Journal of Algebra 324 (2010) 1290–1313

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Algebras and groups defined by permutation relations of alternating type ☆

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ARTICLE INFO

Article history:
Received 14 November 2008
Available online 27 July 2010
Communicated by Efim Zelmanov

MSC:
primary 16S15, 16S36, 20M05
secondary 20M25, 16N20

Keywords:
Semigroup ring
Finitely presented
Semigroup
Jacobson radical
Semiprimitive

ABSTRACT

The class of finitely presented algebras over a field K with a set of generators $a_1,\dots,a_n$ and defined by homogeneous relations of the form $a_1a_2\cdots a_n = a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}$, where σ runs through Alt_n, the alternating group of degree n, is considered. The associated group, defined by the same (group) presentation, is described. A description of the Jacobson radical of the algebra is found. It turns out that the radical is a finitely generated ideal that is nilpotent and it is determined by a congruence on the underlying monoid, defined by the same presentation.

1. Introduction

In recent literature a lot of attention is given to concrete classes of finitely presented algebras A over a field K defined by homogeneous semigroup relations, that is, relations of the form $w = v$, where w and v are words of the same length in a generating set of the algebra. Of course such an algebra is a semigroup algebra $K[S]$, where S is the monoid generated by the same presentation. Particular classes show up in different areas of research. For example, algebras yielding set theo-

☆ Research partially supported by grants of MICIN-FEDER (Spain) MTM2008-06201-C02-01, Generalitat de Catalunya 2005SGR00206, Onderzoeksraad of Vrije Universiteit Brussel, Fonds voor Wetenschappelijk Onderzoek (Belgium), Flemish–Polish bilateral agreement BIL2005/VUB/06 and MNiSW research grant N201 004 32/0088 (Poland).

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.07.014

retic solutions of the Yang–Baxter equation (see for example [7,9,10,12,18]) or algebras related to Young diagrams, representation theory and algebraic combinatorics (see for example [1,5,8,11,14]). In all the mentioned algebras there are strong connections between the structure of the algebra $K[S]$, the underlying semigroup S and the underlying group G, defined by the same presentation as the algebra.

In [3] the authors introduced and initiated a study of combinatorial and algebraic aspects of the following new class of finitely presented algebras over a field K:

$$A = K\langle a_1, a_2, \dots, a_n \mid a_1 a_2 \cdots a_n = a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)},\ \sigma \in H\rangle,$$

where H is a subset of the symmetric group Sym_n of degree n. So $A = K[S_n(H)]$ where

$$S_n(H) = \langle a_1, a_2, \dots, a_n \mid a_1 a_2 \cdots a_n = a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)},\ \sigma \in H\rangle,$$

the monoid with the "same" presentation as the algebra. By $G_n(H)$ we denote the group defined by this presentation. So

$$G_n(H) = \mathrm{gr}(a_1, a_2, \dots, a_n \mid a_1 a_2 \cdots a_n = a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)},\ \sigma \in H).$$

Two obvious examples are: the free K-algebra $K[S_n(\{1\})] = K\langle a_1, \dots, a_n\rangle$ with $H = \{1\}$ and $S_n(\{1\}) = \mathrm{FM}_n$, the rank n free monoid, and the commutative polynomial algebra $K[S_2(\mathrm{Sym}_2)] = K[a_1, a_2]$ with $H = \mathrm{Sym}_2$ and $S_n(H) = \mathrm{FaM}_2$, the rank 2 free abelian monoid. For $M = S_n(\mathrm{Sym}_n)$, the latter can be extended as follows [3, Proposition 3.1]: the algebra $K[M]$ is the subdirect product of the commutative polynomial algebra $K[a_1, \dots, a_n]$ and a primitive monomial algebra that is isomorphic to $K[M]/K[Mz]$, with $z = a_1 a_2 \cdots a_n$, a central element.

On the other hand, let $M = S_n(H)$ where $H = \mathrm{gr}(\{(1, 2, \dots, n)\})$, a cyclic group of order n. Then [3, Theorem 2.2] the monoid M is cancellative and it has a group G of fractions of the form $G = M\langle a_1 \cdots a_n\rangle^{-1} \cong F \times C$, where $F = \mathrm{gr}(a_1, \dots, a_{n-1})$ is a free group of rank $n-1$ and $C = \mathrm{gr}(a_1 \cdots a_n)$ is a cyclic infinite group. The algebra $K[M]$ is a domain and it is semiprimitive. Moreover [3, Theorem 2.1], a normal form of elements of the algebra can be given. It is worthwhile mentioning that the group G is an example of a cyclically presented group. Such groups arise in a very natural way as fundamental groups of certain 3-manifolds [6], and their algebraic structure also receives a lot of attention; for a recent work and some references see for example [2].

In this paper we continue the investigations on the algebras $K[S_n(H)]$ and the groups $G_n(H)$. First we will prove some general results and next we will give a detailed account in case H is the alternating group Alt_n of degree n. It turns out that the structure of the group $G_n(H)$ can be completely determined and the algebra $K[S_n(H)]$ has some remarkable properties. In order to state our main result we fix some notation. Throughout the paper K is a field. If $b_1, \dots, b_m$ are elements of a monoid M then we denote by $\langle b_1, \dots, b_m\rangle$ the submonoid generated by $b_1, \dots, b_m$. If M is a group then $\mathrm{gr}(b_1, \dots, b_m)$ denotes the subgroup of M generated by $b_1, \dots, b_m$. Clearly, the defining relations of an arbitrary $S_n(H)$ are homogeneous. Hence, it has a natural degree or length function. This will be used freely throughout the paper. By $\rho = \rho_S$ we denote the least cancellative congruence on a semigroup S. If η is a congruence on S then $I(\eta) = \mathrm{lin}_K\{s - t \mid s, t \in M,\ (s,t) \in \eta\}$ is the kernel of the natural epimorphism $K[S] \to K[S/\eta]$. For a ring R, we denote by $\mathcal{J}(R)$ its Jacobson radical and by $\mathcal{B}(R)$ its prime radical. Our main result reads as follows.

Theorem 1.1. *Suppose K is a field and $n \geqslant 4$. Let $M = S_n(\mathrm{Alt}_n)$, $z = a_1 a_2 \cdots a_n \in M$ and $G = G_n(\mathrm{Alt}_n)$. The following properties hold.*

(i) *$C = \{1, a_1 a_2 a_1^{-1} a_2^{-1}\}$ is a nontrivial central subgroup of G and G/C is a free abelian group of rank n. Moreover $D = \mathrm{gr}(a_i^2 \mid i = 1, \dots, n)$ is a central subgroup of G with $G/(CD) \cong (\mathbb{Z}/2\mathbb{Z})^n$.*

(ii) *$K[G]$ is a noetherian algebra satisfying a polynomial identity (PI, for short). If K has characteristic $\neq 2$, then $\mathcal{J}(K[G])=0$. If K has characteristic 2, then $\mathcal{J}(K[G])=(1-a_1a_2a_1^{-1}a_2^{-1})K[G]$ and $\mathcal{J}(K[G])^2=0$.*
(iii) *The element z^2 is central in M and z^2M is a cancellative ideal of M such that $G\cong (z^2M)\langle z^2\rangle^{-1}$. Furthermore, $K[M/\rho]$ is a noetherian PI-algebra and $\mathcal{J}(K[M])$ is nilpotent.*
(iv) *Suppose n is odd. Then z is central in M and $0\neq \mathcal{J}(K[M])=I(\eta)$ for a congruence η on M and $\mathcal{J}(K[M])$ is a finitely generated ideal.*
(v) *Suppose n is even and $n\geqslant 6$. If K has characteristic $\neq 2$, then $\mathcal{J}(K[M])=0$. If K has characteristic 2, then $0\neq \mathcal{J}(K[M])=I(\eta)$ for a congruence η on M and $\mathcal{J}(K[M])$ is a finitely generated ideal.*

Part (v) of Theorem 1.1 is also true for $n=4$, but its proof is quite long for this case and it requires additional technical lemmas. (The interested reader can find a proof of this in [4].)

So, in particular, the Jacobson radical is determined by a congruence relation on the semigroup $S_n(\mathrm{Alt}_n)$, it is nilpotent and finitely generated as an ideal. In [3] the question was asked whether these properties hold for all algebras $K[S_n(H)]$, for subgroups H of Sym_n.

2. General results

In this section we prove some preparatory general properties of the monoid algebra $K[S_n(H)]$ for an arbitrary subset H of Sym_n with $n\geqslant 3$. To simplify notation, throughout this section we put

$$M=\langle a_1,a_2,\dots,a_n \mid a_1a_2\cdots a_n=a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)},\ \sigma\in H\rangle. \tag{1}$$

If $\alpha=\sum_{x\in M}k_x x\in K[M]$, with each $k_x\in K$, then the finite set $\{x\in M \mid k_x\neq 0\}$ we denote by $\mathrm{supp}(\alpha)$. It is called the support of α.

Proposition 2.1. *Suppose that there exists k such that $1<k<n$ and, for all $\sigma\in H$, $\sigma(1)\neq k$ and $\sigma(n)\neq k$. Then $\mathcal{J}(K[M])=0$.*

Proof. Suppose that $\mathcal{J}(K[M])\neq 0$. Let $\alpha\in\mathcal{J}(K[M])$ be a nonzero element. Because, by assumption, $\sigma(1)\neq k$ for all $\sigma\in H$, we clearly get that $a_k^2\alpha\neq 0$. As $a_k^2\alpha\in\mathcal{J}(K[M])$, there exists $\beta\in K[M]$ such that $a_k^2\alpha+\beta+\beta a_k^2\alpha=0$. Obviously, $\beta\notin K$. Let α_1, β_1 be the homogeneous components (for the natural $\mathbb{Z}$-gradation of $K[M]$) of α and β of maximum degree respectively. Then $\beta_1a_k^2\alpha_1=0$. In particular, there exist w_1,w_2 in the support of β_1 and w_1',w_2' in the support of α_1 such that

$$w_1a_k^2w_1'=w_2a_k^2w_2'$$

and either $w_1\neq w_2$ or $a_k^2w_1'\neq a_k^2w_2'$. But, because $\sigma(n)\neq k$ for all $\sigma\in H$, this is impossible. Therefore $\mathcal{J}(K[M])=0$. □

Corollary 2.2. *If H is a subgroup of Sym_n and $\mathcal{J}(K[M])\neq 0$ then H is a transitive subgroup of Sym_n.*

Proof. Suppose that H is a subgroup of Sym_n and $\mathcal{J}(K[M])\neq 0$. By Proposition 2.1, for all k there exists $\sigma\in H$ such that either $\sigma(1)=k$ or $\sigma(n)=k$. Suppose that H is not transitive. Then there exists $1\leqslant j\leqslant n$ such that $j\notin\{\sigma(1)\mid\sigma\in H\}$. Hence there exists $\sigma\in H$ such that $\sigma(n)=j$. Thus the orbits $I_1=\{\sigma(1)\mid\sigma\in H\}$ and $I_2=\{\sigma(n)\mid\sigma\in H\}$ are disjoint nonempty sets such that $I_1\cup I_2=\{1,2,\dots,n\}$. So, there are no defining relations of the form $a_1\cdots=a_n\cdots$, nor of the form $\cdots a_1=\cdots a_n$. Consequently, if $0\neq\alpha\in K[M]$ then $a_n^2\alpha\neq 0$ and $\alpha a_1^2\neq 0$.

Let $\alpha\in\mathcal{J}(K[M])$ be a nonzero element. Then, $a_1^2a_n^2\alpha\neq 0$, and there exists $\beta\in K[M]$ such that $a_1^2a_n^2\alpha+\beta+\beta a_1^2a_n^2\alpha=0$. Clearly, it follows that $\beta\notin K$. Let α_1, β_1 be the homogeneous components of α and β of maximum degree respectively. We obtain that $\beta_1a_1^2a_n^2\alpha_1=0$. In particular, there exist

w_1, w_2 in the support of β_1 and w_1', w_2' in the support of α_1 such that $(w_1, w_1') \neq (w_2, w_2')$ and

$$w_1 a_1^2 a_n^2 w_1' = w_2 a_1^2 a_n^2 w_2'.$$

Again, because there are no defining relations of the form $a_1 \cdots = a_n \cdots$ nor of the form $\cdots a_1 = \cdots a_n$, this yields a contradiction. Therefore H is transitive. □

Let $z = a_1 a_2 \cdots a_n \in M$. The fact that z is central in $M = S_n(H)$ for the case of the cyclic group H generated by $(1, 2, \ldots, n)$ was an important tool in [3]. In Section 4 we will show that z^2 is central if $M = S_n(\mathrm{Alt}_n)$. We start by showing that the centrality of z^m, for some positive integer m, has some impact on the algebraic structure of M and $K[M]$ and we determine when z is central in case H is a subgroup of Sym_n.

Proposition 2.3. *Suppose H is a subgroup of* Sym_n *and put $z = a_1 a_2 \cdots a_n$. The following conditions are equivalent.*

(i) *z is central in $M = S_n(H)$,*
(ii) *$a_1 z = z a_1$,*
(iii) *H contains the subgroup of* Sym_n *generated by the cycle* $(1, 2, \ldots, n)$.

Proof. Let H_0 denote the subgroup of Sym_n generated by the cycle $(1, 2, \ldots, n)$. Assume $H_0 \subseteq H$. Then $M = S_n(H)$ is an epimorphic image of $S_n(H_0)$. As $a_1 a_2 \cdots a_n$ is central in $S_n(H_0)$, it follows that z indeed is central in M.

Assume now that $a_1 z = z a_1$. We need to show that $H_0 \subseteq H$. Every defining relation can be written in the form: $z = a_k c_k$, with $1 \leqslant k \leqslant n$, $c_k = \prod_{i=1,\, i \neq k} a_{\tau(i)}$, $\tau \in \mathrm{Sym}(\{1, \ldots, n\} \setminus \{k\}) \subseteq \mathrm{Sym}_n$. By assumption, $a_1^2 a_2 \cdots a_n = a_1 z = z a_1 = a_k c_k a_1$. Since $a_k c_k$ is a product of distinct generators, there must exist a relation of the form $c_k a_1 = z$. Since also z is a product of distinct generators, it follows that $k = 1$. Thus $z = a_1 c_1 = c_1 a_1$. The former equality yields that $\sigma_1 = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 1 & \tau(2) & \ldots & \tau(n-1) & \tau(n) \end{pmatrix} \in H$ and the equality $z = c_1 a_1$ gives $\sigma_2 = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ \tau(2) & \tau(3) & \ldots & \tau(n) & 1 \end{pmatrix} \in H$. Hence $(1, 2, \ldots, n) = \sigma_1^{-1} \sigma_2 \in H$ and so $H_0 \subseteq H$. The result follows. □

Assume now that z^m is central, for some positive integer m. Note that then the binary relation ρ' on M, defined by $s \rho' t$ if and only if there exists a nonnegative integer i such that $s z^i = t z^i$, is a congruence on M. We now show that $G_n(H)$ is the group of fractions of $\overline{M} = M/\rho'$. We denote by $\bar{a}$ the image in $\overline{M}$ of $a \in M$ under the natural map $M \to \overline{M}$.

Lemma 2.4. *Suppose that z^m is central for some positive integer m. Then, $\rho' = \rho$ is the least cancellative congruence on M and $Ma \cap aM \cap \langle z^m \rangle \neq \emptyset$ for every $a \in M$.*

In particular, $\overline{M} = M/\rho$ is a cancellative monoid and $G = \overline{M} \langle \bar{z}^m \rangle^{-1}$ is the group of fractions of $\overline{M}$. Moreover, $G \cong G_n(H) = \mathrm{gr}(a_1, \ldots, a_n \mid a_1 a_2 \cdots a_n = a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)},\ \sigma \in H)$.

Proof. Since z^m is central, we already know that the binary relation ρ' is a congruence on M. Let $a = a_{i_1} a_{i_2} \cdots a_{i_k} \in M$. We shall prove that $aM \cap \langle z^m \rangle \neq \emptyset$ by induction on k. For $k = 0$, this is clear. Suppose that $k > 0$ and that $bM \cap \langle z^m \rangle \neq \emptyset$ for all $b \in M$ of degree less than k. Thus there exists $r \in M$ such that $a_{i_1} \cdots a_{i_{k-1}} r \in \langle z^m \rangle$. Since $a_{i_k} z^m = z^m a_{i_k}$, it follows easily from the type of the defining relations for M that there exists $w \in M$ such that $a_{i_k} w = z$. We thus get that $a w z^{m-1} r = a_{i_1} \ldots a_{i_{k-1}} z^m r = a_{i_1} \ldots a_{i_{k-1}} r z^m \in \langle z^m \rangle$. Similarly we see that $Ma \cap \langle z^m \rangle \neq \emptyset$. Therefore ρ' is the least cancellative congruence on M and $\overline{M} \langle \bar{z}^m \rangle^{-1}$ is the group of fractions of $\overline{M}$ and the second assertion also follows. □

Proposition 2.5. *Suppose that z^m is central for some positive integer m. Let $\alpha_1, \ldots, \alpha_k \in I(\rho) \cap K[Mz^m]$. Then the ideal $\sum_{i=1}^{k} K[M] \alpha_i K[M]$ is nilpotent. In particular, $I(\rho) \cap K[Mz^m] \subseteq \mathcal{B}(K[M])$.*

Proof. Let $\alpha_1, \dots, \alpha_k \in I(\rho) \cap K[Mz^m]$. Clearly there exists a positive integer N such that $\alpha_i z^{mN} = 0$, for all $i = 1, \dots, k$. Since z^m is central and $\alpha_i \in K[Mz^m]$, we have that $(\sum_{i=1}^m K[M]\alpha_i K[M])^{N+1} = 0$ and the result follows. □

Proposition 2.6. *The following properties hold.*

(i) $\mathcal{J}(K[M]/K[MzM]) = 0$.
(ii) $\mathcal{J}(K[M]) \subseteq K[Mz \cup zM]$.
(iii) $\mathcal{J}(K[M])^3 \subseteq K[Mz^2MzM \cup MzMz^2M] \subseteq K[Mz^2M]$.

If, furthermore, z^m is central and Mz^kM is cancellative for some positive integers m, k, and $\mathrm{char}(K) = 0$ *then $K[Mz^kM]$ has no nonzero nil ideal. In particular, $\mathcal{B}(K[Mz^kM]) = 0$. Furthermore, if $k = 2$ then $\mathcal{B}(K[M])^3 = 0$.*

Proof. To prove the first part, let X be the free monoid with basis $x_1, x_2, \dots, x_n$. Then

$$K[M]/K[MzM] \cong K[X]/K[J],$$

where $J = \bigcup_{\sigma \in H \cup \{1\}} X x_{\sigma(1)} \cdots x_{\sigma(n)} X$. Note that X/J has no nonzero nilideal. Hence, by [15, Corollary 24.7], $K[M]/K[MzM]$ is semiprimitive. Therefore $\mathcal{J}(K[M]/K[MzM]) = 0$.

To prove the second and third part, suppose that $\alpha = \sum_{i=1}^q \lambda_i s_i \in \mathcal{J}(K[M])$, with $\mathrm{supp}(\alpha) = \{s_1, \dots, s_q\}$ of cardinality q and $\lambda_i \in K$, is a homogeneous element (with respect to the gradation defined by the natural length function on M). Then α is nilpotent (see for example [17, Theorem 22.6]). Suppose that $s_1 \notin zM$ and $s_1 \notin Mz$. Let i, j be such that $s_1 \in a_iM \cap Ma_j$. Then, for every $l \geqslant 1$, the element $(s_1 a_j a_i)^l = s_1 a_j a_i s_1 a_j a_i \cdots$ can only be rewritten in M in the form $(s' a_j a_i)^l$, where $s' \in M$ is such that $s' = s_1$. Therefore, $\alpha a_j a_i \in \mathcal{J}(K[M])$ is not nilpotent, a contradiction. It follows that s_1, and similarly every $s_i \in Mz \cup zM$. Again by [17, Theorem 22.6], we know that $\mathcal{J}(K[M])$ is a homogeneous ideal. This implies that $\mathcal{J}(K[M]) \subseteq K[Mz \cup zM]$. Hence $\mathcal{J}(K[M])^3 \subseteq Mz\mathcal{J}(K[M])zM \subseteq K[Mz^2MzM \cup MzMz^2M]$. This finishes the proof of statements (ii) and (iii).

To prove the last part, assume $\mathrm{char}(K) = 0$, Mz^kM is cancellative and z^m is central for some positive integers m, k. Since $a_i z^m = z^m a_i$, it follows from the type of the defining relations for M that $z \in a_iM \cap Ma_i$ for every $1 \leqslant i \leqslant n$. Hence, by Lemma 2.4, we know that Mz^kM has a group of fractions G (that is obtained by inverting the powers of the central element z^{km}). Let I be a nil ideal of $K[Mz^kM]$. Then $K[G]IK[G] = I\langle z^{-km}\rangle$ is a nil ideal of $K[G]$. Since, by assumption, $\mathrm{char}(K) = 0$, we know from [16, Theorem 2.3.1] that then $I = 0$. So, if $k = 2$ then, by the first part of the result, $\mathcal{B}(K[M])^3 \subseteq K[Mz^2M] \cap \mathcal{B}(K[M])$. Since $K[Mz^2M] \cap \mathcal{B}(K[M])$ is a nil ideal of $K[Mz^2M]$, the result follows. □

Corollary 2.7. *Suppose z is central. The following properties hold.*

(i) *If $\mathcal{J}(K[\overline{M}]) = 0$ then $\mathcal{J}(K[M]) = I(\rho) \cap K[Mz]$.*
(ii) *If $\mathcal{B}(K[\overline{M}]) = 0$ then $\mathcal{B}(K[M]) = I(\rho) \cap K[Mz]$.*
(iii) *If $\mathcal{B}(K[M]) = 0$ then Mz is cancellative. The converse holds provided* $\mathrm{char}(K) = 0$.

Proof. (i) By Proposition 2.6, $\mathcal{J}(K[M]) \subseteq K[Mz]$. Note that $K[\overline{M}] = K[M/\rho] = K[M]/I(\rho)$. Hence, if $\mathcal{J}(K[\overline{M}]) = 0$, we get that $\mathcal{J}(K[M]) \subseteq I(\rho) \cap K[Mz]$. By Proposition 2.5 we thus obtain that $\mathcal{J}(K[M]) = I(\rho) \cap K[Mz]$.

(ii) If $K[\overline{M}]$ is semiprime, then, by Proposition 2.6, $\mathcal{B}(K[M]) \subseteq I(\rho) \cap K[Mz]$. Thus, by Proposition 2.5, $\mathcal{B}(K[M]) = I(\rho) \cap K[Mz]$.

(iii) Because of Proposition 2.5, we know that $I(\rho) \cap K[Mz] \subseteq \mathcal{B}(K[M])$. Suppose now that $\mathcal{B}(K[M]) = 0$. Then, ρ restricted to Mz must be the trivial relation, i.e., Mz is cancellative. Conversely, assume that $\mathrm{char}(K) = 0$ and Mz is cancellative. Then, by Proposition 2.6, $\mathcal{B}(K[M])$ is a nil ideal of $K[Mz]$, and thus (also by Proposition 2.6) $\mathcal{B}(K[M]) = 0$, as desired. □

3. The monoid $S_n(\mathrm{Alt}_n)$

In this section we investigate the monoid $S_n(\mathrm{Alt}_n)$ with $n \geqslant 4$. The information obtained is essential to prove our main result, Theorem 1.1. Note that the cycle $(1, 2, \ldots, n) \in \mathrm{Alt}_n$ if and only if n is odd. Hence by Proposition 2.3, $z = a_1 a_2 \cdots a_n$ is central if and only if n is odd. However, for arbitrary n, we will show that z^2 is central and that the ideal $S_n(\mathrm{Alt}_n)z^2$ is cancellative as a semigroup and we also will determine the structure of its group of fractions $G_n(\mathrm{Alt}_n)$. This information will be useful to determine the radical of the algebra $K[S_n(\mathrm{Alt}_n)]$.

Throughout this section $n \geqslant 4$, $M = S_n(\mathrm{Alt}_n)$ and $G = G_n(\mathrm{Alt}_n)$. Let $\sigma \in \mathrm{Alt}_n$. Since the set of defining relations of M (of G, respectively) is σ-invariant, σ determines the automorphism of M (of G respectively) defined by $\sigma(a_{i_1}^{n_1} \cdots a_{i_m}^{n_m}) = a_{\sigma(i_1)}^{n_1} \cdots a_{\sigma(i_m)}^{n_m}$.

We will use the same notation for the generators of the free monoid FM_n and the generators of M, if unambiguous. Throughout the rest of the paper, z denotes the element $z = a_1 a_2 \cdots a_n \in M$.

Let $w = a_{i_1} a_{i_2} \cdots a_{i_m}$ be a nontrivial word in the free monoid FM_n on the set $\{a_1, a_2, \ldots, a_n\}$. Let $1 \leqslant p, q \leqslant m$ and r, s be nonnegative integers such that $p + r, q + s \leqslant m$. We say that the subwords $a_{i_p} a_{i_{p+1}} \cdots a_{i_{p+r}}$ and $a_{i_q} a_{i_{q+1}} \cdots a_{i_{q+s}}$ overlap in w if either $p \leqslant q \leqslant p + r$ or $q \leqslant p \leqslant q + s$. For example, in the word $a_3 a_2 a_1 a_3 a_4$ the subwords $a_2 a_1 a_3$ and $a_1 a_3 a_4$ overlap and the subwords $a_3 a_2$ and $a_1 a_3$ do not overlap. Let u, u' be words in the free monoid FM_n. We say that u' is a one step rewrite of u if there exist $u_1, u_2, u_3, u_2' \in \mathrm{FM}_n$ such that u_2 and u_2' represent z in M, and $u = u_1 u_2 u_3$ and $u' = u_1 u_2' u_3$.

Lemma 3.1. *Let $z = a_1 a_2 \cdots a_n \in M$.*

(i) *If $n \geqslant 4$ then $a_i a_j z = z a_i a_j$, for any different integers $1 \leqslant i, j \leqslant n$.*
(ii) *If $n \geqslant 5$ then $a_i a_j a_k z = a_j a_k a_i z$ and $z a_i a_j a_k = z a_j a_k a_i$, for any three different integers $1 \leqslant i, j, k \leqslant n$.*
(iii) *If $n = 4$ and $1 \leqslant i, j, k \leqslant n$ are three different integers then*
  1. *if $a_i a_j a_k a_l = z$ then $a_i a_j a_k z = a_j a_k a_i z = a_k a_i a_j z = z a_k a_j a_i$,*
  2. *if $a_l a_i a_j a_k = z$ then $z a_i a_j a_k = z a_j a_k a_i = z a_k a_i a_j = a_k a_j a_i z$.*
(iv) *If $n \geqslant 6$ is even then $a_i a_j a_k z = z a_j a_i a_k$, for any three different integers $1 \leqslant i, j, k \leqslant n$.*

Proof. (i) If $1 \leqslant i, j \leqslant n$ are different then there exists $\sigma \in \mathrm{Alt}_n$ such that $\sigma(1) = i$ and $\sigma(2) = j$. Hence

$$\begin{aligned} a_i a_j z &= a_i a_j \sigma(1, 2, \ldots, n)^2 (a_1 a_2 \cdots a_n) \\ &= a_i a_j a_{\sigma(3)} \cdots a_{\sigma(n)} a_{\sigma(1)} a_{\sigma(2)} = z a_i a_j. \end{aligned}$$

(ii) and (iv) Suppose that $n \geqslant 5$. In this case, for any three different integers $1 \leqslant i, j, k \leqslant n$ there exists $\sigma \in \mathrm{Alt}_n$ such that $\sigma(1) = i, \sigma(2) = j, \sigma(3) = k$. Let $\tau = \tau_n \in \mathrm{Sym}_n$ be defined by $\tau = id$ if n is odd, and $\tau = (i, j)$ if n is even. So $\tau\sigma(1, 2, \ldots, n)^3 \in \mathrm{Alt}_n$. Hence in M we get

$$\begin{aligned} a_i a_j a_k z &= a_i a_j a_k \tau\sigma(1, 2, \ldots, n)^3 (a_1 a_2 \cdots a_n) \\ &= (a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)})(a_{\sigma(4)} \cdots a_{\sigma(n)} a_{\tau(i)} a_{\tau(j)} a_k) \\ &= \sigma(z) a_{\tau(i)} a_{\tau(j)} a_k. \end{aligned}$$

In particular, (iv) follows. Since $(1, 2, 3) \in \mathrm{Alt}_n$, this yields

$$\begin{aligned} a_i a_j a_k z &= \big(\sigma(1, 2, 3)(a_1 a_2 \cdots a_n)\big) a_{\tau(i)} a_{\tau(j)} a_k \\ &= (a_j a_k a_i) a_{\sigma(4)} \cdots a_{\sigma(n)} a_{\tau(i)} a_{\tau(j)} a_k = a_j a_k a_i z. \end{aligned}$$

Similarly one proves that

$$za_ia_ja_k = za_ja_ka_i,$$

for $n \geqslant 5$.

(iii) Suppose that $n = 4$. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then either $a_ia_ja_ka_l = z$ or $a_la_ia_ja_k = z$. If $a_ia_ja_ka_l = z$, then

$$z = a_ia_ja_ka_l = a_ja_ka_ia_l = a_ka_ia_ja_l,$$

and, since $z \in a_lM$, we get

$$a_ia_ja_kz = a_ja_ka_iz = a_ka_ia_jz.$$

Clearly, $a_ia_ja_kz = a_ia_ja_k(a_la_ka_ja_i) = za_ka_ja_i$.

Similarly, if $a_la_ia_ja_k = z$, we get

$$za_ia_ja_k = za_ja_ka_i = za_ka_ia_j = (a_ka_ja_ia_l)a_ka_ia_j = a_ka_ja_iz. \qquad \square$$

Lemma 3.2. *Let $z = a_1a_2 \cdots a_n \in M$. Then z^2 is central in M.*

Proof. If $n \geqslant 6$ and n is even then

$$\begin{aligned} z^2a_1 &= za_1a_2 \cdots a_na_1 = a_1a_2a_3a_4za_5 \cdots a_na_1 \quad \bigl(\text{by Lemma 3.1(i)}\bigr) \\ &= a_1a_2a_3a_4\bigl((1,5)(2,3)(2,3,\ldots,n)^3(a_1a_2\cdots a_n)\bigr)a_5\cdots a_na_1 \\ &= a_1a_2a_3a_4(a_5a_1a_6\cdots a_na_3a_2a_4)a_5\cdots a_na_1 \\ &= a_1\bigl((1,2,3,4,5)(a_1a_2\cdots a_n)\bigr)(2,3)(1,2,\ldots,n)(a_1a_2\ldots a_n) \\ &= a_1z^2. \end{aligned}$$

If $n = 4$ then

$$\begin{aligned} a_1z^2 &= a_1(a_3a_4a_1a_2)z = a_1a_3a_4za_1a_2 \quad (\text{by Lemma 3.1}) \\ &= a_1a_3a_4(a_2a_3a_1a_4)a_1a_2 = za_3a_1a_4a_1a_2 \\ &= a_3a_1za_4a_1a_2 \quad (\text{by Lemma 3.1}) \\ &= a_3a_1(a_2a_1a_4a_3)a_4a_1a_2 = a_3a_1a_2a_1a_4(a_3a_2a_4a_1) \\ &= a_3a_1za_2a_4a_1 = za_3a_1a_2a_4a_1 \quad (\text{by Lemma 3.1}) \\ &= z^2a_1. \end{aligned}$$

Since Alt_n is transitive, we get that z^2 is central for all even n. Since z is central in M for all odd n, the assertion follows. $\square$

Lemma 3.3. *For $n = 4$, $a_1a_2a_4a_3z = \sigma(a_1a_2a_4a_3)z$, for all $\sigma \in \mathrm{Alt}_4$, and it is central in M. In particular, $\sigma(z)z = z\sigma(z) = z\gamma(z) = \gamma(z)z$ for any $\sigma, \gamma \in \mathrm{Sym}_4$ of the same parity.*

Proof. By Lemma 3.1, we have

$$a_1a_2a_4a_3z = a_1(a_3a_2a_4)z = a_1(a_4a_3a_2)z,$$

and also

$$a_1a_2a_4a_3z = za_1a_2a_4a_3 = z(a_2a_4a_1)a_3 = a_2a_4a_1a_3z,$$

$$a_1a_2a_4a_3z = za_1a_2a_4a_3 = z(a_4a_1a_2)a_3 = a_4a_1a_2a_3z,$$

$$a_1a_2a_4a_3z = a_1a_3a_2a_4z = za_1a_3a_2a_4 = z(a_3a_2a_1)a_4 = a_3a_2a_1a_4z.$$

Thus $a_1a_2a_4a_3z = \sigma(a_1a_2a_4a_3)z$ for all $\sigma \in \mathrm{Alt}_4$. In particular, $\sigma(z)z = \gamma(z)z$ for odd permutations σ, γ. Of course such an equality also holds if γ, σ are even. Note that, because of Lemma 3.1, $z\sigma(z) = \sigma(z)z$ for any permutation σ.

In order to prove that $a_1a_2a_4a_3z$ is central we only need to show that $a_1a_2a_4a_3za_1 = a_1a_1a_2a_4a_3z$. By Lemma 3.1, we have

$$\begin{aligned} a_1a_2a_4a_3za_1 &= a_1a_2za_4a_3a_1 = a_1a_2z(a_1a_4a_3) \\ &= a_1a_2a_1a_4za_3 = a_1(a_1a_4a_2)za_3 \\ &= a_1a_1za_4a_2a_3 = a_1a_1(a_2a_4a_3a_1)a_4a_2a_3 \\ &= a_1a_1a_2a_4a_3z. \qquad \square \end{aligned}$$

Lemma 3.4. *Let $z = a_1a_2\cdots a_n \in M$.*

(i) *If $n \geqslant 6$ is even then $a_i^2a_j(a_ka_la_rz) = a_ja_i^2(a_ka_la_rz)$ and $a_ia_ja_ia_j(a_ka_la_rz) = a_ja_ia_ja_i(a_ka_la_rz)$, for all $1 \leqslant i, j \leqslant n$ and for any three different integers $1 \leqslant k, l, r \leqslant n$.*

(ii) *If $n \geqslant 4$, then $a_i^2a_jz^2 = a_ja_i^2z^2$ and $a_ia_ja_ia_jz^2 = a_ja_ia_ja_iz^2$, for all $1 \leqslant i, j \leqslant n$.*

Proof. (i) Suppose that $n \geqslant 6$ is even. Applying Lemma 3.1 several times, we get

$$\begin{aligned} a_1a_1a_2a_1a_2a_3z &= a_1a_1a_2(a_3a_1a_2z) = a_1a_1a_2(za_3a_2a_1) \\ &= a_1(za_1a_2)a_3a_2a_1 = a_1(za_2a_3a_1)a_2a_1 \\ &= a_1(a_2a_3z)a_1a_2a_1 = (za_2a_1a_3)a_1a_2a_1 \\ &= (a_2a_1z)a_3a_1a_2a_1 = a_2a_1(za_1a_2a_3)a_1 \\ &= a_2a_1(a_1a_2a_3a_1z) = a_2a_1a_1(a_1a_2a_3z), \\ a_1a_2a_1a_2a_1a_2a_3z &= a_1a_2a_1a_2(za_3a_2a_1) = a_1a_2(za_1a_2)a_3a_2a_1 \\ &= a_1a_2(a_3a_2a_1z)a_2a_1 = a_1a_2a_3(za_2a_1)a_2a_1 \\ &= (za_2a_1a_3)a_2a_1a_2a_1 = (a_2a_1z)a_3a_2a_1a_2a_1 \\ &= a_2a_1(za_2a_1a_3)a_2a_1 = a_2a_1(a_2a_1z)a_3a_2a_1 \\ &= a_2a_1a_2a_1(a_1a_2a_3z) \end{aligned}$$

and, for every $i \in \{1, 2, \ldots, n\} \setminus \{3, 4\}$,

$$\begin{aligned}
a_1a_1a_2a_ia_3a_4z &= a_1a_1a_2(a_3a_4a_iz) = a_1a_1a_2a_3(za_4a_i)\\
&= a_1(za_2a_1a_3)a_4a_i = a_1(za_3a_2a_1)a_4a_i\\
&= a_1(a_3a_2z)a_1a_4a_i = (a_2a_1a_3z)a_1a_4a_i\\
&= a_2a_1a_3(a_1a_4z)a_i = a_2a_1(a_1a_4a_3z)a_i\\
&= a_2a_1a_1(za_4a_3)a_i = a_2a_1a_1(a_ia_3a_4z),\\
a_1a_2a_1a_2a_ia_3a_4z &= a_1a_2a_1a_2(za_4a_3a_i) = a_1a_2(za_1a_2)a_4a_3a_i\\
&= a_1a_2(a_4a_2a_1z)a_3a_i = a_1a_2a_4(za_2a_1)a_3a_i\\
&= (za_2a_1a_4)a_2a_1a_3a_i = (a_2a_1z)a_4a_2a_1a_3a_i\\
&= a_2a_1(za_2a_1a_4)a_3a_i = a_2a_1(a_2a_1z)a_4a_3a_i\\
&= a_2a_1a_2a_1(a_ia_3a_4z).
\end{aligned}$$

Hence, in each case applying an appropriate $\sigma \in \mathrm{Alt}_n$ and using Lemma 3.1, we obtain $a_i^2a_j(a_ka_la_rz) = a_ja_i^2(a_ka_la_rz)$ and $a_ia_ja_ia_j(a_ka_la_rz) = a_ja_ia_ja_i(a_ka_la_rz)$, for all $1 \leqslant i, j \leqslant n$ and for any three different integers $1 \leqslant k, l, r \leqslant n$.

(ii) Suppose that n is odd. Let $z' = a_6a_7\cdots a_n$ (so z' is the identity element if $n = 5$). Since z is central in M, by Lemma 3.1, we have

$$\begin{aligned}
a_1a_1a_2z^2 &= a_1a_1a_2(a_3a_4a_5a_1a_2)z'z = a_1(a_2a_3a_1)a_4a_5a_1a_2z'z\\
&= (a_2a_3a_1)(a_4a_5a_1)a_1a_2z'z = a_2(a_1a_4a_3)a_5a_1a_1a_2z'z\\
&= a_2a_1a_4(a_1a_3a_5)a_1a_2z'z = a_2a_1(a_1a_3a_4)a_5a_1a_2z'z\\
&= a_2a_1a_1z^2
\end{aligned}$$

and

$$\begin{aligned}
a_1a_2a_1a_2z^2 &= a_1a_2a_1a_2(a_3a_5a_4a_2a_1)z'z = a_1a_2(a_3a_1a_2)a_5a_4a_2a_1z'z\\
&= a_1a_2a_3(a_5a_1a_2)a_4a_2a_1z'z = a_1(a_5a_2a_3)(a_2a_4a_1)a_2a_1z'z\\
&= (a_2a_1a_5)a_3a_2a_4a_1a_2a_1z'z = a_2a_1(a_2a_5a_3)(a_1a_2a_4)a_1z'z\\
&= a_2a_1a_2(a_1a_5a_3)a_2a_4a_1z'z = a_2a_1a_2a_1z^2.
\end{aligned}$$

For $n = 4$, we have

$$\begin{aligned}
a_1a_1a_2z^2 &= a_1z^2a_1a_2 = a_1(a_2a_3a_1a_4)(a_2a_3a_1a_4)a_1a_2\\
&= a_1a_2a_3(a_4a_2a_1a_3)a_1a_4a_1a_2 = (a_2a_1a_4a_3)a_2a_1a_3a_1a_4a_1a_2\\
&= a_2a_1(a_1a_3a_4a_2)a_3a_1a_4a_1a_2 = a_2a_1a_1a_3a_4(a_1a_2a_3a_4)a_1a_2\\
&= a_2a_1a_1z^2
\end{aligned}$$

and, by Lemmas 3.1 and 3.3,

$$\begin{aligned} a_1a_2a_1a_2z^2 &= za_1a_2za_1a_2 = a_2a_1(a_4a_3a_1a_2z)a_1a_2 \\ &= a_2a_1(a_2a_1a_3a_4z)a_1a_2 = a_2a_1a_2a_1a_3a_4a_1a_2z \\ &= a_2a_1a_2a_1z^2. \end{aligned}$$

Hence, if n is odd or $n = 4$ and for $\sigma \in \mathrm{Alt}_n$ we have that

$$a_{\sigma(1)}a_{\sigma(1)}a_{\sigma(2)}z^2 = a_{\sigma(2)}a_{\sigma(1)}a_{\sigma(1)}z^2$$

and

$$a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(1)}a_{\sigma(2)}z^2 = a_{\sigma(2)}a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(1)}z^2.$$

So

$$a_i^2a_jz^2 = a_ja_i^2z^2 \quad \text{and} \quad a_ia_ja_ia_jz^2 = a_ja_ia_ja_iz^2$$

for all $1 \leqslant i, j \leqslant n$, and (ii) follows.

Suppose that $n \geqslant 6$ is even. By Lemma 3.1, $z^2 = a_2a_1a_3za_4a_5\cdots a_n$. Consequently, by (i), we get that

$$a_i^2a_jz^2 = a_ja_i^2z^2 \quad \text{and} \quad a_ia_ja_ia_jz^2 = a_ja_ia_ja_iz^2$$

for all $1 \leqslant i, j \leqslant n$, as desired. □

We define the map $f : \mathrm{FM}_n \to \{-1, 1\}$ by

$$f(a_{i_1}\cdots a_{i_m}) = \prod_{\substack{1\leqslant j<k\leqslant m \\ i_j\neq i_k}} \frac{i_k - i_j}{|i_k - i_j|}. \tag{2}$$

Note that if two words $w, w' \in \mathrm{FM}_n$ represent the same element in M then $f(w) = f(w')$.

Lemma 3.5. *Let $z = a_1a_2\cdots a_n \in M$. Let t be a positive integer. For $1 \leqslant i < j \leqslant n$, let $F_{ij} = \langle a_i, a_j\rangle$. Then*

(i) *The elements in $z^{2t}F_{ij}$ are of the form*

$$z^{2t}a_i^{2n_1}a_j^{2n_2}w, \tag{3}$$

where $w \in \{1, a_i, a_j, a_ia_j, a_ja_i, a_ia_ja_i, a_ja_ia_j, a_ia_ja_ia_j\}$ and n_1, n_2 are nonnegative integers.

(ii) *The elements in $z^{2t}(M \setminus \bigcup_{1\leqslant i<j\leqslant n} F_{ij})$ are of the form*

$$z^{2t}a_{i_1}^{2n_1}a_{i_2}^{2n_2}wa_{i_2+1}^{m_1}a_{i_2+2}^{m_2}\cdots a_n^{m_{n-i_2}}, \tag{4}$$

where $i_1 < i_2 < n$, $a_{i_1}^{2n_1}a_{i_2}^{2n_2}w \in F_{i_1i_2} \setminus (\langle a_{i_1}\rangle \cup \langle a_{i_2}\rangle)$, $w \in \{1, a_{i_1}, a_{i_2}, a_{i_1}a_{i_2}, a_{i_2}a_{i_1}, a_{i_1}a_{i_2}a_{i_1}, a_{i_2}a_{i_1}a_{i_2}, a_{i_1}a_{i_2}a_{i_1}a_{i_2}\}$, $n_1, n_2, m_1, m_2, \ldots, m_{n-i_2}$ are nonnegative integers, and $\sum_{j=1}^{n-i_2} m_j > 0$.

Furthermore, every element $s \in z^{2t}M$ has a unique representation as a product of the form (3) or (4).

Proof. (i) We may assume that $t=1$. By Lemma 3.2, z^2 is central in M. Now, by Lemma 3.4(ii), we have

$$a_ia_ja_ia_ja_iz^2=(a_ja_ia_ja_i)a_iz^2,$$
$$a_ia_ja_ia_ja_ia_jz^2=(a_ja_ia_ja_i)a_ia_jz^2=a_ja_ia_ja_ja_i^2z^2,$$
$$a_ia_ja_ia_ja_ia_ja_iz^2=(a_ja_ia_ja_i)a_ia_ja_iz^2=a_ja_i^4a_j^2z^2,$$
$$(a_ia_j)^4z^2=(a_ja_i)^2(a_ia_j)^2z^2=a_i^4a_j^4z^2.$$

Therefore

$$(a_ia_j)^2a_iz^2=a_ja_ia_ja_i^2z^2,$$
$$(a_ia_j)^3z^2=a_ja_ia_j^2a_i^2z^2=a_ja_ia_i^2a_j^2z^2,$$
$$(a_ia_j)^3a_iz^2=a_ja_i^4a_j^2z^2=a_ja_j^2a_i^4z^2,$$
$$(a_ia_j)^4z^2=a_i^4a_j^4z^2=a_j^4a_i^4z^2,$$

for all $1\leqslant i,j\leqslant n$. The above easily implies that the elements in z^2F_{ij} are of the form

$$z^2a_i^{2n_1}a_j^{2n_2}w,$$

where $w\in\{1,a_i,a_j,a_ia_j,a_ja_i,a_ia_ja_i,a_ja_ia_j,a_ia_ja_ia_j\}$ and n_1,n_2 are nonnegative integers. Hence (i) follows.

(ii) We may assume that $t=1$. Let $1\leqslant i<j\leqslant n$ and $m\in\{1,\dots,n\}\setminus\{i,j\}$. Then, by (3) and Lemmas 3.1, 3.2 and 3.4(ii), it is easy to see that

$$a_mz^2\bigl(F_{ij}\setminus\bigl(\langle a_i\rangle\cup\langle a_j\rangle\bigr)\bigr)=z^2\bigl(F_{ij}\setminus\bigl(\langle a_i\rangle\cup\langle a_j\rangle\bigr)\bigr)a_m, \tag{5}$$

for all $n\geqslant 4$.

Let $s\in M\setminus\bigcup_{1\leqslant i<j\leqslant n}F_{ij}$. Then $s=a_{j_1}a_{j_2}\cdots a_{j_k}$, where $\{j_1,\dots,j_k\}$ is a subset of $\{1,\dots,n\}$ of cardinality $\geqslant 3$. We shall prove that z^2s is of the form (4) by induction on the total degree $k\geqslant 3$ of s. For $k=3$, we have that j_1,j_2,j_3 are three different elements and, by Lemmas 3.1 and 3.2,

$$z^2a_{j_1}a_{j_2}a_{j_3}=z^2a_{j_2}a_{j_3}a_{j_1}=z^2a_{j_3}a_{j_1}a_{j_2},$$

thus the result follows in this case.

Suppose that $k>3$ and that the result is true for all elements in $M\setminus\bigcup_{1\leqslant i<j\leqslant n}F_{ij}$ of total degree less than k. Then either $a_{j_2}\cdots a_{j_k}\in F_{i_1i_2}\setminus(\langle a_{i_1}\rangle\cup\langle a_{i_2}\rangle)$, for some $i_1<i_2$, or $a_{j_2}\cdots a_{j_k}\in M\setminus\bigcup_{1\leqslant i<j\leqslant n}F_{ij}$. Thus, by (i) and by the induction hypothesis

$$z^2a_{j_2}\cdots a_{j_k}=z^2a_{i_1}^{2n_1}a_{i_2}^{2n_2}wa_{i_2+1}^{m_1}a_{i_2+2}^{m_2}\cdots a_n^{m_{n-i_2}},$$

where $i_1<i_2\leqslant n$, $a_{i_1}^{2n_1}a_{i_2}^{2n_2}w\in F_{i_1i_2}\setminus(\langle a_{i_1}\rangle\cup\langle a_{i_2}\rangle)$, $w\in\{1,a_{i_1},a_{i_2},a_{i_1}a_{i_2}, a_{i_2}a_{i_1},a_{i_1}a_{i_2}a_{i_1},a_{i_2}a_{i_1}a_{i_2}, a_{i_1}a_{i_2}a_{i_1}a_{i_2}\}$ and $n_1,n_2,m_1,\dots,m_{n-i_2}\geqslant 0$. By (i) and Lemma 3.2, we may assume that $j_1\notin\{i_1,i_2\}$ and, by (5), we also may assume that

$$z^2s=z^2a_{i_1}^{2n_1}a_{i_2}^{2n_2}wa_{j_1}a_{i_2+1}^{m_1}a_{i_2+2}^{m_2}\cdots a_n^{m_{n-i_2}}.$$

Suppose that $j_1 < i_2$. Note that in this case, by Lemmas 3.1, 3.2 and 3.4(ii), we get

$$\begin{aligned}
z^2 a_{i_1}^{2n_1} a_{i_2}^{2n_2} a_{j_1} &= z^2 a_{i_1}^{2n_1} a_{j_1} a_{i_2}^{2n_2},\\
z^2 a_{i_1}^{2n_1} a_{i_2}^{2n_2} a_{i_1} a_{j_1} &= z^2 a_{i_1}^{2n_1} a_{i_1} a_{j_1} a_{i_2}^{2n_2},\\
z^2 a_{i_1}^{2n_1} a_{i_2}^{2n_2} a_{i_2} a_{j_1} &= z^2 a_{i_1}^{2n_1} a_{i_2} a_{j_1} a_{i_2}^{2n_2} = z^2 a_{i_1}^{2(n_1-1)} a_{j_1} a_{i_1} a_{i_2}^{2n_2+1},\\
z^2 a_{i_1}^{2n_1} a_{i_2}^{2n_2} a_{i_1} a_{i_2} a_{j_1} &= z^2 a_{i_1}^{2n_1} a_{i_1} a_{i_2} a_{j_1} a_{i_2}^{2n_2} = z^2 a_{i_1}^{2n_1} a_{j_1} a_{i_1} a_{i_2}^{2n_2+1},\\
z^2 a_{i_1}^{2n_1} a_{i_2}^{2n_2} a_{i_2} a_{i_1} a_{j_1} &= z^2 a_{i_1}^{2n_1} a_{i_2} a_{i_1} a_{j_1} a_{i_2}^{2n_2} = z^2 a_{i_1}^{2n_1} a_{i_1} a_{j_1} a_{i_2}^{2n_2+1},\\
z^2 a_{i_1}^{2n_1} a_{i_2}^{2n_2} a_{i_1} a_{i_2} a_{i_1} a_{j_1} &= z^2 a_{i_1}^{2n_1} a_{i_1} a_{i_2} a_{i_1} a_{j_1} a_{i_2}^{2n_2} = z^2 a_{i_1}^{2(n_1+1)} a_{j_1} a_{i_2}^{2n_2+1},\\
z^2 a_{i_1}^{2n_1} a_{i_2}^{2n_2} a_{i_2} a_{i_1} a_{i_2} a_{j_1} &= z^2 a_{i_1}^{2n_1} a_{i_2} a_{i_1} a_{i_2} a_{j_1} a_{i_2}^{2n_2} = z^2 a_{i_1}^{2n_1} a_{i_2} a_{j_1} a_{i_1} a_{i_2}^{2n_2+1}\\
&= z^2 a_{i_1}^{2n_1} a_{j_1} a_{i_1} a_{i_2}^{2n_2+2},\\
z^2 a_{i_1}^{2n_1} a_{i_2}^{2n_2} (a_{i_1} a_{i_2})^2 a_{j_1} &= z^2 a_{i_1}^{2n_1} (a_{i_1} a_{i_2})^2 a_{j_1} a_{i_2}^{2n_2}\\
&= z^2 a_{i_1}^{2n_1} a_{i_1} a_{i_2} a_{j_1} a_{i_1} a_{i_2}^{2n_2+1} = z^2 a_{i_1}^{2n_1} a_{i_1} a_{j_1} a_{i_1} a_{i_2}^{2n_2+2}.
\end{aligned}$$

Hence

$$z^2 s = z^2 a_{i_1}^{2n'_1} w' a_{i_2}^{m_0} a_{i_2+1}^{m_1} a_{i_2+2}^{m_2} \cdots a_n^{m_{n-i_2}},$$

for some $w' \in \{a_{j_1}, a_{i_1} a_{j_1}, a_{j_1} a_{i_1}, a_{i_1} a_{j_1} a_{i_1}\}$ and some nonnegative integers n'_1 and m_0, and therefore $z^2 s$ is of the form (4) (if $i_1 < j_1 < i_2$ then we take the pair (i_1, j_1) in place of the pair (i_1, i_2) in formula (4), and if $j_1 < i_1 < i_2$, then the degree of s with respect to a_{j_1} is equal to 1, so by taking the pair (j_1, i_1) in place of the pair (i_1, i_2) in formula (4) we also get that $z^2 s$ is of the form (4) because we can then write $z^2 s = z^2 a_{j_1}^{2\cdot 0} a_{i_1}^{2n'_1} w' a_{i_1+1}^0 \cdots a_{i_2-1}^0 a_{i_2}^{m_0} a_{i_2+1}^{m_1} a_{i_2+2}^{m_2} \cdots a_n^{m_{n-i_2}}$).

Suppose that $j_1 > i_2$. By Lemma 3.4(ii), $z^2 a_{i_1}^{2n_1} a_{i_2}^{2n_2} w \in (z^2 F_{i_1 i_2} a_{i_1} a_{i_2}) \cup (z^2 F_{i_1 i_2} a_{i_2} a_{i_1})$. Note that if $i_2 < l < j_1$, then, by Lemmas 3.1 and 3.2,

$$z^2 a_{i_1} a_{i_2} a_{j_1} a_l = z^2 a_{i_1} a_l a_{i_2} a_{j_1} = z^2 a_l a_{i_2} a_{i_1} a_{j_1} = a_l z^2 a_{i_2} a_{i_1} a_{j_1}$$

and

$$z^2 a_{i_2} a_{i_1} a_{j_1} a_l = z^2 a_{i_2} a_l a_{i_1} a_{j_1} = z^2 a_l a_{i_1} a_{i_2} a_{j_1} = a_l z^2 a_{i_1} a_{i_2} a_{j_1}.$$

Therefore, by using (5) and Lemmas 3.4(ii), 3.2 and 3.1, we can move the a_{j_1} of

$$z^2 s = z^2 a_{i_1}^{2n_1} a_{i_2}^{2n_2} w a_{j_1} a_{i_2+1}^{m_1} a_{i_2+2}^{m_2} \cdots a_n^{m_{n-i_2}}$$

to the right. Hence, if $j_1 = i_2 + p$, then

$$z^2 s = z^2 a_{i_1}^{2n'_1} a_{i_2}^{2n'_2} w' a_{i_2+1}^{m_1} \cdots a_{i_2+p}^{m_p+1} \cdots a_n^{m_{n-i_2}},$$

where $a_{i_1}^{2n'_1} a_{i_2}^{2n'_2} w' \in F_{i_1 i_2} \setminus (\langle a_{i_1} \rangle \cup \langle a_{i_2} \rangle)$, $w' \in \{1, a_{i_1}, a_{i_2}, a_{i_1} a_{i_2}, a_{i_2} a_{i_1}, a_{i_1} a_{i_2} a_{i_1}, a_{i_2} a_{i_1} a_{i_2}, a_{i_1} a_{i_2} a_{i_1} a_{i_2}\}$ and n'_1, n'_2 are nonnegative integers. Thus (ii) follows by induction.

Let

$$w_1 = a_{i_1}^{2n_1} a_{i_2}^{2n_2} w a_{i_2+1}^{m_1} a_{i_2+2}^{m_2} \cdots a_n^{m_{n-i_2}}$$

and

$$w_2 = a_{i'_1}^{2n'_1} a_{i'_2}^{2n'_2} w' a_{i'_2+1}^{m'_1} a_{i'_2+2}^{m'_2} \cdots a_n^{m'_{n-i'_2}}$$

be two words in FM_n such that $i_1 < i_2 \leqslant n$, $i'_1 < i'_2 \leqslant n$,

$$w \in \{1, a_{i_1}, a_{i_2}, a_{i_1}a_{i_2}, a_{i_2}a_{i_1}, a_{i_1}a_{i_2}a_{i_1}, a_{i_2}a_{i_1}a_{i_2}, a_{i_1}a_{i_2}a_{i_1}a_{i_2}\}$$

and

$$w' \in \{1, a_{i'_1}, a_{i'_2}, a_{i'_1}a_{i'_2}, a_{i'_2}a_{i'_1}, a_{i'_1}a_{i'_2}a_{i'_1}, a_{i'_2}a_{i'_1}a_{i'_2}, a_{i'_1}a_{i'_2}a_{i'_1}a_{i'_2}\}.$$

Suppose that if $\sum_{j=1}^{n-i_2} m_j > 0$, then $a_{i_1}^{2n_1} a_{i_2}^{2n_2} w \in \langle a_{i_1}, a_{i_2}\rangle \setminus (\langle a_{i_1}\rangle \cup \langle a_{i_2}\rangle)$, and if $\sum_{j=1}^{n-i'_2} m'_j > 0$, then $a_{i'_1}^{2n'_1} a_{i'_2}^{2n'_2} w' \in \langle a_{i'_1}, a_{i'_2}\rangle \setminus (\langle a_{i'_1}\rangle \cup \langle a_{i'_2}\rangle)$. Suppose that $(a_1a_2\cdots a_n)^{2t} w_1$ and $(a_1a_2\cdots a_n)^{2t} w_2$ represent the same element in M.

In order to prove the last part of the lemma it is sufficient to prove that $w_1 = w_2$. Note that the degree of w_1 in a_i is equal to the degree of w_2 in a_i, for all $i = 1, \ldots n$. Let f be the map defined by (2). By the definition of f, we have that $f(w_1) = f(w_2)$.

Note that if $\sum_{j=1}^{n-i_2} m_j > 0$, then $\sum_{j=1}^{n-i'_2} m'_j > 0$, $i_1 = i'_1$, $i_2 = i'_2$ and $m_j = m'_j$, for all $1 \leqslant j \leqslant n - i_2$. Furthermore, since $f(w_1) = f(w_2)$, by the definition of f, we have that $f(a_{i_1}^{2n_1} a_{i_2}^{2n_2} w) = f(a_{i'_1}^{2n'_1} a_{i'_2}^{2n'_2} w')$ in this case. Thus we may assume that

$$w_1 = a_{i_1}^{2n_1} a_{i_2}^{2n_2} w \quad \text{and} \quad w_2 = a_{i'_1}^{2n'_1} a_{i'_2}^{2n'_2} w'.$$

Then the definition of f implies that $f(w_1) = f(w)$ and $f(w_2) = f(w')$. Hence $f(w) = f(w')$.

Suppose that $1 = f(w) = f(w')$. In this case, $w \in \{1, a_{i_1}, a_{i_2}, a_{i_1}a_{i_2}\}$ and $w' \in \{1, a_{i'_1}, a_{i'_2}, a_{i'_1}a_{i'_2}\}$. If $w = 1$, then the degree of w_1 in each generator is even. Since w_1, w_2 have the same degree in each generator, we have $w' = 1$ and $w_1 = w_2$ in this case.

If $w = a_{i_1}a_{i_2}$, then the degree of w_1 in a_{i_1} is odd and the degree of w_1 in a_{i_2} is odd. Since $i_1 < i_2$ and $i'_1 < i'_2$ and w_1, w_2 have the same degree in each generator, we have that $i_1 = i'_1$, $i_2 = i'_2$, $w' = a_{i_1}a_{i_2}$ and $w_1 = w_2$, in this case.

If $w = a_{i_1}$ and $n_{i_2} = 0$, then clearly $w_1 = w_2 \in \langle a_{i_1}\rangle$.

If $w = a_{i_1}$ and $n_{i_2} \neq 0$, then by a degree argument it is easy to see that $i_1 = i'_1$, $i_2 = i'_2$, $w' = a_{i_1}$ and $w_1 = w_2$, in this case.

Similarly, if $w = a_{i_2}$, we can see that $w_1 = w_2$.

Suppose that $-1 = f(w) = f(w')$. In this case,

$$w \in \{a_{i_2}a_{i_1}, a_{i_1}a_{i_2}a_{i_1}, a_{i_2}a_{i_1}a_{i_2}, a_{i_1}a_{i_2}a_{i_1}a_{i_2}\}$$

and

$$w' \in \{a_{i'_2}a_{i'_1}, a_{i'_1}a_{i'_2}a_{i'_1}, a_{i'_2}a_{i'_1}a_{i'_2}, a_{i'_1}a_{i'_2}a_{i'_1}a_{i'_2}\}.$$

As above, using f and a degree argument we can also see that $w_1 = w_2$.

Therefore the result follows. □

Lemma 3.6. *Suppose that $n \geqslant 6$ is even. Let t be a nonnegative integer. Let $z = a_1a_2\cdots a_n \in M$. For $1 \leqslant i < j \leqslant n$, let $F_{ij} = \langle a_i, a_j\rangle$. Let k, l, r be three different integers such that $1 \leqslant k, l, r \leqslant n$. Then*

(i) $(a_ka_la_rz)a_i = a_i(a_ka_la_rz)$, *for all* $i \in \{1, 2, \dots, n\} \setminus \{k, l, r\}$.
(ii) $(a_ka_la_rz)a_i = a_i(a_la_ka_rz)$, *for all* $i \in \{k, l, r\}$.
(iii) *The elements in* $z^{2t}a_ka_la_rzF_{ij}$ *are of the form*

$$z^{2t}a_ka_la_rza_i^{2n_1}a_j^{2n_2}w, \tag{6}$$

where $w \in \{1, a_i, a_j, a_ia_j, a_ja_i, a_ia_ja_i, a_ja_ia_j, a_ia_ja_ia_j\}$ *and* n_1, n_2 *are nonnegative integers.*
(iv) *The elements in* $z^{2t}a_ka_la_rz(M \setminus \bigcup_{1\leqslant i<j\leqslant n} F_{ij})$ *are of the form*

$$z^{2t}a_ka_la_rza_{i_1}^{2n_1}a_{i_2}^{2n_2}wa_{i_2+1}^{m_1}a_{i_2+2}^{m_2}\cdots a_n^{m_{n-i_2}}, \tag{7}$$

where $i_1 < i_2 < n$, $a_{i_1}^{2n_1}a_{i_2}^{2n_2}w \in F_{i_1i_2} \setminus (\langle a_{i_1}\rangle \cup \langle a_{i_2}\rangle)$, $w \in \{1, a_{i_1}, a_{i_2}, a_{i_1}a_{i_2}, a_{i_2}a_{i_1}, a_{i_1}a_{i_2}a_{i_1}, a_{i_2}a_{i_1}a_{i_2}, a_{i_1}a_{i_2}a_{i_1}a_{i_2}\}$, $n_1, n_2, m_1, m_2, \dots, m_{n-i_2}$ *are nonnegative integers, and* $\sum_{j=1}^{n-i_2} m_j > 0$.

Furthermore every element $s \in z^{2t}a_ka_la_rzM$ *has a unique representation as a product of the form* (6) *or* (7).

Proof. (i) Let $i \in \{1, 2, \dots, n\} \setminus \{k, l, r\}$. By Lemma 3.1, we have

$$\begin{aligned}(a_ka_la_rz)a_i &= a_k(za_la_r)a_i = a_kz(a_ia_la_r) = a_k(a_ia_lz)a_r \\ &= (a_ia_la_k)za_r = a_i(za_la_k)a_r = a_i(a_ka_la_rz).\end{aligned}$$

(ii) Let $i \in \{k, l, r\}$. By Lemma 3.1, we may assume that $i = k$, and we have

$$(a_ka_la_rz)a_k = a_k(za_la_r)a_k = a_kz(a_ka_la_r) = a_k(a_la_ka_rz).$$

To prove (iii) and (iv) we may assume that $t = 0$. Then the proof of (iii) and (iv) is similar to the proof of Lemma 3.5. Namely, it is obtained by using (i) and (ii) in place of the fact that z^2 is central and using Lemma 3.4(i) in place of Lemma 3.4(ii). The proof of the last part of the lemma is similar to the proof of the last part of Lemma 3.5. □

Lemma 3.7. *Suppose that $n \geqslant 6$ is even. Then*

$$\bigcup_{1\leqslant r\leqslant n}(Mz \cap Mza_r) = \bigcup_{1\leqslant i<j<k\leqslant n}(Ma_ia_ja_kz \cup Ma_ja_ia_kz).$$

Proof. By Lemma 3.1, we have that

$$a_1a_2a_3z = za_2a_1a_3 = a_2a_1za_3.$$

Note that if $1 \leqslant i, j, k \leqslant n$ are three different integers then, since $n \geqslant 6$, there exists $\sigma \in \mathrm{Alt}_n$ such that $\sigma(1) = i$, $\sigma(2) = j$ and $\sigma(3) = k$. Therefore

$$a_i a_j a_k z \in \bigcup_{1 \leqslant r \leqslant n} (Mz \cap Mza_r), \tag{8}$$

for all different $1 \leqslant i, j, k \leqslant n$.

Suppose that $\bigcup_{1\leqslant r\leqslant n}(Mz \cap Mza_r) \nsubseteq \bigcup_{1\leqslant i<j<k\leqslant n}(Ma_ia_ja_kz \cup Ma_ja_ia_kz)$. Let $s \in \bigcup_{1\leqslant r\leqslant n}(Mz \cap Mza_r) \setminus \bigcup_{1\leqslant i<j<k\leqslant n}(Ma_ia_ja_kz \cup Ma_ja_ia_kz)$ be an element of minimal length. There exist $1 \leqslant r \leqslant n$, $s' = a_{j_1} \cdots a_{j_{k-1}} \in M$ and $s'' = a_{i_1}a_{i_2} \cdots a_{i_k}$ such that $s = s'za_r = s''z$. Thus there exist $w_1, w_2, \ldots, w_m$ in the free monoid FM_n on $\{a_1, \ldots, a_n\}$, such that $w_1 = a_{j_1} \cdots a_{j_{k-1}} a_1 a_2 \cdots a_n a_r$, $w_m = a_{i_1} \cdots a_{i_k} a_1 a_2 \cdots a_n$ and $w_i = w_{1,i}w_{2,i}w_{3,i} = w'_{1,i}w'_{2,i}w'_{3,i}$, where $w_{2,i}$ and $w'_{2,i}$ represent the element z in M for all $i = 1, \ldots, m$, and $w_{1,j} = w'_{1,j+1}$ and $w_{3,j} = w'_{3,j+1}$, for all $j = 1, \ldots, m-1$.

Let $g\colon \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n+k\} \to \{1, 2, \ldots, n\}$ be such that $w_i = a_{g(i,1)}a_{g(i,2)} \cdots a_{g(i,n+k)}$ for all $i = 1, \ldots, m$. Let t be the least positive integer such that $a_{g(t,k+1)}a_{g(t,k+2)} \cdots a_{g(t,n+k)}$ represents z in M. Since n is even, by Proposition 2.3, $t > 1$ and $g(i, n+k) = r$, for all $i = 1, \ldots, t$. Hence

$$a_{g(1,1)}a_{g(1,2)} \cdots a_{g(1,n+k-1)}, \ldots, a_{g(t-1,1)}a_{g(t-1,2)} \cdots a_{g(t-1,n+k-1)}$$

represent the same element in M. Furthermore, the length of $w_{3,t-1}$ is less than n and greater than 0.

Suppose that $w_{3,t-1} = a_r$. In this case, $w'_{2,t}a_r = a_{g(t,k)} \cdots a_{g(t,n+k)}$ and $w_{2,t-1}a_r$ represent the same element in M, but, by Proposition 2.3, in M we have that $za_r \neq a_{g(t,k)}z$, a contradiction. Therefore the length of $w_{3,t-1}$ is greater than 1.

Suppose that $w_{3,t-1} = a_{g(t-1,n+k-1)}a_r$. In this case, $w_{2,t-1}a_{g(t-1,n+k-1)}a_r$ and $w'_{2,t}a_{g(t-1,n+k-1)}a_r = a_{g(t,k-1)}a_{g(t,k)} \cdots a_{g(t,n+k)}$ represent the same element in M. Since

$$a_{g(1,1)}a_{g(1,2)} \cdots a_{g(1,n+k-1)}, \ldots, a_{g(t-1,1)}a_{g(t-1,2)} \cdots a_{g(t-1,n+k-1)}$$

represent the same element in M, we have in M that

$$\begin{aligned} a_{g(1,1)} \cdots a_{g(1,k-1)}z &= a_{g(1,1)}a_{g(1,2)} \cdots a_{g(1,n+k-1)} \\ &= a_{g(t-1,1)}a_{g(t-1,2)} \cdots a_{g(t-1,n+k-1)} \\ &= a_{g(t-1,1)}a_{g(t-1,2)} \cdots a_{g(t-1,k-2)}za_{g(t-1,n+k-1)}. \end{aligned}$$

Thus $a_{g(1,1)} \cdots a_{g(1,k-1)}z \in Mz \cap Mza_{g(t-1,n+k-1)}$. By the choice of s, we have that

$$a_{g(1,1)} \cdots a_{g(1,k-1)}z \in \bigcup_{1\leqslant i<j<k\leqslant n} (Ma_ia_ja_kz \cup Ma_ja_ia_kz).$$

Since $s = a_{g(1,1)} \cdots a_{g(1,k-1)}za_r$, by Lemma 3.6(i) and (ii),

$$s \in \bigcup_{1\leqslant i<j<k\leqslant n} (Ma_ia_ja_kz \cup Ma_ja_ia_kz),$$

a contradiction. Therefore the length of $w_{3,t-1}$ is greater than 2.

Thus $w_{3,t-1} = a_{g(t-1,n+k-l)} \cdots a_{g(t-1,n+k-1)}a_r$ for some $1 < l < n$. In this case,

$$w'_{2,t}a_{g(t-1,n+k-l)} \cdots a_{g(t-1,n+k-1)}a_r = a_{g(t,k-l)} \cdots a_{g(t,k-1)}a_{g(t,k)} \cdots a_{g(t,n+k)}.$$

Hence $s \in Mza_{g(t-1,n+k-l)} \cdots a_{g(t-1,n+k-1)}a_r$. Since $a_{g(t,k+1)} \cdots a_{g(t,n+k)}$ represents z in M and $l < n$, we have that $g(t-1, n+k-l), \ldots, g(t-1, n+k-1), r$ are $l+1$ different integers. By Lemma 3.1,

$$s \in \bigcup_{1 \leqslant i < j < k \leqslant n} (Ma_ia_ja_kz \cup Ma_ja_ia_kz),$$

a contradiction. Therefore

$$\bigcup_{1 \leqslant r \leqslant n} (Mz \cap Mza_r) \subseteq \bigcup_{1 \leqslant i < j < k \leqslant n} (Ma_ia_ja_kz \cup Ma_ja_ia_kz).$$

By (8), the result follows. □

Lemma 3.8. *Suppose that $n \geqslant 6$ is even. Let $s = a_{j_1}a_{j_2}\cdots a_{j_m} \in M \setminus MzM$ be such that*

$$sz \notin \bigcup_{1 \leqslant i < j < k \leqslant n} (Ma_ia_ja_kz \cup Ma_ja_ia_kz).$$

Then, for $s_1, s_2 \in M$, $sz = s_1zs_2$ implies that $s_1s_2 = s$.

Proof. Let $s_1, s_2 \in M$ be such that $sz = s_1zs_2$. Then, by an easy degree argument, $s_1 = a_{i_1}\cdots a_{i_k}$ and $s_2 = a_{i_{k+1}}\cdots a_{i_m}$ for some k and some $a_{i_1}, \dots, a_{i_m}$. Thus there exist $w_1, w_2, \dots, w_t$ in the free monoid FM_n on $\{a_1, \dots, a_n\}$, such that $w_i = w_{1,i}w_{2,i}w_{3,i} = w'_{1,i}w'_{2,i}w'_{3,i}$, where $w_{2,i}$ and $w'_{2,i}$ represent the element z in M for all $i = 1, \dots, t$, $w_{1,j} = w'_{1,j+1}$ and $w_{3,j} = w'_{3,j+1}$, for all $j = 1, \dots, t-1$, and $w'_{1,1} = a_{j_1}a_{j_2}\cdots a_{j_m}$, $w'_{3,1} = 1$, $w_{1,t} = a_{i_1}\cdots a_{i_k}$ and $w_{3,t} = a_{i_{k+1}}\cdots a_{i_m}$. Thus, $w_1 = a_{j_1}\cdots a_{j_m}w'_{2,1}$ and $w_t = a_{i_1}\cdots a_{i_k}w_{2,t}a_{i_{k+1}}\cdots a_{i_m}$. It is enough to prove that $w_{1,i}w_{3,i} = a_{j_1}\cdots a_{j_m}$, for all $i = 1, \dots, t$, by induction on i.

If the two subwords $w_{2,1}$ and $w'_{2,1}$ of the word $w_1 = a_{j_1}a_{j_2}\cdots a_{j_m}w'_{2,1} = w_{1,1}w_{2,1}w_{3,1}$ do not overlap, then $w_{2,1}$ is a subword of $a_{j_1}\cdots a_{j_m}$, which is not possible because the latter represents s in M and $s \notin MzM$. Therefore they overlap and hence the degree of $w_{3,1}$ is less than n and $w_{3,1}$ is a product of distinct letters. Since w_1 represents sz in M and $sz \notin \bigcup_{1 \leqslant i<j<k \leqslant n}(Ma_ia_ja_kz \cup Ma_ja_ia_kz)$, it follows that $w_{3,1}$ cannot have degree 1 by Lemma 3.7 and it cannot have degree greater than 2 by Lemma 3.1. Hence, the degree of $w_{3,1}$ is 0 or 2. In the former case, clearly $w_{1,1}w_{3,1} = w_{1,1} = w'_{1,1} = a_{j_1}a_{j_2}\cdots a_{j_m}$. Suppose that $w_{3,1}$ has degree 2. From the equality of words $a_{j_1}a_{j_2}\cdots a_{j_m}w'_{2,1} = w_{1,1}w_{2,1}w_{3,1}$ it follows that $a_{j_{m-1}}a_{j_m}w'_{2,1} = w_{2,1}w_{3,1}$. Let $w'_{2,1} = a_{j_{m+1}}\cdots a_{j_{m+n}}$. Then there exist $\sigma, \tau \in \mathrm{Alt}_n$ such that

$$\sigma(1) = j_{m-1},\ \sigma(2) = j_m,\ \dots,\ \sigma(n) = j_{m+n-2}$$

and

$$\tau(1) = j_{m+1},\ \tau(2) = j_{m+2},\ \dots,\ \tau(n) = j_{m+n}.$$

Hence

$$(1, \dots, n)^{-2}\sigma^{-1}\tau(1) = 1, \dots, (1, \dots, n)^{-2}\sigma^{-1}\tau(n-2) = n-2.$$

Since n is even and $(1, \dots, n)^{-2}\sigma^{-1}\tau \in \mathrm{Alt}_n$, we have that $(1, \dots, n)^{-2}\sigma^{-1}\tau = \mathrm{id}$. Therefore $\tau = \sigma \circ (1, \dots, n)^2$ and then

$$j_{m+n-1} = \tau(n-1) = \sigma(1) = j_{m-1} \quad \text{and} \quad j_{m+n} = \tau(n) = \sigma(2) = j_m.$$

Thus $w_{3,1} = a_{j_{m+n-1}}a_{j_{m+n}} = a_{j_{m-1}}a_{j_m}$ and clearly $w_{1,1} = a_{j_1}a_{j_2}\cdots a_{j_{m-2}}$. Hence, also in this case, $w_{1,1}w_{3,1} = a_{j_1}\cdots a_{j_m}$.

Suppose that $t \geqslant i > 1$ and $w_{1,j}w_{3,j} = a_{j_1}\cdots a_{j_m}$, for all $j = 1,\dots,i-1$. We have that $w'_{1,i} = w_{1,i-1} = a_{j_1}\cdots a_{j_q}$ and $w'_{3,i} = w_{3,i-1} = a_{j_{q+1}}\cdots a_{j_m}$, for some $0 \leqslant q \leqslant m$. Hence $w_i = a_{j_1}\cdots a_{j_q}w'_{2,i}a_{j_{q+1}}\cdots a_{j_m} = w_{1,i}w_{2,i}w_{3,i}$. Since $a_{j_1}\cdots a_{j_m} \notin MzM$ by the hypothesis, as above we get that the subwords $w_{2,i}$ and $w'_{2,i}$ of the word w_i have to overlap.

Let r be the absolute value of the difference of the lengths of the words $w_{1,i}$ and $w'_{1,i}$. Then $r < n$. The equality of words $w'_{1,i}w'_{2,i}w'_{3,i} = w_{1,i}w_{2,i}w_{3,i}$ implies that either $w'_{2,i}u' = uw_{2,i}$ or $u'w'_{2,i} = w_{2,i}u$ for some words u, u' of length r. Then all the generators involved in u (and also in u') are different. Since w_i represents sz in M and $sz \notin \bigcup_{1\leqslant i<j<k\leqslant n}(Ma_ia_ja_kz \cup Ma_ja_ia_kz)$, by Lemma 3.1 we get that $r \leqslant 2$. If $r = 1$ then we get $za_p = a_pz$ in M for some p, which is impossible since n is even. Hence, $r = 0$ or $r = 2$. As above we can see in the both cases that $w_{1,t}w_{3,t} = a_{j_1}\cdots a_{j_m}$. The result follows. □

Let $I = \{s \in M \mid sM \subseteq Mz\}$ and $I' = \{s \in M \mid Ms \subseteq zM\}$. Clearly, I and I' are ideals of M. Let $I_1 = \{s \in Mz \mid sa_i \in Mz,\ \text{for all } i = 1,2,\dots,n\}$, $I'_1 = \{s \in zM \mid a_is \in zM,\ \text{for all } i = 1,2,\dots,n\}$ and

$$T = \bigcup_{1\leqslant i<j<k\leqslant n}(Ma_ia_ja_kz \cup Ma_ja_ia_kz).$$

Lemma 3.9. *Suppose that $n \geqslant 6$ is even. Then $I = I' = I_1 = I'_1 = T$.*

Proof. From Lemma 3.6(i) and (ii), it follows that T is an ideal of M. Hence $T \subseteq I$. Suppose that these two ideals are different. Let $s \in I \setminus T$. Since $s \in I$, there exists $s' \in M$ such that $s = s'z$. We consider two cases.

Case 1. $s' \in MzM$.

Then let $s'' \in M$ be an element of minimal degree such that $s' \in Mzs''$. Thus there exists $t \in M$ such that $s' = tzs''$. Since $s = tzs''z \notin T$, we have that s'' has degree greater than or equal to 2. Let $s'' = a_{j_1}\cdots a_{j_m}$. By the choice of s'', we know that $s'' \notin MzM$. Since $s = s'z = tzs''z \notin T$, clearly $s''z \notin T$. Hence, by Lemma 3.8, the words in the free monoid FM_n on $\{a_1,\dots,a_n\}$ that represent $s''z$ in M are of the form

$$a_{j_1}\cdots a_{j_q}wa_{j_{q+1}}\cdots a_{j_m}, \tag{9}$$

where w represents z in M. Note that $z^2, za_{j_1}z \in T$. Therefore, since $s = tzs''z \notin T$ and T is an ideal of M, we have that $q \geqslant 2$ in (9). By Lemma 3.1 and the choice of s'', $j_1 = j_2$. Note also that by the choice of s'', the words in FM_n that represent $s' = tzs''$ in M are of the form

$$w'a_{j_1}\cdots a_{j_m}, \tag{10}$$

where w' represents tz in M. It follows from the form of the words (9) and (10) that the words in FM_n that represent $s = tzs''z$ in M are of the form

$$w'a_{j_1}\cdots a_{j_q}wa_{j_{q+1}}\cdots a_{j_m}, \tag{11}$$

where w' represents tz in M, w represents z in M and $q \geqslant 2$. Since n is even, we know by Proposition 2.3, that $za_{j_m} \notin Mz$. Therefore, by the form of the words (11) that represent s in M, the words that represent sa_{j_m} in M are of the form

$$w'a_{j_1}\cdots a_{j_q}wa_{j_{q+1}}\cdots a_{j_m}a_{j_m},$$

where w' represents tz in M, w represents z in M and $q \geqslant 2$. In particular, it follows that $sa_{j_m} \notin Mz$, a contradiction since $s \in I$.

Case 2. $s' \notin MzM$.

Let $s' = a_{j_1} \cdots a_{j_m}$ for some $m \geqslant 0$. Since $s = s'z \notin T$, by Lemma 3.8 we get that the words in FM_n that represent s in M are of the form

$$a_{j_1} \cdots a_{j_q} w a_{j_{q+1}} \cdots a_{j_m},$$

where w represents z in M. If $m = 0$ then $s = z$ and $sa_1 = za_1 \notin Mz$, a contradiction, because $s \in I$. Hence $m > 0$. Since $za_{j_m} \notin Mz$, by the form of the words that represent s, it is easy to see that the words that represent sa_{j_m} are of the form

$$a_{j_1} \cdots a_{j_q} w a_{j_{q+1}} \cdots a_{j_m} a_{j_m},$$

where w represents z in M. Therefore $sa_{j_m} \notin Mz$, a contradiction since $s \in I$. Therefore $I = T$.

Clearly, we have $I \subseteq I_1$. Let $s \in I_1$ and let $t \in M \setminus \{1\}$. Then $t = a_r t'$ for some $1 \leqslant r \leqslant n$ and $t' \in M$. Since $sa_r \in Mz \cap Mza_r$, by Lemma 3.7 it follows that $st = sa_r t' \in Tt' \subseteq T \subseteq Mz$. Therefore $s \in I$ and so $I = I_1$.

By Lemma 3.1 and Lemma 3.6(i) and (ii),

$$\bigcup_{1 \leqslant i<j<k \leqslant n} (za_i a_j a_k M \cup za_j a_i a_k M) = \bigcup_{1 \leqslant i<j<k \leqslant n} (Ma_i a_j a_k z \cup Ma_j a_i a_k z) = T.$$

Thus, by symmetry,

$$I = I' = I_1 = I'_1 = T. \qquad \square$$

4. Proof of Theorem 1.1

In this section we prove our main result, Theorem 1.1. So again, $n \geqslant 4$, $M = S_n(\mathrm{Alt}_n)$ and $G = G_n(\mathrm{Alt}_n)$.

Recall that ρ' is the binary relation on M, defined by $s\rho' t$ if and only if there exists a nonnegative integer i such that $sz^i = tz^i$. By Lemma 3.2, z^2 is central in M. By Lemma 2.4, $\rho' = \rho$ is the least cancellative congruence on M.

Proof of (i). Let $\{i, j, k\}$ be a subset of $\{1, 2, \ldots, n\}$ of cardinality three. By Lemma 3.1, in G, we have

$$a_i a_j a_k = a_j a_k a_i = a_k a_i a_j. \tag{12}$$

By Lemma 3.4, in G, we also have $a_i a_j a_i a_j = a_j a_i a_j a_i$ and $a_i^2 a_j = a_j a_i^2$, for all $1 \leqslant i, j \leqslant n$. Therefore $(a_i a_j a_i^{-1} a_j^{-1})^2 = 1$.

Let $\tau \in \mathrm{Sym}_n \setminus \mathrm{Alt}_n$. Suppose that $n = \tau(j)$. If $j = n-1$ then, by (12),

$$a_{\tau(1)} a_{\tau(2)} \cdots a_{\tau(n)} = a_{\tau(1)} a_{\tau(2)} \cdots a_{\tau(n-3)} a_{\tau(n)} a_{\tau(n-2)} a_n.$$

If $n - j$ is even and greater than 1 then, by (12),

$$a_{\tau(1)} a_{\tau(2)} \cdots a_{\tau(n)} = a_{\tau(1)} \cdots a_{\tau(j-1)} a_{\tau(j+1)} \cdots a_{\tau(n)} a_n.$$

If $n-j$ is odd and greater than 1 then, by (12),

$$a_{\tau(1)}a_{\tau(2)}\cdots a_{\tau(n)} = a_{\tau(1)}\cdots a_{\tau(j-1)}a_{\tau(j+1)}\cdots a_{\tau(n-2)}a_{\tau(n)}a_{\tau(n-1)}a_n.$$

So, we have shown that $a_{\tau(1)}a_{\tau(2)}\cdots a_{\tau(n)} = a_{\sigma(1)}\cdots a_{\sigma(n-1)}a_n$ for some $\sigma \in \mathrm{Sym}_{n-1}$. Repeating the above argument at most $n-3$ times, we get that $a_{\tau(1)}a_{\tau(2)}\cdots a_{\tau(n)} = a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}a_4a_5\cdots a_n$ for some $\sigma \in \mathrm{Sym}_3$. Because τ is odd, it follows from (2), that also σ is odd. Hence, σ is a transposition, and thus, again using (12), $a_{\tau(1)}a_{\tau(2)}\cdots a_{\tau(n)} = a_1a_3a_2a_4a_5\cdots a_n$.

Hence we have shown that, in G,

$$a_{\tau(1)}a_{\tau(2)}\cdots a_{\tau(n)} = a_1a_3a_2a_4a_5\cdots a_n,$$

for all $\tau \in \mathrm{Sym}_n \setminus \mathrm{Alt}_n$.

Hence we have the following presentations of the group G.

$$\begin{aligned} G &= \mathrm{gr}(a_1,\dots,a_n \mid a_1a_2\cdots a_n = a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)},\ \sigma \in \mathrm{Alt}_n) \\ &= \mathrm{gr}(a_1,\dots,a_n \mid a_1a_2\cdots a_n = a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}, a_1a_3a_2a_4\cdots a_n = a_{\tau(1)}a_{\tau(2)}\cdots a_{\tau(n)}, \\ &\qquad \sigma \in \mathrm{Alt}_n,\ \tau \in \mathrm{Sym}_n \setminus \mathrm{Alt}_n). \end{aligned}$$

Note that, by (12), $a_i(a_1a_2a_1^{-1}a_2^{-1})a_i^{-1} = a_1a_2a_1^{-1}a_2^{-1}$, for all $2 < i \leqslant n$. Furthermore

$$\begin{aligned} a_1\big(a_1a_2a_1^{-1}a_2^{-1}\big)a_1^{-1} &= a_1(a_1a_2a_3)\big(a_3^{-1}a_1^{-1}a_2^{-1}\big)a_1^{-1} \\ &= a_1(a_2a_3a_1)\big(a_1^{-1}a_2^{-1}a_3^{-1}\big)a_1^{-1} \quad \text{by (12)} \\ &= a_1a_2a_3a_2^{-1}a_3^{-1}a_1^{-1} \\ &= a_1a_2a_3\big(a_3^{-1}a_1^{-1}a_2^{-1}\big) \quad \text{by (12)} \\ &= a_1a_2a_1^{-1}a_2^{-1} \end{aligned}$$

and

$$\begin{aligned} a_2\big(a_1a_2a_1^{-1}a_2^{-1}\big)a_2^{-1} &= a_2(a_1a_2a_3)\big(a_3^{-1}a_1^{-1}a_2^{-1}\big)a_2^{-1} \\ &= a_2(a_3a_1a_2)\big(a_2^{-1}a_3^{-1}a_1^{-1}\big)a_2^{-1} \quad \text{by (12)} \\ &= a_2a_3a_1a_3^{-1}a_1^{-1}a_2^{-1} \\ &= (a_1a_2a_3)a_3^{-1}a_1^{-1}a_2^{-1} \quad \text{by (12)} \\ &= a_1a_2a_1^{-1}a_2^{-1}. \end{aligned}$$

Therefore $a_1a_2a_1^{-1}a_2^{-1}$ is a central element of order at most 2 in G. Let C be the central subgroup $C = \{1, a_1a_2a_1^{-1}a_2^{-1}\}$. Then G/C has the following presentations.

$$\begin{aligned} G/C &= \mathrm{gr}(b_1,\dots,b_n \mid b_1b_2 = b_2b_1,\ b_1b_2\cdots b_n = b_{\sigma(1)}b_{\sigma(2)}\cdots b_{\sigma(n)}, \\ &\qquad b_1b_3b_2b_4\cdots b_n = b_{\tau(1)}b_{\tau(2)}\cdots b_{\tau(n)},\ \sigma \in \mathrm{Alt}_n,\ \tau \in \mathrm{Sym}_n \setminus \mathrm{Alt}_n) \\ &= \mathrm{gr}(b_1,\dots,b_n \mid b_1b_2\cdots b_n = b_{\sigma(1)}b_{\sigma(2)}\cdots b_{\sigma(n)},\ \sigma \in \mathrm{Sym}_n). \end{aligned}$$

Hence G/C is a free abelian group of rank n and, since $C = G'$, in G we have

$$a_1a_2a_1^{-1}a_2^{-1} = a_ia_ja_i^{-1}a_j^{-1},$$

for all $i \neq j$, because there exists $\sigma \in \mathrm{Alt}_n$ such that $\sigma(1) = i$ and $\sigma(2) = j$.

We now show that $a_1a_2a_1^{-1}a_2^{-1} \neq 1$. Let f be the map defined by (2). Note that if two words $w, w' \in \mathrm{FM}_n$ represent the same element in M, then $f(w) = f(w')$. In particular,

$$a_1a_2z^m \neq a_2a_1z^m$$

in M, for all m. Now, by Lemmas 3.2 and 2.4, we have that $a_1a_2 \neq a_2a_1$ in G, as desired.

By Lemma 3.4, every a_i^2 is a central element in G. Let D be the central subgroup of G generated by $a_1^2, \dots, a_n^2$. Now $G/(CD) \cong (\mathbb{Z}/2\mathbb{Z})^n$. Hence (i) follows. □

Proof of (ii). By (i) and [16, Lemma 5.1.11, Corollary 10.2.8], $K[G]$ is a noetherian PI-algebra for any field K. Furthermore, by [16, Theorem 7.3.1] $\mathcal{J}(K[G]) \subseteq \mathcal{J}(K[C])K[G]$. Thus, if K is a field of characteristic $\neq 2$, then $\mathcal{J}(K[G]) = 0$. If K is a field of characteristic 2, then $\mathcal{J}(K[G]) = (1 - a_1a_2a_1^{-1}a_2^{-1})K[G]$, and $\mathcal{J}(K[G])^2 = 0$. □

Proof of (iii). By Lemma 3.2, z^2 is central in M. Thus, it follows from Lemma 2.4 that every nonempty right ideal of M contains z^{2k} for some positive integer k. Therefore, if $sx = tx$ for some $s, t \in z^2M$ and $x \in M$, then $sz^{2k} = tz^{2k}$, for some k. Since z^2 is central and $s, t \in z^2M$, by Lemma 3.5, we get $s = t$. This and a symmetric argument show that z^2M is cancellative and also that the ideal z^2M embeds into M/ρ. Hence, again by Lemma 2.4, $G = (z^2M)\langle z^2\rangle^{-1}$.

Since $K[G]$ is a PI-algebra and G is the group of fractions of M/ρ by Lemma 2.4, $K[M/\rho]$ is a finitely generated PI-algebra. Let $\overline{M} = M/\rho$. It follows from part (i) that G is a nilpotent group and it is abelian-by-finite, thus from [13, Theorem 4.3.3, and the comment following it] we know that $K[\overline{M}]$ is noetherian. By [15, Theorem 18.1], $\mathcal{J}(K[\overline{M}])$ is nilpotent. Therefore, there exists a positive integer m such that $\mathcal{J}(K[M])^m \subseteq I(\rho)$. By Proposition 2.6, $\mathcal{J}(K[M])^3 \subseteq K[z^2M]$. Since z^2M is cancellative, $I(\rho) \cap K[z^2M] = 0$. Hence $\mathcal{J}(K[M])$ is nilpotent. □

Proof of (iv). Suppose that $n \geqslant 4$ is odd. We shall see that $a_1a_1a_2z \neq a_2a_1a_1z$ in M.

Let $w_0 = a_1a_1a_2a_1 \cdots a_n \in \mathrm{FM}_n$ and let $w \in \mathrm{FM}_n$ be a word representing the element $a_1a_1a_2z \in M$. Then there exist $w_1, \dots, w_r \in \mathrm{FM}_n$ with $w_r = w$ and $w_i = w_{1,i}w_{2,i}w_{3,i} = w'_{1,i}w'_{2,i}w'_{3,i}$ such that $w_{2,i}$ and $w'_{2,i}$ represent the element z in M, for all $i = 0, 1, \dots, r$, and $w_{1,j} = w'_{1,j+1}$ and $w_{3,i} = w'_{3,i+1}$, for all $j = 0, \dots, r-1$. We shall prove, by induction on r, that $w_{1,i}w_{3,i} = a_1a_1a_2$ for all $i = 0, 1, \dots, r$. It is clear that $w_{1,0} = a_1a_1a_2$ and $w_{3,0} = 1$, thus $w_{1,0}w_{3,0} = a_1a_1a_2$. Suppose that $i \geqslant 0$ and $w_{1,i}w_{3,i} = a_1a_1a_2$. Then $w_{1,i} \in \{1, a_1, a_1a_1, a_1a_1a_2\}$. We shall deal with four cases separately.

Case 1. $w_{1,i} = 1$. In this case, $w_{3,i} = w'_{3,i+1} = a_1a_1a_2$. Since $w_{i+1} = w_{1,i+1}w_{2,i+1}w_{3,i+1} = w'_{2,i+1}a_1a_1a_2$ and $w_{2,i+1}$ and $w'_{2,i+1}$ represent $z \in M$, we have that $w_{3,i+1} \in \{a_1a_1a_2, a_1a_2\}$. If $w_{3,i+1} = a_1a_1a_2$, then clearly $w_{1,i+1} = 1$ and $w_{1,i+1}w_{3,i+1} = a_1a_1a_2$. Suppose that $w_{3,i+1} = a_1a_2$. Since the degree in a_1 of w_{i+1} is 3 and the degree in a_1 of $w_{2,i+1}$ is 1, we have that $w_{1,i+1} = a_1$. Hence $w_{1,i+1}w_{3,i+1} = a_1a_1a_2$ in this case.

Case 2. $w_{1,i} = a_1$. In this case, $w_{3,i} = w'_{3,i+1} = a_1a_2$. Since $w_{i+1} = w_{1,i+1}w_{2,i+1}w_{3,i+1} = a_1w'_{2,i+1}a_1a_2$, we have that either $w_{1,i+1} = 1$ or $w_{1,i+1}$ begins with a_1. If $w_{1,i+1} = 1$ then, using the degree in a_1 and that $w_{3,i+1}$ finishes with a_1a_2, we see that $w_{3,i+1} = a_1a_1a_2$ and $w_{1,i+1}w_{3,i+1} = a_1a_1a_2$. Suppose that $w_{1,i+1}$ begins with a_1. Then $w_{1,i+1} = a_1u$ for some $u \in \mathrm{FM}_n$. Thus $uw_{2,i+1}w_{3,i+1} = w'_{2,i+1}a_1a_2$. Now $w_{3,i+1} \in \{1, a_2, a_1a_2\}$. If $w_{3,i+1} \in \{a_2, a_1a_2\}$, then using the degree in a_1, we have that $uw_{3,i+1} = a_1a_2$. Suppose that $w_{3,i+1} = 1$. Then $uw_{2,i+1} = w'_{2,i+1}a_1a_2$ and, using the degree in a_1 and in a_2, we have that $u \in \{a_1a_2, a_2a_1\}$. Since $w_{2,i+1}$ and $w'_{2,i+1}$ represent $z \in M$, $f(a_2a_1w_{2,i+1}) = 1$ and $f(w'_{2,i+1}a_1a_2) = -1$, where f is the map defined by (2), thus $u = a_1a_2$. Hence $w_{1,i+1}w_{3,i+1} = a_1a_1a_2$ in this case.

Case 3. $w_{1,i} = a_1a_1$. In this case, $w_{3,i} = w'_{3,i+1} = a_2$. Since $w_{i+1} = w_{1,i+1}w_{2,i+1}w_{3,i+1} = a_1a_1w'_{2,i+1}a_2$ and $w_{2,i+1}$ represents $z \in M$, we have that $w_{1,i+1}$ begins with a_1. Then $w_{1,i+1} = a_1u$ for some $u \in \mathrm{FM}_n$, and $uw_{2,i+1}w_{3,i+1} = a_1w'_{2,i+1}a_2$. Thus, using the degree in a_1 and in a_2, we have $uw_{3,i+1} = a_1a_2$. Hence $w_{1,i+1}w_{3,i+1} = a_1a_1a_2$ in this case.

Case 4. $w_{1,i} = a_1a_1a_2$. In this case, $w_{3,i} = w'_{3,i+1} = 1$. Since $w_{i+1} = w_{1,i+1}w_{2,i+1}w_{3,i+1} = a_1a_1a_2w'_{2,i+1}$ and $w_{2,i+1}$ represents $z \in M$, we have that $w_{1,i+1}$ begins with a_1. Then, as in Case 2, $w_{1,i+1} = a_1u$ for some $u \in \mathrm{FM}_n$, and $uw_{3,i+1} = a_1a_2$. Hence $w_{1,i+1}w_{3,i+1} = a_1a_1a_2$ in this case.

Therefore, we indeed have shown in each of the four cases that $w_{1,i}w_{3,i} = a_1a_1a_2$, for all $i = 0, 1, \ldots, r$. In particular, $a_1a_1a_2z \neq a_2a_1a_1z$ in M.

Note that if $1 \leqslant i, j \leqslant n$ are different then there exists $\sigma \in \mathrm{Alt}_n$ such that $\sigma(1) = i$ and $\sigma(2) = j$. Therefore

$$a_ia_ia_jz \neq a_ja_ia_iz,$$

for all $i \neq j$, in M.

Since n is odd, z is central in M and, by Lemma 3.4, $(a_ia_ia_j - a_ja_ia_i)z^2 = 0$. Therefore $(a_ia_ia_j - a_ja_ia_i)z \in \mathcal{B}(K[M]) \setminus \{0\}$, for all $i \neq j$ and for any field K.

Let $\overline{\rho} = \rho \cap (zM \times zM)$. So $I(\overline{\rho}) = \mathrm{lin}_K\{s - t \mid s, t \in zM \text{ and } \exists i \geqslant 0,\ sz^i = tz^i\}$. Since z^2M is cancellative, it follows that $I(\overline{\rho})^2 = 0$.

Suppose that K has characteristic different from 2. We have that $\mathcal{J}(K[G]) = 0$. Since $\mathcal{J}(K[\overline{M}])$ is nilpotent and G is a central localization of $\overline{M}$, we get $\mathcal{J}(K[\overline{M}]) = \mathcal{B}(K[\overline{M}]) \subseteq \mathcal{J}(K[G])$. Hence $\mathcal{J}(K[\overline{M}]) = 0$. Then $\mathcal{J}(K[M]) \subseteq I(\rho)$, and by Corollary 2.7

$$\mathcal{J}\big(K[M]\big) = \mathcal{B}\big(K[M]\big) = I(\overline{\rho}).$$

Thus $\mathcal{J}(K[M])^2 = 0$.

Assume that $s, t \in M$ are such that $(s, t) \in \rho$. Because z^2M is cancellative, we know that $z^2s = z^2t$. Note that in the proof of Lemma 3.5, in order to obtain the form (3) or (4) of z^2s, we only use the centrality of z^2 and the relations $z^2a_ia_ja_k = z^2a_ja_ka_i$, $z^2a_ia_j^2 = z^2a_j^2a_i$ and $z^2a_ia_ja_ia_j = z^2a_ja_ia_ja_i$, for $1 \leqslant i, j, k \leqslant n$, three distinct elements. Since $z^2s = z^2t$, it follows that $s - t \in K[M]YK[M]$, where

$$Y = \bigcup_{\substack{1 \leqslant i,j,k \leqslant n \\ |\{i,j,k\}| = 3}} \big\{a_ia_ja_k - a_ja_ka_i,\ a_i^2a_j - a_ja_i^2,\ (a_ia_j)^2 - (a_ja_i)^2\big\}.$$

This implies that Y generates $I(\rho)$ as a two-sided ideal. Now, if $s'z, t'z \in zM$ are ρ-related, then also $(s', t') \in \rho$, so by the previous $s'z - t'z \in K[M]YzK[M]$ because z is central. In particular, $I(\overline{\rho}) = \mathcal{J}(K[M])$ is a finitely generated ideal.

Suppose that K has characteristic 2. By Proposition 2.6, $\mathcal{J}(K[M]) \subseteq K[zM]$. Thus $\mathcal{J}(K[M]) = \mathcal{J}(K[zM])$. Note that $zM/\overline{\rho}$ is a cancellative semigroup and G is its group of fractions. Furthermore, $K[zM/\overline{\rho}] = K[zM]/I(\overline{\rho})$. By (iii), we have that $\mathcal{J}(K[M])$ is nilpotent. Hence

$$\mathcal{J}\big(K[M]\big)/I(\overline{\rho}) = \mathcal{B}\big(K[zM]\big)/I(\overline{\rho}) = \mathcal{B}\big(K[zM/\overline{\rho}]\big) = \mathcal{B}\big(K[G]\big) \cap K[zM/\overline{\rho}],$$

see [15, Corollary 11.5]. Let $\pi : zM \to G/C$ be the composition of the natural maps

$$zM \hookrightarrow M \to G \to G/C.$$

Let η be the congruence defined on M by $s\eta t$ if and only if either $s = t$ or $s, t \in zM$ and $\pi(s) = \pi(t)$. Since $\mathcal{B}(K[G]) = \omega(K[C])K[G]$, it follows that $\mathcal{J}(K[M]) = I(\eta)$. In particular, $z(a_i a_j - a_j a_i) \in \mathcal{J}(K[M])$ for all i, j. Let Q be the ideal of $K[M]$ generated by all such elements. Then the set of all elements of the form $za_1^{i_1} \cdots a_n^{i_n}$, for nonnegative integers $i_1, \ldots, i_n$, forms a basis of the algebra $K[zM]/Q$. Therefore this algebra embeds into the algebra $K[G/C]$, which is a commutative domain. It follows that $\mathcal{J}(K[M]) = Q$ and hence it is finitely generated. □

Proof of (v). Suppose that $n \geqslant 6$ is and n is even. We shall prove that $\mathcal{J}(K[M]) \subseteq K[T]$ (where T is as in Lemma 3.9). Suppose that $\mathcal{J}(K[M]) \nsubseteq K[T]$. Let $\alpha \in \mathcal{J}(K[M]) \setminus K[T]$ with $|\operatorname{supp}(\alpha)| = m$. Let $\operatorname{supp}(\alpha) = \{s_1, \ldots, s_m\}$. By Proposition 2.6, $s_i \in zM \cup Mz$. In particular, the degree of s_i is greater than or equal to n. We may assume that $s_1 \notin T$. Then, by Lemma 3.9, there exist i, j such that $s_1 a_j \notin Mz$ and $a_i s_1 \notin zM$. Hence, since the degree of s_1 is greater than or equal to n, we have that $a_i s_1 a_j \notin zM \cup Mz$ and $a_i s_1 a_j \in \operatorname{supp}(a_i \alpha a_j)$. But this is in contradiction with Proposition 2.6. Hence $\mathcal{J}(K[M]) \subseteq K[T]$.

Now we shall prove that $K[T] \cap I(\rho) = 0$, i.e., T is cancellative. Let $s, t \in T$ be such that $s\rho t$. It is sufficient to see that $s = t$. In order to prove this, we first shall verify that there exist three different integers $1 \leqslant i, j, k \leqslant n$ such that $s, t \in Ma_i a_j a_k z$.

Since $s, t \in T$, there exist integers i, j, k, l, p, q and $s', t' \in M$ such that $s = s' a_i a_j a_k z$, $t = t' a_l a_p a_q z$, $|\{i, j, k\}| = 3$ and $|\{l, p, q\}| = 3$. We claim that $t \in Ma_i a_j a_k z$. First we deal with the case that $l \notin \{i, j, k\}$ and $i \notin \{l, p, q\}$. Since $s\rho t$, we have that s and t have the same degrees with respect to every generator. Therefore $t' \in Ma_i M$. Let $t_1, t_2 \in M$ be elements such that $t' = t_1 a_i t_2$. Thus $t = t_1 a_i t_2 a_l a_p a_q z$. By Lemma 3.6(i) and (ii) and Lemma 3.1,

$$\begin{aligned} t &= t_1 a_i t_2 a_l a_p a_q z = t_1 a_i a_l a_{p'} a_{q'} z t_2 = t_1 z a_i a_l a_{p'} a_{q'} t_2 \\ &= t_1 z a_l a_{p'} a_i a_{q'} t_2 = t_1 a_l a_{p'} a_i a_{q'} z t_2 = t_1 a_l t_2 a_i a_{p''} a_{q''} z, \end{aligned}$$

where $\{p, q\} = \{p', q'\} = \{p'', q''\}$. Therefore $t \in Ma_i a_{p''} a_{q''} z$. Now, if $p'' \notin \{j, k\}$ and $j \notin \{p'', q''\}$, then, since i, p'', q'' are three different integers, we can apply the same argument to get that $t \in Ma_j a_{u'} a_{v'} z$, with $\{u', v'\} = \{i, q''\}$. Thus, applying this argument at most one more time, we get that $t \in Ma_{i'} a_{j'} a_{k'} z$, with $\{i, j, k\} = \{i', j', k'\}$. By Lemma 3.1, we may assume that $i' = i$. If $(j', k') = (j, k)$ then we have proved the claim. So we may also assume that $(j', k') = (k, j)$. Thus there exists $t'' \in M$ such that $t = t'' a_i a_k a_j z$.

If $t'' \in \bigcup_{1 \leqslant r \leqslant n} \langle a_r \rangle$ then, since s and t have the same degrees with respect to every generator, there exists $1 \leqslant r \leqslant n$, and a nonnegative integer v such that $s' = t'' = a_r^v$. Hence, since $s\rho t$, we have that $s = t$ in G. Therefore $a_j a_k = a_k a_j$ in G, a contradiction. So, $t'' \notin \bigcup_{1 \leqslant r \leqslant n} \langle a_r \rangle$. Hence there exist different $1 \leqslant r, p \leqslant n$ and $t'_1, t'_2 \in M$ such that $t'' = t'_1 a_r a_p t'_2$. We denote by $\deg_r(x)$ the degree in a_r of $x \in M$. Let $u = \deg_i(t'_2) + \deg_j(t'_2) + \deg_k(t'_2)$. If u is odd and $i \notin \{r, p\}$ then, by Lemma 3.6(i) and (ii) and Lemma 3.1,

$$\begin{aligned} t &= t'_1 a_r a_p t'_2 a_i a_k a_j z = t'_1 a_r a_p a_i a_j a_k z t'_2 = t'_1 a_r a_p a_i z a_j a_k t'_2 \\ &= t'_1 z a_p a_r a_i a_j a_k t'_2 = t'_1 a_p a_r z a_i a_j a_k t'_2 = t'_1 a_p a_r a_i a_k a_j z t'_2 \\ &= t'_1 a_p a_r t'_2 a_i a_j a_k z \in M a_i a_j a_k z. \end{aligned}$$

If u is odd and $j \notin \{r, p\}$ then, by Lemma 3.6(i) and (ii) and Lemma 3.1,

$$\begin{aligned} t &= t_1' a_r a_p t_2' a_i a_k a_j z = t_1' a_r a_p a_j a_k a_i z t_2' = t_1' a_r a_p a_j z a_k a_i t_2' \\ &= t_1' z a_p a_r a_j a_k a_i t_2' = t_1' a_p a_r z a_j a_k a_i t_2' = t_1' a_p a_r a_i a_k a_j z t_2' \\ &= t_1' a_p a_r t_2' a_i a_j a_k z \in M a_i a_j a_k z. \end{aligned}$$

If u is odd and $k \notin \{r, p\}$ then, similarly, we get that $t \in M a_i a_j a_k z$. If u is even then, by Lemma 3.6(i) and (ii) and Lemma 3.1, we also get that $t \in M a_i a_j a_k z$.

Therefore $s, t \in M a_i a_k a_j z$, as claimed.

Hence, we have that $s = s' a_i a_j a_k z$ and $t = \widetilde{t} a_i a_j a_k z$, for some $s', \widetilde{t} \in M$. Since $s \rho t$, there exists a nonnegative integer l such that $z^{2l} s = z^{2l} t$. Let $v = \deg_i(s') + \deg_j(s') + \deg_k(s')$. Since s and t have the same degrees with respect to every generator, it follows that $v = \deg_i(\widetilde{t}) + \deg_j(\widetilde{t}) + \deg_k(\widetilde{t})$. If v is odd then, by Lemma 3.6(i) and (ii),

$$\begin{aligned} s &= s' a_i a_j a_k z = a_i a_k a_j z s', \\ t &= \widetilde{t} a_i a_j a_k z = a_i a_k a_j z \widetilde{t}, \end{aligned}$$

and if v is even then, by Lemma 3.6(i) and (ii),

$$\begin{aligned} s &= s' a_i a_j a_k z = a_i a_j a_k z s', \\ t &= \widetilde{t} a_i a_j a_k z = a_i a_j a_k z \widetilde{t}. \end{aligned}$$

Therefore, if v is odd, since $z^{2l} a_i a_k a_j z s' = z^{2l} a_i a_k a_j z \widetilde{t}$, from Lemma 3.6 it follows that $a_i a_k a_j z s' = a_i a_k a_j z \widetilde{t}$, i.e. $s = t$. Similarly, if v is even we also get $s = t$. Hence $K[T] \cap I(\rho) = 0$.

Note that, since T is a cancellative ideal in M, we have that $G = T \langle z^2 \rangle^{-1}$. Furthermore, $\mathcal{J}(K[M]) \subseteq K[T]$ implies that $\mathcal{J}(K[M]) = \mathcal{J}(K[T])$. Therefore by [15, Corollary 11.5],

$$\mathcal{J}\big(K[M]\big) = \mathcal{B}\big(K[M]\big) = \mathcal{B}\big(K[T]\big) = \mathcal{J}\big(K[G]\big) \cap K[T].$$

If K has characteristic $\neq 2$, then we know (from part (ii) of Theorem 1.1) that $\mathcal{J}(K[G]) = 0$. Thus, in this case, it follows that $\mathcal{J}(K[M]) = 0$.

Suppose that K has characteristic 2. Let $\pi : T \to G/C$ be the composition of the natural maps

$$T \hookrightarrow G \to G/C.$$

Let η be the congruence defined on M by $s \eta t$ if and only if either $s = t$ or $s, t \in T$ and $\pi(s) = \pi(t)$. Since $\mathcal{B}(K[G]) = \omega(K[C])K[G]$, it follows that $\mathcal{J}(K[M]) = I(\eta)$. Note that $a_1 a_2 a_3 z - a_2 a_1 a_3 z \in K[T] \cap \mathcal{B}(K[G])$ and it is nonzero (because $a_1 a_2 \neq a_2 a_1$ in G).

Let P be the ideal of $K[M]$ generated by all elements of the form $a_j a_i a_k z - a_i a_j a_k z$, with different i, j, k. Let $s, t \in T$ be such that $s \eta t$. We shall prove that $s - t \in P$, and therefore $\mathcal{J}(K[M]) = I(\eta) = P$ is finitely generated.

By the definition of η it is clear that s and t have the same degrees with respect to every a_l. Thus, as in the proof of the cancellativity of T, one can see that there exist three different integers i, j, k and $w, w' \in M$ such that $s = a_i a_j a_k z w$ and $t = a_{i'} a_{j'} a_{k'} z w'$, where $\{i, j, k\} = \{i', j', k'\}$. Note that in $K[M]$, it follows by Lemma 3.1 that

$$\begin{aligned} a_i a_j a_k z a_i - a_i^2 a_j a_k z &= a_i a_j a_k z a_i - a_i a_j a_k a_i z \\ &= a_i a_j a_k z a_i - z a_i a_j a_k a_i \\ &= a_i a_j a_k z a_i - a_j a_i a_k z a_i \in P. \end{aligned}$$

Therefore by Lemma 3.1 and Lemma 3.6(i), $a_i a_j a_k z$ is central in $K[M]/P$.

Let $\deg_r(w) = m_r$. We shall prove that $s - a_i a_j a_k z a_1^{m_1} \cdots a_n^{m_n} \in P$. Since $a_i a_j a_k z$ is central in $K[M]/P$, it is sufficient to prove that $a_i a_j a_k z a_p a_q - a_i a_j a_k z a_q a_p \in P$, for all p, q. Clearly, we may assume that $p \neq q$. Then by Lemma 3.1, we may assume that $k \notin \{p, q\}$, and again by Lemma 3.1,

$$a_i a_j a_k z a_p a_q - a_i a_j a_k z a_q a_p = a_i a_j (a_k a_p a_q z - a_k a_q a_p z) \in P,$$

as desired.

Similarly we obtain that $t - a_{i'} a_{j'} a_{k'} z a_1^{m_1} \cdots a_n^{m_n} \in P$, because $\deg_r(w') = m_r$. Thus, since $a_i a_j a_k z - a_{i'} a_{j'} a_{k'} z \in P$, it follows that

$$s - t = s - a_i a_j a_k z a_1^{m_1} \cdots a_n^{m_n} - \left(t - a_{i'} a_{j'} a_{k'} z a_1^{m_1} \cdots a_n^{m_n}\right) + (a_i a_j a_k z - a_{i'} a_{j'} a_{k'} z) a_1^{m_1} \cdots a_n^{m_n} \in P.$$

Therefore assertion (v) follows. □

This finishes the proof of Theorem 1.1.

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