r, b$ is a shortest path between a and b in $T(\Gamma(R))$, then there is a path $a, z'_1, \ldots, z'_n, b$ in $Reg(\Gamma(R))$, where $n \leq r$. If $r = 1$ or $r = 2$, then as we discussed before, one can replace z_i's with regular elements. Therefore assume that $r \geqslant 3$. If there exists some j, $1 \leq j \leq r$ such that $z_j \in Reg(R)$, then using induction (two times), we obtain a path of length at most $r + 1$ between a and b in $Reg(\Gamma(R))$. Thus we may assume that all z_i are zero-divisors. If $z_1 + z_3 \in Z(R)$, then we obtain the path $a, z_1, z_3, z_4, \ldots, z_r, b$, a contradiction. So $z_1 + z_3 \in Reg(R)$. Now, consider the path $a, z_1, -(z_1 + z_3), z_3, \ldots, z_r, b$ and the proof is complete. □

In [8] it was proved that if R is a commutative ring such that $T(\Gamma(R))$ is connected and $Z(R)$ is not an ideal, then $\mathrm{diam}(T(\Gamma(R))) \leqslant \mathrm{diam}(Reg(\Gamma(R))) + 2$. Hence we have the following corollary.

Corollary 1. *Let R be a commutative Noetherian ring. If $T(\Gamma(R))$ is connected with diameter d, then $d - 2 \leqslant \mathrm{diam}(Reg(\Gamma(R))) \leqslant d$.*

Note that there are some commutative Noetherian rings R, for which the equality $\mathrm{diam}(Reg(\Gamma(R))) = \mathrm{diam}(T(\Gamma(R)))$ does not hold. For instance $\mathrm{diam}(Reg(\Gamma(\mathbb{Z}_6))) = 1$, but $\mathrm{diam}(T(\Gamma(\mathbb{Z}_6))) = 2$.

3. Structural results

Anderson and Livingston in [7] proved that if $Z(R)$ is finite, then R is finite. Commutative rings with a few zero-divisors have been characterized in [7]. Indeed, there are just two rings, $\mathbb{Z}_4$ and $\mathbb{Z}_2[x]/(x^2)$ such that $|Z(R)| = 2$. Now, using this result we want to classify all commutative Noetherian rings with at most two regular elements. We want to show that if R is a commutative Noetherian ring and $Reg(R)$ is finite, then R is finite.

Theorem 2. *Let R be a commutative ring and I_i be a proper ideal of R for $i = 1, \ldots, n$. If $R \setminus \bigcup_{i=1}^{n} I_i$ is finite, then R is finite.*

Proof. Clearly, one can assume that each I_i is a maximal ideal. Assume that $\bigcup_{i=1,i\neq j}^{n} I_i \neq \bigcup_{i=1}^{n} I_i$ for any j, $1 \leqslant j \leqslant n$. By contradiction assume that R is infinite and n is the least integer such that $R \setminus \bigcup_{i=1}^{n} I_i$ is finite (note that $R \setminus I_1$ is infinite). Thus $n \geqslant 2$. Since $R \setminus \bigcup_{i=1}^{n-1} I_i = (R \setminus \bigcup_{i=1}^{n} I_i) \bigcup (I_n \setminus \bigcup_{i=1}^{n-1} I_i)$ and $R \setminus \bigcup_{i=1}^{n-1} I_i$ is infinite, so $I_n \setminus \bigcup_{i=1}^{n-1} I_i$ is infinite. Let $x_i \in I_i \setminus I_n$ for $i = 1, \ldots, n-1$. Clearly, if $X = I_n \setminus \bigcup_{i=1}^{n-1} I_i$, then $X + x_1 \cdots x_{n-1} \subseteq R \setminus \bigcup_{i=1}^{n} I_i$. Therefore $R \setminus \bigcup_{i=1}^{n} I_i$ is infinite, which is a contradiction. The proof is complete. □

Corollary 2. *Let R be a commutative Noetherian ring. If $Reg(R)$ is finite, then R is finite.*

Proof. Since R is a Noetherian ring, there are prime ideals $P_1, \ldots, P_n$ such that $Z(R) = \bigcup_{i=1}^{n} P_i$. Now, by the previous theorem the result holds. □

Remark 1. If in the previous theorem, the number of ideals is infinite, then the assertion is not true. To see this, consider the direct product of infinitely many $\mathbb{Z}_2$. The set of zero-divisors of this ring is a union of prime ideals (it is well known that the set of zero-divisors of every commutative ring is a union of a family of prime ideals), but it has just one regular element.

Lemma 1. *Let R be a commutative Noetherian ring such that $|Reg(R)| \leqslant 2$. Then R is isomorphic to one of the following rings:*

$$\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2), {\mathbb{Z}_2}^r, {\mathbb{Z}_2}^r \times \mathbb{Z}_3, {\mathbb{Z}_2}^r \times \mathbb{Z}_4, {\mathbb{Z}_2}^r \times \mathbb{Z}_2[x]/(x^2),$$

where r is a natural number.

Proof. By Corollary 2, R is finite. Therefore by Theorem 8.7, of [9, p.90], $R \simeq R_1 \times \cdots \times R_n$, where R_i is a local commutative ring. We claim that if (S, m) is a finite commutative local ring and $|Reg(S)| \leqslant 2$, then S is isomorphic to one of the rings: $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2)$.

We know that $Reg(S) = S \setminus m$. Thus $|S \setminus m| \leqslant 2$ and this implies that $|S| \leqslant 4$. Since S has unity, if $|S| = 2$ or $|S| = 3$, then clearly $S \simeq \mathbb{Z}_2$ or $\mathbb{Z}_3$, respectively. Thus assume that $|S| = 4$. Thus $|Z(R)| = 2$. Now, by [7], $S \simeq \mathbb{Z}_4$ or $S \simeq \mathbb{Z}_2[x]/(x^2)$, and the claim is proved. By assumption $|Reg(R)| \leqslant 2$, and so either $|Reg(R_i)| = 1$, for $i = 1, \ldots, n$ or $|Reg(R_j)| = 2$ and $|Reg(R_i)| = 1$, for $i \in \{1, \ldots, n\} \setminus \{j\}$. This completes the proof. □

4. Total graph and regular graph are Hamiltonian

Now, we want to determine when the total graph and the regular graph of a finite commutative ring are Hamiltonian. H. Maimani conjectured that if R is a finite commutative ring and $Z(R)$ is not an ideal, then $T(\Gamma(R))$ is Hamiltonian. Here, we give an affirmative answer to this conjecture.

Lemma 2. *Let (R_1, m_1) and (R_2, m_2) be two finite commutative local rings such that $\mathrm{char}(\frac{R_1}{m_1}) \neq 2$ and $\mathrm{char}(\frac{R_2}{m_2}) \neq 2$. Then $T(\Gamma(R_1 \times R_2))$ and $Reg(\Gamma(R_1 \times R_2))$ are Hamiltonian graphs.*

Proof. We have $Z(R_i) = m_i$, for $i = 1, 2$. Note that there are elements $a_i \in R$, $i = 1, \ldots, n$ such that $Reg(R_1) = \bigcup_{i=1}^{n} (\pm a_i + m_1)$, where $a_i \notin m_1$ and the cosets are distinct. On the other hand one can write $Reg(R_2) = \{r_1, \ldots, r_s\}$, where $r_s = -r_1$. Let $m_1 = \{b_1, \ldots, b_t\}$. It is easy to see that the induced subgraph on the vertices $(a_i + m_1) \cup (-a_i + m_1)$ is a complete bipartite graph with two parts $a_i + m_1$ and $-a_i + m_1$, for every i, $i = 1, \ldots, n$. So we have the following Hamilton cycle in $Reg(\Gamma(R))$:

$$\begin{array}{l}(a_1 + b_1, r_1), (-a_1 + b_1, r_1) \cdots (a_1 + b_t, r_1), (-a_1 + b_t, r_1) \cdots (a_1 + b_t, r_s), (-a_1 + b_t, r_s),\\ (a_2 + b_1, r_1), (-a_2 + b_1, r_1) \cdots (a_2 + b_t, r_1), (-a_2 + b_t, r_1) \cdots (a_2 + b_t, r_s), (-a_2 + b_t, r_s),\\ \cdots \\ (a_n + b_1, r_1), (-a_n + b_1, r_1) \cdots (a_n + b_t, r_1), (-a_n + b_t, r_1) \cdots (a_n + b_t, r_s), (-a_n + b_t, r_s).\end{array}$$

Note that in each row of the above table, we have $2st$ elements. Now, assume that $R_2 = \{r_1, \ldots, r_s\}$, where $r_s = -r_1$. By putting the sequence

$$(b_1, r_1), \ldots, (b_1, r_s), (b_2, r_1), \ldots, (b_2, r_s), (b_t, r_1), \ldots, (b_t, r_s)$$

in the above table, we will find a Hamilton cycle for $T(\Gamma(R))$ and the proof is complete. □

Lemma 3. *Let (R_1, m_1) and (R_2, m_2) be two finite commutative local rings such that $\mathrm{char}(\frac{R_1}{m_1}) = 2$ and $\mathrm{char}(\frac{R_2}{m_2}) \neq 2$. Then the following hold:*

(i) *If $R_1 \times R_2$ is not isomorphic to $\mathbb{Z}_6$, then $Reg(\Gamma(R_1 \times R_2))$ is a Hamiltonian graph.*
(ii) *$T(\Gamma(R_1 \times R_2))$ is a Hamiltonian graph.*

Proof. (i) One can write $Reg(R_2) = \{r_1, \dots, r_s\}$, where $r_1 = -r_s$ and $Reg(R_1) = \bigcup_{i=1}^{n}(a_i + m_1)$, where $a_i \notin m_1$. Since $\mathrm{char}(\frac{R_1}{m_1}) = 2$ and $\mathrm{char}(\frac{R_2}{m_2}) \neq 2$, by Lemma 1, we have $s \geq 3$. Assume that $m_1 = \{b_1, \dots, b_t\}$. It is easy to see that the induced subgraph on the vertices $a_i + m_1$ is a complete graph, for $i = 1, \dots, n$. So we have the following Hamilton cycle:

$$\begin{array}{lllllllll}(a_1+b_1,r_1), & (a_1+b_2,r_1) & \cdots & (a_1+b_t,r_1) & \cdots & (a_1+b_1,r_s) & \cdots & (a_1+b_t,r_s),\\ (a_2+b_1,r_1), & (a_2+b_2,r_1) & \cdots & (a_2+b_t,r_1) & \cdots & (a_2+b_1,r_s) & \cdots & (a_2+b_t,r_s),\\ & & \cdots & & \cdots & \cdots & & \\ (a_n+b_1,r_1), & (a_n+b_2,r_1) & \cdots & (a_n+b_t,r_1) & \cdots & (a_n+b_1,r_s) & \cdots & (a_n+b_t,r_s).\end{array}$$

(ii) The proof for the total graph is similar to (i). □

Lemma 4. *Let (R_1, m_1) and (R_2, m_2) be two finite commutative local rings such that $\mathrm{char}(\frac{R_1}{m_1}) = 2$ and $\mathrm{char}(\frac{R_2}{m_2}) = 2$. Then the following hold:*

(i) *If $R_1 \times R_2$ is isomorphic to none of the rings, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, then $Reg(\Gamma(R_1 \times R_2))$ is a Hamiltonian graph.*

(ii) *$T(\Gamma(R_1 \times R_2))$ is a Hamiltonian graph.*

Proof. (i) We have $Reg(R_1) = \bigcup_{i=1}^{n}(a_i + m_1)$, where $a_i \notin m_1$. Let $Reg(R_2) = \{r_1, \dots, r_s\}$. Since $\mathrm{char}(\frac{R_1}{m_1}) = 2$ and $\mathrm{char}(\frac{R_2}{m_2}) = 2$, by Lemma 1, we have $s \geq 3$. The induced subgraph on the vertices $a_i + m_1$ is a complete graph for each $i, i = 1, \dots, n$. If m_1 or m_2, say m_1, is non-zero, then we can set the elements $(a_i + m_1) \times Reg(R_2)$ in a sequence such as x_{ij}, $1 \leq j \leq l = s|m_1|$ such that the second components of x_{i1} and x_{il} be r_1. Now, consider the following sequence which forms a Hamilton cycle for $Reg(\Gamma(R_1 \times R_2))$:

$$x_{11}, \dots, x_{1l}, x_{21}, \dots, x_{2l}, \dots, x_{n1}, \dots, x_{nl}.$$

If $m_1 = m_2 = 0$, then R_1 and R_2 are fields. It is easy to see that $Reg(\Gamma(R_1 \times R_2)) \simeq K_m \times K_n$. So it is a Hamiltonian graph, see [10].

(ii) The proof for the total graph is similar to (i). □

Lemma 5. *Let R_1 and R_2 be two finite commutative rings. Then the following hold:*

(i) *If $T(\Gamma(R_1))$ is a Hamiltonian graph, then $T(\Gamma(R_1 \times R_2))$ is a Hamiltonian graph.*
(ii) *If $Reg(\Gamma(R_1))$ is a Hamiltonian graph, then $Reg(\Gamma(R_1 \times R_2))$ is a Hamiltonian graph.*

Proof. (i) Let $R_1 = \{r_1, \dots, r_n\}$ and $R_2 = \{r'_1, \dots, r'_m\}$ such that the sequence $r_1, \dots, r_n$ is a Hamilton cycle. So $r_1 + r_n \in Z(R_1)$. So we have the following Hamilton cycle in $T(\Gamma(R_1 \times R_2))$.

$$\begin{array}{lll}(r_1, r'_1), & \dots, & (r_n, r'_1),\\ (r_1, r'_2), & \dots, & (r_n, r'_2),\\ & \dots & \\ (r_1, r'_m), & \dots, & (r_n, r'_m).\end{array}$$

(ii) Assume that $Reg(R_1) = \{r_1, \dots, r_n\}$ and $Reg(R_2) = \{r'_1, \dots, r'_m\}$. The above sequence is a Hamilton cycle for $Reg(\Gamma(R_1 \times R_2))$. □

Theorem 3. *Let R be a finite commutative ring such that $Z(R)$ is not an ideal. Then the following hold:*

(i) *$T(\Gamma(R))$ is a Hamiltonian graph.*

(ii) *$Reg(\Gamma(R))$ is a Hamiltonian graph if and only if R is isomorphic to none of the rings:*

$$\mathbb{Z}_2{}^{n+1}, \mathbb{Z}_2{}^{n} \times \mathbb{Z}_3, \mathbb{Z}_2{}^{n} \times \mathbb{Z}_4, \mathbb{Z}_2{}^{n} \times \mathbb{Z}_2[x]/(x^2),$$

where n is a natural number.

Proof. (i) Let $R \simeq R_1 \times R_2 \times \cdots \times R_n$, where R_i is a local ring ($1 \leqslant i \leqslant n$). Since $Z(R)$ is not an ideal, then $n \geqslant 2$. If $n = 2$, then by the previous lemmas the result holds. If $n > 2$, then set $R' = R_3 \times \cdots \times R_n$. By the previous lemma $T(\Gamma((R_1 \times R_2) \times R'))$ is a Hamiltonian graph.

(ii) We have $Reg(R) \simeq Reg(R_1) \times Reg(R_2) \times \cdots \times Reg(R_n)$, where R_i is a local ring for $i = 1, \dots, n$. Since $Z(R)$ is not an ideal, then $n \geqslant 2$. If $n = 2$, then by the previous lemmas the result holds. If $n > 2$, then set $R' = R_3 \times \cdots \times R_n$. By the previous lemma $Reg(\Gamma((R_1 \times R_2) \times R'))$ is a Hamiltonian graph. Conversely, if R isomorphic to one of the rings given in the statement of theorem, then $|Reg(R)| \leq 2$ and so $Reg(\Gamma(R))$ is not a Hamiltonian graph. □

Remark 2. If R is a commutative local ring, then $Z(R)$ is an ideal and $T(\Gamma(R))$ is not connected by Theorem 2.1 of [8]. Therefore $T(\Gamma(R))$ is not a Hamiltonian graph.

Finally, we find a lower bound for $\kappa(T(\Gamma(R_1 \times R_2)))$, where R_1 and R_2 are finite commutative rings. The following theorem due to Menger is used to obtain this result.

Theorem A *([11, Corollary 4.2.19]). $\kappa(G) \geqslant k$ if and only if there are at least k internally vertex disjoint (x, y)-paths for every $x, y \in V(G)$.*

Theorem 4. *Let $R = R_1 \times R_2$ be a finite commutative ring. Then $\kappa(T(\Gamma(R))) \geqslant |R_1| + |R_2| - 4$.*

Proof. Let (a, b) and (a', b') be two distinct elements of R. If $a \neq a', b' \neq \pm b, \lambda \notin \{b, -b, b', -b'\}$, then consider the paths $(a, b), (-a, \lambda), (-a', -\lambda), (a', b')$ for $\lambda \in R_2$. If $\eta \in R_1$, $(a, b) \neq (\eta, -b)$ and $(a', b') \neq (-\eta, -b')$, then consider the paths $(a, b), (\eta, -b), (-\eta, -b'), (a', b')$. If $(a, b) \neq (\eta, -b)$ and $(a', b') = (-\eta, -b')$, then consider the paths $(a, b), (\eta, -b), (a', b')$. If $(a, b) = (\eta, -b)$ and $(a', b') \neq (-\eta, -b')$, then consider the paths $(a, b), (-\eta, -b'), (a', b')$. If $(a, b) = (\eta, -b)$ and $(a', b') = (-\eta, -b')$ for some η, then consider the paths $(a, b), (a', b')$ and $(a, b), (\eta, -b), (a', b')$ for $\eta \neq a$. So there are at least $|R_1| + |R_2| - 4$ disjoint paths joining (a, b) and (a', b').

Let $a \neq a', b' \neq b$ and $b' = -b$. Then the paths $(a, b), (-a, \lambda), (-a', -\lambda), (a', -b)$ for $\lambda \in R_2 \setminus \{\pm b\}$ and the paths $(a, b), (\eta, -b), (-\eta, b), (a', -b)$ for $\eta \in R_1 \setminus \{-a, a'\}$ are $|R_1| + |R_2| - 4$ disjoint paths.

Let $a \neq a'$ and $b = b'$. Consider the paths $(a, b), (-a, \lambda), (-a', -\lambda), (a', b)$, where $\lambda \in R_2 \setminus \{b, -b\}$ and $(a, b), (\eta, -b), (a', b)$ for $\eta \in R_1 \setminus \{a, a'\}$.

If $a = a'$, since (a, b) and (a', b') are distinct, then $b \neq b'$ and the proof is the same as the case $a \neq a'$ and $b = b'$. □

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